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SINGULAR PERTURBATIONS OF DIRICHLET AND NEUMANN DOMAINS AND RESONANCES FOR OBSTACLE SCATTERING

Peter D. Hislop [1]

1. Introduction

Some of the work reported in this article is joint with R.M. Brown, University of Kentucky, and A. Martinez, Université de Paris XIII. We want to describe some recent results concerning the existence and estimation of the poles of the S-matrix for the scattering of waves by a single, compact obstacle. The details of the calculations appear in [6], [12], [11]. We are interested in the scattering poles for a class of obstacles known as Helmholtz resonators. These obstacles are characterized by a large cavity $\mathcal{C}$ which is coupled to the (unbounded) exterior $\mathcal{E}$ by means of a tube $T(\varepsilon)$ of diameter ε. The waves propagate in $\Omega(\varepsilon) \equiv Int(\overline{\mathcal{C} \cup T(\varepsilon) \cup \mathcal{E}})$ and we consider either Dirichlet or Neumann boundary conditions (DBC or NBC) on the boundary of $\Omega(\varepsilon), \partial\Omega(\varepsilon)$. We consider two classes of problems : (1) local in energy : for a fixed compact subset $K \subset \mathbf{C}$, intersecting the real axis $\mathbf{R}$, describe and estimate the position of all scattering poles in K for all ε sufficiently small; (2) global in energy : for a fixed ε (say $\varepsilon = 1$), consider the high energy behavior of the scattering poles and show that there exists a sequence of poles converging to the real axis.

The problem of a local characterization of scattering poles for a Helmholtz resonator has been considered by Beale [4] and Arsen'ev [3]. For the case of DBC, the poles arise from either eigenvalues of the cavity Laplacian $-\Delta_{\mathcal{C}}$ with DBC or resonances of the exterior Laplacian $-\Delta_{\mathcal{E}}$ with DBC. In particular for K as above, they prove that there exists $\varepsilon_K > 0$ such that for all $\varepsilon < \varepsilon_K$, there exists a bijection between the scattering poles in K and the set consisting of the eigenvalues of $-\Delta_{\mathcal{C}}$ in K and the resonances of $-\Delta_{\mathcal{E}}$ in K (including multiplicities).

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When there are NBC and $T(\varepsilon)$ is a straight tube $D_\varepsilon \times [0,1], D_\varepsilon = \varepsilon D_1$ and $D_1 = \{x' \epsilon \mathbf{R}^{n-1} \mid |x'| \leq 1\}$, there is an addition set of poles coming from the longitudinal modes of the tube. We reprove these results and give precise upper bounds on the displacement of the poles from the cavity eigenvalues or exterior resonances as a function of ε. For the case of DBC these are exponentially small in ε. For the NBC case, the upper bound is $\mathcal{O}(\varepsilon^\beta)$ where $\beta = 1/2$ for dimension $n \geq 4$ and $0 < \beta < 1/2$ for $n = 3$.

In order to derive these results, we also study the effect of adding a small tube $T(\varepsilon)$ to the cavity $\mathcal{C}$ on the eigenvalues of $-\Delta_{\mathcal{C}}$. We consider both DBC and NBC. In the DBC case, we find that the shift of the eigenvalues is bounded above by $\mathcal{O}(\varepsilon^\beta)$ where $\beta = 1/2$ for $n \geq 3$ and $0 < \beta < 1/2$ for $n = 2$.

In the NBC case, we must restrict ourselves to a straight tube. We find a similar estimate for the shift of the eigenvalues. We mention that singular perturbations of NBC have been recently discussed by several authors, for example [2], [10], [16].

The second type of problem is related to a conjecture of Lax and Phillips [18] concerning the behavior of scattering poles in the case that the obstacle has trapped rays. They conjectured that if an obstacle, like a Helmholtz resonator, has trapped rays, then there is a sequence of scattering poles converging to the real axis as the energy diverges to infinity. Although this conjecture is false, as shown by Ikawa [13] for the case of two bounded, convex obstacles with a single trapped hyperbolic ray, we show that it holds for a class of symmetric Helmholtz resonators (see section 4). In the case studied by Ikawa and, later, by Gérard [9], there is an infinite number of scattering poles but they are bounded a fixed distance from the real axis. This may be a manifestation of the instability of the trapped ray in this example. Indeed, Ikawa [14] later showed that if the obstacles are sufficiently flat in the neighborhood of the trapped ray, there is a sequence of poles converging to the real axis. A similar situation of stability occurs in an example studied by Ralston [20]. He examined the poles for scattering in spherically symmetric inhomogeneous media for which there is an infinite family of stable, trapped rays. Again in this case, there is a sequence of poles converging exponentially fast to the real axis. This model can also be treated by the methods of section 4.

The outline of this paper is as follows. In sections 2 and 3 we discuss the local in energy problem for the Helmholtz resonator. Section 2 is devoted to the DBC case and section 3 to the NBC case. In section 4 we turn to the global in energy problem and sketch the proof of the Lax-Phillips conjecture on the existence of a sequence of scattering poles converging to the real axis for a family of symmetric Helmholtz resonators.

Finally, we mention that a scattering pole is also a pole of the meromorphic continuation of matrix elements of the resolvent of $-\Delta_{\Omega(\varepsilon)}$ for vectors in a certain dense set. Hence they are resonance of the operator $-\Delta_{\Omega(\varepsilon)}$ on $L^2(\Omega(\varepsilon))$. We will freely use the results of the theory of quantum resonances and spectral deformation below. In particular, we will assume the application of spectral deformation techniques as discussed in [12].

2. Perturbation of Dirichlet Domains and Resonances

The first situation for which we will consider the local resonance structure is the Helmholtz resonator with DBC. This material has already been published so we will be brief and simply review the results. The notation and general ideas, however, will be used in the other sections. To be more specific about the geometry, let $\widetilde{\Omega} \subset \mathbf{R}^n$ be an open set with C^2-boundary admitting a decomposition into two disjoint components $\mathcal{C}$, the cavity, and $\mathcal{E}$, the exterior, such that $\mathcal{C} \subset \mathbf{R}^n \backslash \mathcal{E}$ and $\mathcal{C}$ is bounded. Let $x_0 \epsilon \partial\mathcal{C}$ and $x_1 \epsilon \partial\mathcal{E}$. We join these two points by a tube $T(\varepsilon)$ which is an open subset of $\mathbf{R}^n \backslash \widetilde{\Omega}$ diffeomorphic to the standard tube $D_\varepsilon \times [0,1]$ where $D_\varepsilon = \varepsilon D_1$ and $D_1 \equiv \{x' \epsilon \mathbf{R}^{n-1} \mid |x'| \leq 1\}$. As in the introduction, we set $\Omega(\varepsilon) \equiv Int(\overline{\mathcal{C} \cup T(\varepsilon) \cup \mathcal{E}})$ and consider the Laplacian $H_\varepsilon = -\Delta$ on $\Omega(\varepsilon)$ with DBC on $\partial\Omega(\varepsilon)$. Our main result is to characterize the resonances of H_ε in a compact complex set K intersecting $\mathbf{R}$ for all ε sufficiently small.

To this end, we need a preliminary estimate of some interest in itself. Consider the cavity $\mathcal{C}$ and the cavity with the tube $T(\varepsilon)$ attached : $\mathcal{C}(\varepsilon) \equiv Int(\overline{\mathcal{C} \cup T(\varepsilon)})$, both with DBC. We want to know by how much the eigenvalues of the Dirichlet Laplacian $-\Delta_{\mathcal{C}}$ shift when the tube is adjoined to the cavity. By the Poincaré inequality for $-\Delta_{T(\varepsilon)}$, one expects that the effect is small.

PROPOSITION **2.1.** *Let $\lambda_0 \epsilon\ \sigma(-\Delta_{\mathcal{C}})$ with multiplicity N_0. Then there exists $\varepsilon_0 > 0, c > 0$ such that for all $\varepsilon < \varepsilon_0, -\Delta_{\mathcal{C}(\varepsilon)}$ has N_0 eigenvalues (counting multiplicity) $\lambda_1(\varepsilon), \ldots, \lambda_{N_0}(\varepsilon)$, satisfying for all $j = 1, \ldots, N_0$:*

$$|\lambda_0 - \lambda_j(\varepsilon)| \leq c\varepsilon^\beta$$

where $\beta = 1/2$ for $n \geq 3$ and $0 < \beta < 1/2$ for $n = 2$.

