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ON RIEMANN ZETA-FUNCTION AND ALLIED QUESTIONS

K. RAMACHANDRA

TO PROFESSOR R. BALASUBRAMANIAN
ON HIS FORTIETH BIRTHDAY

§ 1. **Introduction.** I will divide my lecture as follows.

§ 1. Introduction

§ 2. Conjectures 1 and 2.

§ 3. Conjecture 3 and a further question.

§ 4. Balasubramanian-Ramachandra (sufficient) condition for the validity of Conjecture 3.

§ 5. The Balasubramanian-Ramachandra condition is not always satisfied (Counterexample : a power of the Kahane series).

§ 6. Progress towards Conjectures 1 and 2 (Titchmarsh Series-I, Weak Titchmarsh Series, and Titchmarsh Series-II)

§ 7. Applications of Theorems 1 to 4 of § 6.

§ 8. Consequences of Conjectures 1 and 2 without Riemann Hypothesis.

§ 9. Further Conjectures (which would follow from Conjectures 1 and 2 and Riemann Hypothesis).

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As will be seen in § 7 Conjectures 1 and 2 are more intimately connected with the Riemann zeta-function (and also with zeta and L-functions of algebraic number fields, zeta-functions of ray class fields and so on). But we emphasise only on the Riemann zeta-function.

§ 2. **Conjectures 1 and 2.** For all $N \geq H \geq 1000$ and all N-tuples $a_1 = 1, a_2, \cdots, a_N$ of complex numbers prove (or disprove!) that

Conjecture 1.

$$\frac{1}{H}\int_0^H \Big|\sum_{n\leq N} a_n n^{it}\Big| \, dt \geq 10^{-1000}, \tag{1}$$

Conjecture 2.

$$\frac{1}{H}\int_0^H \Big|\sum_{n\leq N} a_n n^{it}\Big|^2 dt \geq (\log H)^{-1000} \sum_{n\leq M} |a_n|^2 \tag{2}$$

where $M = H(\log H)^{-2}$.

Remark 1. We can formulate conjectures where RHS in (1) and (2) are bigger than the present ones. For example we can replace RHS of (2) by

$$C_1 \sum_{n\leq C_2 H} |a_n|^2$$

where $C_1 > 0$ and $C_2 > 0$ are numerical constants (in fact with any $C_1 < 1$).

Remark 2. Let $1 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ be any sequence of real numbers with $\lambda_{n+1} - \lambda_n$ bounded both above and below. Then we can formulate more general conjectures where $\sum_{n\leq N} a_n n^{it}$ is replaced by $\sum_{n\leq N} a_n \lambda_n^{it}$.

§ 3. **Conjecture 3 and a further question.** Let $a_1 = \lambda_1 = 1$, $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$, $\frac{1}{C} \leq \lambda_{n+1} - \lambda_n \leq C$, $\{a_n\}$ $(n = 1, 2, 3, \cdots)$ a sequence of complex numbers such that $F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ is convergent for some $s = \sigma + it$ in the complex plane and can be continued analytically in $(\sigma \geq \beta, T \leq t \leq 2T)$ where β is a positive constant $< \frac{1}{2}$ and there the maximum of $|F(s)| \leq T^A$ where A and C are positive constants ≥ 1. Suppose $\sum_{n\leq x} |a_n|^2 \gg_\varepsilon x^{1-\varepsilon}$ for every $\varepsilon > 0$ and every $x \geq 1$. Then prove (or disprove) the

Conjecture 3. *The number of zeros of* $F(s)$ *in* $(\sigma \geq \beta, T \leq t \leq 2T)$ *is*

$$\gg T \log T. \tag{3}$$

Remark. A simple application of Jensen's Theorem (see $[T]_1$, page 125) shows that the number of zeros of $F(s)$ in $(\sigma \geq (\beta + \frac{1}{2})/2, T + D \leq t \leq 2T - D)$ is $\ll T \log T$, provided D is a large positive constant. Thus speaking roughly, the order of magnitude of the number of zeros in question is $T\log T$ (if the Conjecture 3 is true).

A further question. *Let $\beta(T) < \frac{1}{2}$ and $\beta(T) \to \frac{1}{2}$ as $T \to \infty$. Then study the lower bound for the number of zeros of $F(s)$ in*

$$(\sigma \geq \beta(T), T \leq t \leq 2T). \tag{4}$$

§ 4. **Balasubramanian-Ramachandra (sufficient) condition for the validity of Conjecture 3.**

Sufficient Condition. *There should exist a positive constant $\delta < \frac{1}{2} - \beta$ such that*

$$\frac{1}{T}\int_T^{2T} |F(\frac{1}{2} - \delta + it)|^2 \, dt \ll \left(\frac{1}{T}\int_T^{2T} |F(\frac{1}{2} - \delta + it)| \, dt\right)^2. \tag{5}$$

Remark 1. We will show in this section that the condition (5) is sufficient for the validity of Conjecture 3. (Note that the inequality which is the opposite of (5) is always true). From our proof it will be plain that the condition

$$\left(\frac{1}{T}\int_T^{2T} |F(\frac{1}{2} - \delta + it)|^g \, dt\right)^h \ll \left(\frac{1}{T}\int_T^{2T} |F(\frac{1}{2} - \delta + it)|^h \, dt\right)^g \tag{5'}$$

for some constants g, h with $0 < h < g$ is also sufficient.

