FRANÇOIS LAUDENBACH

Appendix. On the Thom-Smale complex

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Morse theory has been much studied and still it is the source of very interesting papers (Witten [W], Floer [F1], [F2] ; see also the review and comments by Bott [B]). Therefore, it seems very hard to write down any new ideas on the subject. Nevertheless, the generic structure of the gradient field of a Morse function is always hidden, though it should be very simple. The aim of this paper is to uncover this simplicity, at least partially. Then some applications to de Rham currents are given. The bifurcation theory in 1-parameter families of gradient fields is also considered.

From now on, M is a C^∞ closed manifold (i.e. compact, without boundary), $f : M \to \mathbb{R}$ is a Morse function and X is the gradient field of $-f$ with respect to a metric on TM. If x is a critical point, $W^u(x)$ (resp. $W^s(x)$) will denote the unstable (resp. stable) manifold of x for the vector field X. We recall that $W^u(x)$ is a submanifold (non closed), diffeomorphic to an open ball whose dimension is the index $i(x)$ of f at x. In the sequel, we make the assumption (T), which is generically satisfied in the space of gradient vector fields [S] :

(T) *For any pair* x, y *of critical points, the manifolds* $W^u(x)$ *and* $W^s(y)$ *are transversal.*

A gradient vector field X satisfying (T) will be said to be Morse-Smale. Then it is known [R] that the closure $\overline{W}^u(x)$ of $W^u(x)$ is obtained by adding a union of unstable manifolds of smaller index. This will be proved again in a special case. For an arbitrary Morse-Smale vector field, this closure may be very complicated ; but when the vector field is gradient and is of *special Morse* type near the singularities (see the condition (SM) below), the structure of $\overline{W}^u(x)$ is very simple and we will describe it.

a) Submanifolds with conical singularities

We define submanifolds with conical singularities (abridged : *smcs*) of dimension k in a smooth manifold N^p of dimension p by recursion on the dimension k. For $k = 0$, it is a discrete set of points. A stratified set $\Sigma = (\Sigma_0, \Sigma_1, \ldots, \Sigma_k)$ in a manifold N^p is a *smcs* of dimension k if the following conditions are satisfied.

(1) For any $i \leq k$, $\Sigma_i - \Sigma_{i+1}$ is a smooth submanifold of dimension $k - i$.

(2) For any point $x \in \Sigma_i - \Sigma_{i+1}$, there exist a neigbourhood V diffeomorphic to a product of discs $D^{k-i} \times D^{p-k+i}$ and a *smcs* $T = (T_0, \ldots, T_i)$ of dimension i in D^{p-k+i} such that :

$$V \cap (\Sigma_0, \Sigma_1, \ldots, \Sigma_k) = D^{k-i} \times (T_0, \ldots, T_i, \emptyset, \ldots, \emptyset).$$

(3) If $x \in \Sigma_k$, there is a C^1 p-ball B centered at x such that :

$\Sigma' = \Sigma \cap \partial B$ is a *smcs* of dimension $(k-1)$ in the $(p-1)$ -sphere ,

and

$$(B, B \cap \Sigma_0, \ldots, B \cap \Sigma_{k-1}) = (B, c\Sigma'_0, \ldots, c\Sigma'_{k-1}),$$

where $c\Sigma'_i$ denotes the cone on Σ'_i with respect to the linear structure of the C^1-parametrized ball B.

Of course, a submanifold with boundary is a *smcs*. Also the singular locus of Σ lies in Σ_1, but some strata of Σ_1 may consist of regular points of Σ. When one does not need to label each stratum, one denotes a *smcs* by Σ or by (Σ_0, Σ_1).

The following facts may be easily proved by recursion on the dimension :

(4) There exists a neighbourhood V of Σ_1 in N and a deformation retract of $(V, V \cap \Sigma_0)$ onto Σ_1.

A submanifold S is said to be *transversal* to a *smcs* Σ if S is transversal to each stratum.

Lemma 1. *1) If a submanifold S of codimension q in N^p is transversal to $\Sigma = (\Sigma_0, \ldots, \Sigma_k)$, then $(S \cap \Sigma_0, \ldots, S \cap \Sigma_{k-q})$ is a smcs of dimension $k - q$ in S.*

2) Suppose that S has a product neighbourhood $S \times D^q$ in N^p, with $S = S \times \{0\}$. Then there exists a germ of diffeomorphisms H of $S \times D^q$ along $S \times \{0\}$ commuting with the projection on D^q, such that $H(\Sigma) \subset (\Sigma \cap S) \times D^q$.