The proof of this theorem begins with Green's formula expressing the difference of the two Laplacians, $-\Delta_{\mathcal{C}} \oplus -\Delta_{T(\varepsilon)}$ and $-\Delta_{\mathcal{C}(\varepsilon)}$, in terms of normal derivatives and surface integrals. These integrals are then estimated using Sobolev embedding and trace theorems.

The basis for the existence of resonances in K is the fact that a narrow tube with Dirichlet boundary conditions cannot support states with energy in K if ε is sufficiently small. Consequently, the coupling between the cavity and the exterior is very weak. This weak coupling, however, is sufficient to change the bound states of $-\Delta_{\mathcal{C}}$ to resonances of H_ε and to shift the resonances of $-\Delta_{\mathcal{E}}$ a small amount to become resonances of H_ε. We note that $\sigma(H_\varepsilon) = [0,\infty)$ and is absolutely continuous whereas the spectrum of the operator obtained when $\varepsilon = 0$, a direct sum, has eigenvalues embedded in the continuous spectrum.

As described in the introduction, the poles of the S-matrix are characterized also as the complex eigenvalues of the spectrally deformed Hamiltonian. We denote by $H_\varepsilon(\mu), H_\varepsilon^{ext}(\mu)$ and $H^D_{ext,\varepsilon}(\mu)$ the spectrally deformed operators obtained from $H_\varepsilon, -\Delta_{\mathcal{E}(\varepsilon)}$ and $-\Delta_{T(\varepsilon)} \oplus -\Delta_{\mathcal{E}}$, respectively, where $\mathcal{E}(\varepsilon) \equiv Int(\overline{\mathcal{E} \cup T(\varepsilon)})$. There is a result for the shift of the resonances of $-\Delta_{\mathcal{E}}$ by the addition of $T(\varepsilon)$, which is the analog of Proposition 2.1.

PROPOSITION **2.2.** *Let λ_0 be a resonance of $-\Delta_{\mathcal{E}}$ for some $\mu \epsilon\ i]0,1[$ of (algebraic) multiplicity N_0. Then there exists $\varepsilon_0 > 0, c > 0$ such that for all $\varepsilon < \varepsilon_0, H_\varepsilon^{ext}(\mu)$ has N_0 eigenvalues $\lambda_1(\varepsilon), \ldots, \lambda_{N_0}(\varepsilon)$ satisfying for all $j = 1, \ldots, N_0$:*

$$|\lambda_0 - \lambda_j(\varepsilon)| \le c\varepsilon^\beta$$

where $\beta = 1/2$ for $n \ge 3$ and $0 < \beta < 1/2$ for $n = 2$.

To prove that H_ε has resonances in some fixed $K \subset \mathbf{C}$, for all small ε, and that these resonances are precisely, those coming from the eigenvalues of $-\Delta_{\mathcal{C}}$ in K and the resonances of $-\Delta_{\mathcal{E}}$ in K, we show that for z in a neighborhood of any of these latter points, the difference of the resolvents of $H_{\varepsilon,\mu}$ and of $-\Delta_{\mathcal{C}(\varepsilon)} \oplus H_\varepsilon^{ext}(\mu)$, $\mu \epsilon\ i]0,1[$ vanishes as $\varepsilon \to 0$. Note that $\mathcal{C}(\varepsilon) \cap \mathcal{E}(\varepsilon) = T(\varepsilon)$ and it is in this region where states of energy in K are, in fact, exponentially small (see below). To quantify this idea, we use geometric perturbation theory. Let (J_1, J_2) be a partition of unity covering $\Omega(\varepsilon)$, independent of ε, such that $\text{supp}|\nabla J_i|$ is well inside the tube. Indeed, if $d(x,\Omega) \equiv$ Euclidean distance from x to Ω, then we take

$$J_1 | \{x | d(x,\mathcal{E}) \ge 2\delta\} = 1$$
$$J_2 | \{x | d(x,\mathcal{E}) \ge \delta\} = 1$$

so $\text{supp}|\nabla J_i| \subset \{x | \delta \le d(x,\mathcal{E}) \le 2\delta\}$. Set $\mathcal{H}_0 \equiv L^2(\mathcal{C}(\varepsilon)) \oplus L^2(\mathcal{E}(\varepsilon))$ and $\mathcal{H} \equiv L^2(\Omega(\varepsilon))$ and define $J : \mathcal{H} \to \mathcal{H}_0$ by

$$Ju = J_1 u \oplus J_2 u$$

so that $J^\star J = 1_{\mathcal{H}}$. Let $R(z) \equiv (H_{\varepsilon,\mu} - z)^{-1}$ and $R_0(z) = (-\Delta_{\mathcal{C}(\varepsilon)} \oplus H_\varepsilon^{ext}(\mu) - z)^{-1}$. Then for z in the intersection of the resolvent sets, we have the geometric resolvent equation on $\mathcal{H}$:

$$R(z) = J^\star R_0(z) J + R(z) M R_0(z) J$$

where $M : \mathcal{H}_0 \to \mathcal{H}$ is given by

$$M(u_1 \oplus u_2) = [-\Delta, J_1]u_1 + [-\Delta, J_2]u_2,$$

for $u_1 \epsilon H^1(\mathcal{C}(\varepsilon)), u_2 \epsilon H^1(\mathcal{E}(\varepsilon))$. We want to show that $||JMR_0(z)||$ vanishes as $\varepsilon \to 0$.

LEMMA **2.3.** *Let $\lambda_0 \epsilon \sigma(-\Delta_{\mathcal{C}})$ and let Γ_ε be a simple closed contour about λ_0 of radius $2c\varepsilon^\beta$, where β is defined in Prop.2.1. Let $\chi \epsilon C_0^\infty$ (supp$|\nabla J_i|$). Then for each $\delta > 0 \exists c_\delta > 0$ such that uniformly on Γ_ε,*

$$||\chi(-\Delta_{\mathcal{C}(\varepsilon)} - z)^{-1}|| \leq c_\delta \varepsilon^{2-\delta}$$
$$||\nabla\chi(-\Delta_{\mathcal{C}(\varepsilon)} - z)^{-1}|| \leq c_\delta \varepsilon^{1-\delta}$$

Similar estimates hold for $H_\varepsilon^{ext}(\mu)$.

Idea of the Proof. The proof is based on the inequality

$$| < \chi u, (-\Delta_{\mathcal{C}(\varepsilon)} - z)\chi u > | \geq \; ||\nabla(\chi u)||^2_{L^2(T(\varepsilon))} - |z| \; ||\chi u||^2_{L^2(T(\varepsilon))} \cdot \qquad (2.1)$$

Now, the Poincaré inequality states that for any $\phi \epsilon H_0^1(T(\varepsilon))$,

$$\int_{T(\varepsilon)} |\phi|^2 \leq c\varepsilon^2 \int_{T(\varepsilon)} |\nabla\phi|^2.$$

Applying this to the right side of (2.1), we obtain

$$| < \chi u, (-\Delta_{\mathcal{C}(\varepsilon)} - z)\chi u > | \geq (c\varepsilon^{-2} - |z|) \; ||\chi u||^2_{L^2(T(\varepsilon))} \cdot$$

Finally, we take $u = (-\Delta_{\mathcal{C}(\varepsilon)} - z)^{-1}v$ and compute the left side of (2.1). $\diamondsuit$

COROLLARY **2.4.** *Let Γ_ε be as in Lemma 2.3. Define $K(z) : \mathcal{H}_0 \to \mathcal{H}_0$ as $JMR_0(z)$. Then for any $\delta > 0 \exists c_\delta > 0, \varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$, and uniformly on Γ_ε*

$$||K(z)|| \leq c_\delta \varepsilon^{2-\delta} .$$

We use this result to solve the geometric resolvent equation for $z \epsilon \Gamma_\varepsilon$ and ε sufficiently small. This gives

$$R(z) - J^\star R_0(z)J = J^\star R_0(z)(1 - K(z))^{-1}K(z)J . \qquad (2.2)$$

We can now state our main theorem on the existence of resonances.

