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AVY SOFFER

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On the Many Body Problem in Quantum Mechanics

Avy Soffer*

Section 1. Introduction

The aim of these lectures is to describe some of the modern mathematical techniques of N-body Scattering and with particular mention of their relations to other fields of analysis.

Consider a system of N quantum particles moving in $\mathbb{R}^n$, interacting with each other via the pair potentials V_α; the Hamiltonian (with center of mass removed) for such a system is given by

$$H = -\Delta + \sum_{i<j} V_{ij}(x_i - x_j) \quad \text{on} \quad L^2(\mathbb{R}^{n\,N-n}) \ .$$

Here $1 \le i, j \le N$, $x_i \in \mathbb{R}^n$. $-\Delta$ is the Laplacian on $L^2(\mathbb{R}^{n\,N-n})$ with metric

$$x \cdot y = \sum_{i=1}^{N} m_i x_i \cdot y_i \qquad ; \quad m_i > 0 \ .$$

The m_i are the masses of the particles. The main problem of scattering theory is to describe the spectral properties of H and find the asymptotic behavior of $e^{-iHt}\varphi$ for $\varphi \in L^2$, as $t \to \pm\infty$.

There are two reasons for that: one, the behavior is much simpler as $t \to \pm\infty$. Secondly it determines the full properties of the system. Since the

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sum $\sum_{i<j} V_{ij}$ does not vanish as $|x| \to \infty$ in certain directions, the perturbation of $-\Delta$ is not negligible at infinity. The spectral properties and asymptotic behavior of H are therefore radically different than that of $-\Delta$.

This is the generic multichannel problem. There are many different asymptotic behaviors possible, depending on the choice of φ. Thus the main theorem can be phrased as: given $\varphi \in L^2(\mathbb{R}^n)$, find hamiltonians H_a and functions $\varphi_a^{\pm}$, s.t.

$$e^{-iHt}\varphi - \sum_a e^{-iH_a t}\varphi_a^{\pm} \approx 0 \qquad \text{as } t \to \pm\infty .$$

Accepting the physicist's dogma that every state of the system is described asymptotically in terms of *particles* (or bound clusters of particles) we conclude that the only possible H_a are the subhamiltonians of the system:

$$\begin{aligned} H_a &= H - I_a \\ I_a &\equiv \sum_{(i,j)\subset a} V_{ij}(x_i - x_j) \end{aligned}$$

and a stands for arbitrary disjoint cluster decomposition of $\{1, 2, \ldots, N\}$.

I_a is called the intercluster interaction. The Hamiltonian that describes the bound clusters of a decomposition a, is denoted by H^a. Not much is known for Multichannel Non Linear Scattering; see however [Sof-We and cited ref.].

The approach to studying $e^{-iHt}\psi$ for large $|t|$ is by first reducing the problem via channel decoupling (or other methods) to the study of the localization in the phase space of $e^{-iHt}\psi$. Then, we develop a theory of propagation in the phase space for H. The channel decoupling is achieved by constructing a partition of unity of the space, with two main properties: one, on the support of each member of the partition the motion $e^{-iHt}\psi$ is simple (= one channel) and can be described by one fixed hamiltonian. The second property is that the boundary of the partitions is localized in regions where we can prove that no propagation of $e^{-iHt}\psi$ is possible there for large times; in this way we conclude that no switching back and forth between channels is possible as $|t| \to \infty$ which implies the desired results.

The first part, based on the construction of partitions of unity relies mainly on geometric analysis combined with the kinematics of (freely) moving particles. Different techniques are now known, each with its own importance, and I will describe some of the main constructions. The second part of the proof is analytic; it provides an approach to finding the asymptotic behavior of $e^{-iHt}\psi$ as $|t| \to \infty$, which is *complementary to that of stationary phase.* As I will describe below it replaces the (central) notion of *oscillation* by that of *microlocal monotonicity.* The distinctive feature of this approach allows the study of general pseudo differential operators H on equal footing with constant coefficient operators.

The first proof of Asymptotic Completeness (AC) for N-body systems along these lines was given in [Sig-Sof1]. Since then, different proofs were developed, with new useful implications [Der1, Kit, Gr, Ta] (see also [En2, Ger2-3]). Further developments concentrated on the long range problem. The three body case was first solved by Enss [En2]. (See also [Sig-Sof3].) Local decay and minimal and maximal velocity bounds were proved for N-body hamiltonian, including ones with time dependent potentials in [Sig-Sof2]. This approach is further utilized in [Sk, FrL, Ger2 , Ger-Sig, H-Sk]. A method of dealing with the problem of AC for long range many body scattering is developed in [Sig-Sof4,5]; the case of $N = 4$ is solved there.

A final comment; the phase space approach to N-body scattering originated with the fundamental works of Enss [En1,2]. A comprehensive description of the Enss method can be found in [Pe], including applications to many problems in spectral theory. References of many of by now classical results, including the works of Mourre, until about 1983 can be found in [CFKS]. We refer the reader to this book also as the basic reference used here on spectral and scattering theory.

Section 2. Microlocal Propagation Theory

Let H be a self adjoint operator on $L^2(\mathbb{R}^n)$ arising from the quantization of a classical Hamiltonian h. By solving the Hamilton-Jacoby equations for h it makes sense to talk about the classical trajectories (or bi-characteristics) of h (or H). As $t \to \pm\infty$ the (unbounded) trajectories concentrate, in general, in a certain set of the phase space.

DEFINITION 2.1. A bounded p.d.o. j with symbol homogeneous of degree 0 in x is said to be supported away from the *propagation set* (at energy E) of H if the following estimate holds

$$\int_{\pm 1}^{\pm\infty} \| \frac{1}{\langle x\rangle^{1/2}} j\, e^{-iHt}\psi\|^2 dt \leq c\|\psi\|^2 \quad \text{for all} \quad \psi = E_\Omega(H)\psi \; .$$

Here $\langle x\rangle^2 \equiv 1 + x^2$, $E_\Omega(H)$ is the spectral projection of H, with Ω any sufficiently small interval containing E.

Our aim is to identify the (conical) set PS_E of the phase space, with the property that any j is supported away from the propagation set in the sense of the above definition if and only if it is supported away from PS_E. We can therefore think of PS_E as the *propagation set of* H *at energy* E.

The main tool to proving that a given conical set $\tilde{K}$ is away from the propagation set PS_E will be to prove (microlocal) monotonicity of the flow generated by H in $\tilde{K}$.

The claim is that the *classical* flow generated by H is moving out of any such $\tilde{K}$ monotonically in t, for large t. By finding a lower bound for this monotone flow in $\tilde{K}$ we can then absorb the effects of quantization and other potential perturbations of H.

I chose to describe the above approach first when applied to $H = -\Delta$, and along the way prove some known and new smoothing estimates for $-\Delta$. The proofs are easy but allow the introduction of some of the other fundamental notions and arguments repeatedly used later.

DEFINITION 2.2. The Heisenberg derivative of an operator family $F(t)$, $DF(t)$, w.r.t. to H is defined by

$$DF(t) \equiv i[H, F] + \frac{\partial F}{\partial t} \; ;$$

DEFINITION 2.3. A bounded family of linear operators $F(t)$ on $L^2(\mathbb{R}^n)$ is called a *propagation observable* for H if its Heisenberg derivative is positive –

lower order terms. For some $\theta > 0$:

$$DF(t) \geq \theta B^*B - O((B^*B)^{1-\varepsilon}) - O(L^1(dt))$$

we then say that $DF(t)$ majorates B^*B ($DF(t) \geq \theta B^*B$).

Basic Lemma 2.4 *Let $F(t)$ be a propagation observable which majorates B^*B. Then*

$$\int_{\pm 1}^{\pm\infty} \|Be^{-iHt}\psi\|^2 dt \leq c\|\psi\|^2 \ .$$

The proof follows by the fundamental theorem of calculus and Heisenberg equations of motion:

$$\frac{d}{dt}(e^{iHt}\psi, F\,e^{-iHt}\psi) = (e^{iHt}\psi, DF\,e^{-iHt}\psi) \ . \qquad \blacksquare$$

The Basic Lemma reduces the proof that a given j is supported away from PS to finding a propagation observable majorating $j^*\langle x\rangle^{-1}j$. When F is chosen to be a p.d.o., one can often use Görding's inequality to check majoration, which reduces the problem to finding a lower bound for the Poisson bracket $\{h, f\}$.

Theorem 2.5 (Microlocal Smoothing Estimate) *Let j be a bounded homogeneous of degree* 0 *(in x) symbol, with support away from*

$$PS \equiv \{(x,\xi) \in T^*X \mid x\|\xi\} \ .$$

Then

a) $$\int_0^T \left\| \frac{1}{\langle x\rangle^{1/2}} J\langle p\rangle^{1/2} e^{+i\Delta t}\psi\right\|^2 dt \leq C_T\|\psi\|^2 \quad \psi \in L^2(\mathbb{R}^n)$$

b) $$\int_0^T \|\frac{1}{\langle x\rangle^{1/2+\varepsilon}}\langle p\rangle^{1/2} e^{+i\Delta t}\psi\|^2 dt \leq c_T\|\psi\|^2 \ .$$

Furthermore, in case the dimension $n \geq 3$, C_T can be chosen independent of T. The same is true in any dimension if $\hat{\psi}(\xi)$ is supported away from 0.

REMARK. Part b) of the theorem is known as local smoothing estimate. If was proved in [Co-S, Sj, V] (see also [Be-K, G-V2, Ka-Ya]) and found since then many important applications in both linear and nonlinear PDE see e.g. [JSS], [KPV].

PROOF. The proof for a general H replacing Δ is given in [So2]. Here I sketch the main steps: By the Basic Lemma we have to find operators F_1, F_2 bounded and s.t.

$$DF_1 \geq \langle x\rangle^{-1/2} J\langle p\rangle J^* \langle x\rangle^{-1/2} + O(1)$$

and

$$DF_2 \geq \langle x\rangle^{-1/2-\varepsilon/2} \langle p\rangle \langle x\rangle^{-1/2-\varepsilon/2} + O(1)$$

where $O(1)$ stands for an operator of order zero (in ξ). Using p.d. calculus it is easy to check that

$$F_i \equiv \hat{\gamma}_i \equiv (\hat{x}_i \cdot \hat{p} + \hat{p} \cdot \hat{x}_i)\frac{1}{2} \qquad i = 1, 2$$

satisfy both of the above;

$$\hat{x}_i \equiv x/(1 + x^2 + g_i(x))^{1/2} \qquad \hat{p} \equiv p/\langle p\rangle \quad g_1(x) \equiv 0, g_2(x) \equiv |x|^{2-\varepsilon}\ .$$

REMARK 1.. The original proofs of b) uses stationary phase analysis, and therefore does not extend to cases where the kernel of e^{-itH} is not explicitly constructible, e.g. $H = -\Delta + V$, V singular. The above argument trivially extends to such general H.