Remark 2. A class of examples of $F(s)$ satisfying the condition (5) were studied in $[\text{BR}]_3, [\text{BR}]_4, [\text{R}_5]$, and to some extent $[\text{BR}]_5$. We content here by saying that a simple example of $F(s)$ satisfying (5) is $\zeta(s) + \sum_{n=1}^{\infty}(\chi(n)n^{-s})$ where $\chi(n)$ is any sequence of complex numbers with $\sum_{n\leq x}\chi(n) = O(1)$. For the series $\zeta(s) + \sum_{n=1}^{\infty}(\chi(n)n^{-s})$ where $\sum_{n\leq x}\chi(n) = O(x^{\frac{1}{2}-\eta})$ for some constant $\eta > 0$ and further $\chi(n) = O(1)$ the lower bound for the number of zeros in $(\sigma \geq (1-\eta)/2, T \leq t \leq 2T)$ is $\gg T(\log T)(\log\log T)^{-1}$. Both these results have been generalised to generalised Dirichlet series of the form $F(s) = \sum_{n=1}^{\infty}(a_n b_n \lambda_n^{-s})$ where $\{a_n\}$ and $\{b_n\}$ are somewhat general sequences of complex numbers and $\{\lambda_n\}$ is a somewhat general sequence of real numbers (see $[\text{BR}]_3, [\text{BR}]_4, [\text{R}]_5, [\text{BR}]_5$,). Less satisfactory results than Conjecture 3 were obtained in $[\text{R}]_3$, and $[\text{R}]_4$.

Remark 3. For suitable $\beta(t)$ the lower bounds for the number of zeros in (4) of $F(s)$ were studied in $[\text{R}]_6, [\text{BR}]_6$, and $[\text{BR}]_7$. For example in $[\text{R}]_6$, it was shown in great generality that $F(s)$ has $\gg T^{1-\varepsilon}$ zeros in $(\sigma \geq \frac{1}{2} - C_0(\log\log T)^{-1}, T \leq t \leq 2T)$. In $[\text{BR}]_6$, the functions mentioned in Remark 2 were considered. For example it was proved that the number of zeros of

$F(s) = \zeta(s) + \sum_{n=1}^{\infty}(\chi(n)n^{-s})$ $\left(\text{with } \chi(n) = O(1) \text{ and } \sum_{n\leq x}\chi(n) = O\left(x^{\frac{1}{2}-\delta}\right)\right)$ in $(\sigma \geq \frac{1}{2} - C_0(\log\log T)^{\frac{3}{2}}(\log T)^{-\frac{1}{2}}, T \leq t \leq 2T)$ is $\gg T(\log\log T)^{-1}$. In $[\mathrm{BR}]_7$, it was shown that (for $(\log T)^C \leq H \leq T$ where C is a large constant) in $(\sigma \geq \frac{1}{2} - C_0(\log\log T)(\log H)^{-1}, T \leq t \leq T+H)$ $\zeta(s)$ has $\gg H(\log\log\log T)^{-1}$ zeros. The interesting fact was that only the Euler product and the analytic continuation (in fact for the lower bound $\gg \cdots$ the upper bound $\log|F(s)| \ll (\log T)^C$ is enough)

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^{\infty}\left(n^{-s} - \int_n^{n+1} u^{-s}du\right), (\sigma > 0),$$

were used in the proof. Using these things only it is well known that we can prove that when $H = T$, the number of zeros is $\leq T^{\frac{3}{2}-\sigma}(\log T)^C$.

We now resume the Remark 1. We will show that (5) implies that there are points $t_1, t_2, \cdots, t_N$ (where $N \gg T$) with $|t_j - t_{j'}|$ bounded below whenever $j \neq j'$ and further

$$|F(\frac{1}{2} - \delta + it_j)| > T^{\frac{\delta}{10}}, \ (T \leq t_j \leq 2T). \tag{6}$$

After this we have only to apply Theorem 3 of § 4 in $[\mathrm{BR}]_3$ to obtain the proof of Conjecture 3. To deduce (6) from (5) we write

$$\psi(T) = \frac{1}{T}\int_T^{2T} |F(\frac{1}{2} - \delta + it)|\, dt. \tag{7}$$

Now RHS of (7) is

$$\frac{1}{T}\sum_I \int_I |F(\frac{1}{2} - \delta + it)|\, dt \tag{8}$$

where $\{I\}$ is a division of $[T, 2T]$ into disjoint (but abutting intervals I of equal length the length being both above and below). Plainly the expression (8) is

$$\leq \frac{2}{T}\sum_I{}' \int_I |F(\frac{1}{2} - \delta + it)|\, dt \tag{9}$$

where the accent denotes the restriction of the sum to intervals I for which the integral is not less than $\varepsilon\ \psi(T)$ for a small constant $\varepsilon > 0$. Now

$$\begin{aligned}
\psi(T) &\leq \tfrac{2}{T}\sum_I{}' \int_I |F(\frac{1}{2} - \delta + it)|\, dt \\
&\leq \tfrac{2}{T}\left(\sum_I{}' 1\right)^{\frac{1}{2}}\left(\sum_I{}'\left(\int_I |F(\frac{1}{2} - \delta + it)|\, dt\right)^2\right)^{\frac{1}{2}} \\
&\ll \tfrac{1}{T}\left(\sum_I{}' 1\right)^{\frac{1}{2}}\left(\sum_I{}' \int_I |F(\frac{1}{2} - \delta + it)|^2\, dt\right)^{\frac{1}{2}}.
\end{aligned}$$