THEOREM **2.5.** (1) *Let $\lambda_0 \epsilon \sigma(-\Delta_{\mathcal{C}})$ with multiplicity N_0 and let $\lambda_j(\varepsilon) \epsilon \sigma$ $(-\Delta_{\mathcal{C}(\varepsilon)}), j = 1, \ldots, N_0$, be such that $|\lambda_0 - \lambda_j(\varepsilon)| \leq c\varepsilon^\beta, \beta \leq 1/2$ according to n.*

Then $\exists c > 0$ *and* $\varepsilon_0 > 0$ *such that for all* $\varepsilon < \varepsilon_0, H_\varepsilon(\mu)$ *has* N_0 *eigenvalues (counting multiplicity)* $\rho_1(\varepsilon), \ldots, \rho_{N_0}(\varepsilon)$ *and* $\forall_j, k = 1, \ldots, N_0$*:*

$$|\lambda_j(\varepsilon) - \rho_k(\varepsilon)| \leq c\varepsilon^\beta .$$

(2) *Fix* $\mu \epsilon]0,1[$ *and let* λ_0 *be a resonance of* $-\Delta_{\mathcal{E}}$ *of multiplicity* N_0*. Let* $\lambda_j(\varepsilon)\epsilon\sigma_d(H_\varepsilon^{ext}(\mu))$ *be the eigenvalues satisfying* $|\lambda_0 - \lambda_j(\varepsilon)| \leq c\varepsilon^\beta$*. Then* $\exists c > 0$ *and* $\varepsilon_0 > 0$ *such that* $\forall \varepsilon < \varepsilon_0, H_\varepsilon(\mu)$ *has* N_0 *eigenvalues* $\rho_1(\varepsilon), \ldots, \rho_{N_0}(\varepsilon)$ *such that* $\forall_j, k = 1, \ldots, N_0$,

$$|\lambda_j(\varepsilon) - \rho_k(\varepsilon)| \leq c\varepsilon^\beta .$$

As a final part of this description of the DBC case, we want to make precise the location of the resonances of H_ε. That is, we show that the shifts between $\lambda_j(\varepsilon)$ and $\rho_k(\varepsilon)$, as described in Theorem 2.5, are exponentially small in ε. The key to this is the fact that eigenfunctions of $-\Delta_{\mathcal{C}(\varepsilon)}$ and of $H_\varepsilon^{ext}(\mu)$ decay exponentially in $T(\varepsilon)$.

PROPOSITION **2.6.** *Let* $\lambda(\varepsilon)\epsilon\sigma(-\Delta_{\mathcal{C}(\varepsilon)})$ *be such that* $\lambda(\varepsilon) \to \lambda_0\epsilon\sigma(-\Delta_{\mathcal{C}})$ *as* $\varepsilon \to 0$ *and let* u_ε *be the corresponding eigenfunction with* $||u_\varepsilon|| = 1$*. Then for all* $\alpha\epsilon\mathbf{N}^n$*, for all* $\delta > 0, \exists \widetilde{c}_{\alpha,\delta} > 0$ *such that for all* ε *small enough:*

$$||e^{(1-\delta)\widetilde{d}_\varepsilon(\cdot,x_0)/\varepsilon}\partial^\alpha u_\varepsilon||_{L^2(T(\varepsilon))} \leq \widetilde{c}_{\alpha,\delta}\varepsilon^{-c_{\alpha,\delta}}$$

and $c_{0,\delta} = 0, c_{1,\delta} = 1$, *and* $\widetilde{d}_\varepsilon(x,y)$ *is the minimum distance from* x *to* y *along paths lying in* $\Omega(\varepsilon)$*. A similar estimate holds for eigenfunctions of* $H_\varepsilon^{ext}(\mu)$ *corresponding to eigenvalues* $\lambda(\varepsilon) \to \lambda_0$, *a resonance of* $-\Delta_{\mathcal{E}}$*.*

THEOREM **2.7.** *Let* $\lambda_j(\varepsilon)$ *and* $\rho_k(\varepsilon)$ *be as in Theorem 2.5. For each* $j, j = 1, \ldots, N_0, \exists$ *a permutation* k *of* $\{1, \ldots, N_0\}$ *such that*

$$|\rho_{k(j)}(\varepsilon) - \lambda_j(\varepsilon)| \leq c\exp[-2(1-\delta)S(\varepsilon,\delta)/\varepsilon]$$

where $S(\varepsilon,\delta) \equiv \max\{\widetilde{d}_\varepsilon(x,y)|\ x,y\epsilon\overline{T(\varepsilon)}, d(x,\mathcal{E}\cup\mathcal{C}) \geq \delta, d(y,\mathcal{E}\cup\mathcal{C}) \geq \delta\}$ *and* $\widetilde{d}_\varepsilon$ *is defined in Prop.2.6.*

The proof of this theorem follows from the construction of an approximate basis for the invariant subspace of H_ε corresponding to $\{\rho_j(\varepsilon)\}$ using the eigenfunctions of $-\Delta_{\mathcal{C}(\varepsilon)}$ or $H_\varepsilon^{ext}(\mu)$. In this basis, H_ε is diagonal up to exponentially small terms.

3. Perturbation of Neumann Domains and Resonances

In this section, we first consider the effect of adding a small tube $T(\varepsilon)$ to the cavity $\mathcal{C}$ when NBC are imposed on the boundary. We then consider the determination and estimation of resonances for Helmholtz resonators in the NBC case. The geometry is as in the DBC case with one additional requirement. The tube $T(\varepsilon)$ must be straight, i.e. $T(\varepsilon) \subset D_\varepsilon \times \mathbf{R}$. We fix coordinates $(x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$ so that $(0,0) \in \partial\mathcal{C} \cap T(\varepsilon)$ and $(0,1) \in \partial\mathcal{E} \cap T(\varepsilon)$. As above, we let $\mathcal{C}(\varepsilon) \equiv \mathcal{C} \cup T(\varepsilon)$ and $\mathcal{E}(\varepsilon) \equiv \mathcal{E} \cup T(\varepsilon)$. We must make an assumption on the smoothness of $\partial\mathcal{C}(\varepsilon)$ and $\partial\mathcal{E}(\varepsilon)$.

Boundary regularity assumption: $\partial\mathcal{C}(\varepsilon)$ and $\partial\mathcal{E}(\varepsilon)$ are in $C^{0,1}(\mathbf{R}^{n-1})$, i.e. they are both locally the graph of a Lipschitz continuous function.

We note that this implies that the tube $T(\varepsilon)$ is bounded by Lipschitz surfaces. Moreover, both $-\Delta_{\mathcal{C}}$ and $-\Delta_{\mathcal{C}(\varepsilon)}$ have compact resolvents. We recall [15] that if $\mathcal{C}$ is a region such that $\partial\mathcal{C} \epsilon C^{0,1}(\mathbf{R}^{n-1})$, then the Neumann resolvent $R_N(z)$ followed by restriction to the boundary maps $L^2(\mathcal{C})$ to $H^1(\partial\mathcal{C})$.

We denote by D_ε^0 and D_ε^1 subsets of $T(\varepsilon)$ given by $\partial\mathcal{C} \cap T(\varepsilon)$ and $\partial\mathcal{E} \cap T(\varepsilon)$, respectively. We will consider the eigenvalues of $\mathcal{C}(\varepsilon)$ as $\varepsilon \to 0$ in two cases : (1) NBC everywhere on $\partial\mathcal{C}(\varepsilon)$, and (2) NBC on $\partial\mathcal{C}(\varepsilon)\backslash D_\varepsilon^1$ and DBC on D_ε^1. Case 2 will be important for the study of resonances given in the next section. As in the DBC case, the eigenvalues of $\mathcal{C}(\varepsilon)$ should be well (locally) approximated by eigenvalues of $\mathcal{C}$ with NBC and of $T(\varepsilon)$, where $T(\varepsilon)$ has NBC on $T(\varepsilon)\backslash(D_\varepsilon^0 \cup D_\varepsilon^1)$, DBC on D_ε^0, and either NBC or DBC on D_ε^1, for case 1 or 2, respectively. We introduce the following operators indexed by $i = 1, 2$ depending on NBC or DBC on D_ε^1.

