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MODULAR FORMS AND ALGEBRAIC K-THEORY

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In this paper, which follows closely the talk given at the conference, I will sketch an example of a non-trivial element of K_2 of a certain threefold, whose existence is related to the vanishing of an *incomplete* L-function of a modular form at $s = 1$. To explain how this fits into a general picture, we begin with a simple account, for the non-specialist, of some of the conjectures (mostly due to Beilinson) which relate ranks of K-groups and orders of L-functions, supplemented by examples coming from modular forms. The picture presented is in some respects wildly distorted; among the important topics which are given little mention are:

(i) the connection between special values of L-functions and higher regulators, which is at the heart of the Beilinson conjectures;

(ii) the conjectures of Birch and Swinnerton-Dyer, and their generalisation by Beilinson and Bloch;

(iii) the theory of (mixed) motives, which underlies the constructions of the last section.

But I hope that it may be of some use as a gentle introduction to the subject, and to prepare the reader for a more comprehensive account (see for example [**9,17,18,21**] and above all [**1**]).

1. Beginnings

The story begins with Dirichlet's unit theorem: if F is a number field with ring of integers $\mathfrak{o}_F$, then

$$\operatorname{rk} \mathfrak{o}_F^* = r_1 + r_2 - 1 = \operatorname{ord}_{s=0} \zeta_F(s)$$

and there is the analytic class number formula, which at $s = 0$ reads:

$$\zeta_F^*(0) = -\frac{h_F R_F}{w_F} \tag{1}$$

where $\zeta_F^*(0)$ denotes the leading coefficient in the Taylor series of $\zeta_F(s)$ at $s = 0$. More generally, let S be a finite set of primes of F, and $\mathfrak{o}_{F,S}$ the ring of S-integers of F. Then the S-unit theorem says

$$\begin{aligned}\operatorname{rk} \mathfrak{o}_{F,S}^* &= r_1 + r_2 - 1 + \#S \\ &= \operatorname{ord}_{s=0} \zeta_{F,S}(s)\end{aligned}$$

where $\zeta_{F,S}(s)$ is the *incomplete* zeta function:

$$\zeta_{F,S}(s) = \prod_{\mathfrak{p} \notin S} (1 - N\mathfrak{p}^{-s})^{-1}.$$

and the analogue of (1) is the S-class number formula.

Borel found a generalisation of these results to the zeta function at arbitrary negative integers:

Theorem. [5] *Let $l > 0$ be an integer. Then $K_{2l}\mathfrak{o}_F$ is finite, and*

$$\begin{aligned}\operatorname{rk} K_{2l+1}\mathfrak{o}_F &= \begin{cases} r_1 + r_2 & l \text{ even} \\ r_2 & l \text{ odd} \end{cases} \\ &= \operatorname{ord}_{s=-l} \zeta_F(s).\end{aligned}$$

Moreover the leading coefficient $\zeta_F^*(-l)$ is equal, up to a non-zero rational factor, to a "higher regulator".

Remarks: (i) Here $K_i\mathfrak{o}_F$ are the higher K-groups of F, as defined by Quillen (see section 2). This is a natural generalisation of the unit theorem since $K_1\mathfrak{o}_F = \mathfrak{o}_F^*$. The fact that $K_i\mathfrak{o}_F$ are finitely generated was proved by Quillen.

(ii) The higher regulator is the determinant of a certain natural homomorphism

$$K_{2l+1}\mathfrak{o}_F \otimes \mathbf{R} \to \mathbf{R}^{m_l}, \quad m_l = \operatorname{ord}_{s=-l} \zeta_F(s).$$

(iii) The analogue of the S-unit theorem for these higher K-groups is uninteresting; on the one hand, one has

$$K_q\mathfrak{o}_{F,S} \otimes \mathbf{Q} = K_q\mathfrak{o}_F \otimes \mathbf{Q} = K_qF \otimes \mathbf{Q} \tag{2}$$

for every $q > 1$ (cf. section 2); on the other, the individual Euler factors in $\zeta_F(s)$ have no poles at negative integer points, so

$$\operatorname{ord}_{s=-l} \zeta_F(s) = \operatorname{ord}_{s=-l} \zeta_{F,S}(s)$$

for any finite set S of primes and any $l > 0$.