We now sum over all I in the last sum and use (5). We get

$$\psi(T) \ll \frac{1}{T}\left(\sum_I{}'1\right)^{\frac{1}{2}}(T\psi^2)^{\frac{1}{2}}$$

and this in turn gives

$$\sum_I{}'1 \gg T.$$

This gives (6) (which as we have already stated yields the conjecture), provided we prove that $\psi(T) \gg T^{\frac{\delta}{9}}$. By (5) and (7) it suffices to prove that

$$\frac{1}{T}\int_T^{2T}|F(\frac{1}{2}-\delta+it)|^2\,dt \gg T^{\frac{2\delta}{9}}. \tag{10}$$

This will follow unconditionally from Theorem 2 of § 6 of the present paper.

§ 5. **The Balasubramanian-Ramachandra condition is not always satisfied (Counterexample : a power of the Kahane series).** In this section we will show that a suitable (bounded) positive integral power of the "KAHANE SERIES" serves as a counterexample to (5). Kahane series are series of the form

$$K(s) = \sum_{n=1}^{\infty}\varepsilon_n((2n-1)^{-s}-(2n)^{-s}) \tag{11}$$

where ε_n are arbitrary subject to $\varepsilon = \pm 1$. By an ingenious probabilistic argument J.-P. Kahane showed that there exists a sequence $\{\varepsilon_n\}$ for which the Lindelöf function $\mu(\sigma)$ equals $1-\sigma$ for $0<\sigma<1$. In particular given a fixed δ with $0<\delta<\frac{1}{2}$ there exists a sequence $\{t_n\}$ with $t_n = t_n(\delta)\to\infty$ such that

$$|K(\frac{1}{2}-\delta+it_n)| \gg_{\varepsilon,\delta} t_n^{\delta+\frac{1}{2}-\varepsilon} \tag{12}$$

(For reference see Theorem 2.2 equation (30) on page 139 of [K]). It is not hard to show (whatever be the choice of ε_n) that $K(s)$ is analytic in $\sigma \geq \varepsilon > 0$ and there it is $O(t)$. Also all the derivatives upto any fixed order are $O(t)$ (for instance by Cauchy's theorem). Moreover with some work it is not difficult to show (see for example Theorem 11 of $[\mathrm{R}]_5$) that

$$\frac{1}{T}\int_T^{2T}\left|K(\frac{1}{2}-\delta+it)\right|^2 dt \ll T^{2\delta} \tag{13}$$

and so

$$\frac{1}{T}\int_T^{2T}|K(\frac{1}{2}-\delta+it)|\,dt \ll T^{\delta}. \tag{14}$$

By iterating (5) we obtain

$$\frac{1}{T}\int_T^{2T}|K(\frac{1}{2}-\delta+it)|^k\,dt \ll \left(\frac{1}{T}\int_T^{2T}|K(\frac{1}{2}-\delta+it)|\,dt\right)^k \tag{15}$$

for all fixed $k = 2^m$ where m is an integer ≥ 1. Now for $|t - t_n| \leq t_n^{-1} (= \lambda \text{ say})$, we have $|K(\frac{1}{2} - \delta + it)| \gg_{\delta,\varepsilon} t^{\delta + \frac{1}{2} - \varepsilon}$ since as remarked above the derivative of $K(\frac{1}{2} - \delta + it)$ is $O(t)$. Hence

$$\int_{t_n-\lambda}^{t_n+\lambda} \left|K(\frac{1}{2} - \delta + it)\right|^k dt \gg_{\delta,\varepsilon} t_n^{k(\delta+\frac{1}{2}-\varepsilon)-1}. \tag{16}$$

Thus if (15) is true we obtain for $T = t_n - 1$ the estimate

$$T^{k(\delta+\frac{1}{2}-\varepsilon)-2} \ll_{\varepsilon,\delta} \frac{1}{T}\int_T^{2T} \left|K(\frac{1}{2} - \delta + it)\right|^k dt \ll T^{k\delta} \tag{17}$$

which is false for $k = 8$. Thus (15) is false for $k = 8$ and so at some stage of iteration (5) is false i.e. (5) is false for $F(s) = K(s), (K(s))^2$ or $(K(s))^4$. We can prove that (5) is false for $(K(s))$ or $(K(s))^2$ by a slight sharpening of the method above.

Remark. The procedure adopted in this section resembles to some extent an argument on page 329 of $[\mathrm{T}]_2$.

