SHMUEL AGMON

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1. Introduction

In this paper we describe a representation theorem for solutions of the differential equation

$$\Delta u + \lambda q(x) u = 0 \tag{1.1}$$

on certain non-compact real analytic Riemannian manifolds. Here Δ is the Laplace-Beltrami operator, λ a complex number and $q(x)$ is a positive real-analytic function. The theorem is a generalization of a representation theorem for solutions of the Helmholtz equation on hyperbolic space proved by Helgason [3; 4] and Minemura [5]. By way of introduction we recall this special representation theorem.

We take for the hyperbolic n-space the Poincaré model of the unit ball $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ with the Riemannian metric

$$ds^2 = (\frac{1 - |x|^2}{2})^{-2} |dx|^2. \tag{1.2}$$

$\mathbb{B}^n$ is a complete non-compact Riemannian manifold with an ideal boundary $\partial\mathbb{B}^n$ identified with the sphere $S^{n-1} \subset \mathbb{R}^n$. The Laplace-Beltrami operator on $\mathbb{B}^n$, denoted by Δ_h, is given in Euclidean global coordinates by

$$\Delta_h = (\frac{1 - |x|^2}{2})^2 \Delta + (n-2) \frac{1 - |x|^2}{2} \sum_{i=1}^{n} x_i \partial_i \tag{1.3}$$

where Δ is the Laplacian on $\mathbb{R}^n$, $\quad \partial_i = \partial/\partial x_i$.

Consider the equation

$$\Delta_h u + \lambda u = 0 \text{ in } \mathbb{B}^n. \tag{1.4}$$

The Helmholtz equation (1.4) has a distinguished class of solutions known as the generalized eigenfunctions of $-\Delta_h$. Given any $s \in \mathbb{C}$ and $\omega \in \partial\mathbb{B}^n$ there is a unique (normalized) generalized eigenfunction denoted by $E(x, \omega; s)$, $\quad x \in \mathbb{B}^n$. In Euclidean coordinates it has the explicit form

$$E(x, \omega; s) = \left(\frac{1 - |x|^2}{|x - \omega|}\right)^s \tag{1.5}$$

for $|x| < 1$, $\quad \omega \in S^{n-1}$. The function $u(x) = E(x, \omega; s)$ is a solution of equation (1.4) with $\lambda = s(n-1-s)$. The problem arises whether any solution u of equation (1.4) can be represented by an integral formula of the form

$$u(x) = \int_{S^{n-1}} \Phi(\omega) E(x, \omega; s) d\omega,$$

for s satisfying $s(n-1-s) = \lambda$, where Φ is some generalized function on S^{n-1}. This problem was solved in the affirmative by Helgason [3;4] and by Minemura [5]. Their main result can be stated as follows,

THEOREM 1.1. *Let $u(x)$ be a solution of the Helmholtz equation*

$$\Delta_h u + s(n-1-s)u = 0 \textit{ in } \mathbb{B}^n \tag{1.6}$$

where s is some complex number such that $s \neq (n-1-j)/2$ for $j = 1, 2, \ldots$ Then there exists a unique hyperfunction Φ_u on S^{n-1} such that

$$u(x) = \langle \Phi_u, E(x, \cdot; s) \rangle \tag{1.7}$$

for $x \in \mathbb{B}^n$. Moreover, the map: $u \to \Phi_u$ is a bijection of the space of solutions of (1.6) on the space of hyperfunctions on S^{n-1}.