2. K-THEORY

For any scheme X there is a Grothendieck group K_0X. It is defined as the abelian group generated by symbols $[\mathcal{E}]$, where $\mathcal{E}$ runs over all isomorphism classes of vector bundles on X, with relations of the form

$$[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}'']$$

for every exact sequence $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$. For a ring R one can define K_0R to be $K_0 \operatorname{Spec} R$, or (which amounts to the same thing) as the Grothendieck group of projective R-modules, with relations $[M \oplus N] = [M] + [N]$.

In a similar way one also has the group $K_0'X$, generated by $[\mathcal{E}]$ for arbitrary coherent sheaves $\mathcal{E}$, with relations from exact sequences of coherent sheaves.

Quillen showed that K_0X and $K_0'X$ are part of an infinite sequence of groups K_qX, $K_q'X$ for $q \geq 0$, constructed as the higher homotopy groups π_{q+1} of certain spaces attached to X. For some of the different ways to define them, see **[10,16,22]**.

Among the important properties of these groups are:

(i) There are cup-products $K_pX \times K_qX \to K_{p+q}X$;
(ii) For X regular (e.g. a smooth variety) $K_q'X = K_qX$;
(iii) For $Y \subset X$ a closed subscheme, there is a long exact sequence (the localisation sequence)

$$\cdots \to K_q'Y \to K_q'X \to K_q'(X - Y) \to K_{q-1}'Y \to \dots$$

(iv) $\mathcal{O}^*(X)$ injects into K_1X, with equality if $X = \operatorname{Spec} F$ is the spectrum of a field.
(v) The K-groups of finite fields are finite (of known order).

For a number field F the localisation sequence gives

$$\cdots \to K_q\mathfrak{o}_F \to K_q\mathfrak{o}_{F,S} \to \coprod_{\mathfrak{p}\in S} K_{q-1}\mathfrak{o}_F/\mathfrak{p} \to K_{q-1}\mathfrak{o}_F \to \dots$$

which together with (v) gives (2).

3. L-FUNCTIONS OF AN ALGEBRAIC VARIETY

Consider a smooth, projective algebraic variety X over $\mathbf{Q}$. Since any variety over a number field may be regarded—by restriction of scalars *à la* Grothendieck—as a variety over $\mathbf{Q}$ (in general, not geometrically connected) the restriction to ground field $\mathbf{Q}$ is not serious.

For each integer i in the range $0 \leq i \leq 2\dim X$ there is an L-function $L(h^i(X), s)$, which is an Euler product:

$$L(h^i(X), s) = \prod_p P_p^{(i)}(p^{-s})^{-1}.$$

The polynomials $P_p^{(i)}(t)$ here are defined as follows. Pick a prime $\ell \neq p$, and let $H_\ell^i(X)$ be the ℓ-adic cohomology of $X/\overline{\mathbf{Q}}$, which is a finite-dimensional $\mathbf{Q}_\ell$-vector space on which $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts continuously. Let $I_p \subset D_p \subset \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ be inertia and decomposition subgroups at a prime of $\overline{\mathbf{Q}}$ over p, and $\mathrm{Frob}_p = \phi_p^{-1} \in D_p/I_p$ the inverse of the Frobenius substitution. Then

$$P_p^{(i)}(t) = \det\bigl(1 - t\,\mathrm{Frob}_p \bigm| H_\ell^i(X)^{I_p}\bigr)$$

is the characteristic polynomial of Frob_p (the "geometric Frobenius") acting on the inertia invariants.