§ 6. Progress towards Conjectures 1 and 2 (Titchmarsh Series-I, Weak Titchmarsh Series, and Titchmarsh Series-II)

My paper $[\mathrm{R}]_7$ contains a result which is sufficient for all practical applications. Here we mention six theorems which solved the conjectures proposed by me in $[\mathrm{R}]_7$. These six theorems form further progress towards the conjectures 1 and 2 of § 2 of the present paper. We begin with a definition.

Titchmarsh series. *Let $a_1 = \lambda_1 = 1$, $\{a_n\}$ $(n = 1, 2, 3, \cdots)$ a sequence of complex numbers, $\{\lambda_n\}$ $(n = 1, 2, 3, \cdots)$ a sequence of real numbers with $\frac{1}{C} \leq \lambda_{n+1} - \lambda_n \leq C$, where C (≥ 1) is a constant. Let $H \geq 10$ be a real parameter and $|a_n| \leq (nH)^A$ for all n, where A is a positive integer constant. Suppose $F(s) = \sum_{n=1}^{\infty}(a_n\lambda_n^{-s}), (\sigma \geq A + 2)$ can be continued analytically in $(\sigma \geq 0,\ 0 \leq t \leq H)$. Such a function $F(s)$ is called Titchmarsh series.*

Theorem 1. *Put $u = H^{\frac{7}{8}} + 50(A + 2)\log\log(K + 1)$, where $K \geq 30$ and suppose that there exist T_1 and T_2 with $0 \leq T_1 \leq U, H - U \leq T_2 \leq H$ such that the Titchmarsh series $F(s)$ satisfies the inequality*

$$|F(\sigma + iT_1)| + |F(\sigma + iT_2)| \leq K \tag{18}$$

uniformly in σ for $0 \leq \sigma \leq A + 2$. Also let

$$H \geq (3456000A^2C^3)^{640000A} + (240000A)^{20}\log\log(K + 1). \tag{19}$$

Then, we have,

$$\int_0^H |F(it)|dt \geq H - 8000AH^{\frac{7}{8}} - 240000A^2\log\log(K + 1). \tag{20}$$

Theorem 2. *Suppose that (for some $K \geq 30$) there exist T_1 and T_2 with $0 \leq T_1 \leq H^{\frac{7}{8}}, H - H^{\frac{7}{8}} \leq T_2 \leq H$, such that the Titchmarsh series $F(s)$ satisfies the inequality*

$$|F(\sigma + iT_1)| + |F(\sigma + iT_2)| \leq K \tag{21}$$

uniformly in σ for $0 \leq \sigma \leq A + 2$. Also let

$$H \geq (4C)^{9000A^2} + 520000A^2 \log\log K_1 \tag{22}$$

where $K_1 = (HC)^{12A}K$. Then, we have,

$$\int_0^H |F(it)|^2 dt \geq \sum_{n \leq \alpha H} (H - (3C)^{1000A} H^{\frac{7}{8}} - 130000A^2 \log\log K_1 - 100C^2 n) |a_n|^2 \tag{23}$$

where $\alpha = (200C^2)^{-1} 2^{-8A-20}$.

Remark. These theorems are consequences of the second and the third main theorems of the paper [BR]$_8$. We can get Theorem 1 above by putting $\varepsilon = \frac{1}{2}$, $r = 800A$ in the second main theorem.

Our next two theorems deal with

Weak Titchmarsh Series. *Let $0 \leq \varepsilon < 1, D \geq 1, C \geq 1$ and $H \geq 10$. Put $R = H^\varepsilon$. Instead of $|a_n| \leq (nH)^A$ suppose that*

$$\sum_{\lambda_n \leq X} |a_n| \leq D(\log X)^R \tag{24}$$

holds for all $X \geq 3C$. Then, for complex $s = \sigma + it, \sigma > 0$ we refer to $F(s) = \sum_{n=1}^{\infty} (a_n \lambda_n^{-s})$ as a weak Titchmarsh series associated with the parameters occuring in this definition

Theorem 3. *For a weak Titchmarsh series $F(s)$ with $H \geq 36C^2H^\varepsilon$, we have*

$$\liminf_{\sigma \to 0+} \int_0^H |F(\sigma + it)|\, dt \geq H - 36C^2H^\varepsilon - 12CD. \tag{25}$$

Theorem 4.
For a weak Titchmarsh series $F(s)$ with $\log H \geq 4320C^2(1-\varepsilon)^{-5}$, we have

$$\liminf_{\sigma \to 0+} \int_0^H |F(\sigma + it)|^2\, dt \geq \sum_{n \leq M} \left(H - \frac{H}{\log H} - 100C^2 n\right) |a_n|^2 - 2D^2 \tag{26}$$

where $M = (36C^2)^{-1} H^{1-\varepsilon} (\log H)^{-4}$.

Remark. The four theorems mentioned above are enough for all applications in the present state of knowledge. For Theorems 3 and 4 see [R]$_8$.

The next two theorems represent a further progress towards conjectures 1 and 2 although *no new applications* are possible in the present state of knowledge. These two theorems form the subject matter of the paper $[BR]_9$.