Cavity $-\Delta_{\mathcal{C}} \geq 0$ Cavity Laplacian with NBC on $\partial\mathcal{C}$
$$R_{\mathcal{C}}(z) \equiv (-\Delta_{\mathcal{C}} - z)^{-1}$$

Tube $-\Delta_T^i \geq 0, i = 1, 2$. Tube Laplacian
$$R_T^i(z) \equiv (-\Delta_T - z)^{-1}$$

Unperturbed $\Delta_0^i \equiv \Delta_{\mathcal{C}} \oplus \Delta_T^i,\ i = 1, 2$
$$R_0^i(z) \equiv (-\Delta_0^i - z)^{-1} = (-\Delta_{\mathcal{C}} - z)^{-1} \oplus (-\Delta_T^i - z)^{-1}$$

Coupled $-\Delta_N^i \geq 0$ Laplacian for $\mathcal{C}(\varepsilon)$ in case 1 (NBC on D_ε^1) or case 2 (DBC on D_ε^1)

$$R_N^i(z) \equiv (-\Delta_N^i - z)^{-1}.$$

We will omit the index i when the results hold in both cases. We note that $\sigma(-\Delta_0^i) = \sigma(-\Delta_C) \cup \sigma(-\Delta_T^i)$ and that $\sigma(-\Delta_C)$ is independent of ε and $\lambda_1 = 0$. Unlike the Dirichlet case, we can write $\sigma(-\Delta_T^i) = \sigma_L^i \cup \sigma_T^i$, where σ_L^i, the longitudinal modes, consists of those eigenvalues which differ by $\mathcal{O}(\varepsilon^{1/2})$ from the eigenvalues of the boundary value problem on $[0,1]$:

$$\begin{cases} -u'' = \lambda u \text{ on } [0,1] \\ u(0) = 0 \\ u(1) = 0 \text{ case 1 or } u'(1) = 0 \text{ case 2 .} \end{cases}$$

In case 1, $\sigma_L^1 = \{(n\pi)^2 | n\epsilon \mathbf{Z}^+\}$ and in case 2, $\sigma_L^2 = \{((2n+1)\pi/2)^2 | n\epsilon\mathbf{Z}\}$ (up to $\mathcal{O}(\varepsilon^{1/2})$), which are both independent of ε and each eigenvalue has multiplicity one. Hence we expect these eigenvalues to contribute to $\sigma(-\Delta_N^i)$. The other set of tube eigenvalues, σ_T^i, consists of transverse mode eigenvalues and satisfy a Poincaré-type inequality reminiscent of the Dirichlet case: $\lambda \geq c_0\varepsilon^{-2}$. Accordingly, these do not contribute to the local spectrum of $-\Delta_N^i$ in any compact subset of $\mathbf{R}^+$ for ε sufficiently small.

The case $n = 2$ requires some special treatment so we omit it here (see [6]).

THEOREM **3.1.** *Let $n \geq 3$ and $\mathcal{C}(\varepsilon) = \mathcal{C} \cup T(\varepsilon)$ as described above.*

1) *Let $\lambda_0 \epsilon \sigma(-\Delta_C)$, $\lambda_0 \notin \sigma_L^i$, and let N_0 be the multiplicity of λ_0. Then $-\Delta_N^i$ has N_0 eigenvalues (counting multiplicity) $\lambda_k(\varepsilon) \to \lambda_0$ as $\varepsilon \to 0$ such that for all ε sufficiently small $\exists c_0 > 0$ s.t.*

$$|\lambda_k(\varepsilon) - \lambda_0| \leq c_0 \varepsilon^\alpha$$

where $\alpha = 1/2$ for $n \geq 4$ and $0 < \alpha < 1/2$ for $n = 3$.

2) *$\lambda_0 \epsilon \sigma_L^i, \lambda_0 \notin \sigma(-\Delta_C)$. Then $-\Delta_N^i$ has exactly one eigenvalue $\lambda_0(\varepsilon) \to \lambda_0$ as $\varepsilon \to 0$ and the bound in (1) is satisfied. If it happens that $\lambda_0 \epsilon \sigma_L^i \cap \sigma(-\Delta_C)$ then $-\Delta_N^i$ has $N_0 + 1$ eigenvalues $\lambda_k(\varepsilon) \to \lambda_0$ as $\varepsilon \to 0$ (where N_0 is the multiplicity of λ_0 in $\sigma(-\Delta_C)$) and satisfying the bound in (1).*

We remark that the convergence of Neumann eigenvalues under singular perturbations of the domain has been considered by several researchers. For $n = 2$, Hempel, Seco and Simon [10] show norm resolvent convergence of $R^i(z)$ to $R_0^i(z)$ (and for unperturbed resolvents corresponding to more than one tube) as $\varepsilon \to 0$ but they do not give any estimate on the rate of convergence. Jimbo [16] considers a similar problem in $\mathbf{R}^n$ and gives pointwise asymptotics on the eigenfunctions as $\varepsilon \to 0$. Arrieta, Hale and Han [2] consider more singular perturbations of Neumann domains for which the attached region is shrinking at different rates in different directions.

Theorem 3.1 follows from the main technical lemma which estimates the convergence of resolvents.

LEMMA **3.2.** *Under the hypotheses of Theorem 3.1, for any* $z \epsilon \rho(-\Delta_N^i) \cup \rho(-\Delta_0^i)$, *we have*

$$||R^i(z) - R_0^i(z)|| \leq c_z \varepsilon^\alpha (1 + ||R^i(z)||)(1 + ||R_0^i(z)||)$$

where $c_z \equiv c_0(1+|z|)^{3/2}, c_0$ *dependents on the smoothness of* $\partial\mathcal{C}$, *and* $\alpha = 1/2,\ n \geq 4$, *and* $\alpha < 1/2,\ n = 3$.

We sketch the proof of this lemma. As in the Dirichlet case, it is based on Green's formula.

Proof of Lemma 3.2.

The basic formula is the following. Let $w = w_\mathcal{C} \oplus w_T \epsilon D(-\Delta_0)$ and let $u \epsilon D(-\Delta_N)$ (we drop the index i). Then

$$D \equiv \int_{\mathcal{C}(\varepsilon)} w\Delta_N u - u\Delta_0 w = \sum_{X=\mathcal{C},T} \int_{\partial X} \left(u \frac{\partial w_X}{\partial \nu} - w_X \frac{\partial u}{\partial \nu} \right)$$

where $\partial/\partial\nu$ denotes the outward normal derivative from $X = T$ or $\mathcal{C}$. Applying the various BC, we obtain :

$$D = -\int_{D_\varepsilon^0} w_\mathcal{C} \frac{\partial u}{\partial \nu} + \int_{D_\varepsilon^0} u \frac{\partial w_T}{\partial \nu} \tag{3.1}$$

in both cases. If we recall that $w = R_0(z)g, g\epsilon L^2(\mathcal{C}) \oplus L^2(T(\varepsilon))$ and $u = R(z)f, f\epsilon L^2(\mathcal{C}(\varepsilon))$, then estimates on the integrals in D give directly an upper bound on $R(z) - R_0(z)$. To estimate the integral involving w_T we will use the following two facts.

1) $||u||_{L^2(D_\varepsilon^0)} \leq c_0 \varepsilon^\alpha (1 + ||R(z)||)||f||_{L^2(\mathcal{C}(\varepsilon))}$,
where $u = R(z)f$ and α is as in the lemma.

2) $||\frac{\partial w_T}{\partial \nu}||_{L^2(D_\varepsilon^0)} \leq c_z(1 + ||R_T(z)||)||g||_{L^2(T(\varepsilon))}$,
where $w_T = R_T(z)g$.

It is then clear that we obtain an estimate of the type on the right side of (3.1) for the w_T term in D. The estimate on the $w_\mathcal{C}$ term is more involved. Here we use $(H^{1/2}(\partial\mathcal{C}),\ H^{-1/2}(\partial\mathcal{C}))$ duality. Let η be a smooth cut-off function such that $|\nabla\eta| \leq c_0\varepsilon^{-1}$ and $\chi_{D_\varepsilon^0} \leq \eta \leq \chi_{D_{2\varepsilon}^0}$, where $\chi_{D_\varepsilon^0}$ is the characteristic function on D_ε^0. We show that

3) $\|\eta w_{\mathcal{C}}\|_{H^{1/2}(\partial\mathcal{C})} \leq c_z \varepsilon^{\alpha}(1 + \|R_{\mathcal{C}}(z)\|)\|g\|_{L^2(\mathcal{C})}$,
   where $w_{\mathcal{C}} = R_{\mathcal{C}}(z)g,\ g \epsilon L^2(\mathcal{C})$.

4) $\left\|\frac{\partial u}{\partial \nu}\right\|_{H^{-1/2}(\partial\mathcal{C})} \leq c(1 + \|R(z)\|)\|f\|_{L^2(\mathcal{C}(\varepsilon))}$,
   where $u = R(z)f,\ f \epsilon L^2(\mathcal{C}(\varepsilon))$.