In this paper we generalize Theorem 1.1 and show that a similar representation theorem holds for solutions of equations (1.1) on a general class of non-compact Riemannian manifolds of which hyperbolic space is a special case. We use a P.D.E. oriented approach. When restricted to the special situation of Theorem 1.1 our approach yields a new proof of the theorem which is not using the special structure of $\mathbb{B}^n$ as a symmetric space (see also [1]). The general set up of our study is as follows. Let X be a real-analytic compact Riemannian manifold with a boundary ∂X. Let g denote the Riemannian metric on X and let Δ_g denote the corresponding Laplace-Beltrami operator. Set

$$\overset{\circ}{X} = X \setminus \partial X.$$

Introduce on $\overset{\circ}{X}$ a new Riemannian metric h, conformal with g, defined by

$$h = \rho^{-2} g \tag{1.8}$$

where $\rho(x)$ is a real-analytic function on X such that

$$\begin{aligned} &\rho(x) > 0 \text{ on } \overset{\circ}{X}, \\ &\rho(x) = 0 \text{ and } d\rho \neq 0 \text{ on } \partial X. \end{aligned} \tag{1.9}$$

Denote by Δ_h the Laplace-Beltrami operator on $\overset{\circ}{X}$ in the metric h. It is given by

$$\Delta_h = \rho^2 \Delta_g - (n-2)\rho(\nabla_g \rho) \tag{1.10}$$

where throughout the paper n denotes the dimension of X and where $\nabla_g \rho$ denotes the gradient vector field in the metric g. As usual $\nabla_g \rho$ is identified with a first order differential operator given in local coordinates by

$$\nabla_g \rho = \sum_{i,j} g^{ij} \frac{\partial \rho}{\partial x_i} \frac{\partial}{\partial x_j}. \tag{1.11}$$

We denote by $|\nabla_g \rho(x)|$, the norm of the vector $\nabla_g \rho(x)$ induced by g. In local coordinates

$$|\nabla_g \rho(x)|^2 = \sum_{i,j} g^{ij}(x) \frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial x_j}. \tag{1.12}$$

We consider solutions of the differential equation

$$\Delta_h u + \lambda q(x) u = 0 \text{ in } \overset{\circ}{X} \tag{1.13}$$

where $q(x)$ is a positive real-analytic function on X and λ is a complex number. We shall derive a representation theorem similar to Theorem 1.1 for solutions of (1.13). It will involve the generalized eigenfunctions of the operator $q^{-1}\Delta_h$ which will be defined in section 2.

REMARK: We note that the main result of this paper (the representation theorem) holds under weaker smoothness assumptions than those imposed above. The result holds if one assumes for instance that X, ρ and q are of class C^2 and that in addition X, ρ and q are real analytic in some neighborhood of ∂X.

In this paper we are going to impose on the function q a boundary condition. We shall assume that

$$q(x) = |\nabla_g \rho(x)|^2 \text{ on } \partial X. \tag{1.14}$$

We note that this condition is not necessary for the validity of the main representation theorem. However assumption (1.14) simplifies considerably many details in the proof of the theorem. Observe that equation (1.6) on hyperbolic n-space belongs to the class of equations introduced above. We conclude this introduction by noting that the representation theorem described in this paper for solutions of (1.13) can be shown to hold for solutions of a much wider class of equations of the form

$$\rho^2 \Delta_g u + \rho B u + C u = 0 \text{ in } \overset{\circ}{X}$$

where B is a real-analytic vector field on X satisfying some conditions on ∂X and C is a real-analytic function on X.

The main part of this paper is divided into two sections. In section 2 we discuss the Green's function associated with equation (1.13). The asymptotic and related real-analyticity properties of the Green's function play a crucial role in our study. These are described in Theorem 2.1. Using the theorem we define the generalized eigenfunctions which form a distinguished class of solutions of equation (1.13) and which are the building blocks in the representation theorem for any solution of that equation. The representation theorem is stated and proved in section 3. We note that the proof of the theorem is composed of the following two main ingredients. (i) Asymptotic and real-analyticity properties of the Green's function described in Theorem 2.1. (ii) A theorem of Baouendi and Goulaouic [2] on the solvability of the Cauchy problem on a characteristic initial hypersurface for certain P.D.E. of Fuchsian type.