If X has a good reduction X_p at p, then $P_p^{(i)}$ has integer coefficients, and does not depend on ℓ, by Deligne's proof of the Weil conjectures [**6**]; moreover in this case the zeroes of $P_p^{(i)}(t)$ all have absolute value $p^{-i/2}$. For general p it is conjectured that $P_p^{(i)}(t)$ has integer coefficients, is independent of ℓ, and that its roots have absolute values $p^{-j/2}$ for various integers $j \leq i$. This is known in very few cases (curves, a class of surfaces and some sporadic higher-dimensional examples). For the conjectures that follow to make sense, we must assume these local properties are true. It is then conjectured that $L(h^i(X), s)$—which is analytic and non-zero for $\Re(s) > i/2 + 1$, by the Euler product—has a meromorphic continuation satisfying a functional equation for the substitution $s \mapsto 1 + i - s$.

4. General conjectures

The part of Beilinson's conjecture related to orders of L-functions can now be approximately stated:

Let m be an integer satisfying $m \leq \frac{1+i}{2}$. Write $q = 1 + i - 2m$. Then the order of $L(h^i(X), s)$ at $s = m$ is equal to the dimension of a certain subspace of $K_q(X)_{\mathbf{Z}} \otimes \mathbf{Q}$. More precisely, for $q > 0$

$$\dim K_q X_{/\mathbf{Z}} \otimes \mathbf{Q} = \sum_{\substack{(i,m) \\ 1+i-2m=q}} \mathrm{ord}_{s=m} L(h^i(X), s).$$

Remarks: (i) The group $K_q X_{/\mathbf{Z}}$ is defined as follows. Let $\mathcal{X}$ be a regular model for X over $\mathbf{Z}$; in other words, $\mathcal{X}$ is a regular scheme, proper over $\mathrm{Spec}\,\mathbf{Z}$, such that $\mathcal{X} \otimes \mathbf{Q} = X$. Then

$$K_q X_{/\mathbf{Z}} = \mathrm{Image}\bigl(K_q \mathcal{X} \to K_q X\bigr).$$

It would be wrong to take K_qX by itself; this can be seen already in the case of $X = \operatorname{Spec} F$, $i = m = 0$ (so that $q = 1$). For then $K_1X = F^*$ has infinite rank, but $K_1F_{/\mathbf{Z}} = \mathfrak{o}_F^*$ has the correct, finite rank. It was Bloch and Grayson who observed that in higher dimensions, and for higher q, it might still be necessary to impose a similar integrality condition (see section 5 below).

(ii) (dimension 0) In the case $X = \operatorname{Spec} F$ the conjecture is a consequence of Borel's theorem; as there is only one L-function ($i = 0$) there is no splitting up of the K-groups. These are essentially the only L-functions for which the conjecture is known to be true.

(iii) (dimension 1) If X is a curve, the conjectural picture is still quite simple. There are three L-functions: $L(h^0(X), s)$ and $L(h^2(X), s)$, which are respectively $\zeta_F(s)$ and $\zeta_F(s-1)$ (if X is irreducible with constant field F), and the Hasse-Weil L-function $L(h^1(X), s)$. There is a parity condition $q \equiv 1 + i \pmod 2$. Therefore the even K-groups K_qX are expected to contribute to the order of the Hasse-Weil L-function at the points $s = 1 - q/2$; whereas the contribution of the odd groups should be to $L(h^0(X), s)$ and $L(h^2(X), s)$, and this should be accounted for by Borel's theorem.

For varieties of higher dimension it becomes necessary to specify a decomposition of the K-groups into pieces corresponding to the various L-functions. There are in fact two (conjecturally equivalent) ways to do this. The first rests on certain conjectures on algebraic cycles (which are only known in a few cases). Suppose that the decomposition of the cohomology $H_\ell^*(X)$ into its graded pieces is algebraic, in the following strong sense: regard the projectors $\pi_i\colon H_\ell^*(X) \to H_\ell^i(X)$ (for $0 \le i \le 2\dim X$) as cohomology classes in $H_\ell^{2\dim X}(X \times X)$. Then one wants algebraic cycles Π_i on $X \times X$ whose cohomology classes are π_i, and whose images in the ring of correspondences $CH^{\dim X}(X \times X) \otimes \mathbf{Q}$ form a complete set of orthogonal idempotents.