These two estimates allow us to establish (note that $\partial u/\partial x_n = 0$ on $\partial\mathcal{C}\backslash D^0_\varepsilon$):

$$\left|\int_{D^0_\varepsilon} w_{\mathcal{C}} \frac{\partial u}{\partial \nu}\right| \leq \int_{\partial\mathcal{C}} \left|\eta w_{\mathcal{C}} \frac{\partial u}{\partial \nu}\right| \leq \|\eta w_{\mathcal{C}}\|_{H^{1/2}(\partial\mathcal{C})} \left\|\frac{\partial u}{\partial \nu}\right\|_{H^{-1/2}(\partial\mathcal{C})}$$
$$\leq c_0 \varepsilon^{\alpha}(1 + \|R_{\mathcal{C}}(z)\|)(1 + \|R(z)\|)\|g\|_{L^2(\mathcal{C})}\|f\|_{L^2(\mathcal{C}(\varepsilon))}$$

This result, plus the similar estimate for the integral involving w_T, proves the lemma. It remains to prove (1)-(4). We sketch their proof below. $\diamondsuit$

The proofs of statements (1), (3) and (4) relay on various, but standard, trace estimates and Sobolev embedding theorems. The Lipschitz condition on $\partial\mathcal{C}$ insures that Sobolev embedding theorems hold in our case. For example, to prove (1), note that $u|\partial\mathcal{C}\epsilon H^1(\partial\mathcal{C})$ and, consequently, $u|\partial\mathcal{C}\epsilon H^{1/2}(\partial\mathcal{C})$ by a natural embedding. A Sobolev embedding theorem states in this case $H^{1/2}(\partial\mathcal{C}) \hookrightarrow L^q(\partial\mathcal{C})$, where $q \equiv 2(n-1)(n-2)^{-1},\ n \geq 3$. By this and the Hölder inequality, we obtain

$$\int_{D^0_\varepsilon} u^2 \leq c\varepsilon \left[\int_{\partial\mathcal{C}} u^q\right]^{2/q} \leq c\varepsilon \|u\|^2_{H^{1/2}(\partial\mathcal{C})}$$
$$\leq c\varepsilon \|u\|^2_{H^{3/2}(\mathcal{C})} \leq c\varepsilon(1 + \|R(z)\|)^2 \|f\|^2_{L^2(\mathcal{C}(\varepsilon))} \cdot$$

The proof of (3) follows a similar line. We obtain the estimate

$$\|\eta w\|_{H^s(\partial\mathcal{C})} \leq c\varepsilon^{\alpha'-s}\|w\|_{H^1(\partial\mathcal{C})}$$

for any $0 \leq s \leq 1, w\epsilon H^1(\partial\mathcal{C})$ and $\alpha' = 1, n \geq 4, \alpha' = 1/2$ for $n = 3$. Combining this with trace estimates gives (3). The proof of (4) follows from an application of the divergence theorem to the integral

$$\int_{\partial\mathcal{C}} \phi.\frac{\partial u}{\partial \nu}$$

as a sum of integrals over $\mathcal{C}$. Here $\phi\epsilon H^{1/2}(\partial\mathcal{C})$ has an extension $\widetilde{\phi}\epsilon H^1(\mathcal{C})$.

Finally, we consider (2). Let α be a C^1 vector field in a neighborhood of $T(\varepsilon)$ such that $0 < \delta < \alpha.\nu < 1$ on D^0_ε, $\alpha.\nu = 0$ on $\partial T(\varepsilon)\backslash D^0_\varepsilon$ and $\alpha = 0$ on D^0_ε. Here ν is the normal vector field. Such a vector field can be constructed by cutting-off the vector field in the x_n-direction. We easily verify the identity on $\partial T(\varepsilon)$:

$$\left|\frac{\partial w_T}{\partial \nu}\right|^2 (\alpha.\nu) = 2\left(\frac{\partial w_T}{\partial \alpha}\right)\left(\overline{\frac{\partial w_T}{\partial \nu}}\right) - (\alpha.\nu)|\nabla w_T|^2$$

where $\partial/\partial\alpha$ is the directional derivative for α. Finally, we write

$$\begin{aligned}\delta\int_{D_\varepsilon^0}\left|\frac{\partial w_T}{\partial\nu}\right|^2 &\leq \int_{D_\varepsilon^0}\left|\frac{\partial w_T}{\partial\nu}\right|^2(\alpha.\nu)\\ &\leq Re\int_{\partial T(\varepsilon)}\left[2\left(\frac{\partial w_T}{\partial\nu}\right)\left(\overline{\frac{\partial w_T}{\partial\alpha}}\right)-(\alpha.\nu)|\nabla w_T|^2\right]\\ &= Re\int_{T(\varepsilon)}\nabla.\left[2\nabla w_T\left(\overline{\frac{\partial w_T}{\partial\alpha}}\right)-\alpha|\nabla w_T|^2\right]\\ &\leq Re\int_{T(\varepsilon)}\left[-2(g+zw_T)\frac{\partial\overline{w}_T}{\partial\alpha}+c_0|\nabla w_T|^2\right]\quad .\end{aligned}$$

Here $w_T = R_T(z)g, g\epsilon L^2(T(\varepsilon))$. The terms involving ∇w_T can be estimated by $(1+\|R_T(z)\|)\|g\|_{L^2(T)}$. Noting that α and its derivatives are independent of ε, this proves the result. (Actually, the integration by parts requires a certain regularity of ∇w_T. See [6] for the details).$\diamondsuit$

We now turn to the existence of resonances in the case of NBC. This case is different from the Dirichlet case since, as we will show below, none of the eigenfunctions of $-\Delta_0^i$ decay exponentially in $T(\varepsilon)$ and there are the longitudinal modes, σ_L^i, of the tube which should become resonances. Because we do not have eigenfunction decay in $T(\varepsilon)$, we do not expect that the localized resolvents satisfy estimates like those in Lemma 2.3. However, with the use of suitable projection operators, one can prove the existence of resonances for $-\Delta_{\Omega(\varepsilon)}$ with NBC near the eigenvalues of $-\Delta_C$ and near the resonances of the exterior Laplacian $-\Delta_\varepsilon$. The proof of this follows as in section 2.

We wish to consider the cavity eigenvalues and the longitudinal modes of $T(\varepsilon)$, with DBC on D_ε^0 and D_ε^1, on an equal footing. Consequently, we take as an approximate Laplacian $H_{0,\varepsilon}\equiv -\Delta_{C(\varepsilon)}\oplus-\Delta_{\mathcal{E}}$, where $-\Delta_{C(\varepsilon)}$ has NBC on $\partial\mathcal{C}(\varepsilon)\backslash D_\varepsilon^1$ and DBC on D_ε^1 and $-\Delta_{\mathcal{E}}$ is the exterior Laplacian with mixed boundary conditions : NBC on $\partial\mathcal{E}\backslash D_\varepsilon^1$ and DBC on D_ε^1. After spectral deformation, we find that $R_0(z)\equiv(H_{0,\varepsilon}(\mu)-z)^{-1}$ and $R(z)\equiv(-\Delta_{\Omega(\varepsilon),\mu}-z)^{-1}$ satisfy an estimate similar to that in Lemma 3.2.

LEMMA **3.3.** *Under the hypotheses describe above and for $\mu\epsilon i]0,1[$ fixed we have for any $z\epsilon\rho(H_{0,\varepsilon}(\mu))\cap\rho(-\Delta_{\Omega(\varepsilon),\mu})$,*

$$\|R(z)-R_0(z)\|\leq c_z\varepsilon^\alpha(1+\|R(z)\|)(1+\|R_0(z)\|)\ ,$$

where $c_z = c_0(1+|z|)^{3/2}$ and $\alpha = 1/2$ for $n\geq 4, 0<\alpha<1/2$ for $n=3$.