Theorem 5. *Let* $0 \le \varepsilon < 1$, $C \ge 1$, $D \ge 1$, $E = \frac{1}{1-\varepsilon}$, $a_1 = 1 = \lambda_1$, $\frac{1}{C} \le \lambda_{n+1} - \lambda_n \le C$ *(for* $n = 1, 2, 3, \cdots$*) and for* $n \ge 2$ *(instead of* $|a_n| \le (nH)^A$*)*

$$|a_n| \le \mathrm{Exp}\{(DH^\varepsilon - 100C - 1)\log \lambda_n\}$$

where $H \ge 10$. *Let* $F(s) = \sum_{n=1}^{\infty}(a_n \lambda_n^{-s})$ *(convergent absolutely in* $\sigma \ge DH^\varepsilon$*) admit an analytic continuation in* $(\sigma \ge 0, 0 \le t \le H)$. *Assume that there exist* T_1, T_2 *with* $0 \le T_1 \le \frac{1}{8}H$, $\frac{7}{8}H \le T_2 \le H$, *such that, for some* $K \ge 30$,

$$|F(\sigma + iT_1)| + |F(\sigma + iT_2)| \le K$$

holds uniformly in $\sigma \ge 0$. *Let finally*

$$H \ge \max\{100D \ \log\log K)^E, 100D(100DE)^{3E}\}. \tag{27}$$

Then

$$200 \int_0^H |F(it)|\, dt \ge H. \tag{28}$$

Theorem 6. *Let* $0 \le \varepsilon < 1$, $C \ge 1$, $D \ge 2560C^2$, $E = \frac{1}{1-\varepsilon}$. *Let* $\{a_n\}$ *and* $\{\lambda_n\}$ *be as in Theorem 5,*

$$H \ge \max\{(256\ D\ \log\log K)^E, (24000\ C^6 DE)^{3E}\}. \tag{29}$$

Suppose $F(s)$ *has the properties stated in Theorem 5 with* K *defined exactly as in Theorem 5. Then, we have,*

$$\frac{10^8}{H} \int_0^H |F(it)|^2\, dt \ge \sum_{n \le M_1} |a_n|^2 \tag{30}$$

where $M_1 = [(8000C^6 D)^{-1} H^{1-\varepsilon}]$.

§ 7. **Applications of Theorems 1 to 4 of §6.** Theorems 1 to 4 have a lot of applications. We mention only a few.

Theorem 7. *For all integers* $k \ge 1$ *and for all* H *satisfying* $C(k)\log\log T \le H \le T$, *we have,*

$$\frac{1}{H}\int_T^{T+H} \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt \ge C_k'(\log H)^{k^2}\left(1 + O\left(\frac{\log\log T}{H} + \frac{1}{\log H}\right)\right) \tag{31}$$

where $C(k) (> 0)$ *depends only on* k *and*

$$C_k' = (\Gamma(k^2+1))^{-1} \prod_p \left\{(1 - p^{-1})^{k^2} \sum_{m=0}^{\infty} \left|\frac{\Gamma(k+m)}{m!\,\Gamma(k)}\right|^2 p^{-m}\right\}.$$

Proof. In Theorem 2 take $F(s) = (\zeta(\frac{1}{2}+s+iT))^k$. Theorem 2 shows that the LHS of (31) is

$$\geq \sum_{n\leq \alpha H}\left(1+O\left(\frac{\log\log T}{H}+\frac{1}{\log H}\right)\right)\frac{|d_k(n)|^2}{n}. \tag{32}$$

Using the well-known result that $\sum_{n\leq x}(|d_k(n)|^2 n^{-1}) = C'_k(\log x)^{k^2}\left(1+O\left(\frac{1}{\log x}\right)\right)$ we obtain Theorem 7.

Remark 1. If $C(k,\varepsilon)\leq H\leq C(k)\log\log T$ the only result known is

$$\int_{|t-t_0|\leq C(k,\varepsilon)}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}dt > t_0^{-\varepsilon}.$$

Plainly we can make ε a function of t_0 (and k) which tends to zero as $t_0\to\infty$ for fixed k. For this and other results see $[\mathrm{BR}]_{10}$.

Remark 2. The result

$$\frac{1}{H}\int_T^{T+H}\left|\frac{d^\ell}{dt^\ell}\left(\zeta\left(\frac{1}{2}+it\right)\right)^k\right|dt \gg (\log H)^{\frac{1}{4}k^2+\ell}$$

was proved for positive integral k in $[\mathrm{R}]_{10}$. (In $[\mathrm{R}]_9$ a weaker result was proved; but it dealt with positive real k). The result

$$\frac{1}{H}\int_T^{T+H}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}dt \gg (\log H)^{k^2}$$

was proved for all positive rational constants k in $[\mathrm{R}]_{12}$ by using an idea of D.R. Heath-Brown [H]. Later this was extended in $[\mathrm{R}]_{13}$ for positive irrational constants k but with the lower bound $\gg \left(\frac{\log H}{\log\log H}\right)^{k^2}$. In $[\mathrm{R}]_{14}$ this was sharpened (the result involved the s.c.f. expansion of k). For example as a special case of a general result it was shown that

$$\frac{1}{H}\int_T^{T+H}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2\sqrt{2}}dt \gg (\log H)^2(\log\log H)^{-1}.$$