This would follow from Grothendieck's standard conjectures; it is the decomposition of the "motive" $h(X)$ into submotives $h^i(X)$. It is known for curves and surfaces: see **[15]** for more details.

The ring $CH^{\dim X}(X \times X) \otimes \mathbf{Q}$ acts on $K_*X \otimes \mathbf{Q}$. So if the projectors Π_i exist, one can write $K_qX \otimes \mathbf{Q} = \oplus K_q h^i(X)$, where $K_q h^i(X) = \Pi_i(K_qX \otimes \mathbf{Q})$. Let $K_q h^i(X)_{/\mathbf{Z}}$ be the image of the composite:

$$K_q\mathcal{X} \otimes \mathbf{Q} \to K_qX \otimes \mathbf{Q} \to K_q h^i(X).$$

The precise conjecture would then be:

$$\dim K_q h^i(X)_{/\mathbf{Z}} = \operatorname{ord}_{s=m} L(h^i(X), s) \quad \text{for } q = 1 + i - 2m > 0.$$

Beilinson actually uses an alternative description of the decomposition, which is not conjectural, and gives a reasonably computable theory (for example, it is compatible with the maps in the localisation sequence when suitably interpreted). There are certain operators ψ^p (Adams operators) acting on the groups K_qX, coming from the exterior power operation on vector bundles. Define $K_q^{(n)}X$ to be the subspace of $K_qX \otimes \mathbf{Q}$ on which ψ^p acts as multiplication by p^n, with $p > 1$. It is known that this is independent of $p > 1$ and that one has a direct sum decomposition:

$$K_qX \otimes \mathbf{Q} = \oplus_{n \geq 0} K_q^{(n)}X.$$

Defining $K_q^{(n)}X_{/\mathbf{Z}}$ to be the image of $K_q\mathcal{X} \otimes \mathbf{Q}$ in $K_q^{(n)}X$, Beilinson's precise conjecture reads:

Conjecture 4.1. [1]

$$\dim K_q^{(n)}X_{/\mathbf{Z}} = \operatorname{ord}_{s=m} L(h^i(X), s)$$

for $q = 1 + i - 2m > 0$ *and* $n = 1 + i - m = q + m$.

Remark: The relation between these two decompositions is almost completely conjectural. It is only over a number field that one expects the two decompositions to be the same—this is apparent even in the case $X = \operatorname{Spec} F$.

To formulate an S-integral version of the conjecture, let S be a finite set of rational primes, and $\mathbf{Z}_S = \mathbf{Z}[\{p^{-1}\}_{p \in S}]$, as in the first section. Let $\mathcal{X}_S = \mathcal{X} \otimes \mathbf{Z}_S$ be the restriction of the regular model to $\operatorname{Spec} \mathbf{Z}_S$, and define

$$K_q^{(n)}X_{/\mathbf{Z}_S} = \operatorname{Image}(K_q\mathcal{X}_S \otimes \mathbf{Q} \to K_q^{(n)}X)$$

Conjecture 4.2. *Let $L_S(h^i(X), s)$ be the incomplete L-function (i.e. with the Euler factors for $p \in S$ removed). Then:*

$$\dim K_q^{(n)}X_{/\mathbf{Z}_S} = \operatorname{ord}_{s=m} L_S(h^i(X), s)$$

for $q = 1 + i - 2m > 0$ *and* $n = 1 + i - m = q + m$.