REMARK 2. The above theorem shows that the notion of propagation set is relevant also for finite time behavior of $e^{-iHt}\psi$

The operator $\hat{\gamma}$ comes from regularizing the operator $\gamma = \frac{1}{2}(\hat{x} \cdot p + p \cdot \hat{x})$. Different versions of γ appeared in scattering theory [L, M1-2, M-R-S].

Its centrality for the N–body problem was first realized in [Sig-Sof1]. To see its importance, let us compute the Poisson bracket of ξ^2 with the symbol of γ

$$\{\xi^2, \hat{x}\cdot\xi\}_{PB} = 2\xi\cdot\nabla_i(\hat{x}\cdot\xi) = \frac{2}{\langle x\rangle}(\xi^2 - (\hat{x}\cdot\xi)^2)$$

and it is clear that the above bracket is positive $(0(\frac{1}{\langle x\rangle}\xi^2))$ iff (x,ξ) is localized away from $\{\hat{x}\|\xi\}$. We can therefore identify the *PS* of $-\Delta$ with $\{\hat{x}\|\xi\}$, which is not surprising since $x(t) = x_0 + 2\xi t$ are the classical trajectories of $-\Delta$, and they concentrate where $\frac{x}{t} = 2\xi$.

Section 3. Hamiltonians and Kinematics

Consider an N–body system in the physical space R^ν. The configuration space in the center–of–mass frame is

$$X = \{x \in R^{\nu N} \mid \sum_{i=1}^{N} m_i x_i = 0\} \tag{3.1}$$

where $x = (x_1, \ldots, x_N)$ with $x_i \in \mathbb{R}^\nu$, with the inner product

$$\langle x, y\rangle = 2\sum_{i=1}^{N} m_i x_i \cdot y_i \ . \tag{3.2}$$

Here $m_i > 0$ are masses of the particles in question. The *Schrödinger operator* of such a system is

$$H = -\Delta + V(x) \quad \text{on} \quad L^2(X)\ .$$

Here Δ is the Laplacian on X and

$$V(x) = \Sigma V_{ij}(x_i - x_j)\ ,$$

where (ij) runs through all the pairs satisfying $i < j$.

We assume that the potentials V_{ij} are real and obey: $V_{ij}(y)$ are Δ_y–compact. It is shown in [Com] (see also [CFKS]) that under this condition Kato theorem applies and H is self–adjoint on $D(H) = D(\Delta)$. Moreover, by a simple application of Hölder and Young inequalities and by a standard approximation argument (see [CFKS]) one shows that if V_{ij} are Kato potentials, i.e.

$$V_{ij} \in L^r(R^\nu) + (L^\infty(R^\nu))_\varepsilon \text{ , where } r > \frac{\nu}{2} \text{ if } \nu > 4 \text{ and } r = 2 \text{ if } \nu \leq 3 \text{ ,}$$

and the subscript ε indicate that the L^∞–component can be taken arbitrarily small, then V_{ij} is Laplacian compact.

Now we describe the decomposed system. Denote by $a, b, \ldots,$ *partitions* of the set $\{1, \ldots, N\}$ into non–empty disjoint subsets, called clusters. The relation $b < a$ means that b is a refinement of a and $b \neq a$. Then $a_{\min}$ is the partition into N clusters $(1), \ldots, (N)$. Usually, we assume that partitions have at least two clusters. $\#(a) \equiv |a|$ denotes the number of clusters in a. We also identify pairs $\ell = (ij)$ with partitions having $N-1$ clusters: $(ij) \leftrightarrow \{(ij)(1)\ldots(i)\ldots(j)\ldots(N)\}$. We emphasize that the relation $\ell \subsetneq a$ (resp. $\ell \subseteq a$) with $\ell = (ij)$ is equivalent to saying that i and j belong to different clusters (resp. to same cluster) of a.

We define the *intercluster interaction* for a partition a as $I_a =$ sum of all potentials linking different clusters in a, i.e.

$$I_a = \sum_{\ell \not\subseteq a} V_\ell \ . \tag{3.3}$$

For each a we introduce the truncated Hamiltonian:

$$H_a = H = I_a \ . \tag{3.4}$$

These operators are clearly self–adjoint. They describe the motion of the original system broken into non–interacting clusters of particles.

For each cluster decomposition a, define the configuration space of relative motion of the clusters in a:

$$X_a = \{x \in X \mid x_i = x_j \text{ if } i \text{ and } j \text{ belong to same cluster of } a\}$$

and the configuration space of the internal motion within those clusters:

$$X^a = \{x \in X \mid \sum_{j \in C_i} m_j x_j = 0 \text{ for all } C_i \in a\} \ .$$

Clearly X_a and X^a are orthogonal (in our inner product) and they span X:

$$X = X^a \oplus X_a \ .$$

Given generic vector $x \in X$, its projections on X^a and X_a will be denoted by x^a and x_a, respectively.

If i and j belong to some cluster of a, then $x_i - x_j = (\pi^a x)_i - (\pi^a x)_j$, where π^a is the orthogonal projection in X on X^a. This elementary fact and the fact that $-\Delta = \langle p, p\rangle$ with $p = -i\nabla_x$ (see equation (3.6) and the sentence after it) yield the following decomposition:

$$H_a = H^a \oplus 1 + 1 \oplus T_a \text{ on } L^2(X) = L^2(X^a) \oplus L^2(X_a) \ . \tag{3.5}$$

Here H^a is the Hamiltonian of the non–interacting a–clusters with their centers–of–mass fixed at the originating on $L^2(X^a)$, and $T_a =$ $-$(Laplacian on X_a), the kinetic energy of the center–of–mass motion of those clusters.

The eigenvalues of H^a, whenever they exist, will be denoted by ε^α, where $\alpha = (a, m)$ with m, the number of the eigenvalue in question *counting the multiplicity*. For $a = a_{\min}$, we set $\varepsilon^\alpha = 0$. The set $\{\varepsilon^\alpha, \text{ all } \alpha\}$ is called the *threshold set* of H and ε^α are called the *thresholds* of H. For $\alpha = (a, m)$ we denote $|\alpha| = |\alpha|$ and $a(\alpha) = a$.

Our method is based on localization of operators in the *phase–space* $T^*X = X \times X'$. Hence and henceforth, the prime stands for taking dual of the space in question. The dual (momentum) space X' is identified with

(3.6)

$$X' = \{k \in R^{\nu N} \mid \Sigma k_i = 0\} \text{ with the inner product } \langle k, u\rangle = \Sigma \frac{1}{2m_i} k_i \cdot u_i \ .$$

Thus $|k|^2$ is the symbol of $-\Delta$ and $-\Delta = |p|^2$. We use extensively the natural bilinear form on $X \times X' : \langle x, k\rangle = \Sigma x_i \cdot k_i$. Given generic vector $k \in X'$, its projections on X'_a and $(X^a)'$ will be denoted by k_a and k^a, correspondingly. Accordingly, the momenta canonically conjugate to x_a and x^a and corresponding to k_a and k^a will be denoted by p_a and p^a, respecitvely. Thus $T_a = |p_a|^2$. Using the bilinear form above we define the generator of dilations as

$$A = \frac{1}{2}(\langle p, x\rangle + \langle x, p\rangle)$$

and the self–adjoint operator γ,

$$\gamma = \frac{1}{2}(\langle p, \hat{x}\rangle + \langle \hat{x}, p\rangle) ,$$

associated with the angle between the velocity and coordinate. Again, for decomposed systems A splits into the operator

$$A^a = \frac{1}{2}(\langle p^a, x^a\rangle + \langle x^a, p^a\rangle)$$

corresponding to the internal motion of the clusters, and the operator

$$A_a = \frac{1}{2}(\langle p_a, x_a\rangle + \langle x_a, p_a\rangle)$$

corresponding to the motion of the centers–of–mass of the clusters.

Finally, we mention some notation. We denote $E_\Delta = f(H \in \Delta)$ for an interval $\Delta \subset R$ and set $H_\Omega = HE_\Omega$. P will stand for the orthogonal projection the pure point spectral subspace of H.

Section 4. Partitions of Units

The configuration space of N particles moving in ν dimensions with the center of mass removed is

$$X = \{x \in \mathbb{R}^{\nu N} \mid \sum_{i=1}^{N} m_i x_i = 0\} .$$

Here the m_i's are the masses of the particles and $x_i \in \mathbb{R}^\nu$ their position. Let a be any disjoint cluster decomposition of $\{1, 2, \dots, N\}$. Denote by X_a the subspace of X given by

$$X_a = \{x \mid x_i - x_j = 0 \text{ if } (ij) \text{ belong to the same cluster in } a\} .$$

Define $|x|_a = \min_{(ij)\not\subset a} |x_i - x_j|$. We can now prove the existence of a *two cluster* partition of unity $\{j_a\}$.

Proposition 4.1 *There exists a partition of unity of* X, $\{j_a\}_{\#(a)=2}$ *s.t.*

i) $\sum\limits_a j_a^2(x) = 1$ *on* X.

ii) Each $0 \le j_a(x) \le 1$ is smooth and homogeneous of degree 0 outside the unit ball of X.

iii) $j_a(x) = 1$ for some neighborhood of $X_a/\{|x| \le 1\}$

iv) $j_a(x) = 0$ for $|x|_a \le \varepsilon_a |x|$ for some positive ε_a.

The proof follows by finding a covering of the unit sphere of X by neighborhoods of X_a, $\#(a) = 2$. For each member of the covering we then associate a smooth characteristic function which we then extend by homogeneity to $|x| > 1$ and in a smooth but otherwise arbitrary way to $|x| < 1$. We then normalize these functions so that $\sum_a j_a^2 = 1$.

The partition constructed in the proposition above is called a two–cluster partition of unity and it appeared already in the Haag-Ruelle theory [GJ]. By generalizing the construction above to neighborhoods of X_a, a any two or more cluster decomosition we can construct a k-cluster decomposition $\{j_a\}_{\#(a)\le k}$ to obtain

Proposition 4.2 *There exists a partition of unity of X, $\{j_a\}$ s.t.*

i) $\sum_{\#(a)\le k} j_a^2(x) = 1$

ii) Each $0 \le j_a(x) \le 1$ is smooth and homogeneous of degree zero outside $\{|x| \le 1\}$

iii) $j_a(x) = 1$ for some neighborhood of $X_a/\{|x| \le 1\}$

iv) supp $j_a(x) \subset \{|x|_a \ge \delta |x|\}$ for some $\delta > 0$.