Sketch of the proof. We first note that Green's formula holds for $H_{0,\varepsilon}(\mu)$ and $-\Delta_{\Omega(\varepsilon),\mu}$ provided we choose the vector field for the spectral deformation to be spherically symmetric, which we can always do. For $u = u_{C(\varepsilon)}\oplus u_{\mathcal{E}} =$

$R_0(z)g, g\epsilon L^2(\Omega(\varepsilon))$, and $v = R(z)f,\ f\epsilon L^2(\Omega(\varepsilon))$, we have

$$\int_{\Omega(\varepsilon)} \left[v(H_{0,\varepsilon}(\mu)u) - (-\Delta_{\Omega(\varepsilon),\mu}v)u\right] = \int_{D_\varepsilon^1} v\left[\frac{\partial u_{C(\mathcal{E})}}{\partial\nu} - \frac{\partial u_{\mathcal{E}}}{\partial\nu}\right]$$

Since $-\Delta_{\Omega(\varepsilon),\mu}$ and $H_{0,\varepsilon}(\mu)$ are both analytic families of type A, their resolvents enjoy the same Sobolev space mapping properties as the resolvents of their undeformed counterparts. For the term involving $u_{C(\varepsilon)}$ we utilize L^2-estimates:

1) $\int_{D_\varepsilon^1}\left|\frac{\partial u_{C(\mathcal{E})}}{\partial\nu}(x',1)\right|^2 \leq c_z(1+\|R_{C(\varepsilon)}(z)\|)^2\|g\|^2$

2) $\|\nu\|_{L^2(D_\varepsilon^1)} \leq c_0\varepsilon^\alpha(1+\|R(z)\|)\|f\|$

The proof of (2) follows as above due to the mapping properties of the resolvent just commented upon. The proof of (1) follows by the same integration by parts identity used in the eigenvalue case. As for the term involving the exterior function, we again use the $(H^{-1/2}, H^{1/2})$-duality. Again, the same type of proof implies :

3) $\|\frac{\partial u_{\mathcal{E}}}{\partial\nu}\|_{H^{-1/2}(\partial\mathcal{E})} \leq c_0(1+\|R_{\mathcal{E}}(z)\|)\|g\|$

4) For η as described in the proof of Lemma 3.2,
$\|\eta\nu\|_{H^{-1/2}(\partial\mathcal{E})} \leq c_0\varepsilon^\alpha(1+\|R(z)\|)\|f\|$
where α is as in the lemma.

These estimates prove the result.$\diamondsuit$

Give Lemma 3.3, it is now easy to prove the existence of resonances for $-\Delta_{\Omega(\varepsilon)}$ near the ε-independent eigenvalues of $-\Delta_{C(\varepsilon)}$ with mixed NBC and DBC and near the resonances of $-\Delta_{\mathcal{E}}$, with mixed BC also. We omit the details. We do mention, however, that we do not expect exponentially small upper bounds on the shift of the resonances in the NBC case as in Theorem 2.7. Instead, we can prove $|\rho_k(\varepsilon) - \lambda_j(\varepsilon)| \leq c_0\varepsilon^{1/2}$ for $n \geq 3$. However, we do not have a lower bound.

As a final topic, we give an estimate on the decay of $u - \bar{u}, T(z)$ where u is an eigenfunction of $-\Delta_{C(\varepsilon)}$ and

$$\bar{u}(x_n) = \frac{1}{|D_\varepsilon|}\int_{D_\varepsilon} u(x', x_n)dx'$$

For this result, we must require that $\partial\mathcal{E}\cap T(\varepsilon) = \{(x',1) \ \mid\ \mid x' \mid\leq \varepsilon\}$, i.e. the surface is flat.

THEOREM 3.4. *Let α be the first non-zero eigenvalue of the Laplacian on D_1 with NBC. Then*

$$\|\exp\left[\frac{\alpha}{\varepsilon}\inf(x_1, 1-x_1)\right](u-\bar{u})\|_{L^2(T(\varepsilon))} = \mathcal{O}(\varepsilon^{1-n/2})\ .$$

We remark that this is proved using the following Agmon-type formula:

$$-\left\langle e^{\phi/\varepsilon}\Delta u, e^{\phi/\varepsilon}u\right\rangle = -\int \nabla(e^{2\phi/\varepsilon}(\overline{\nabla u})u) + \int \overline{u}\nabla(e^{2\phi/\varepsilon}).\nabla u + \int e^{2\phi/\varepsilon}|\nabla u|^2$$

and the Poincaré inequality:

$$\|\nabla'(u-\bar{u})\|_{L^2(D_\varepsilon)} \geq (\alpha/\varepsilon)\ \|u-\bar{u}\|_{L^2(D_\varepsilon)} \ .$$

4. The Lax-Phillips Conjecture for Helmholtz Resonators

We now turn to the second class of problems mentioned in the introduction, namely, a description of resonances for a trapping obstacle at all energies. Since we are interested in "global-in-energy" results, we fix the diameter ε of the tube $T(\varepsilon)$ equal to 1 and write $T \equiv T(1), \Omega \equiv \Omega(1)$, etc. In order to obtain results, we must restrict the family of Helmholtz resonators to those in dimension $n \geq 3$ which are symmetric with respect to an axis passing through the tube T, which we call the z-axis. This symmetry allows us to use the eigenvalues $\sigma_\ell > 0$ of the square of the angular momentum operator with respect to the z-axis as a perturbation parameter in the theory. To see this, we introduce generalized cylindrical coordinates $(\rho, \widehat{\Theta}, z)$ where $\rho \equiv \left[\sum_{i=1}^{n-1} x_i^2\right]^{1/2}$ and $\widehat{\Theta}\epsilon S^{n-2}$ are any suitable coordinates on the $(n-2)$-sphere. The Laplacian admits a direct sum decomposition

$$-\Delta_\Omega = \bigoplus_\ell \left(-\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial\rho^2} - \frac{(n-2)}{\rho}\frac{\partial}{\partial\rho} + \frac{\sigma_\ell}{\rho^2}\right)$$

on the Hilbert space

$$L^2(\Omega) = \bigoplus_\ell L^2(\widehat{\Omega}, \rho^{n-2}d\rho dz)$$

where $\widehat{\Omega}$ is defined as follows :

$$\widehat{\Omega} \equiv \left\{(\rho, z)\epsilon\mathbf{R}^+ \times \mathbf{R} | \exists\widehat{\Theta}\epsilon S^{n-2}\ s.t.(\rho, \widehat{\Theta}, z)\epsilon\Omega\right\} \ .$$

In an analogous manner we define $\widehat{\mathcal{C}}, \widehat{T}$ and $\widehat{\mathcal{E}}$. After a unitary transformation, we find that $-\Delta_\Omega$ is equivalent to a direct sum of operators

$$H_\ell = -\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial\rho^2} + \frac{\sigma_\ell}{\rho^2}$$

on the spaces $L^2(\widehat{\Omega}, d\rho dz)$. We will show that for all ℓ sufficiently large, the first eigenvalue of H_ℓ on $\widehat{\Omega}$ with DBC will generate a resonance of $-\Delta_\Omega$.

In this way, we obtain a sequence of resonances $\{z_\ell\}$ of $-\Delta_\Omega$ such that $Re z_\ell \to \infty$ and $\exists c_0 > 0$ and $\alpha > 0$, independent of ℓ, such that $|Im z_\ell| \leq c_0 e^{-\alpha \ell^{-1}}$. The idea of using $\{\sigma_\ell\}$ as a perturbation parameter appears already in [8].

We notice that the classical system has trapped rays running along the interior of the cavity which correspond to the resonances. It appears as if the resonances concentrate more strongly on these trapped rays as the eigenvalue parameter ℓ (and hence the real part of the energy) increases. We mention that R. Lavine [17] has obtained similar results for a resonator formed by a sphere with a small cap removed. We also note that we do not have any results in the 2 dimensional case. The method of proof outlined here also applies to the problem of spherically symmetry media which was considered by Ralston [20].

The family of resonators we consider are constructed as in Section 2 with the tube T centered on the z-axis, which is the axis of symmetry. In particular, T contains the interval $[z_0, z_1]$ where $z_0 \epsilon \overline{\mathcal{C}}$ and $z_1 \epsilon \overline{\mathcal{E}}$. The diameter of the tube is $\rho_0 \equiv \max\{\rho | \exists z \epsilon [z_0, z_1], \widehat{\Theta} \epsilon S^{n-2} \text{ s.t. } (\rho, \widehat{\Theta}, z) \epsilon T\}$. Similarly, we define $\rho_1 \equiv max\{\rho | \exists z, \widehat{\Theta} s.t. (\rho, \widehat{\Theta}, z) \epsilon \mathcal{C}\}$. We require $\rho_1 > \rho_0$, which simply says that the tube is small relative to the cavity. We need a final condition on $\partial\mathcal{E}$ and on T near z_1. Let $D_{ext} \equiv \partial\mathcal{E} \cap \overline{T}$ $(z_1 \epsilon D_{ext})$. Define a neighborhood of T near $\partial\mathcal{E}$ by

$$\mathcal{N}(T, \varepsilon) \equiv \{x \epsilon T | z(x) \epsilon [z_1 - \varepsilon, z_1]\}$$

Exterior non-trapping hypothesis The surface $(\partial\mathcal{E} \backslash D_{ext}) \cup (\partial\mathcal{N}(T, \varepsilon))$ admits an escape function $p(x, \xi)$ for some $\varepsilon > 0$.