Proof. 1) The first part is local. For instance, take $x \in S \cap \Sigma_{k-q}$. By (2), there is a chart near x such that $\Sigma = D^q \times (T_0, \ldots, T_{k-q}$, where $T = (T_0, \ldots, T_{k-q})$ is a *smcs* in D^{p-q}. The projection $p : D^q \times D^{p-q} \to D^{p-q}$ induces a local diffeomorphism $\varphi : S \to D^{p-q}$. In the corresponding chart on S, $S \cap \Sigma = T$, and so $S \cap \Sigma$ is a *smcs*.

2) One has a local stratified projection $\varphi^{-1}p : D^q \times D^{p-q} \to S$; by stratified projection we mean a C^1-map which is the identity on S and preserves the stratification $T_i \to S \cap T_i$.

It is easy to construct a stratified projection π' defined on a small tube U around S glueing together local stratified projections by means of partition of unity.

On the other hand, one has the projection $\pi'' : U \to D^q$ given by the trivialization of the normal bundle of S. Then (π', π'') is a diffeomorphism near S which is the wanted H. $\square$

b) The main result

If x is a critical point of index k of the Morse function f, the Morse lemma states there exist coordinates $x_1, \ldots, x_n$ near x such that

$$f(x_1, \ldots, x_n) = f(x) - x_1^2 \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2. \tag{5}$$

The gradient vector field X is said to be *Special Morse* (SM) if, near every critical point, there exists a coordinate system $(x_1, \ldots, x_n)$ such that f can be written as in (5), and that X is the gradient of $-f$ with respect to the canonical Euclidean metric associated to the coordinates $x_1, \ldots, x_n$.

Proposition 2. *Assume that X verifies (T) and (SM).*

a) If x is a critical point of index k, then $(\overline{W}^u(x), \overline{W}^u(x) - W^u(x))$ is a smcs of dimension k.

b) $\overline{W}^u(x) - W^u(x)$ is stratified by unstable manifolds of critical points of index strictly less than k.

Remark 3. This proposition says that the unstable manifolds give rise to a structure of CW-complex on M, with one cell for each critical point, the attaching maps of the cells being given by the retractions of (4). In **[T]**, René Thom anticipated such a decomposition.

This result can probably be extended to the case where X verifies only (T) . To do this, one needs to change the definition of a *smcs* by delinearizing the cone construction.

Proof of Proposition 2. Let x be a critical point of f. For $a \in \mathbb{R}$, set $S_a(x) = \overline{W}^u(x) \cap \{f = a\}$. Then if $a < f(x)$ is close enough to $f(x)$, $S_a(x)$ is a sphere. As a decreases, this picture remains stable, as long as a does not coincide with the value of f at a critical point x', which, by (T), is such that $i(x') < i(x)$. The set $\overline{W}^u(x) \cap f^{-1}(f(x') - \epsilon)$ is no longer a smooth manifold. However the next lemma states it is a *smcs* and that its structure remains of the same type as we pass the other critical values of f. The singular strata of this set will be also described.

Let $W \subset \mathbb{R}^n$ be the canonical Morse model : it is a cobordism from a level set $V_{-1} \cong S^{i-1} \times D^{n-i}$ to $V_{+1} \cong D^i \times S^{n-i-1}$. It is equipped with the canonical Morse function $q = -x_1^2 \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2$. The gradient field X of $-q$ is calculated with respect to the canonical Euclidean metric. Of course $V_{\pm 1} = \{q = \pm 1\} \cap W$.

Put $S = S^{i-1} \times \{0\}$ in V_{-1} and $S' = \{0\} \times S^{n-i-1}$ in V_{+1}.

Lemma 4. *Let (Σ', Σ'_1) be a smcs of dimension k in V_{+1}, transversal to S' with non empty intersection. Let Σ (resp. Σ_1) be the closure in V_{-1} of the set of points which lie on a gradient line descending from Σ' (resp. Σ'_1). Then Σ contains S and ($\Sigma, \Sigma_1 \bigcup S$) is a smcs of dimension k.*

Proof. In V_{-1} (resp. V_{+1}), we use polar coordinates $(\phi, \psi, r) \in S^{i-1} \times S^{n-i-1} \times [0, 1]$. With these coordinates and when $r > 0$, the map ($V_{+1} - S'$) $\to$ ($V_{-1} - S$) is the identity. Set $K = \Sigma' \cap S'$, which is a *smcs* by the transversality condition.