This kind of partitions will allow us to use induction on number of cluster decompositions. Such partitions were used extensively in [Ag, Sig-Sof 1]. Partitions of unity are the basic tool to decouple channels of propagation from each other. In spectral geometry they are used to decouple different neighborhoods of infinity from each other [FHP1-2], see also [CFKS chapter 11].

Theorem 4.3 (Hunziker Van Winter Zislin) (HVZ)

$$\sigma_{\text{ess}}(H) = \bigcup_a \sigma_{\text{ess}}(H_a) \qquad \#(a) = 2 \ .$$

PROOF. [Sig 3] Using trail functions it is easy to prove that

$$\sigma_{\text{ess}}(H) \supseteq \bigcup_a \sigma_{\text{ess}}(H_a) \ .$$

To prove that $\sigma_{\text{ess}}(H) \subseteq \bigcup_a \sigma_{\text{ess}}(H_a)$ we use the two cluster partition of unity $\{j_a\}_{\#(a)=2}$:

$$\begin{aligned} H &= \frac{1}{2}\sum_a (j_a^2 H + H j_a^2) = \sum_a j_a H j_a + \sum_a j_a [j_a, H] \\ &= \sum_a j_a (H_a + I_a) j_a + \frac{1}{2}\sum_a [j_a, [j_a, H]] \ . \end{aligned}$$

Since $H = -\Delta + V(x)$ it follows that

$$[j_a, [j_a, H]] = [j_a, [j_a, -\Delta]] = O(|x|^{-2})$$

by property (ii) of the partitions j_a. Furthermore, by property (iv) of j_a we conclude that

$$j_a I_a = O(|x|^{-\mu}) \ .$$

Hence

$$H = \sum_a j_a H_a j_a + O(|x|^{-\mu}) + O(|x|^{-1}) \ .$$

Since $O(|x|^{-\mu})$ is $-\Delta$ (and hence H) compact it follows by Weyl's Lemma that

$$\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(\sum_a j_a H_a j_a)$$

from which the result follows by an elementary argument. ∎

In the study of the asymptotic behavior of multichannel systems the decoupling by $j_a(x)$ is not sufficient (see however [D-S,] [Sig3]). This is due to the fact that the flow under H can move through the boundary of one partition into the other and back. Therefore, to achieve a true decoupling between channels a new approach is needed.

This has been done in [Sig-Sof1] by introducing a phase space partition of unity with the boundary of the partitions localized away from the PS_E of H. In this way the decoupling is achieved between channels, modulo quantum oscillations capable of tunneling through the classically forbidden regions. The contribution of such oscillations is then controlled by the microlocal propagation estimates as explained in the previous section.

Proposition 4.4 (phase space partition of unity) *There exists a partition of* $X \times X'(\equiv T^*X), j_{a,E}(x, \xi_a)$ *s.t.*

i) $\sum_{\#(a)\geq 2} j^2_{a,E}(x, \xi_a) = 1$

ii) Each $0 \leq j_a \leq 1$ *is homogeneous of degree zero in* x

iii) supp $j_a(x, \xi_a) \subset \{|x|_a > \delta|x|\}$ *and* $j_a = 1$ *in some neighborhood of* X_a.

iv) supp $\nabla_x j_a(x, \xi_a)$ *is away from* PS_E.

The construction of such partition can be found in [Sig-Sof1]. The main building blocks of such a partition are $\chi_{x_0}(x)$, and $\tilde{\chi}_{\xi_0}(\xi)$; $\chi_{x_0}(x)$ is a (smooth) characteristic function of a cone in x near the direction x_0 and $\tilde{\chi}_{\xi_0}(\xi)$ in ξ near ξ_0; the support of the $\tilde{\chi}_{\xi_0}$ is taken to be either strictly inside that of χ_{x_0} or strictly outside. It is then easy to see that $\nabla_x(\chi_{x_0}(x)\tilde{\chi}_{\xi_0}(\xi))$ is supported where $x \mid\mid \xi$.

The main application of the above partition is

Theorem 4.5 (Channel Decoupling) *Let E be a given non threshold energy of H. Then AC follows from the propagation theorem on PS_E.*

PROOF.

$$
\begin{aligned}
e^{-iHt}\psi &= \sum_a j_a^2 e^{-iHt}\psi \\
&= \sum_a e^{-iH_a t}(e^{iH_a t} j_a^2 e^{-iHt}\psi) \ .
\end{aligned}
$$

It is therefore left to show that

$$
e^{iH_a t} j_a^2 e^{-iHt}\psi \to \psi_a^{\pm} \quad \text{as} \quad t \to \pm\infty \ .
$$

By Cook's method this is reduced to proving that

$$\int \|(H_a j_a^2 - j_a^2 H) e^{-iHt} \psi\|_2 dt \le c\|\psi\| \ .$$

But

$$H_a j_a^2 - j_a^2 H = H_a j_a^2 - j_a^2 H_a - j_a^2 I_a$$

$j_a^2 I_a = 0(|x|^{-\mu})$ and for $\mu > 1$ the above estimate holds by local decay since E is away from the thresholds (see section 7).

$$H_a j_a^2 - j_a^2 H_a = \mathcal{O}(|x|^{-1}) \tilde{j}_a(x, \xi_a)$$

where $\tilde{j}_a(x, \xi_a)$ lives away from the PS_E by property iv). Applying the propagation theorem to this term the result follows. ■

A very interesting partition of unity of X was constructed in [Gr]. (A simpler construction is given in [Der3].) It is an N–cluster partition of unity with further property on the boundary which implies monotonicity of the flow there in a certain sense:

Proposition 4.6 (Monotonic Partition of Unity) *There exists an N–cluster partition of unity $\{q_a\}$; furthermore, the derivative of q_a along x_a on the boundary is nonnegative:*

$$\sum_a x_a \otimes \nabla q_a(x) \ge 0 \ .$$

The idea behind the construction of q_a is the observation that $\nabla F(|x| \le c) = -\nabla F(|x| \ge c)$ and q_a is a product of such F's with $x \to x_b^a$ and $c \to c_b^a$. One can then cancel the negative terms in the sum $\sum\limits_a x_a \otimes \nabla q_a(x)$ by the corresponding positive ones, using that

$$x_a \otimes \nabla F(|x_a^b| \ge c_a^b) + x_b \otimes \nabla F(|x_a^b| \le c_a^b) \ge 0 \ .$$

Using the above partition, one can construct new propagation observables with monotone Heisenberg derivative by "clustering" (see [FHP1] for the first such procedure) the corresponding two body analog: In one channel nonlinear scattering one uses the propagation observable $(p - \frac{x}{t})^2 + V(\varphi)$ ($V(\varphi)$ stands for the nonlinearity), to derive the pseudo-conformal identities for the NLS equation [G-V1]. For N-body systems, using q_a one replaces $\frac{x}{t}$ by $v(x,t)$ where

$$v(x,t) \equiv \sum_a \frac{x_a}{t} w_a$$

where w_a are appropriately scaled (in time) q_a. One can then show that

$$-K \equiv (p - v(x,t))^2 + V(x)$$

is a propagation observable. Other observables can also be constructed, e.g. $\sum_a w_a \frac{A_a}{t} w_a$.

In the study of Long Range Scattering one is led to study the asymptotics of N–body systems at *threshold energies*. This requires zooming on zero velocities (coming from the critical points of the symbols of $E_a + H_a$) which we do by scaling in the time variable (see section 8). A natural partition of unity used in such an analysis is multiscaled [Sig-Sof5]:

Proposition 4.7 (Multiscale partition of unity) *There exists a k–cluster $\{j_a\}$ partition of X, depending on time, s.t. on support $j_a(x,t)$ $|x|_a \geq \delta_1 t^{\alpha(a)}$ and $|x^a| < \delta_2 t^{\beta(a)}$ (where of course $\alpha(a) > \beta(a)$). The partition is multiscaled since $\alpha(a) > \alpha(a')$ if $a \subset a'$.*

Just like with the monotonic partition it is possible to cluster operators using the multiscale partition leading to new propagation observables.

Section 5. The Channel Expansion

Recall that in the two body case the dilation generator A has positive commutator with H ($\equiv -\Delta + V(x)$) for sufficiently regular $V(x)$ and when the commuator is localized away from the thresholds of H. The question arises whether we can "cluster" A to prove similar bounds for the N–body case. (See [Hu1] for the case of classical mechanics.) This was first shown by Mourre in the case $N = 3$ and later generalized for all N in [PSS, FH1, BG2]. In [Sig-Sof1] it is shown that the commuator of H with certain global observables, including A and γ, can be approximated to arbitrary accuracy by a sum of contributions from the *open channels* of the system, at the given energy. This became a central technical result in the study of spectral and scattering of N-body systems. It implies, for example, that the Mourre estimate holds at non threshold energies with *precise lower and upper bounds* on the commutator $i[H, A]$. This section is devoted to proving the theorem using an important simplifying idea of Hunziker [Hu2].

The channel expansion theorem state that certain commutators with H, as well as the identity can be approximated, arbitrarily close, by a finite sum of contributions of open channels only. The approximation gets better as we add more and more *open* channels to the sum and shrink the interval Δ around the energy E. We first need a few definitions.

Let $\{\varepsilon_j(a)\}_{j=1}^{\infty}$ be the eigenvalues of H^a, with corresponding projections $P_j(a)$. Here the $P_j(a)$ are all chosen to be finite dimensional. Denote by $P(a) = \sum_j P_j(a) = 1 - \overline{P}(a)$ the projection on $\mathcal{H}_{p\cdot p}(H^a)$ and $\overline{P}(a)$ is the projection on the continuous spectral subspace of H^a.

$$P^N(a) = \sum_{j=1}^{N} P_j(a) \quad \text{and} \quad \overline{P}^N(a) = 1 - P^N(a) \ .$$

We drop the index a when $H^a \equiv H$ ($\#a = 1$). For the smooth spectral projection F_Δ of H^a, we let

$$F_\Delta^N = F_\Delta \overline{P}^N(a) \ .$$

A cluster decomposition a and a choice of eigenvalue for H^a determines a channel α. So, we let P_α be the projection on the channel bound state, p_α be the channel momentum ($p_\alpha = p_{a(\alpha)}$), T_α its kinetic energy ; $T_\alpha = T_{a(\alpha)} = |p_{a(\alpha)}|^2 = |p_\alpha|^2$, $m(\alpha) -$ order number of the eigenvalue $\varepsilon_j(\alpha)$.