The general result contains as a special case the following result. Let k be any real number satisfying $a\leq k\leq b$ where $0<a\leq b<\infty$. It is easy to prove that there exist integers p,q with $1\leq q\leq 10\ \log\log H$ such that $|k-\frac{p}{q}|\leq(\log\log H)^{-1}$. For any such q we have for $C\ \log\log T\leq H\leq T$,

$$\frac{1}{H}\int_T^{T+H}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}dt > D\left(\frac{\log H}{q}\right)^{k^2}$$

where C and D are certain positive constants depending only on a and b. When the s.c.f. expansion of k has bounded partial quotients, q can always be chosen to lie between two constant multiples of $(\log\log H)^{\frac{1}{2}}$. This leads to the special case mentioned above. All the results mentioned in this remark are subject to $T \geq H \gg \log\log T$. See the final remark at the end of the present paper.

Remark 3. Results like

$$\frac{1}{H}\int_T^{T+H}\left|\zeta^{(\ell)}\left(\frac{1}{2}+it\right)\right|dt \ll (\log T)^{\frac{1}{4}+\ell}$$

proved in $[\mathrm{R}]_{11}$ with $H = T^{\frac{1}{2}+\varepsilon}$ perhaps have some relavance here. It was also proved in $[\mathrm{R}]_{11}$ that the same upper bound holds with $H = T^{\frac{1}{4}+\varepsilon}$ if we assume Riemann hypothesis.

Theorem 8. *For $C_0 \ \log\log T \leq H \leq T$, we have,*

$$\max_{T\leq t\leq T+H}\left|\zeta\left(\frac{1}{2}+it\right)\right| > \mathrm{Exp}\left(\frac{3}{4}\sqrt{\frac{\log\ H}{\log\log H}}\right), \tag{33}$$

where C_0 is an effective positive constant.

Remark. Factors like $(\log H)^{-1000}$ in place of $\frac{1}{2}$ in (34) below are enough for application to prove Theorem 8. Such results were already available in $[\mathrm{R}]_1, [\mathrm{R}]_7$. In fact we mentioned Theorem 8 in $[\mathrm{R}]_7$. All that was needed was the study of $(\mathrm{RHS})^{\frac{1}{2k}}$ as a function of k in $1 \leq k \leq \log H$ which was done in [B] of course by an ingenious development of $[\mathrm{BR}]_1$. In $[\mathrm{BR}]_1$ a smaller constant than $\frac{3}{4}$ was obtained.

Proof. Put $F(s) = (\zeta(\frac{1}{2}+iT+s))^k$ where $k \geq 1$ is an integer. Then $|a_n| \leq n^2\sum_{m=1}^{\infty}(|a_m|\,m^{-2}) \leq n^2(\zeta(2))^k \leq (nH)^{10}$ if $k \leq \log H$. Note that in $(\sigma \geq 0, 0 \leq t \leq H)$ we have $|F(s)| \leq T^{2k}$ so that we can take $K = T^{3k}$. We can take $A = 2$. Notice that for $k \leq \log H$ we have $\log\log\ K_1 \leq 1000(\log\log H + \log\log T)$. This proves that

$$\frac{1}{H}\int_T^{T+H}\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k}dt \geq \frac{1}{2}\sum_{n\leq\alpha H}(d_k(n))^2 n^{-1} \tag{34}$$

provided $1 \leq k \leq \log H$. Here an asymptotic formula was obtained in [B] for the quantity $\max(\log(\mathrm{RHS})^{\frac{1}{2k}})$ the maximum being taken over $1 \leq k \leq \log H$. In fact the quantity in question is asymptotic to $C_3(\log\ H)^{\frac{1}{2}}(\log\log H)^{-\frac{1}{2}}$ where $C_3 = 0.75\cdots > \frac{3}{4}$. This proves Theorem 8.

Theorem 9. *Let $\frac{1}{2} < \sigma < 1$ and $C_0\ \log\log T \leq H \leq T$. Then*

$$\max_{T\leq t\leq T+H}|\zeta(\sigma+it)| > \mathrm{Exp}(C_\sigma(\log H)^{1-\sigma}(\log\log H)^{-1}). \tag{35}$$

Remark. Better inequalities, when we take the maximum over $1 \leq t \leq T$ in Theorem 9 are due to H.L. Montgomery (see [M]) who proved in this case that the maximum $> \text{Exp}\left(\frac{1}{20}\left(\sigma - \frac{1}{2}\right)^{\frac{1}{2}} (\log\ T)^{1-\sigma}(\log\log T)^{-\sigma}\right)$. In [RS] this was improved to $\text{Exp}(C_\sigma(\log T)^{1-\sigma}(\log\log\ T)^{-\sigma})$ where $C_\sigma = C' L^{1-\sigma}(1-\sigma)^{-1}$, C' being a positive numerical constant and $L = \sigma - \frac{1}{2}$ or 1 according as we do not or do assume Riemann hypothesis. The method was a sharpening of the arguments of [M].