The notion of an escape function which we use is given in [MRS]. Whereas some non-trapping condition is necessary on $\partial\mathcal{E} \backslash D_{ext}$ to control the exterior resolvent, the condition on the end of the tube can probably be relaxed. Roughly, the condition states that the boundary of the tube must join smoothly with $\partial\mathcal{E} \backslash D_{ext}$ and that there be no trapped rays in the end of the tube. Given these geometric considerations, we can state the main theorem.

THEOREM 4.1. *Let $\Omega = Int(\overline{\mathcal{C} \cup T \cup \mathcal{E}})$ be a symmetric resonator in $\mathbf{R}^n, n \geq 3$, defined above, satisfying the exterior non-trapping hypothesis. Let $-\Delta_\Omega$ be the Dirichlet Laplacian on Ω. Then $-\Delta_\Omega$ has a sequence of resonances $\{z_\ell\}$ satisfying (1) $0 < Re z_\ell \to \infty$ as $\ell \to \infty$, and (2) $Im z_\ell < 0$ and $\exists c_0 > 0, \alpha > 0$ such that for all ℓ sufficiently large, $|Im z_\ell| \leq c_0 e^{-\alpha \ell}$.*

Sketch of the proof of Theorem 4.1

As above, we will study $-\Delta_{\Omega,\mu} \equiv -U_\mu \Delta_\Omega U_\mu^{-1} \equiv \bigoplus_\ell H_{\ell,\mu}, \mu \epsilon \mathbf{R}$ and it's analytic continuation. Here U_μ is an appropriate spectral deformation group which vanishes inside a ball of radius R (R large enough so that $\widetilde{\Omega} \subset\subset$

$B_R(0))$ and implements the dilations in $\mathbf{R}^n \backslash B_{2R}(0)$. To define the approximate Hamiltonian, we consider two overlapping subsets of Ω. We define $\widetilde{\mathcal{C}} \equiv Int\,[\mathcal{C} \cup \{x\epsilon T | z(x)\epsilon[z_0, z_1 - \varepsilon/2]\}]^{c\ell}$, where $c\ell$ denotes closure and $\varepsilon > 0$ is fixed by the exterior non-trapping hypothesis. Similarly, we define $\widetilde{\mathcal{E}} \equiv Int\,[\mathcal{E} \cup \{x\epsilon T | z(x)\epsilon[z_1 - \varepsilon, z_1]\}]^{c\ell}$. We denote by $\mathcal{O}$ the overlap region : $\mathcal{O} = \{x\epsilon T | z(x)\epsilon[z_1 - \varepsilon/2, z_1 - \varepsilon]\}\} \subset T$. We associate a Dirichlet Laplacian to each $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{E}}, -\Delta_{\widetilde{\mathcal{C}}}$ and $-\Delta_{\widetilde{\mathcal{E}}}$, respectively. Then the approximate Laplacian is $-\Delta_\Omega^0 \equiv -\Delta_{\widetilde{\mathcal{C}}} \oplus -\Delta_{\widetilde{\mathcal{E}},\mu}$, after the exterior Laplacian has been dilated. Finally, we construct a partition of unity depending only on ρ and z, $\{J_i\}_{i=1}^2, \sum_{i=1}^2 J_i = 1$, such that $supp|\nabla J_i| \subset \mathcal{O}, J_i$ is a function of z only in a neighborhood of $\mathcal{O}$ and $J_1|\widetilde{\mathcal{C}}\backslash\mathcal{O} = 1, J_2|\widetilde{\mathcal{E}}\backslash\mathcal{O} = 1$.

1. Interior estimates

Let H_ℓ^{int} denote the Dirichlet Laplacian on $\widehat{\mathcal{C}}$ and write its spectrum as $\{\lambda_n(\ell)\}_{n=1}^\infty$. Then we have $\lambda_n(\ell) \geq \sigma_\ell/\rho_1^2$, due to the effective potential. For perturbation theory uniform in ℓ, we need to know that the gap between the first two eigenvalues of $H_\ell^{int}, \delta_\ell \equiv \lambda_2(\ell) - \lambda_1(\ell)$, does not decrease as $\ell \to \infty$.

LEMMA **4.2.** *There exists ℓ_0 sufficiently large and a constant $c > 0$ such that for all $\ell > \ell_0, \delta_\ell \geq c > 0$.*

This lemma is proved with the help of a lower bound on the gap between the first two eigenvalues of a Schrödinger operator on a convex domain with DBC and with a smooth, non-negative convex potential due to Singer, Wong, Yau and Yau [21]. We find a convex region $K \subset \widehat{\mathcal{C}}$ such that $\partial K \cap \partial\widehat{\mathcal{C}} \neq \phi$. For all large ℓ, the part of K nearest $\rho = 0$ will be in the classically forbidden region for the energy $\lambda_1(\ell)$. Consequently, the eigenfunctions will be exponentially small there [1]. We then apply the Variational Principle with appropriate test functions constructed from the eigenfunctions of H_ℓ^{int} localized to K and the eigenfunctions of $H_\ell^{int}|K$ with DBC.

We also need decay estimates on $R_\ell^{int}(z) \equiv (H_\ell^{int} - z)^{-1}$ localized in the region $\mathcal{O}$. These essentially follow from the Poincaré inequality as in the proof of Lemma 2.3.

LEMMA **4.3.** *Let χ be a smooth characteristic function supported on $\mathcal{O}$. Then for any $z\epsilon\rho(H_\ell^{int})$,*

$$\|\chi R_\ell^{int}(z)\| \leq ((\ell/\rho_0)^2 - |z|)^{-1} d(z)^{-1}(1 + c_z)$$
$$\|\chi \widetilde{\nabla} R_\ell^{int}(z)\| \leq ((\ell/\rho_0)^2 - |z|)^{-1/2} d(z)^{-1}(1 + c_z)$$
$$\|\chi \widetilde{\nabla} R_\ell^{int}(z) \widetilde{\nabla}\chi\| \leq c_z$$

where $\widetilde{\nabla} \equiv (\partial_\rho, \partial_z), d(z) \equiv dist(z, \sigma(H_\ell^{int}))$ and $c_z \equiv c(z, d(z)) \geq 0$.

2. Exterior estimates

We need to estimate $R^{ext}_{\ell,\mu}(z) \equiv (H^{ext}_{\ell,\mu} - z)^{-1}$ for $\mu \in \mathbf{C}$, $Im\mu > 0$, and z in a neighborhood of $\lambda_1(\ell)$, for each ℓ sufficiently large. This estimate must be uniform in ℓ. By the geometric perturbation theory described below in part 3, we will need a priori bounds on $\chi R^{ext}_{\ell,\mu}(z)\chi$, $\chi\widetilde{\nabla} R^{ext}_{\ell,\mu}(z)\chi$ and $\chi\widetilde{\nabla} R^{ext}_{\ell,\mu}\widetilde{\nabla}\chi$, where $supp\ \chi \subset \mathcal{O}$ and $\widetilde{\nabla} \equiv (\partial_\rho,\ \partial_z)$, as above. It is precisely for these estimates that we assume the exterior non-trapping hypothesis. Indeed, [19] show how to obtain such bounds given an escape function for a boundary. We briefly review the main points and refer to [18] and [19] for the details.