Theorem 5.1 (Channel Expansion) *Given $E \in \mathbb{R}$, $\delta, \varepsilon > 0$ there exists integers $\{N_i\}_{i=1}^{N}$ and a* **finite** *set of channels C_ε:*

$$\alpha \in C_\varepsilon \quad \textit{iff} \quad m(a) \leq N_{\#(\alpha)} \ .$$

For each $\alpha \in C_\varepsilon$ there exists a smooth characteristic function of the origin χ_α, with sharpness less than δ, and an interval $\Delta \supset E$ s.t.
i)

$$F_{\Delta_1}^{N_1} i[H,A] F_{\Delta_1}^{N_1} \stackrel{\varepsilon}{=} F_{\Delta_1}^{N_1}\Big(\sum_{\alpha\in C_\varepsilon} \sum_{S=(a_2,\dots,a_k,a(\alpha))} j_S 2p_\alpha^2 \chi_\alpha(p_\alpha^2+\varepsilon(\alpha)-E\Big) j_S^*\Big) F_{\Delta_1}^{N_1}$$

$$(F_{\Delta_1}^{N_1})^2 \stackrel{\varepsilon}{=} F_{\Delta_1}^{N_1}\Big(\sum_{\alpha\in C_\varepsilon} \sum_{S=(a_2,\dots,a_k,a(\alpha))} j_S \chi_\alpha(p_\alpha^2+\varepsilon(\alpha)-E) j_S^*\Big) F_{\Delta_1}^{N_1}$$

where $\Delta_1 \subset \Delta$, and for $S = (a_2, \dots, a_k, a(\alpha))$

$$j_S = \varphi_{a_2}^{a_1} \overline{P}_{a_2}^{N_2} \varphi_{a_3}^{a_2} \overline{P}_{a_3} \dots \varphi_{a_k}^{a_{k-1}} P\alpha \ .$$

PROOF. We sketch the proof before giving the details. The idea is to try to mimic the proof of the HVZ theorem, by reducing the problem to H_a using the two cluster partition of unity j_a. We then get

$$i[H,A] = \sum_{\substack{a \\ \#(a)=2}} j_a i[H,A] j_a + K$$

where K stands for a relatively compact operator (w.r.t. H). Next, we want to replace $i[H,A]$ by $i[H_a,A_a]$, using that

$$H = H_a + I_a \quad \text{and} \quad J_a I_a = K$$

and

$$A = A^a + A_a \ .$$

Doing that, we get

$$i[H,A] = \sum_{a_2} j_a i[H_a,A_a] j_a + \sum_{a_2} j_a i[H_a,A^a] j_a + K \ .$$

The first sum on the r.h.s. involves a "one body" commutator and produces the $2p_a^2$ term. It is left to consider $i[H_a,A^a]$, which we rewrite now as

$$i[H^a \otimes 1 + p_a^2 \otimes 1, A^a] = i[H^a, A^a] \ .$$

This last commutator is exactly the same as $i[H, A]$ but for a subhamiltonian coming from some two cluster decomposition a. We can therefore *assume* it satisfies the theorem and proceed by induction to conclude the proof.

The difficulty lies at this stage: the original commutator is localized with $H \sim E$, but the new one, $i[H^a, A^a]$ has H^a localized near $E - 2p_a^2$ which varies over a large interval, in general, and can hit bound states of H^a for example, where the induction hypothesis is useless. To proceed, we use a resolution of the identity

$$1 = \sum_{j=1}^{N} P_j(a) + \overline{P}^N(a)$$

and study

(H1) $P_j(a) i[H^a, A^a] P_j(a)$

(H2) $\overline{P}^N(a) i[H^a, A^a] \overline{P}^N(a)$

(H3) $\overline{P}^N(a) i[H^a, A^a] P_j(a)$ (and its adjoint).

Case (H1) is shown to contribute zero by applying the virial theorem and localization using F_Δ. Case (H2) is treated by the induction hypothesis. Case (H3) is shown to be small in norm by compactness. The main simplification in the proof below compared to [Sig-Sof1] is that the induction hypothesis is formulated and used for H_a, rather than H^a. In this we follow Hunziker [Hu2].

The induction on clusters begins with $n = N$ and descends to 1. For $n = N$, H is reduced to $-\Delta$ where the proof is straightforward. For the sake of notation we only do the last step of the induction: proving it for H assuming it for all H_a, $\#(a) \geq 2$. Using the IMS localization formula:

$$B_{\Delta_1}^{N_1} \equiv F_{\Delta_1}^{N_1} i[H, A] F_{\Delta_1}^{N_1} \stackrel{\varepsilon}{=} \sum_{\#(a)=2} F_{\Delta_1}^{N_1} j_a i[H_a, A] j_a F_{\Delta_1}^{N_1}$$

for Δ_1 sufficiently small, and N_1 sufficiently large, depending on ε.

$$B_{\Delta_1}^{N_1} \stackrel{\varepsilon}{=} \sum_{a_2} F_{\Delta_1}^{N_1} j_a F_{\Delta_2}(H_a) i[H_a, A] F_{\Delta_2}(H_a) j_a F_{\Delta_1}^{N_1}$$

for all $\Delta_2 \supset \Delta_1$, s.t. $F_{\Delta_2} F_{\Delta_1} = F_{\Delta_1}$.

To prove the last inequality we used that for all $\varepsilon' > 0$

$$\|F_{\Delta_1}^{N_1} j_a F_{\Delta_2}(H_a) - F_{\Delta_1}^{N_1} j_a\| < \varepsilon'$$

for $|\Delta_1| < \delta(\varepsilon')$ and $N_1 > N_1'(\varepsilon)$.

To prove it, observe that since $j_a J_a = K$ and $H = H_a + I_a$ then

$$j_a F_{\Delta_2}(H_a) = j_a F_{\Delta_2}(H) + K = F_{\Delta_2}(H) j_a + K \ .$$

Next, we use

$$i[H_a, A] = i[H^a, A^a] + 2p_a^2 \ .$$

Hence, using the resolution of identity $\sum\limits_1^N P_j + F_\Delta^N = 1$ we get

$$\begin{aligned}
B_{\Delta_1}^{N_1} \overset{\varepsilon}{=} &\sum_{a_2} F_{\Delta_1}^{N_1} j_{a_2} F_{\Delta_2}^{N_2} i[H_a, A] F_{\Delta_2}^{N_2} j_{a_2} F_{\Delta_1}^{N_1} \\
&+ \sum_{\substack{\alpha \\ \#(a)=2 \\ m(\alpha)\le N_1}} F_{\Delta_1}^{N_1} j_a P_\alpha 2p_\alpha^2 F_{\Delta_1}^2 (p_\alpha^2 + \varepsilon_\alpha) P_\alpha j_a F_{\Delta_1}^{N_1} \\
&+ \sum_{\substack{a_2 \\ m(\alpha)\le N_2 \\ m(\alpha')\le N_2 \\ \alpha\ne\alpha'}} F_{\Delta_1}^{N_1} j_{a_2} F_{\Delta_2} P_\alpha i[H_{a_2}, A] P_\alpha F_{\Delta_2} j_{a_2} F_{\Delta_1}^{N_1} \\
&+ \sum_{\substack{a_2 \\ m(\alpha)\le N_2}} F_{\Delta_1}^{N_1} j_{a_2} F_{\Delta_2}^{N_2} i[H_{a_2}, A] P_\alpha F_{\Delta_2} j_{a_2} F_{\Delta_1}^{N_1} + h.c.
\end{aligned}$$

h.c. stands for hermitian conjugate.

The second term on the r.h.s. is derived by using the virial theorem:

$$P_\alpha i[H^a, A] P_\alpha = 0 \ .$$

The third term is zero, since P_α localizes H^a near ε_α (in F_{Δ_2}) and $P_{\alpha'}$ localizes H^a near $\varepsilon_{\alpha'} \ne \varepsilon_\alpha$ (in F_{Δ_2}). Therefore, since $\varepsilon_{\alpha'} \ne \varepsilon_\alpha$, choosing the sharpness of F_{Δ_2} sufficiently small either $F_{\Delta_2} P_\alpha = 0$ or $F_{\Delta_2} P_{\alpha'} = 0$ (or both).

The fourth term and its hermitian conjugate are made smaller than any $\varepsilon > 0$, by observing that

$$F_{\Delta_2}^{N_2} i[H_{a_2}, A] P_\alpha$$

is a (relatively) compact operator, since P_α is compact on $L^2(X^a)$. Hence, letting $|\Delta_2| \downarrow 0$, $N_2 \to \infty$, we see that $F_{\Delta_2}^{N_2} \xrightarrow{s} 0$ and hence

$$\|F_{\Delta_2}^{N_2} i[H_{a_2}, A]P_\alpha\| \to 0 \ .$$

One technical tool used here and in the above compactness arguments is reduction to the subspace $L^2(X^a)$ using the fibre representation for H_a is fibres of p_a:

$$F_\Delta(H_a) = \int_\oplus F_\Delta(H^a + \xi_a^2)\, d\xi_a \ .$$

Since H is semibounded from below the sum over ξ_a extends over a compact set. This allows us to use compactness arguments in $L^2(X^a)$ for each fibre ξ_a.