Remarks: (i) The order of the incomplete L-function at $s = m$ is the sum of the order of the complete L-function and

$$\sum_{p \in S} \dim \ker\left(\operatorname{Frob}_p - p^m \mid H^i_\ell(X)^{I_p}\right)$$

(assuming that the action of Frob_p is semisimple). In particular, if p is a prime of good reduction, then there will be no contribution to the sum unless $m = i/2$. Thus for $m < i/2$ (ie. $q > 1$) the order of L_S stabilises as soon as S contains all bad primes. At the same time, $K_qX_{/\mathbf{Z}_S}$ is the kernel of the boundary map in the localisation sequence:

$$K_qX_{/\mathbf{Z}_S} = \ker\bigl(K_qX \to \coprod_{p\notin S} K'_{q-1}\mathcal{X}_{\mathbf{F}_p}\bigr).$$

For a good prime p, $K'_{q-1}\mathcal{X}_{\mathbf{F}_p} = K_{q-1}\mathcal{X}_{\mathbf{F}_p}$ and Parshin has conjectured that this is torsion if $q-1 \neq 0$. If this conjecture is true, then the left-hand side of conjecture 4.2 also stabilises as soon as S contains all bad primes.

(ii) Conjecture 4.2 was made by Deligne in [**7**]. He also asked for the existence of an S-regulator analogous to the one for units. A general candidate for this has yet to be constructed; for something in this direction see section §4.7 of [**17**].

5. Modular curves

Let Γ be a congruence subgroup of $SL_2(\mathbf{Z})$ of level n, and let U_Γ be the modular curve, whose set of complex points is the non-compact Riemann surface $\Gamma\backslash\mathfrak{H}$. There is the standard compactification

$$X_\Gamma = U_\Gamma \cup (\text{cusps})$$

which has the structure of an irreducible curve over the field $\mathbf{Q}(\zeta_n)$ (although it often can be defined over a smaller field). The Hasse-Weil L-function of X_Γ is a product

$$L(h^1(X_\Gamma), s) = \prod_{i=1}^{g} L(f_i, s)$$

where f_i are certain (not necessarily distinct!) newforms of weight 2 and some level, and $L(f_i, s)$ is the associated Hecke L-series. There is a functional equation relating $L(h^1(X), s)$ and $L(h^1(X), 2-s)$.

At the point $s = m = (1+i)/2 = 1$ one has the conjecture of Birch and Swinnerton-Dyer. This fits into the framework of 4.1 because of the relation between K_0 of a curve and its Jacobian. At other points the functional equation determines the order of vanishing of the L-function, and

$$\operatorname{ord}_{s=m} L(h^1(X), s) = g \quad \text{for } m = 0, -1, -2, \ldots$$

Conjecture 4.1 therefore predicts that $K_{2r}X_{\Gamma/\mathbf{Z}}$ will have rank at least g, for every positive integer r.

Remarks: (i) As defined here, g will equal $\phi(n)$ times the genus of the curve X_Γ. If the chosen field of definition is $\mathbf{Q}$, g will be simply the genus.

(ii) The levels of the forms f_i need not equal n, or even divide n; however they always divide n^2.

The simplest case is the point $s = 0$. Here we have the fundamental result of Beilinson:

Theorem. *(a)* **[1]** *There exists a g-dimensional subspace $\mathcal{P}_\Gamma$ of $K_2X_\Gamma \otimes \mathbf{Q}$; its regulator is a non-zero rational multiple of $L^{(g)}(h^1(X_\Gamma), 0)$.*

(b) **[19]** *$\mathcal{P}_\Gamma$ is contained in $K_2X_{\Gamma/\mathbf{Z}} \otimes \mathbf{Q}$.*

The proof of the theorem involves an explicit construction of elements of $K_2X_\Gamma \otimes \mathbf{Q}$. We indicate here the idea of the construction; for details, including the definition and calculation of the regulator, the reader should consult **[1]**,**[2]** or **[19]**. The basic tool is:

Theorem (Manin-Drinfeld). *Any divisor of degree zero on X_Γ supported on the cusps is of finite order in the Jacobian of X_Γ.*

This guarantees a good supply of elements of $\mathcal{O}^*(U_\Gamma)$, which are the *modular units*; for example the function $\Delta(nz)/\Delta(z)$ is such a function. Now if g, $g' \in \mathcal{O}^*(U_\Gamma)$, we may form the cup product $g \cup g' \in K_2U_\Gamma$. The localisation sequence gives an exact sequence:

$$0 \longrightarrow K_2X_\Gamma \otimes \mathbf{Q} \longrightarrow K_2U_\Gamma \otimes \mathbf{Q} \xrightarrow{\partial} K_1(\text{cusps}) \otimes \mathbf{Q}$$

(it is exact on the left since K_2 of a number field is torsion).