2. Asymptotic properties of Green's functions and related results

When studying solutions of (1.13) it will be convenient to introduce the differential operator P on $\mathring{X}$ defined by

$$P = -q^{-1}\Delta_h. \tag{2.1}$$

We associate with P the measure dm on $\mathring{X}$ defined by

$$dm := q d\mu_h = q\rho^{-n} d\mu_g \tag{2.2}$$

where $d\mu_h$ (resp. $d\mu_g$) is the measure induced by the metric h (resp. g) on $\mathring{X}$. Considering P as a symmetric operator in $L^2(\mathring{X}; dm)$ with domain $C_0^\infty(\mathring{X})$ it is not difficult to show that $\bar{P}$ (the closure of P) is a self-adjoint operator in $L^2(\mathring{X}; dm)$. Furthermore, it can be shown that the spectrum of $\bar{P}$ has the following properties.

(i) $$\sigma_{ess}(\bar{P}) = [(\frac{n-1}{2})^2, \infty), \quad n = dim X.$$

(ii) $\sigma_p(\bar{P})$ consists of a finite number of eigenvalues contained in the interval $[0, (\frac{n-1}{2})^2)$.

Next, it will be convenient to replace the parameter λ in (1.13) by a parameter s related by

$$s(n-1-s) = \lambda. \tag{2.3}$$

Thus we rewrite equation (1.13) in the form

$$Pu - s(n-1-s)u = 0. \tag{2.4}$$

Note that the map: $s \longrightarrow \lambda$ defined by (2.3) takes the half-plane $Re s > (n-1)/2$ onto the domain $\mathbb{C} \setminus \sigma_{ess}(\bar{P})$. We shall denote by $\mathcal{E}$ the set of points $\{s_i\}$ in the half-plane $Re s > (n-1)/2$ such that $s_i(n-1-s_i)$ is an eigenvalue of $\bar{P}$. From (ii) above it follows that $\mathcal{E}$ is a finite set of points contained in the interval $(\frac{n-1}{2}, n-1]$.

From now on we shall assume that s is some fixed number in the half-plane $Re s > (n-1)/2$ such that $s \notin \mathcal{E}$. We shall denote by $G(x, y; s)$ the Green's function associated with equation (2.4) in $\overset{\circ}{X}$. It is the kernel (with respect to the measure dm) of the resolvent operator

$$G(s) = (\bar{P} - s(n-1-s))^{-1}. \tag{2.6}$$

From the ellipticity of P in $\overset{\circ}{X}$ and the real-analyticity of the manifold (X, g) and the functions ρ and g, it follows that $G(x, y; s)$ is real-analytic in $x, y \in \overset{\circ}{X}$ for $x \neq y$. This property can be extended in some generalized sense to the (ideal) boundary of $\overset{\circ}{X}$. In this connection we introduce the following notation. For any two sets $X_i \subset X$, $\quad i = 1, 2$, we define

$$(X_1 \times X_2)' = \{(x_1, x_2) : x_1 \in X_1, x_2 \in X_2, x_1 \neq x_2\}.$$

The following "extension theorem" has a basic role in this paper.

THEOREM 2.1. *Let $F(x, y; s)$ be the real-analytic function on $(\overset{\circ}{X} \times \overset{\circ}{X})'$ defined by*

$$F(x, y; s) = \rho(x)^{-s} \rho(y)^{-s} G(x, y; s). \tag{2.7}$$

Then $F(x, y; s)$ admits a real-analytic extension from $(\overset{\circ}{X} \times \overset{\circ}{X})'$ to $(X \times X)'$.

The proof of Theorem 2.1 is quite long and technical. For reasons of brevity we shall not give the proof in this paper. We plan to give the proof in another publication. Note that in the special case of equation (1.6) on the hyperbolic space $\mathbb{B}^n$ the Green's function is known explicitly and Theorem 2.1 can be verified by inspection.