Similarly, we prove the channel expansion for the identity:

$$\begin{aligned}(F_{\Delta_1}^{N_1})^2 &= \sum_{a_2} F_{\Delta_1}^{N_1} j_{a_2}^2 F_{\Delta_1}^{N_1} \\ &\stackrel{\varepsilon}{=} \sum_{\substack{a \\ \#(a)=2}} F_{\Delta_1}^{N_1} j_a F_{\Delta_2}^2(H_a) j_a F_{\Delta_1}^{N_1} \ .\end{aligned}$$

Since $\|F_{\Delta_1}^{N_1} K F_{\Delta_1}^{N_1}\| \le \varepsilon$ for all sufficiently small $|\Delta_1|$ and large N_1. Writing

$$F_{\Delta_2}^2(H_a) = F_{\Delta_2}^{N_2}(H_a)^2 + (F_{\Delta_2} P^{N_2}(a))^2$$

we set

$$\begin{aligned}(F_{\Delta_1}^{N_1})^2 = &\sum_{\substack{a \\ \#(a)=2}} F_{\Delta_1}^{N_1} j_a F_{\Delta_2}^{N_2}(H_a)^2 j_a F_{\Delta_1}^{N_1} \\ &+ \sum_{\substack{\#(a)=2 \\ m(\alpha)\le N_2}} F_{\Delta_1}^{N_1} j_a F_{\Delta_2}^2(\varepsilon_\alpha + p_\alpha^2) j_a F_{\Delta_1}^{N_1}\end{aligned}$$

and we redefine

$$F_{\Delta_2}^2(\varepsilon_2 + p_\alpha^2) \equiv \chi_\alpha(p_\alpha^2 + \varepsilon_\alpha - E) \ .$$

Using the induction hypothesis and using local compactness to prove

$$F_{\Delta_1}^{N_1} j_a F_{\Delta_2}^{N_2} \stackrel{\varepsilon}{=} F_{\Delta_1}^{N_1} j_a \overline{P}_a^{N_2}$$

the result follows. ■

Next we prove the *Localization Lemma*:

Lemma 5.2 *Let F_i, $i = 1, 2$ be smooth functions s.t. supp $F_1 \subset (-\delta/2, \delta/2)$ and supp $F_2 \subset [\frac{3}{4}\delta, \infty)$. Then*

$$F_1(|p| - \kappa)F_2(\pm\gamma - \kappa) = O(|x|^{-1}) .$$

PROOF. Pick F_3 so that supp $F_3 \subset (-(2/3)\delta, (2/3)\delta)$ and $F_3 = 1$ on supp F_1. Denote $g = F_3(|p| - \kappa)$, $F_1(p) = F_1(|p| - \kappa)$ and $F_2(\gamma) = F_2(\pm\gamma - \kappa)$. The operator $\gamma_g \equiv g\gamma g$ is symmetric and bounded:

$$|\langle \hat{x} \cdot pgu, gu \rangle \leq \|pgu\| \; \|\hat{x}gu\| \leq (\kappa + \frac{2}{3}\delta) \; \|gu\|^2 ,$$

where we have used that

$$\|\hat{x}f\| \leq \|f\| .$$

This shows that

$$\pm g\gamma g \leq \kappa + \frac{2}{3}\delta .$$

Now observe

$$F_1(p)F_2(\gamma) = F_1(p)(F_2(\gamma) - F_2(\gamma_g)) .$$

Using the Fourier representation

$$F(\gamma) = \int \hat{F}_2(s)e^{i\gamma s} \, ds$$

and using a continuity argument in order to extend the following result from $\zeta(\mathbb{R})$-functions to smooth bounded functions F_2 with C_0^∞ derivatives, we obtain

$$\begin{aligned} F_2(\gamma) - F_2(\gamma_g) &= \int_{-\infty}^{\infty} \hat{F}_2(s)(e^{i\gamma s} - e^{i\gamma_g s}) \, ds \\ &= -i \int_{-\infty}^{+\infty} ds \hat{F}_2(s)e^{i\gamma s} \int_0^s du \; e^{-i\gamma u}(\gamma - g\gamma g)e^{i\gamma_g u} . \end{aligned}$$

This implies

$$F_1(p)F_2(\gamma) = -i\int_{-\infty}^{\infty} ds\hat{F}_2(s)\int_0^s duF_1(p)e^{i\gamma(s-u)}(\gamma - g\gamma g)e^{i\gamma_g u} \ .$$

We commute $F_1(p)$ on the r.h.s. of this expression through $e^{i\gamma(s-u)}$. The result is

$$F_1(p)F_2(\gamma) = B_1 + B_2 \ ,$$

where

$$B_1 = -i\int_{-\infty}^{\infty} ds\hat{F}_2(s)\int_0^s du[F_1(p), e^{i\gamma(s-u)}](\gamma - g\gamma g)e^{i\gamma_g u}$$

and

$$B_2 = i\int_{-\infty}^{\infty} ds\hat{F}_2(s)\int_0^s due^{i\gamma(s-u)}F_1(p)\gamma(1-g)e^{i\gamma_g u} \ .$$

Next we show that

$$(\alpha) \qquad [F_1(p), \gamma] = O(|x|^{-1}) \ .$$

Using that

$$[F_1(p), \gamma] = \Sigma[p_i F_1(p), \frac{x_i}{\langle x\rangle}] + O(|x|^{-1})$$

and using the commutator formulas [Section 6] we obtain (α). Now the relation

$$[F_1(p), e^{i\gamma t}] = -i\int_0^t e^{i\gamma(t-s)}[F_1(p), \gamma]e^{i\gamma s}\, ds \ ,$$

Equation (α) and commuator formulas imply that

$$[F_1(p), e^{i\gamma t}] = O(|x|^{-1}\, t^2) \ .$$

Equations (α) and the above equation together with the relations

$$F_1(p)(1-g) = 0$$

and

$$\int_{-\infty}^{\infty} |\hat{F}_2(s)|\ |s|^n\, ds < \infty \quad \text{for} \quad n = 1, 2, 3 \ ,$$

imply

$$B_i = O(|x|^{-1}) \ , \quad i = 1,2 \ .$$

■

Using the Localization Lemma, we can then sharpen the statement of the channel expansion theorem, by replacing $\chi_\alpha(p_\alpha^2 + \varepsilon(\alpha) - E)$ by $F_\alpha \chi_\alpha(p_\alpha^2 + \varepsilon(\alpha) - E)F_\alpha$, with

$$F_\alpha \equiv \chi(\gamma_{a(\alpha)} = \sqrt{E - \varepsilon(\alpha)}) \ .$$

Section 6. Some Operator Calculus

In this section we derive various estimates on functions of self-adjoint operators following [Sig-Sof2]. We begin with a few remarks.

Let A be a self-adjoint operator on $L^2(\mathbb{R}^n)$. If f is a measurable function with integrable Fourier transform then we define

$$f(A) = \int \hat{f}(s) e^{iAs} ds \tag{6.1}$$

where $\hat{f}(s)$ is the Fourier transform of $f(\lambda)$ and the limit defining the integral is taken in the strong sense. With some care this formula can be extended to a broader class of functions. For positive powers of positive operators we use the representation on $D(A^{[\alpha]+1})$

$$A^\alpha = \frac{\sin \pi(\alpha - [\alpha])}{\pi} \int_0^\infty \frac{w^{\alpha-[\alpha]-1}}{A+w} \, dw A^{[\alpha]+1} \ , \tag{6.2}$$

where $[\alpha]$ is the integer part of α.

Expansion of Commutators

Let H and A be self-adjoint operators on the same Hilbert space $\mathcal{H}$. We assume that $D(A) \cap D(H)$ is dense in $\mathcal{H}$, and for some $n \geq 1$

(6.3) $\quad ad_A^k(H)$ extends to a bounded operator for all $1 \leq k \leq n$.

Here $ad_A^k(H) \equiv [\cdots[H, A], A], \cdots A]k$-times are defined initially as forms on $D(A) \cap D(H)$.

Property (F):

We consider a class of smooth functions f whose Fourier transforms $\hat{f}$ obey

$$(6.4) \qquad \|\hat{f}^{(n)}\|_1 \equiv \int |\hat{f}(s)|\,|s|^n ds < \infty \,.$$

Here, $f^{(k)}$ stands for the k^{th} derivative of f. We derive Taylor-type expansions for the commutator $[H, f(A)]$.

Lemma 6.1 (Leibnitz Rule)

Let for some $n > 0$ H obey (6.3)and f obey condition (F). Let $[H, f(A)]$ be defined as a form on $D(A^n)$. Then,

$$(6.5) \qquad [H, f(A)] = \sum_{k=1}^{n-1} \frac{1}{k!} f^{(k)}(A) ad_A^k(H) + R_n(f)$$

in the form sense with the remainder $R_n(f)$ satisfying

$$(6.6) \qquad \|R_n(f)\| \leq C \|\hat{f}^{(n)}\|_1 \|ad_A^n(H)\|$$

Consequently, $[H, f(A)]$ defines an operator on $D(A^{n-1})$.

PROOF. We begin with $f \epsilon C_0^\infty$ functions for which representation (6.1) is well-defined and then extend the expressions obtained to the class of interest. Thus on $D(H) \times D(H)$

$$(6.7) \qquad [H, f(A)] = \int ds \hat{f}(s)[H, e^{iAs}] \,.$$

We have

$$[H, e^{iAs}] = e^{iAs}(e^{-iAs} H e^{iAs} - H) \,.$$

Using that

$$\frac{d}{ds}(e^{-iAs} H e^{iAs}) = i e^{-iAs} ad_A(H) e^{iAs}$$

and the Fundamental Formula of calculus we compute

$$(6.8) \qquad e^{-iAs} H e^{iAs} - H = i \int_0^s du e^{-iAu} ad_A(H) e^{iAu} \,.$$

The above two formulas are first derived with H replaced by $H_\epsilon \equiv \frac{H}{1+i\varepsilon H}$, $\varepsilon > 0$. Then, we let $\varepsilon \downarrow 0$ and use the boundedness of $ad_A(H)$ to prove that this limit exists.

Subtracting from and adding to the integrand $ad_A(H)$ gives

$$e^{-iAs}He^{iAs} - H = is\, ad_A(H) + i\int_0^s du\big(e^{-iAu}ad_A(H)e^{iAu} - ad_A(H)\big)\ .$$

Iterating this relation $n-1$ times we obtain

$$e^{-iAs}He^{iAs} - H = \sum_{k=1}^{n-1}\frac{(is)^k}{k!}ad_A^k(H) + R_n(s)$$

where

(6.9)
$$R_n(s) = \int_0^s du_1 \cdots \int_0^{u_{n-1}} du_n e^{-iAu_n} ad_A^n(H)e^{iAu_n}\ .$$

This together with (6.7) and the relation

$$\int_{-\infty}^{\infty} \hat{f}(s)(is)^k e^{iAs} ds = f^{(k)}(A)$$

yields

(6.10)
$$[H, f(A)] = \sum_{k=1}^{n-1}\frac{1}{k!}f^{(k)}(A)ad_A^k(H) + R_n(f)$$

where

(6.11)
$$R_n(f) = \int_{-\infty}^{\infty} \hat{f}(s)e^{iAs}R_n(s)ds\ .$$

Since $ad_A^n(H)$ is bounded we have that

$$\|R_n(s)\| \le \text{const.}\ |s|^n \|ad_A^n(H)\|$$

which yields

$$(6.12) \qquad \|R_n(f)\| \leq \text{const.} \int_{-\infty}^{\infty} |\hat{f}(s)|\,|s|^n ds \|ad_A^n(H)\| \ .$$

Finally we extend the above analysis to arbitrary function f satisfying condition F.

First assume that H is a bounded operator. Then the form

$$A(f) \equiv i[H, f(A)] - \sum_{k=1}^{n-1} \frac{1}{k!} f^{(k)}(A) ad_A^k(H)$$

is well defined on $D(A^n)$.