Lemma. *Assume that the cusps are rational over the field of constants of X_Γ. Let W be the subspace of $K_2U_\Gamma \otimes \mathbf{Q}$ generated by elements of the form $c \cup h$, with $h \in \mathcal{O}^*(U_\Gamma)$ and c a constant function. Then $\partial(W) = \partial(K_2U_\Gamma \otimes \mathbf{Q})$.*

Accordingly for any cup-product $g \cup g'$, there are $h_\alpha \in \mathcal{O}^*(U_\Gamma) \otimes \mathbf{Q}$ and constant functions c_α such that

$$g \cup g' + \sum_\alpha c_\alpha \cup h_\alpha \in K_2X_\Gamma \otimes \mathbf{Q}.$$

By varying g, g' one thus obtains a subspace $\mathcal{Q}_\Gamma \subset K_2X_\Gamma \otimes \mathbf{Q}$. For $\Gamma' \subset \Gamma$ there is a direct image map:

$$\theta_{\Gamma,\Gamma'} \colon K_2X_{\Gamma'} \otimes \mathbf{Q} \longrightarrow K_2X_\Gamma \otimes \mathbf{Q}$$

and the subspace of the theorem is obtained as

$$\mathcal{P}_\Gamma = \bigcup_{\Gamma'} \theta_{\Gamma,\Gamma'}(\mathcal{Q}_{\Gamma'})$$

where Γ' runs over all congruence subgroups $\Gamma' \subset \Gamma$.

The proof that $\dim(\mathcal{P}_\Gamma) \geq g$ is by finding the regulators of these elements, which reduces to the calculation of a certain Rankin-Selberg integral. The proof that the elements belong to $K_2X_{\Gamma/\mathbf{Z}}$ results from examining the localisation sequence:

$$0 \to K_2X_{\Gamma/\mathbf{Z}} \to K_2X_\Gamma \xrightarrow{\partial} \bigoplus_p K_1'\mathcal{X}_\Gamma \otimes \mathbf{F}_p \tag{3}$$

and using the structure of the reduction modulo p of the modular curves [**8**,**12**]. The key ingredient is the fact that the action of the Hecke algebra on supersingular points in characteristic p can be expressed in terms of the action on suitable (characteristic zero) cusp forms (see for example [**13**]).

For example, consider the first non-trivial case, the modular curve $X = X_0(11)_{/\mathbf{Q}}$, which has $g = 1$. There are just two cusps 0 and ∞, and their difference has order 5 in the Jacobian of $X_0(11)$, so that the group of modular units has rank 1. If g is a generator, then $g \cup g \in K_2$ is torsion (as the cup-product is skew-symmetric). So $\mathcal{Q}_{\Gamma_0(11)} = 0$, and modular units on $\Gamma_0(11)$ do not suffice to give a non-zero element of K_2. However the covering $X_1(11)$ is an elliptic curve with 5 cusps, all of them rational (it is the Weil curve 11A of the tables in [**3**]). This curve is one of a number studied by Bloch and Grayson in [**4**]. By calculating the regulator (numerically) they determined an element of $\mathcal{Q}_{\Gamma_1(11)}$ of infinite order. Since the isogeny $X_1(11) \to X_0(11)$ induces an isomorphism on $K_2 \otimes \mathbf{Q}$, this produces the desired non-zero element of $K_2(X_0(11)) \otimes \mathbf{Q}$. The integrality of this element was also verified by Bloch and Grayson.