Proof. The proof is similar to that of Theorem 8. The study of

$$\max_{1\leq k\leq \log H} \log\left\{\frac{1}{2}\sum_{n\leq \alpha H}((d_k(n))^2 n^{-2\sigma})\right\}^{\frac{1}{2k}}$$

is much simpler. We find that this is $\gg (\log\ H)^{1-\sigma}(\log\log H)^{-1}$, by taking $n_0 = 2.3\cdots p_m \leq \alpha H$ as large possible and ignoring terms with $n \neq n_0$.

Remark. A theorem analogous to Theorems 8 and 9 for $\sigma = 1$ can be proved but we will not state it here. As a corollary to Theorem 4 we prove Theorem 10 which is a much more powerful theorem.

Theorem 10. *Let $z = e^{i\theta} = C + iS$ (C not to be confused with a constant). Put*

$$f(H) = \min_{T\geq 1}\ \max_{T\leq t\leq T+H} |(\zeta(1+it))^z|, \tag{36}$$

and $\lambda(\theta) = \prod_p \lambda_p(\theta)$, where

$$\lambda_p(\theta) = \left(1 - \frac{1}{p}\right)\left(\sqrt{1 - \frac{S^2}{p^2}} - \frac{C}{p}\right)^{-C} \text{Exp}\left(S\ \text{Sin}^{-1}\frac{S}{p}\right). \tag{37}$$

Then (for all $H \geq H_0$, a fixed numerical constant), we have,

$$|f(H)e^{-\gamma}(\lambda(\theta))^{-1} - \log\log H| \leq \log\log\log H + O(1). \tag{38}$$

Remark. We give a rough sketch of the proof. For details see $[\text{R}]_2$. The idea in (a) below can be traced to $[\text{BR}]_1$.

Proof. (a) *Lower bound for $f(H)$.* Put $F(s) = (\zeta(1 + iT + s))^z$. The a_n's are multiplicative. To obtain a lower bound for

$$\max_{1\leq k\leq \log H}\left(\frac{1}{2}\sum_{n\leq x}\left|\frac{a_n}{n}\right|^2\right)^{\frac{1}{2k}}$$

write $a_n = a(n)$. We choose $p_1, p_2, \cdots, p_\ell$ (the first ℓ primes) and exponents $b_1, b_2, \cdots, b_\ell$ such that

(i) $p_1^{b_1} \cdots p_\ell^{b_\ell} \leq x$

and where for each fixed p_j, b_j are so chosen that

(ii) $|a(p_j^{b_j})p_j^{-b_j}|^2$

is maximum. This gives the lower bound for $f(H)e^{-\gamma}(\lambda(\theta))^{-1}$. (For details see [BRS]).

(b) *Upper bound for* $f(H)$. For this we use the following Lemma from $[\mathrm{BR}]_2$ which is quoted in $[\mathrm{R}]_2$ (page 29) and we quote it here.

Lemma. *Let* $T = \mathrm{Exp}((\log H)^2)$ *where* H *exceeds a certain positive constant. Then there exists a sub-interval* I *of* $[T, 2T]$ *of length* $H + 2(\log H)^{10}$ *such that the rectangle* $(\sigma \geq \frac{3}{4}, t \in I)$ *does not contain any zero of* $\zeta(s)$ *and moreover*

$$\max |\log \zeta(\sigma + it)| = O((\log H)^{\frac{1}{4}} (\log\log H)^{-\frac{3}{4}})$$

the maximum being taken over the rectangle referred to (and $\log \zeta(s)$ *is the branch for which* $\log \zeta(s) = \log \frac{\pi^2}{6}$ *is real).*

From this lemma the upper bound for $f(H)$ follows with some further arguments (for details see $[\mathrm{R}]_2$).

(a) and (b) complete the sketch of the proof of Theorem 10.

Theorem 11. *Let* $\zeta(s,a) = \sum_{n=0}^{\infty} (n+a)^{-s}$ *where* $0 < a \leq 1$ *be the Hurwitz zeta-function in* $\sigma > 1$. *Consider its analytic continuation in* $\sigma \geq 1$. *Then, we have,*

$$\min_{T \geq 1} \int_T^{T+H} |\zeta(1+it, a)| \, dt \geq \frac{1}{a} H + O(H). \tag{39}$$

Proof. Follows from Theorem 3 of § 6. For reference see $[\mathrm{R}]_8$.

§ 8. Consequences of Conjectures 1 and 2 without Riemann Hypothesis.

Theorem 12. *We have, for* $C_0 \leq H \leq T$,

$$\frac{1}{H} \int_T^{T+H} |\zeta(\sigma + it)| \, dt \geq C_4, \tag{40}$$

where C_4 *is a positive numerical constant.*

Proof. We approximate to $\zeta(s)$ by $\sum_{n=1}^{\infty} \left(n^{-s} \mathrm{Exp}\left(-\frac{n}{X}\right)\right)$, where X is large and apply Conjecture 1. This gives Theorem 12.