Now, let $f_j \epsilon C_0^\infty(\mathbb{R})$ $j = 1, \dots \infty$ with $f_j \to f$ in the F topology

$$\|f\|_F \equiv \|\hat{f}^{(n)}\|_1 \ .$$

It readily follows that $A(f_j) \to A(f)$ in the form sense since $f_j \xrightarrow{F} f$ implies

$$\|\langle\lambda\rangle^{-n}(f_j(\lambda) - f(\lambda)\|_\infty \to 0 \ .$$

By the estimate (6.12)

$$R_n(f_j) \to R_n(f)$$

with $Rn(f)$ bounded.

Equality (6.10) then implies

$$A(f) - R_n(f) = 0$$

in the form sense on $D(A^n)$.

Since $R_n(f)$ is bounded and $\sum_{k=1}^{n-1} \frac{1}{k!} f^{(k)}(A) ad_A^k(H)$ is an operator defined on $D(A^{n-1})$ the above equality extends to $D(A^{n-1})$. The result for unbounded H now follows by approximating H by $\frac{H}{1+i\varepsilon H}$ and a simple continuity argument. ■

REMARK. For similar expansions, based on resolvents see [B-G1 and cited ref.].

Lemma 6.2 *Let $A(t)$ be a commutative family of self-adjoint operators with common domain $\mathcal{D}$. We assume that $A(t)$ is norm differentiable in t : $A(t +$*

$\delta) - A(t)$ are bounded for δ small and the following norm limit exist:

$$\lim_{\delta\to 0} \frac{1}{\delta}\left(A(t+\delta) - A(t)\right) \equiv \frac{dA(t)}{dt}\ .$$

Then, we have

(a) For all bounded smooth functions f, with $\hat{f}^{(1)}\epsilon L^1$, $f\big(A(t)\big)$ is norm differentiable.

(b) Assume that $A(t) > 0$ for all t. Let $\alpha \geq 0$. Then $A(t)^\alpha f\big(A(t)\big)$ is differentiable in the strong resolvent sense and the chain rule applies for $g(\lambda) \equiv \lambda^\alpha f(\lambda)$:

$$\frac{d}{dt} g\big(A(t)\big) = g'\big(A(t)\big)\frac{dA(t)}{dt}\ .$$

PROOF.

(a) As in the proof of Lemma 6.1 we derive the formula

$$\frac{d}{dt} e^{i\lambda A(t)} = i\lambda \frac{dA(t)}{dt} e^{i\lambda A(t)}\ .$$

Using this formula and equation (6.1) we get

$$\frac{d}{dt} f\big(A(t)\big) = \frac{dA(t)}{dt} f'\big(A(t)\big) \tag{6.13}$$

where the limits defining $\frac{d}{dt}$ are taken in the norm sense.

(b) Since the $A(t)$ have a common domain $\mathcal{D}$, $A(t)/\big(A(s)+i\big)^{-1}$ are bounded for all s and t by the closed graph theorem. We can then compute directly that

$$\frac{d}{dt} A(t)^\alpha = \alpha A(t)^{\alpha-1}\frac{dA(t)}{dt}$$

for all positive integers α. Thus by the Chain rule it suffices to consider the case $0 < \alpha < 1$.

To this end we compute

$$\frac{1}{\delta}\ \left[A(t+\delta)^\alpha - A(t)^\alpha\right] == \frac{1}{\delta}\ A(t)^\alpha\ \left[\big(1+\delta\beta(t)\big)^\alpha - 1\right]$$

where

$$\beta(t) \equiv \frac{\big(A(t+\delta) - A(t)\big)}{\delta} A(t)^{-1}\ .$$

Note that since $A(t) > 0$ and norm differentiable and, $A(t)$ are commuting for different values of $t, \beta(t)$ is a bounded self-adjoint operator for each t. Using that by the spectral theorem

$$(1+\delta\beta)^{\alpha} \; - 1 \; = \; \alpha\delta\beta \; + \; 0(\delta)$$

the result follows.

Some Domain Estimates

Lemma 6.3 *Let H and $A \geq 0$ be self adjoint operators on a Hilbert space $\mathcal{H}$. Assume that $D(H) \cap D(A)$ is dense in $\mathcal{H}$. Then, for $\alpha \geq 0$ s.t.*

$$(6.15) \qquad ad_H^k(A) \text{ are bounded operators for } 1 \leq k \leq [\alpha]+1$$

we have

$$g(H)\colon\ D(A^{\alpha}) \to D(A^{\alpha})$$

for all $g(\lambda)\epsilon C_0^{\infty}(\mathbb{R})$.

PROOF. By interpolation it is sufficient to prove the lemma for integer α.

We first show that $ad_A^k(g(H))$ are bounded for $1 \leq k \leq [\alpha]+1$; using eq. (6.8) we derive, in the sense for forms on $D(H) \cap D(A) \equiv D$,the following equality

$$[A, e^{isH}] = e^{isH}(e^{-isH} A e^{isH} \; - A =)$$

$$(6.16) \qquad = ie^{isH} \int_0^s d\mu e^{-i\mu H} ad_H(A) \quad e^{i\mu H},$$

hence

$$(6.17) \qquad \sup_{\substack{\varphi,\psi\epsilon D \\ \|\phi\|=\|\psi\|=1}} | < \phi, [A, e^{isH}]\psi > | = \|[A, e^{isH}]\| \leq s\|ad_H(A)\| \leq cs$$

due to (6.15) (with $k=1$).

Iterating the formula (6.16) and using the estimate (6.17) we get

$$(6.18) \qquad \|ad_A^n(e^{isH})\| \leq cs^n \qquad\qquad n \leq [a]+1$$

By (6.7) and (6.18)

$$(6.19)\quad \|ad_A^n(g(H))\| = \| \int_{-\infty}^{\infty} ds\, \hat{g}(s) ad_A^n \left(e^{isH}\right)\| \le C \int_{-\infty}^{\infty} |s^n \hat{g}(s)| ds < \infty\,.$$

Since $\|ad_A^n(g(H))\| < \infty$ for $n = [\alpha] + 1$, taking g-real valued we apply Lemma 6.1 (with $f(\lambda) \equiv \lambda^\alpha$, α integer) to get:

$$(6.20)\qquad [g(h), A^\alpha] = \sum_{k=1}^{[\alpha]} \frac{C_k}{k!} ad_A^k(g(H)) A^{\alpha-k} + R_{[\alpha]+1}$$

and $R_{[\alpha]+1}$ is a bounded operator due to (6.6). (Note that since α is an integer, one can derive eq. (6.20) directly, without reference to Lemma 6.1.)

Hence, if we let $u \epsilon D(A^\alpha)$, then using eq. (6.20) we obtain

$$\begin{aligned} A^\alpha g(H) u &= g(H) A^\alpha u + \quad [A^\alpha, g(H)] u \\ &= g(H) A^\alpha u + \sum_{i=1}^{[\alpha]} B_i A^{\alpha-i} u + R_{[\alpha]+1} u \epsilon \mathcal{H} \end{aligned}$$

since $B_i \equiv \frac{C_i}{i!} ad_A i(g(H))$ are all bounded by (6.19). ∎

Section 7. Local Decay, Velocity Bounds and Spectral Theory

Recall the notion of threshold energy: $\mathcal{K}_\alpha = 0$ for some cluster decomposition $a(\alpha)$ and channel α. Here $\mathcal{K}_\alpha = E - \varepsilon_\alpha$ where ε_α is an eigenvalue of H^a. Thus thresholds E are eigenvalues of subhamiltonians H^a. The set of all thresholds is denoted $\mathcal{T}$. It is known that $\mathcal{T}$ is discrete and bounded. Furthermore points of $\mathcal{T}$ can accumulate (at $\mathcal{K} \in \mathcal{T}$) only from below. These properties of H follow from the Mourre estimate [see e.g. CFKS] which we now turn to:

For $E \notin \mathcal{T}$, the channel expansion for H gives (the Mourre estimate) [M1-2, PSS, FH1, BG2]. For more general Hamiltonians see [Der2, Ger1, FHP1].

$$E_\Delta(H) i[H, A] E_\Delta(H) \ge \theta E_\Delta^2(H) + K\ , \quad \theta > 0$$

with K compact ($E \subset \Delta$).

Let $\tilde{\Delta} \subset \Delta$ be s.t. H has no eigenvalues in $\tilde{\Delta}$. Then $E_{\tilde{\Delta}}(H) \xrightarrow{s} 0$ as $|\tilde{\Delta}| \to 0$. Since K is compact, we can choose $\tilde{\Delta}$ sufficiently small s.t.

$$E_{\tilde{\Delta}} i[H,A] E_{\tilde{\Delta}} \geq \theta E_{\tilde{\Delta}}^2(H) - \varepsilon E_{\tilde{\Delta}}^2 > (\theta - \varepsilon) E_{\tilde{\Delta}}^2(H) .$$

A general spectral theory has been developed with the Mourre estimate as the main tool. This theory can be thought of infinitesimal and microlocal version of scaling theory in PDE (see also [L]). The Mourre estimate determines the way an infinitesimal scaling affects the operator H. Let us describe few notable consequences of the Mourre estimate. (A comprehensive analysis of the continuous spectral part of H is done in [ABG], [BG1].) See [Iw] for applications to systems of equations and [We] to nonhomogeneous media.

Theorem 7.1 (Mourre) *Assume H satisfies the Mourre estimate for an interval Δ. Then H has only finitely many eigenvalues in Δ; assume moreover that the commutator $i[[H,A],A]$ is H bounded. Then H has no singular spectrum in Δ.*

Theorem 7.2 (Local Decay) *Assume H satisfies the Mourre estimate for an interval Δ and $ad_A^2(H)$ is H bounded. Then local decay holds:*

$$\int_{-\infty}^{\infty} \|\langle A\rangle^{-\frac{1}{2}-\varepsilon} e^{-iHt}\psi\|_2^2 dt \leq c\|\psi\|_2^2 \quad \text{for all } \psi = E_\Delta \psi .$$

In case A is the dilation generator it is easy to show that $\langle A\rangle$ can be replaced by $\langle x\rangle$, in the above local decay estimate.

Theorem 7.3 (Minimal and Maximal Velocity Bounds) *[Sig-Sof2] Assume as before that the Mourre estimate holds for some energy E. Let θ_m and θ_M be the lower and upper bounds:*

$$\theta_m E_\Delta^2 \leq E_\Delta i[H,A] E_\Delta \leq \theta_M E_\Delta^2 .$$

Assume furthermore that $ad_A^2(H)$ and ad_A^3 are H bounded. Then

$$\int \|F(\frac{A}{t} \leq \theta_m - \varepsilon)e^{-iHt}\psi\|^2 \frac{dt}{t^\alpha} \leq c\|\langle x\rangle\beta(\alpha)\psi\|_2^2 \qquad \beta(\alpha) = \frac{1-\alpha}{2} ,$$

and

$$\int \|F(\frac{A}{t} > \theta_M + \varepsilon)e^{-iHt}\psi\|\frac{dt}{t^\alpha} \leq c\|\langle x\rangle^{\beta(\alpha)}\psi\|_2^2 .$$

Here $\psi = E_\Delta\psi$.