In this setting, conjecture 4.2 states that

$$\mathrm{ord}_{s=0}\, L_S(X_\Gamma, s) = g + \sum_{p \in S} m_p,$$

where m_p is the number of times $(1 - p^{-s})^{-1}$ occurs in the Euler factor of $L(X_\Gamma, s)$ at p. When $g = 1$, then $m_p = 1$ if the reduction mod p of X_Γ has an ordinary double point with rational tangent directions, and is 0 otherwise. This is also precisely the rank of $K_1'\mathcal{X}_{\Gamma/\mathbf{F}_p}$, suggesting that the boundary map ∂ in (3) is surjective, up to torsion. (For a more detailed analysis of a more general situation, see §4.7 of [**17**].) The calculations of Bloch and Grayson exhibit, in many cases, non-integral elements of $K_2(X_\Gamma)$ in partial confirmation of this result.

Consider for example the case of $X_0(11)$. The only bad prime is $p = 11$, where the reduction is split multiplicative, and conjecture 4.2 therefore predicts that $K_2(X_0(11))$ has rank 2. Bloch and Grayson found two independent elements of K_2 by working with functions with divisors with support in the 5 rational points of $X_0(11)$ (only two of which are cusps). However, unlike the case of the integral elements (Beilinson's theorem), there is as yet no general construction of the "extra" elements predicted by conjecture 4.2.

6. Generalisations

The results of the previous section have been generalised in various ways. One is by considering the behaviour of $L(X_\Gamma, s)$ at negative integers $s = -l < 0$. Here Beilinson has proved:

Theorem. [**2**] *There exists a subspace of* $\dim K_{2l+2}(X_\Gamma) \otimes \mathbf{Q}$ *of dimension* $g = \mathrm{ord}_{s=-l} L(X_\Gamma, s)$.

He also proves that elements constructed have the predicted regulators. We should remark that in this case there is, up to torsion, no difference between $K_{2l+2}X_{\Gamma/\mathbf{Z}}$ and $K_{2l+2}X_\Gamma$, since K'_j of a (possibly singular) curve over a finite field is torsion if $j > 1$, by [**11**]. The construction of the elements uses not just the modular curves themselves but also the "Kuga-Sato varieties" (fibre products of the universal families of elliptic curves).

It is possible to generalise these results to cusp forms of weight $k > 2$. If f is such a cusp form (assumed to be a newform of some level), then its L-series $L(f, s)$ occurs in the L-function of V_k, a Kuga-Sato variety of dimension $k-1$. (For weight two, V_2 is X_Γ.) Corresponding to the simple zero of $L(f, s)$ at the point $s = -l \leq 0$, we can construct a non-zero element of $K_{2l+k}(V_k) \otimes \mathbf{Q}$, and determine its regulator. For a precise statement of this and the previous results, and some indications of the proofs, we refer to §5 of [**9**].

There are very few examples of evidence in support of conjecture 4.2. Other than the examples of Bloch-Grayson and the example of the next section, there is only the work of Mestre and Schappacher [**14**]. They consider the symmetric square L-function of an elliptic curve E over $\mathbf{Q}$ at $s = 0$ (where it vanishes to order 2), and exhibit in many cases an experimental relation with $K_3(E \times E)$, generalising all the phenomena observed by Bloch and Grayson.

7. An example

Let $\Gamma < PSL_2(\mathbf{Z})$ be a subgroup of index 7, with two cusps, one of width 5 and another of width two. It is easy to show (by constructing fundamental regions, for example) that up to conjugacy there is exactly one such subgroup: one such is generated by the elements

$$\begin{pmatrix} 1 & 7 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}.$$

The associated modular curve X_Γ has genus zero, and there is a rational parameter t on X_Γ satisfying the equation

$$j = \frac{(t+18)(t^2+t-26)^3}{(7t+1)^2}$$

where j is the modular invariant. (Similar constructions were made by Klein and Fricke: see [**0**] for further examples and references.) Let $\phi : E \to X_\Gamma$ be an elliptic surface with invariant j; such a surface may be obtained by taking the affine equation

$$y^2 + xy = x^3 - \frac{36x+1}{j-1728},$$

although there are others. Finally let V be a nonsingular model over $\mathbf{Q}$ for the fibre product $E \times_{X_\Gamma} E$.