Now define a family of solutions of equation (2.4) as follows. For any $\omega \in \partial X$ and $x \in \overset{\circ}{X}$ set

$$E(x, \omega; s) = \lim_{\substack{y \to \omega \\ y \in \overset{\circ}{X}}} \rho(y)^{-s} G(x, y; s). \tag{2.8}$$

In view of Theorem 2.1 it is clear that $E(x,\omega;s)$ is a well defined real-analytic function of (x,ω) on $\mathring{X}\times\partial X$. Furthermore, for a fixed ω the function $E(x,\omega;s)$ is a solution of equation (2.4) in $\mathring{X}$. We shall refer to the family $E(x,\omega;s)$ (parameterized by $\omega\in\partial X$) as the generalized eigenfunctions of P with eigenvalue $s(n-1-s)$. These functions are the building blocks of the general representation theorem (Theorem 3.1). Note that in the special case of equation (1.6) on hyperbolic n-space the generalized eigenfunctions defined by (2.8) are (up to a multiplicative constant) those defined previously by (1.5).

We conclude this section by introducing some classes of real-analytic functions on ∂X. Let Λ be the Laplace-Beltrami operator on ∂X in the Riemannian metric induced by g. For any number $d>0$ we denote by $\mathcal{A}_d(\partial X)$ the class of C^∞ functions $\varphi(\omega)$ on ∂X satisfying the inequalities

$$|\Lambda^j\varphi(\omega)|\le C(2j)!d^{2j} \text{ for } j=0,1,\ldots, \tag{2.9}$$

and all $\omega\in\partial X$ where C is some constant depending on φ. $\mathcal{A}_d(\partial X)$ is a Banach space under the norm

$$\|\varphi\|_d = \text{smallest constant } C \text{ for which (2.9) holds.}$$

We denote by $\mathcal{A}(\partial X)$ the class of real-analytic functions on ∂X. It is well known that

$$\mathcal{A}_d(\partial X)\subset\mathcal{A}(\partial X) \text{ for all } d>0$$

and that

$$\mathcal{A}(\partial X)=\lim_{d\uparrow\infty}\mathcal{A}_d(\partial X). \tag{2.10}$$

We consider $\mathcal{A}(\partial X)$ as a topological linear space with the inductive limit topology induced by (2.10) and the given topologies on the Banach spaces $\mathcal{A}_d(\partial X)$.

Let $\mathcal{A}'(\partial X)$ be the dual of $\mathcal{A}(\partial X)$. Any member of $\mathcal{A}'(\partial X)$ is called a hyperfunction on ∂X. Thus a hyperfunction on ∂X is a linear functional Φ on $\mathcal{A}(\partial X)$ such that for any $d>0$ and any $\varphi\in\mathcal{A}_d(\partial X)$ the following inequality holds

$$|\langle\Phi,\varphi\rangle|\le C_d\|\varphi\|_d \tag{2.11}$$

where C_d is a constant depending only on Φ and d.

3. The representation theorem.

We come now to the main result of this paper.

THEOREM 3.1. *Let $u(x)$ be any solution of equation (2.4) on $\overset{\circ}{X}$. Then there exists a unique hyperfunction Φ_u on ∂X such that the following representation holds*

$$u(x) = \langle \Phi_u, E(x, \cdot; s) \rangle \tag{3.1}$$

for $x \in \overset{\circ}{X}$. Moreover the map: $u \longrightarrow \Phi_u$ is a bijection of the space of solutions of (2.4) on $\mathcal{A}'(\partial X)$.

REMARK 1: It can be shown that (3.1) holds with Φ_u a Schwartz distribution on ∂X if and only if

$$|u(x)| \leq \text{Const.}\rho(x)^{-N} \text{ on } \overset{\circ}{X} \tag{3.2}$$

for some $N \geq 0$. Moreover, this variant of the representation theorem can be shown to hold under weaker smoothness assumptions. Namely, it is enough to assume that X is a C^∞ Riemannian manifold and that ρ and q are C^∞ functions on X.