REMARK. The upper bound inequality for $i[H, A]$ is called the reverse Mourre estimate. Sharp values of θ_m and θ_M can be found for a general N–body hamiltonian using the Channel Expansion Theorem.

A corollary of Theorems 1 and 2 is a proof of asymptotic completeness for the two body case. Further results can also be established by the analysis leading to the above theorems, e.g. propagation estimates for the region of phase space where $A < 0$ and analytic properties in certain weighted spaces of the resolvent of H. But not less important and impressive are the results about *eigenfunctions* of H and its resonances.

Theorem 7.4 (Froese-Herbst) *Exponential decay of eigenfunctions. Let $H = -\Delta + V$ where V satisfies:*

i) V is $-\Delta$ bounded with bound less than 1

ii) $x \cdot \nabla V$ is bounded from $\mathcal{H}^1$ to $\mathcal{H}^{-2}$.

Suppose that $H\psi = E\psi$. Then $e^{\lambda\langle x\rangle}\psi \in L^2$ for all $\lambda^2 \leq M^\sim(H) - E$. Here $\eta \in M(H)$ iff the Mourre estimate holds at $\eta = E$ (see CFKS, ch.4).

The results on absence of embedded eigenvalues uses:

Theorem 7.5 (Froese-Herbst) *Absence of embedded eigenvalues. Let $H = -\Delta + V$ where V satisfies conditions (i), (ii) of the above theorem and furthermore, $x \cdot \nabla V$ is Δ bounded with bound less than 2. Then, if $e^{\lambda\langle x\rangle}\psi \in L^2$ for all λ real, then $\psi = 0$ (see CFKS, ch. 4).*

There are also interesting results about the existence and characterization of resonances using the Mourre estimate in [Or] and to Nonlinear instability of periodic solutions [Sig2]. See also [FL, J, Na].

PROOF. The proof of the first part of Theorem 7.1 is very simple: Let $\{\psi_n\}_{n=1}^{\infty}$ be eigenfunctions of H with eigenvalues in Δ. Then

$$0 = \langle \psi_n, [H, iA]\psi_n \rangle = \langle \psi_n, E_\Delta [H, iA] E_\Delta \psi_n \rangle \geq \theta \|\psi_n\|^2 + \langle \psi_n, K\psi_n \rangle \ .$$

Now, let $n \to \infty$; then $\langle \psi_n, K\psi_n \rangle \to 0$ since K is compact. We used also the virial theorem to prove the first equality.

The absence of singular continuous spectrum in Δ follows from Local Decay (Theorem 7.2).

The original proof of Theorem 7.2 given by Mourre was based on proving differential inequality for the complex distorted resolvent of H, the distortion is generated by the group $e^{i\lambda A}$.

Later a new proof, more general, was given by the methods of microlocal propagation estimates [Sig-Sof2]. This proof implied also the minimal and maximal velocity bounds as well as pointwise decay estimates (in time) in certain regions of the phase–space [Sig-Sof2, Sk, Ger2, H-Sk, Ger-Sig, Her]. This approach could also be extended to time dependent hamiltonians of the type

$$H(t) = H + W(x, t)$$

that arise in long range scattering theory.

IDEA OF PROOF. (Theorems 7.2,3) We construct a sequence of *negative propagation observables* of singular operators. Each one, when used, implies a propagation estimate which is then used to control the remainder terms of the next, more singular observable. Denote for a moment by $\lambda(t)$ a monotone increasing function of t, and let $\alpha, \beta \geq 0$. Then

$$\phi_{\alpha,\beta}(\lambda, t) \equiv -\Big(\frac{-\lambda(t)}{\tau^\beta}\Big)^\alpha F\Big(\frac{\lambda(t)}{t} < -\delta\Big)$$

satisfies

i) $\phi_{\alpha,\beta} \leq 0$

ii) $\frac{d}{dt}\phi_{\alpha,\beta}(t) \geq 0$ for any $\delta > 0$.

The Mourre estimate then suggest the use of

$$\lambda(t) \sim e^{iHt} A e^{-iHt} .$$

To make the analysis go, we take $\phi_{\alpha,\beta}(A,t) \equiv F_\alpha$ as the propagation observables and work inductively in α.

Using the Leibnitz Rule for operators and the Mourre estimate we can then bound $E_\Delta D F_\alpha E_\Delta$ from below, to prove the theorems 2, 3 by invoking the Basic Lemma.

This approach allows very precise localization of the orbit $e^{-itH}\psi$ in the phase space. For general two body hamiltonians of the PDO type one can prove asymptotic completeness by proving sharp localization of the solution near the classical trajectories [Sig1]. The previous approach to this problem required intricate stationary phase analysis and resolvent estimates due to Agmon; see [Hö IV, last chapter], see also [Kit-Ku], [Comb] for another approach.

The above method of proving theorems 2, 3 suggests a way of getting *finite propagation speed* behavior to Schrödinger type equations and may be useful in the study of propagation of singularities. For some progress in this direction see [Ger-Sig].

Section 8. The N–body Long Range Scattering

The results of this section are based on [Sig-Sof4,5].

Using the minimal velocity bounds we infer that, for large times, $|x| \gtrsim ct$, when the total energy of the state ψ_+ is localized away from the thresholds of H.

In the Long Range case, the two body potentials (at least some of them) vanish like $|x|^{-\mu}$ with $\mu \leq 1$. Therefore $|x_{ij}|^{-\mu} \sim t^{-\mu}$ for large $|t|$ which is not integrable. In particular the proof of AC fails, since now

$$I_a J_a = O(|x|^{-\mu}) \notin L^1(dt) .$$

(by this we mean $(\psi(t), |x|^{-\mu}\psi(t)) \neq L^1(dt)$).
It can be shown that the asymptotic motions of subsystems cannot be free. The modification needed makes the asymptotic hamiltonians *time dependent*.

For each cluster decomposition a we define the hamiltonian

$$H_a(t) \equiv H_a + I_a(x^a, x_a)\,|_{x_a = v_a t} \ .$$

Using $H_a(t)$ instead of H_a in the proof of existence of the Deift-Simon Wave Operators, the $J_a I_a$ term is replaced by

$$J_a(I_a(x^a, x_a) - I_a(x^a, v_a t)) = O(|x|^{-1-\mu})|x_a - v_a t| J_a$$

We therefore need to prove a *sharp propagation estimate*

$$\| J_a |x_a - v_a t| \psi(t) \| = O(t^{\mu - \varepsilon}) \quad \text{for some } \varepsilon > 0$$

to conclude the proof in the Long Range Case.

In practice we modify the time dependent part of $H_a(t)$ further, to include the known minimal and maximal velocity bounds:

$$H_a(t) = H_a + W_a(x,t)$$

where

$$W_a(x,t) = F_{a,E}(x,t) I_a(x^a, v_a t) \ .$$

Here $F_{a,E}(x,t)$ localizes $m \leq |x|/t \leq M$ with m, M depending on a and E. We therefore have

$$|\partial_x^\alpha \partial_t^\beta W| \leq C_{\alpha,\beta}(1 + |x| + |t|)^{-\mu - |\alpha| - |\beta|} \ .$$

Let

$$\phi_a \equiv F(m \leq \frac{|x|}{t} \leq M) j_a(x, p_a)$$

where j_a is a phase–space partition of unity of T^*X. Then it is easy to verify that

Theorem 8.1

a) $\sum\limits_a \phi_a + F(\frac{|x|}{t} \leq m) + F(\frac{|x|}{t} > M) = 1 + O(t^{-1}) \quad |t| > 1$

b) ϕ_a *are supported in* $\{|x|_a > \delta |x|\} \cap \{mt \leq |x| < Mt\} \times \{|k_a| < R\} \equiv \Omega_a$

c) $\langle t \rangle^{|\alpha| + \beta} \partial_x^\alpha \partial_t^\beta \phi_a$ *are supported in* $\Omega_a \backslash PS_E^t$.

Here $PS_E^t \equiv PS_E \cap \{m < \frac{|x|}{t} < M\}$.

We then have (Sig-Sof4)

Theorem 8.2 (Sharp Propagation Estimate) *Assume* $\mu > 0$. *Let* E *be away from the thresholds and eigenvalues of* H. *Then, there exists an interval* Δ *around* E *s.t.*

$$\int_1^\infty \left\| \left|\frac{x_a}{t} - v_a\right|^{1/2} \phi_a \psi_t \right\|^2 \frac{dt}{t} \leq c\|\psi\|^2 \ .$$

The proof follows by studying the following propagation observables. Let

$$\Lambda_a \equiv \left|\frac{x_a}{t} - v_a\right|^2 + t^{-2\beta-2} \quad t \geq 1$$

and define the propagation observables

$$F_a \equiv \phi_a \Lambda_a \phi_a \ .$$

The Heisenberg derivative of F_a consists of two (kinds of) terms:

$$\phi_a(D\Lambda_a)\phi_a \leq 0$$

and

$$(D\phi_a)\Lambda_a\phi_a + \phi_a\Lambda_a D\phi_a \ .$$

This second term lives away from the PS_E^t by the properties of ϕ_a. Hence the original propagation theorem for PS_E and the minimal/maximal velocity bounds show that this term is $L^1(dt)$ which completes the proof. Alternatively, one can use Graf's argument and consider

$$\sum_a \tilde{\phi}_a \Lambda_a \tilde{\phi}_a$$

and try to arrange that, by choosing different $\tilde{\phi}_a$

$$\sum_a (D\tilde{\phi}_a)\Lambda_a\tilde{\phi}_a + \sum_a \tilde{\phi}_a \lambda_a D\tilde{\phi}_a \leq 0 + O(L^1(dt)) \ .$$

This indeed can be done using the monotonic partitions of unity [Gr, Der3]. So far we studied the asymptotic behavior of $e^{-iHt}\psi$ for t large and E away from the thresholds and eigenvalues of H. This establishes *Asymptotic Clustering*

Theorem 8.3 (Asymptotic Clustering) *Let $\mu = 1$ and E be away from the thresholds and eigenvalues of H. Then Asymptotic Clustering holds for any number of particles:*

$$\lim_{t\to\pm\infty} \|e^{-iHt}\psi - \sum_a U_a(t)\psi_a^\pm\| \to 0 \ .$$

The proof follows from the Sharp propagation estimate and construction of the Deift–Simon wave operators as in the short range case [Sig-Sof4]. Recently, by using an intermediate asymptotic dynamics [Ge-Der] improved the above theorem to include all cases of $\frac{1}{2} < \mu$.