Similar fibre varieties were studied in [**20**], and the same methods can be used to show that the interesting part of the L-function $L(h^3(V), s)$ is a Hecke L-series $L(f, s)$, where f is a certain cusp form on $\Gamma_0(35)$ of weight 4. At the bad primes 5 and 7 the Euler factors of the L-series are $(1+5^{1-s})^{-1}$ and $(1-7^{1-s})^{-1}$.

The functional equation shows $L(f,1) \neq 0$, and so the incomplete L-function $L_S(f,s)$ vanishes at $s=1$ if and only if $7 \in S$. Conjecture 4.2 predicts that there is a non-zero element $\xi \in K_2(V) \otimes \mathbf{Q}$, which is non-integral. We now give the construction of such an element.

Let $\infty \in X_\Gamma$ be the cusp $t = \infty$. The fibre E_∞ of ϕ is a Néron polygon, so there is a canonical (up to sign) inclusion $\mathbb{G}_m \hookrightarrow E_\infty$. Therefore (at any rate if the model V is sufficiently carefully chosen) the fibre V_∞ contains a copy of $\mathbb{G}_m \times \mathbb{G}_m = \operatorname{Spec} \mathbf{Q}[x_1, x_1^{-1}, x_2, x_2^{-1}]$. The element

$$x_1 \cup x_2 \in K_2(\mathbb{G}_m \times \mathbb{G}_m) \otimes \mathbf{Q}$$

can be shown to extend to an element of $K_2'(V_\infty) \otimes \mathbf{Q}$. By the functoriality of K'-theory with respect to the inclusion $V_\infty \hookrightarrow V$ we obtain an element

$$\xi \in K_2'(V) \otimes \mathbf{Q} = K_2(V) \otimes \mathbf{Q}.$$

Theorem. *ξ is non-zero.*

We can only give a vague idea of the proof here. It relies on the existence of the ℓ-adic regulator map

$$K_q(V) \to H^{2j-q}_{\text{ét}}(V, \mathbf{Z}_\ell(j))$$

which takes values in the ℓ-adic cohomology of $V/\mathbf{Q}$ (as distinct from that of $V/\overline{\mathbf{Q}}$). Here as usual $\mathbf{Z}_\ell(j)$ denotes the $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ module which is dual to the module of ℓ-power roots of unity, tensored with itself j times. In this case we obtain a class in $H^4(V, \mathbf{Q}_\ell(3))$, and from the Hochschild-Serre spectral sequence this maps to an element

$$\xi_\ell \in H^1(\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}), H^3_\ell(V) \otimes \mathbf{Q}_\ell(3))$$

or equivalently, to a class of extensions of Galois modules

$$0 \to H^3_{\text{ét}}(\overline{V}, \mathbf{Q}_\ell) \to (\text{extension}) \to \mathbf{Q}_\ell(-3) \to 0$$

This extension class is realised by a subquotient of the cohomology of the open variety $V - V_\infty - V'_\infty$ (where V'_∞ is the fibre at the other cusp $t = -1/7$). Using the theory of vanishing cycles, one then shows that the action of $\mathrm{Gal}(\overline{\mathbf{Q}}_7/\mathbf{Q}_7)$ on this cohomology is highly non-trivial, which is enough to prove the non-vanishing of ξ. Full details will appear elsewhere.

Remark: It should be noted that Γ is not a congruence subgroup. Indeed, for congruence subgroups the analogous elements to ξ are always trivial. This is an example of the "Manin-Drinfeld principle", and was proved by Beilinson in **[2]** by explicitly constructing elements of K-theory of the open varieties, analogous to modular units. I know of *no* examples of non-integral elements of the K-groups of Kuga-Sato varieties for congruence subgroups, and it would be of great interest to have a general construction of them.

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