REMARK 2: Using Theorem 2.1 and some related estimates one can show that $E(x, \omega; s)$ is a meromorphic function of s in the half-plane $Re s > (n-1)/2$ with simple poles contained in $\mathcal{E}$. Furthermore, it can be shown that $E(x, \omega; s)$ admits a meromorphic continuation in s into the whole complex plane. The last (deep) result can be used to extend Theorem 3.1 to all complex values of the parameter s which are not poles of $E(x, \omega; s)$. Thus <u>in general</u> solutions of equation (2.4) admit two representations of the form (3.1). One representation involves the family of solutions $E(x, \omega; s)$ and the other representation involves the family $E(x, \omega; n-1-s)$.

The proof of Theorem 3.1 will be based on Theorem 2.1 and on Theorem 3.2 below which deals with the analytic Cauchy problem for a differential equation related to (2.4) with initial data given on the characteristic manifold

∂X. Before stating the result we introduce the following notation. For any $\varepsilon > 0$ we denote by Ω_ε a neighborhood of ∂X in X defined by

$$\Omega_\varepsilon = \{x \in X : 0 \le \rho(x) \le \varepsilon\}.$$

We also set

$$X_\varepsilon = X \setminus \Omega_\varepsilon = \{x \in X : \rho(x) > \varepsilon\}.$$

We now state

THEOREM 3.2. *Given $\varphi \in \mathcal{A}_d(\partial X)$, $d > 0$, there exists a unique function $v_\varphi(x)$, defined and real-analytic in Ω_δ for some $\delta = \delta(d) > 0$, (δ depending on d but not on φ) such that the following holds:*

(i) v_φ is a solution in Ω_δ of the differential equation

$$\rho^{-s}P(\rho^s v) - s(n-1-s)v = 0. \tag{3.3}$$

(ii) v_φ satisfies the initial condition

$$v_\varphi = \varphi \text{ on } \partial X. \tag{3.4}$$

Moreover, the map: $\varphi \to v_\varphi$ is a continuous map from $\mathcal{A}_d(\partial X)$ to $C^k(\Omega_\delta)$ for $k = 0, 1, \ldots$

Theorem 3.2 follows as an easy corollary from a general theorem dealing with the initial value problem for Fuchsian type partial differential equations proved by Baouendi and Goulaouic ([2]; see Theorem 3 with $m = 2, k = 1$ and $h = 0$). In this connection note that the two indicial exponents associated with equation (2.4) at the boundary are s and $n - 1 - s$. This implies that equation (3.3) can be written in the form

$$\rho\Delta_g v + Bv + Cv = 0 \text{ in } \Omega_\delta \tag{3.5}$$

where B is a real-analytic field on X and C is a real-analytic function on X. It is this form of equation (3.3) which allows one to deduce Theorem 3.2 from the results of [2].

We turn to the

PROOF OF THEOREM 3.1: Observe that since $E(x,\omega;s)$ is a real-analytic function in (x,ω) on $\overset{\circ}{X} \times \partial X$ it is clear that the function $u(x) = \langle \Phi, E(x,\cdot;s)\rangle$ is a well defined solution of (2.4) in $\overset{\circ}{X}$ for any $\Phi \in \mathcal{A}'(\partial X)$. To establish the converse we introduce some notation.

For any $\varphi \in \mathcal{A}(\partial X)$ we set

$$w_\varphi(x) = \rho^s v_\varphi(x) \tag{3.6}$$

where $v_\varphi(x)$ is the solution of the initial value problem described in Theorem 3.2. With no loss of generality we shall assume in the following that $\varphi \in \mathcal{A}_d(\partial X)$ for some $d > 0$ and that $v_\varphi(x)$ is defined in $\Omega_{\delta(d)}$ for some $\delta(d) > 0$. We shall also assume that $\delta(d) \le \delta_0$ where $\delta_0 > 0$ is chosen sufficiently small so that $(d\rho)(x) \neq 0$ for $x \in \Omega_{\delta_0}$. It follows form (3.6) and (3.3) that $w_\varphi(x)$ is a well defined solution of (2.4) in $\operatorname{int}(\Omega_{\delta(d)})$.