We now turn to the problem of Asymptotic Completeness in the Long Range Case. The new feature is the need to analyze the asymptotic behavior of an N–body system with time dependent perturbation $W(x,t)$ added. In this case we have to redevelop all the local decay, velocity bounds, etc. for such hamiltonians. The first problem we are faced with is that the energy is not conserved by time dependent hamiltonians: $E_\Delta(H_a)U_a(t) \neq U_a(t)E_\Delta(H_a)$.

Asymptotic Energy Operators

To treat the lack of energy conservation, we need the method of asymptotic microlocalization. We show that asymptotically the energy distribution is constant and we will microlocalize using the asymptotic observables build from the energy projections [Sig-Sof2-5] see also [Sol]. Let Ω be any bounded interval. Then

$$E_\Omega^\pm(H) \equiv s - \lim_{t\to\pm\infty} U(t)^* E_\Omega(H) U(t)$$

exists, for any $\mu > 0$.

Here $U(t)$ is generated by $H + W(x,t)$ and $W(x,t)$ satisfies the conditions of this section. Moreover

$$\|U(t)E_\Omega^\pm - E_\Omega U(t)\| \leq C|\Omega|^{-1}\langle t\rangle^{-\mu} \ .$$

The proof is simple; it uses Cook's argument to $U(t)^* E_\Delta U(t)$ and the decay properties of $W(x,t)$. From now on, we refer to $U_a(t)$ as $U(t)$ (with $H_a \to H$). By considering $E \notin \mathcal{T}$ (the threshold set of H) and letting

$$\psi^\pm = E^\pm_\Delta(H)\psi \quad \Delta \supset E, |\Delta| \text{ small enough}$$

we can prove now asymptotic clustering for $U(t)E^\pm_\Delta$. This is because the local decay and minimal and maximal velocity bounds can be proved for $H(t)$ by the methods of [Sig-Sof2] as for the independent case. The same is true for the sharp propagation estimates. The main difference now is that we have to estimate $D_{H(t)}F$ instead of $D_H F$ for the propagation observables F. Furthermore, we need to localize using $E^\pm_\Delta$ instead of E_Δ. Both of these can be achieved using the decay properties of W. It is left to consider states in the range of the *singular asymptotic projections*: $E^\pm_0$:

Let $E \in \mathcal{T}$ and $\Omega \supset E$. Then, as before $E^\pm_\Omega$ exists. Since $\mathcal{T}$ is discrete, by density argument we can reduce the problem to an arbitrary small interval around E. We then are left to consider

$$\mathcal{H}^\pm_{\text{thres.}}(H) = \bigcup_{E \in \mathcal{T}} \left\{ \overline{\lim_{\substack{|\Omega| \downarrow 0 \\ \Omega \supset E}}} E^\pm_\Omega(H)\psi \mid \psi \in L^2 \right\} .$$

The scattering theory for initial states in $\mathcal{H}^\pm_{\text{thres.}}(H)$ is fundamentally different than that of states in the orthogonal complement. Such states, if they exist, can only diffuse in certain channels (open) rather than scatter, because they are localized on threshold energies. Consequently the propagation theory is very different. To begin with, the Mourre estimate does not hold and therefore local decay, velocity bounds fail.

Asymptotic Microlocalization and Propagation

We defined the space of (asymptotic) thresholds $\mathcal{H}_{\text{thres.}}(H)$ in terms of the singular projections $\overline{\lim}_{|\Delta| \downarrow 0} E^\pm_\Delta(H)$ $\Delta \supset E$, $E \in \mathcal{T}$. Since we cannot expect $|x| \sim ct$ for such states we need another way of getting some local decay. The first step is then the following time dependent decomposition of the space:

(A) $|x| < ct^{\alpha}$

(B) $|x| \geq ct^{\alpha}$

for some $\alpha < 1$.

In region (A) scattering is not possible. However we treat states in this region using the following wave–operator argument:

$$W(x,t) = W(x,t) - W(0,t) + W(0,t) = O(\frac{|x|}{t^2}) + W(0,t) = 0(t^{-2+\alpha}) + W(0,t)$$

in the region (A). Hence we expect the following wave operator to exist:

$$U_D(t)^* \Big(\frac{|x|}{t^{\alpha}} \leq 1\Big) U(t) \xrightarrow{s} \Omega_D^{\pm} \ .$$

By Cook's argument and the observation above, it is reduced to proving that $DF(\frac{|x|}{t^{\alpha}} \leq 1) \in L^1(dt)$. Since DF lives in the region $\frac{|x|}{t^{\alpha}} \sim 1$, the problem is reduced to the region $\frac{|x|}{t^{\alpha}} \geq 1$, where scattering is expected.

In region (B) $|x| \geq t^{\alpha}$, but the momentum can be arbitrarily close to zero. This suggests another sharp decomposition:

(I) $|p| > ct^{-\beta}$

(II) $|p| \leq ct^{-\beta}$.

Our aim is now to get positive commutators in the region (I), (II) by using that for free flow $x_a \sim v_a t$. Let us consider a simple example to illustrate this approach. Assume that $U(t)$ is generated by a three particle Hamiltonian plus $W(x,t)$. (This is the hard case for $N = 4$). Let $E \in \mathcal{T}$ be negative. In this case, there is no propagation on the three cluster decomposition, by energy conservation.

Consider the following propagation observable, for any two cluster decomposition a:

$$F_a = F_1\Big(\frac{|x^a|}{|x|} \leq \varepsilon^a\Big) F_2\Big(\frac{|x_a|}{t^{\alpha}} \geq 1\Big) F_3(|p_a| < \alpha t^{\alpha-1}) \ .$$

We estimate $D_0 F_a \equiv D_{-\Delta} F_a$. Clearly

$$D_0 F_3 \leq 0 \quad \text{since} \quad [p_a, \Delta] = 0 \ .$$

$$D_0 F_2 = F_2' t^{-\alpha}\left(\gamma_a - \frac{\alpha|x_a|}{t}\right) + O(t^{-2\alpha}) \ .$$

for $\alpha > \frac{1}{2}$, $O(t^{-2\alpha}) \in L^1(dt)$. F_2' localizes $\frac{|x_a|}{t^\alpha} \sim 1$, hence

$$-\frac{\alpha|x_a|}{t} \sim -\alpha t^{\alpha-1} \ .$$

Finally,

$$|\gamma_a| F_3(|p_a| < \alpha t^{\alpha-1}) < \alpha t^{\alpha-1} + O(t^{-1})$$

by the Localization Lemma. Combining all the above we conclude that

$$D_0 F_2 \le -\delta t^{-1} + O(t^{-2\alpha}) \quad \text{for some positive } \delta \ ,$$

on support of F_3.

Finally, observe that DF_1 lives where $|x^a| \sim \varepsilon^a |x|$ and therefore on the free channel (recall $\#(a) = 2$ and the total number of particles is 3). There is no propagation on support DF_1, since $E < 0$. We therefore conclude that

$$D_0 F_a \le \frac{-\delta}{t} F_1 F_2' F_3 + O(L^1(dt)) \ .$$

To show that the potential parts do not change the monotonicity estimate, observe that F_a lives in the region $\{|x|_a \ge t^\alpha, \frac{|x^a|}{|x|} \le \varepsilon^a\}$ which is a two cluster decomposition a with $|x| \ge ct^\alpha$. Furthermore F_a, as a phase space operator is independent of p^a. Hence the commutator with V decays like $|t|^{-(1+\mu)\alpha} \in L^1(dt)$. The commutator with $W(x,t)$ is estimated by the formulas for the commutator of functions of operators to give a contribution of order

$$O(t^{-1-\mu} t^{1-\alpha}) = O(t^{-\mu-\alpha}) \in L^1(dt) \ .$$

We therefore conclude that $-F_a$ is a propagation observable for $H(t)$, when $H(t) = H + W(x,t)$ and H is a three particle hamiltonian.

The resulting propagation estimate and similar analysis for $|p_a| \ge \alpha t^{\alpha-1}$ and a time dependent two cluster partition of unity, shows that there is no propagation in the region $\frac{|x|}{t^\alpha} = 1$. Other types of estimates are needed to complete the proof for $N = 4$.

Let us consider another case. Let $E = 0$. Then $E \in \mathcal{T}$ and the system can propagate on 3 cluster decompositions, so the previous observable does not work, since now we do not know if $DF_1 \in L^1(dt)$. However, in this case, we compute

$$DF_1 = \frac{1}{\langle x\rangle} F_1'(\gamma^a - \frac{|x^a|}{|x|}\gamma) + O(|x|^{-2}) .$$

We use F_2 to conclude that

$$O(|x|^{-2})F_2 = O(t^{-2\alpha}) \in L^1(dt) .$$

Furthermore, F_1' localizes on three cluster decompositions and $\frac{|x^a|}{|x|} \sim \varepsilon^a$. On three cluster decompositions $\sum\limits_{i<j} V = 0(|x|^{-\mu})$.

Hence, if we pull a sharp energy localization projection of the type $F(|H| < t^{-\beta})$ from $\psi \in \mathcal{H}_{\text{thres.}}$, (Recall that now $E = 0$) we get

$$F_1' F(|H| < t^{-\beta})F_2 = F_1' F(|p^2| < t^{-\beta})F_2 + O(t^{-\mu\alpha+\beta}) .$$

Using now that $|p^2| < t^{-\beta}$ we use the localization Lemma to conclude that

$$\langle x\rangle^{-1}\gamma^a \text{ and } \langle x\rangle^{-1}\gamma \text{ are both of order } t^{-\alpha}t^{-\beta/2} \in L^1(dt)$$

if we choose $\alpha + \beta/2 > 1$. A complete solution of the 4 body problem along these lines is given in [Sig-Sof5].

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Note Added: After completion of this survey the following results have been obtained.

1) Asymptotic Completeness for N-body Long Range Scattering is proved by extending the methods described, here by I. M. Sigal, A. Soffer, preprint by J. Derezinski, in preparation, and by L. Zielinski, in preparation.

2) Asymptotic Completeness for 3-Particle Short Range Systems was proved in D. Yafaev, "Radiation Condition and Scattering Theory for Three-Particle Hamiltonians," preprint.

Avy Soffer
Department of Mathematics
Princeton University
Princeton, NJ 08544, U.S.A