Theorem 13. *In Theorems* 7, 8 *and* 9 *the condition on* H *can be relaxed to* $C(k) \leq H \leq T, C_0 \leq H \leq T$ *and* $C_0 \leq H \leq T$ *respectively. The results are still valid.*

Proof. We approximate to $(\zeta(s))^k$ by $\sum_{n=1}^{\infty} (d_k(n) n^{-s} \mathrm{Exp}(-\frac{n}{X}))$ where X (a function of T) is chosen to be large enough. The rest of the proof is the same. We have simply to use Conjecture 2.

§ 9. Further Conjectures which follow from Conjectures 1,2 and Riemann Hypothesis.

Theorem 14. *We have, with* $z = e^{i\theta}$,

$$\max_{T \leq t \leq T+H} \left| \left(\zeta \left(\frac{1}{2} + it \right) \right)^z \right| \geq \operatorname{Exp} \left(\frac{3}{4} \sqrt{\frac{\log H}{\log \log H}} \right) \tag{41}$$

provided $C_0 \leq H \leq T$. *Also under the same conditions*

$$\max_{T \leq t \leq T+H} |(\zeta(\sigma + it))^z| \geq \operatorname{Exp} \left(C_\sigma \frac{(\log H)^{1-\sigma}}{\log \log H} \right) \tag{42}$$

where σ *is fixed subject to* $\frac{1}{2} < \sigma < 1$.

Remark 1. We interpret $(\zeta(s))^z$ as its analytic continuation from $\sigma \geq 2$ along lines parallel to the real axis which are free from zeros of $\zeta(s)$.

Remark 2. For some unconditional (weaker) results in the direction of Theorem 14 see [BRS], (to appear).

Proof. Put $F(s) = (\zeta(\frac{1}{2} + iT + s))^{zk}$ and proceed as in the proof of Theorem 8. The investigations of [B] necessary for (41) are obviously valid.

Theorem 15. *Theorem 7 is valid for complex constants* k *with* $|(\zeta(\frac{1}{2} + it))^k|^2$ *in place of* $|\zeta(\frac{1}{2} + it)|^{2k}$, *provided in (31) and also in the definition of* C'_k *we replace* k^2 *by* $|k^2|$ *wherever* k^2 *occurs. Moreover the condition on* H *can be relaxed to* $C(k) \leq H \leq T$.

Final remark. Most of the theorems published earlier depended on the condition $(\log T)^\varepsilon \leq H \leq T$, (see [I] in particular pages 231-249). But it is a routine matter to change the kernel $\operatorname{Exp}(z^{4\alpha+2})$ to $\operatorname{Exp}((\operatorname{Sin} z)^2)$ and obtain the same results in the wider range $C_0 \log \log T \leq H \leq T$.

Postscript. The previous history of Theorem 7 is due to J.B. CONREY AND A. GHOSH who proved that if $k > 0$ is real then

$$\frac{1}{T} \int_0^T \left| \zeta(\frac{1}{2} + it) \right|^{2k} dt \geq (C'_k + o(1))(\log T)^{k^2}$$

provided we assume Riemann hypothesis. (See p.181 of $[T]_2$ or [CG]. It should be mentioned that $C'_1 = 1$ and $C'_2 = (4\pi^2)^{-1}$ and that when $k = 2$, LHS $\sim 2C'_2(\log T)^4$.

Regarding the "further question" figuring after Conjecture 3, R. BALASUBRAMANIAN and K. RAMACHANDRA have made considerable progress after their paper $[BR]_7$. We can reach $\beta(T) = \frac{c \log\log T}{\log T}$, $\beta(T) = \frac{c \log\log\log T}{\log T}$ and even $\beta(T) = \frac{c}{\log T}$ without assuming an Euler product hypothesis. See the forthcoming papers $[BR]_{11}, [BR]_{12}$ and $[R]_{15}$.

Actually the paper $[BR]_{12}$ deals also with the zeros of a class of generalised Dirichlet series in $\sigma \geq \frac{1}{2} + \delta$ where $\delta(> 0)$ is a constant. The proof of such results makes use of the result of the $[BR]_{13}$.

Continuing the work $[BR]_{12}$, myself and R. BALASUBRAMANIAN have recently succeeded in proving the following two theorems.

Theorem 16. *Let μ be any constant satisfying $0 < \mu < 1$, and let a be any complex constant. Then there exists a constant $\delta > 0$ such that $\zeta'(s) - a$ has $\gg T^\mu$ well-spaced zeros in $(\sigma \geq \frac{1}{2} + \delta,\ T \leq t \leq T + T^\mu)$ for all T exceeding a certain positive constant T_0.*

Remark. The term "well-spaced set of points" means, as usual, that the differences taken positively, between the ordinates of any two points is bounded below by a positive constant.

Theorem 17. *Let δ and μ be any two positive constant satisfying $0 < \delta < 1$ and $0 < \mu < 1$. Let a be any non-zero complex constant. Then $\zeta(s) - a$ has $\gg T^\mu$ well-spaced zeros in $(\sigma \geq 1 - \delta,\ T \leq t \leq T + T^\mu)$ for all T exceeding a certain positive constant T_0.*

Remark. The results in theorems 16 and 17 have actually been proved to hold good for more general Dirichlet series in place of $\zeta'(s) - a$ and $\zeta(s) - a$. The references to these works are respectively $[BR]_{14}$ and $[BR]_{15}$.

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