Let now $u(x)$ be a given solution of equation (2.4) in $\overset{\circ}{X}$. For any $\varphi \in \mathcal{A}_d(\partial X)$ and $0 < \varepsilon < \delta(d)$, we set

$$I_u^\varepsilon(\varphi) := \int\limits_{\partial X_\varepsilon} (w_\varphi D_\nu u - u D_\nu w_\varphi) d\mu_h^\varepsilon(x) \tag{3.7}$$

where D_ν denotes a derivation in the direction of the outward unit normal vector (in the metric h) at the boundary ∂X_ε. Here $d\mu_h^\varepsilon$ denotes the measure on ∂X_ε induced by $d\mu_h$. We claim that $I_u^\varepsilon(\varphi)$ is independent of ε; i.e.

$$I_u^{\varepsilon_1}(\varphi) = I_u^{\varepsilon_2}(\varphi) \text{ for } 0 < \varepsilon_1 < \varepsilon_2 < \delta(d). \tag{3.8}$$

Indeed, we have

$$w_\varphi \Delta_h u - u \Delta_h w_\varphi = 0 \text{ in } \operatorname{int}(\Omega_{\delta(d)}). \tag{3.9}$$

Integrating (3.9) on the domain $X_{\varepsilon_1} \setminus X_{\varepsilon_2}$, applying Green's formula, one obtains (3.8).

Next we define a hyperfunction Φ_u on ∂X as follows: For any $\varphi \in \mathcal{A}(\partial X)$ we set

$$\langle \Phi_u, \varphi \rangle := \lim_{\varepsilon \to 0} I_u^\varepsilon(\varphi). \tag{3.10}$$

That Φ_u is a well defined linear functional on $\mathcal{A}(\partial X)$ is clear in view of (3.8) and the linearity of w_φ in φ. That Φ_u is also continuous on $\mathcal{A}(\partial X)$ one sees by noting that for a given $d > 0$ there exists a $\delta = \delta(d) > 0$ such that for any $\varepsilon \in (0, \delta)$ I_u^ε is a well defined continuous linear functional on $\mathcal{A}_d(\partial X)$. This observation follows easily from the definition of I_u^ε and Theorem 3.2.

Finally we shall show that the hyperfunction Φ_u defined by (3.10) yields the representation (3.1). To this end fix a point $y \in \overset{\circ}{X}$ and set

$$\psi(\omega) = E(y, \omega; s). \tag{3.11}$$

From (2.8), Theorem 2.1 and Theorem 3.2 it follows that the unique solution v_ψ of equation (3.3) with the initial data $\psi(\omega)$ on ∂X is given by

$$v_\psi(x) = \rho(x)^{-s} G(x, y; s),$$

so that

$$w_\psi(x) = G(x, y; s) \text{ for } x \in int(\Omega_\delta) \tag{3.12}$$

(we can take $\delta = \rho(y)$). Combining (3.7) to (3.12), taking ε sufficiently small, we get

$$\begin{aligned} \langle \Phi_u, \psi \rangle &= I_u^\varepsilon(\psi) \\ &= \int\limits_{\partial X_\varepsilon} (G(x, y; s) D_\nu u(x) - u(x) D_\nu G(x, y; s)) d\mu_h^\varepsilon(x) \\ &= u(y), \end{aligned} \tag{3.13}$$

where the last equality follows by application of Green's formula to u and the Green's function. This yields formula (3.1) and proves the existence part of Theorem 3.1.