PROPOSITION 4.4. *Assume that an exterior domain $\mathcal{D}$ admits an escape function. Then the local energy decays as $t \to \infty$ (at least as $\mathcal{O}(t^{-1})$) for all solutions of the wave equation on the exterior domain $\mathcal{D}$ with initial conditions $B_R(0) \cap \mathcal{D}$ (for R large enough). Let B be the generator of the Lax-Phillips semigroup $Z(t) = P_+\ U(t)\ P_-$, $t \geq 0$. Then there exists $\alpha > 0$ such that $(\mu - B)^{-1}$ is holomorphic on $Re\ \mu > -\alpha$.*

This proposition and standard spectral deformation results imply that $R^{ext}_{\ell,\mu}(z)$ is holomorphic in the region

$$\mathcal{O}_{\alpha,\mu} \equiv \{z \in \mathbf{C} \mid Im\ z > -\alpha \text{ and } arg\ z > -2\ arg(1+\mu)\}.$$

Next, from the construction of $Z(t)$, if f, g are initial conditions for the wave equation with support in $\widetilde{\mathcal{E}} \cap B_R(0)$, it follows that [18]

$$((\mu - B)^{-1} f, g)_E = ((\mu - A)^{-1} f, g)_E$$

where $(.,.)_E$ is the energy inner product and A is the generator of $U(t)$, the unitary evolution group for the wave equation. Consequently, by Proposition 4.4., we get an analytic continuation of $((\mu - A)^{-1} f, g)_E$ into $Re\ \mu > -\alpha$. It follows by a simple calculation and suitable choice of initial conditions that $(\chi R^{ext}_{\ell,\mu}(z)\chi\phi,\ \psi)$ can be bounded for $z \in \mathcal{O}_{\alpha,\mu}$, uniformly in the L^2-norms of ϕ and ψ, by $\|(\mu - B)^{-1}\|_E$ for $Re\ \mu > -\alpha$. This latter norm is bounded as follows. The local energy decay implies $\exists c > 0,\ \alpha > 0$ such that

$$\|Z(t)\| \leq ce^{-\alpha t},\ t > 0.$$

Hence, the Laplace transform of $Z(t)$ converges for $Re\ \mu > -\alpha$:

$$(\mu - B)^{-1} = \int_0^\infty e^{-\mu t} Z(t)dt$$

and thus, for some $c_0 > 0$:

$$\|(\mu - B)^{-1}\| \leq c_0(\alpha + Re\ \mu)^{-1}$$

provided $Re\ \mu > -\alpha$. Consequently, we derive that

$$\|\chi R^{ext}_{\ell,\mu}(z)\chi\| \leq c_0[Im\ z+\alpha]^{-1}$$

for $z \in \mathcal{O}_{\alpha,\mu}$, uniformly in ℓ. Combining this a priori estimate with a Poincaré inequality valid for the region $\mathcal{O}$, we obtain an analog of Lemma 4.3.

LEMMA 4.5. *Let χ be a smooth characteristic function with support in $\mathcal{O}$ and let*
$\rho_2 \equiv max\{\rho \mid \rho(x) \in \mathcal{O}\}$. Then for any $z \in \mathcal{O}_{\alpha,\mu}$

$$\|\chi R^{ext}_{\ell,\mu}(z)\chi\| \leq c_1((\ell/\rho_2)^2 - \mid z \mid)^{-1}$$
$$\|\chi \widetilde{\nabla} R^{ext}_{\ell,\mu}(z)\chi\| \leq c_2((\ell/\rho_2)^2 - \mid z \mid)^{-1/2}$$
$$\|\chi \widetilde{\nabla} R^{ext}_{\ell,\mu}(z)\widetilde{\nabla}\chi\| \leq c_3$$

where the constants c_i depend only on $\mathcal{O}_{\alpha,\mu}$ and are uniform in ℓ.

We remark that the proof of the third estimate requires some machinery of [5] (see [11] and below).

3. Geometric perturbation theory

We use the methods of [5] (see also [7]) to prove that $H_{\ell,\mu}$ has an eigenvalue near $\lambda_1(\ell) \in \sigma(H^{int}_\ell)$. This more detailed form of pertubation theory is necessary since we only have the localized resolvent estimates of Lemmas 4.3 and 4.5. As in section 2, define $\mathcal{H}_0 \equiv L^2(\widetilde{\mathcal{C}}) \oplus L^2(\widetilde{\mathcal{E}})$, $\mathcal{H} \equiv L^2(\widehat{\Omega})$ and $J\ :\ \mathcal{H}_0 \to \mathcal{H}$ by $J(u_1 \oplus u_2) = J_1\ u_1 + J_2\ u_2$, where $\{J_i\}^2_{i=1}$ is the partition of unity introduced above such that $supp \mid \nabla J_i \mid \subset \mathcal{O}$. Let $\{\widetilde{J}_i\}^2_{i=1}$ be another pair of functions such that $\widetilde{J}_i\ J_i = J_i$, $\widetilde{J}_i \mid supp\ J_i = 1$, and define $\widetilde{J}\ :\ \mathcal{H}_0 \to \mathcal{H}$ as above. Then $J\ \widetilde{J}^\star = 1_{\mathcal{H}}$. We will supress the indices (ℓ,μ) when the meaning is clear. As in section 2, we obtain a geometric resolvent equation

$$R(z)J = JR_0(z) + R(z)JMR_0(z)$$

where M : $\mathcal{H}_0 \to \mathcal{H}_0$ is given by

$$M(u_1 \oplus u_2) \equiv (\widetilde{\nabla} J'_1 + J'_1\widetilde{\nabla})u_1 \oplus (\widetilde{\nabla} J'_2 + J'_2\widetilde{\nabla})u_2$$

with a prime denoting the z derivative and $\widetilde{\nabla} \equiv (\partial_\rho,\ \partial_z)$. We factorize M with the aid of two auxiliary operators $M_i\ :\ \mathcal{H}_0 \to \mathcal{H}_0 \oplus \mathcal{H}_0$:

$$M_1(u_1 \oplus u_2) \equiv (J'_1 u_1 \oplus \chi\widetilde{\nabla} u_1) \oplus (J'_2 u_2 \oplus \chi\widetilde{\nabla} u_2)$$
$$M_2(v_1 \oplus v_2) \equiv (\chi\widetilde{\nabla} v_1 \oplus J'_1 v_1) \oplus (\chi\widetilde{\nabla} v_2 \oplus J'_2 v_2)$$

so that $M = -M_2^\star M_1$. Using this factorization and solving for $R(z)J\ M_2^\star$, we obtain

$$R(z) = JR_0(z)\widetilde{J}^\star - JR_0(z)M_2^\star(1 - K(z))^{-1}M_1\ R_0(z)\widetilde{J}^\star \tag{4.1}$$

where $K(z)\ :\ \mathcal{H}_0 \oplus \mathcal{H}_0 \to \mathcal{H}_0 \oplus \mathcal{H}_0$ is defined by

$$K(z) \equiv -M_1\ R_0(z)\ M_2^\star$$

For (4.1) to be valid, it is sufficient that $\|K(z)\| < 1$. If we write out the form of $K(z)$, we see that the estimates of Lemmas 4.3 and 4.5 guarantee this for all large ℓ and z on a contour Γ_ℓ about $\lambda_1(\ell)$, for each ℓ large enough. It is a consequence of Lemma 4.2 that we can take $rad(\Gamma_\ell) = \mathcal{O}(1)$ for all ℓ. We now integrate both sides of (4.1) about Γ_ℓ. The estimates of Lemma 4.3 and the holomorphy of $R_{\ell,\mu}^{ext}(z)$ on and inside Γ_ℓ allow us to prove

$$\|(z\pi i)^{-1} \oint_{\Gamma_\ell} R(z)dz - JP_0\widetilde{J}^\star\| \leq c_0\ \ell^{-1/2},$$

where P_0 is the projection onto the eigenspace of $\lambda_1(\ell)$. The estimates discussed in section 4 below insure that for any $\varepsilon > 0$, $\|JP_0\widetilde{J}^\star\| \geq 1 - \varepsilon$ for all large ℓ. Consequently, $H_{\ell,\mu}$ has an eigenvalue near $\lambda_1(\ell)$ with the same algebraic multiplicity as $\lambda_1(\ell)$. This proves the existence of resonances for Δ_Ω near $\lambda_1(\ell)$ for all large ℓ.

4. Exponential decay

Estimates on the resonance width come from exponential decay estimates on the eigenfunctions of H_ℓ^{int} in the tube region and bounds on the interior resolvents as in Lemma 4.3. The procedure is as in [12]. Agmon-type calculations as presented there give the necessary decay estimates with constants uniform in ℓ. These estimates also allow us to establish the uniform bound on $\|JP_0\widetilde{J}^\star\|$ mentioned above. Note that the decay of an eigenfunction of H_ℓ^{int} is due to the fact that the tube T and a neighborhood of $\rho.= 0$ in $\hat{e}$ lie in the classically forbidden region for $\lambda_1(\ell)$ for all large ℓ. $\diamondsuit$

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