It remains to show that the representation (3.1) is unique. This is an easy consequence of the following

LEMMA 3.3. *Given $\varphi \in \mathcal{A}(\partial X)$ there exists a function $f \in C_0^\infty(\overset{\circ}{X})$ such that*

$$\varphi(\omega) = \int\limits_{\overset{\circ}{X}} f(x) E(x, \omega; s) dm(x). \tag{3.14}$$

Deferring the proof of the lemma we establish the uniqueness of the representation (3.1) by showing that if $\Phi \in \mathcal{A}'(\partial X)$ satisfies

$$\langle \Phi, E(x;\cdot;s)\rangle = 0 \text{ for all } x \in \overset{\circ}{X} \tag{3.15}$$

then

$$\langle \Phi, \varphi\rangle = 0 \text{ for all } \varphi \in \mathcal{A}(\partial X), \tag{3.15 $'$}$$

Indeed, it follows from (3.15) that for any function $f \in C_0^\infty(\overset{\circ}{X})$ we have

$$\begin{aligned} 0 &= \int_{\overset{\circ}{X}} f(x)\langle \Phi, E(x,\cdot;s)\rangle dm(x) \\ &= \langle \Phi, \int_{\overset{\circ}{X}} f(x)E(x,\cdot;s)dm(x)\rangle, \end{aligned} \tag{3.16}$$

where the change of order of "integrations" in (3.16) is easily justified. Combining (3.16) with Lemma 3.3 we obtain (3.15$'$). This establishes uniqueness and completes the proof of Theorem 3.1.

We conclude with the

PROOF OF LEMMA 3.3: As before we shall associate with the given function $\varphi \in \mathcal{A}(\partial X)$ the solution $v_\varphi(x)$ of the initial value problem described in Theorem 3.2. Thus in particular v_φ is a real-analytic function defined in some Ω_δ, $\delta > 0$. Next we pick a function $\zeta(x) \in C^\infty(X)$ such that

$$\zeta(x) = 1 \text{ for } x \in \Omega_{\delta/3}, \quad \zeta(x) = 0 \text{ for } x \in X \setminus \Omega_{\delta/2} \tag{3.17}$$

and define a function $w \in C^\infty(\overset{\circ}{X})$ by

$$\begin{aligned} w(x) &= \zeta(x)\rho(x)^s v_\varphi(x) \text{ for } x \in \Omega_{\delta/2} \setminus \partial X, \\ w(x) &= 0 \text{ for } x \in X \setminus \Omega_{\delta/2}. \end{aligned} \tag{3.18}$$

Set

$$f(x) := (P - s(n-1-s))w(x). \tag{3.19}$$

Since v_φ is a solution of (3.3) in Ω_δ it follows from (3.19), (3.18) and (3.17) that $f(x) = 0$ in $\Omega_{\delta/3}$ and thus $f \in C_0^\infty(\mathring{X})$. We also observe that $w \in L^2(\mathring{X}; dm)$ (since $Res > (n-1)/2$). These remarks and (3.19) imply that

$$w = G(s)f \tag{3.20}$$

where $G(s)$ denotes the resolvent operator (2.6). Rewriting (3.20) in terms of the Green's function (the kernel of $G(s)$), using (3.17), (3.18) and the symmetry of the Green's function, we find that for any $y \in \Omega_{\delta/3}/\partial X$ the following formula holds

$$v_\varphi(y) = \int_{\mathring{X}} f(x)G(x, y; s)\rho(y)^{-s} dm(x). \tag{3.21}$$

Now fix a point $\omega \in \partial X$ and let $y \to \omega$ in (3.21). Using (3.4), (2.8) and Theorem 2.1 we find that

$$\begin{aligned}\varphi(\omega) = \lim_{y\to\omega} v_\varphi(y) &= \lim_{y\to\infty} \int_{\mathring{X}} f(x)G(x, y; s)\rho(y)^{-s} dm(x)\\ &= \int_{\mathring{X}} f(x)E(x, \omega; s)dm(x).\end{aligned}$$

This proves the lemma.

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Shmuel Agmon
The Hebrew University of Jerusalem
and
University of Virginia