First, suppose that Σ' is $D^i \times K \subset D^i \times S^{n-i-1}$, that is :

$$\Sigma' - K = \{(\phi, \psi, r) | \phi \in S^{i-1}, \psi \in K, r > 0\}.$$

In V_{-1}, $\Sigma - S$ is given by the same formula and therefore one has : $\Sigma = S^{i-1} \times cK$, which is a cone fibration, whose vertices lie in S. More generally, by Lemma 1, there is a diffeomorphism H of the form $H(\varphi, \psi, r) = (\varphi, \overline{\psi}(\varphi, \psi, r), r)$ with

$\overline{\psi}(\varphi, \psi, 0) = \psi$, such that $H(D^i \times K) = \Sigma$ near $\{0\} \times K$. Then Σ can be expressed locally as the image of $S^{i-1} \times cK$ by the map $\widetilde{H}$, which is the map H considered as a map from $S^{i-1} \times D^{n-i}$ into itself. Because the radial derivatives of $\widetilde{H}$ exist and are continuous, one verifies easily that $\widetilde{H}$ is C^1-diffeomorphism. Therefore Σ' is a *smcs*. □

Remark 5. 1) Σ is not transversal to S, both sides of the cobordism don't play the same role.

2) The proof of the lemma shows that each stratum of $\overline{W}^u(x)$ is C^∞. However the way in which strata adhere to each other may only be C^1.

Now, we prove Proposition 2. By condition (T), $\overline{W}^u(x) \cap f^{-1}(f(x') + \epsilon)$ is transversal to the sphere S' of the Morse model of x'. Then $\overline{W}^u(x) \cap f^{-1}(f(x') - \epsilon)$ is a *smcs* with a new singular stratum. One then proceed by recursion. The proof of Proposition 2 is completed.

□

c) The Thom-Smale complex

In this section, we make the same assumptions as in Proposition 2. An orientation is chosen on each $W^u(x)$.

For critical points x and y of f, with $i(y) = i(x) - 1$, we define the integer $n(x, y)$ as follows: $n(x, y) = 0$ when $W^u(y)$ does not lie in the closure of $W^u(x)$; otherwise, near $W^u(y)$, $W^u(x)$ consists of $n_+ + n_-$ connected components, $W^u(y)$ being the oriented boundary of n_+ of these. Then $n(x, y) = n_+ - n_-$.

Here is an alternative definition for $n(x, y)$. As $W^s(y)$ is co-oriented (i.e. transversally oriented), to each gradient line in $W^u(x) \cap W^s(y)$ (which is the union of a finite number of gradient lines), one can attach a sign and $n(x, y)$ is the sum of these signs.

Let C_k denote the free abelian group generated by the critical points of index k. The boundary operator $\partial : C_k \to C_{k-1}$ is defined by

$$\partial x = \Sigma n(x, y) y, \tag{6}$$

the sum being over all critical points of index $i(x)-1$. On the other hand, as the geometry of $W^u(x)$ is "finite" near its boundary, we can consider the oriented $\overline{W}^u(x)'s$ as currents, and we have the following Stokes formula.

Proposition 6. *For any smooth differential form ω of degree $k-1$ on M, one has :*

$$\int_{\overline{W}^u(x)} d\omega = \sum_y n(x,y) \int_{\overline{W}^u(y)} \omega\,. \tag{7}$$

Proof. Let U be a neighborhood of $\overline{W}^u(x) - W^u(x)$ which has property (4) in Section a). We apply Stokes theorem to ω on $\overline{W}^u(x) - U$. As we let U shrink, the Stokes formula is seen to converge to the right-hand side of (7), because the singular locus of $\overline{W}^u(x) - W^u(x)$ is negligible with respect to the $(k-1)$-dimensional Lebesgue measure. □

Corollary. $\partial \circ \partial = 0$.

Proof. For any critical point y of index $k-2$, there exists a $(k-2)$-form whose integral over $W^u(y)$ is nonzero and which vanishes over the other $(k-2)$-unstable manifolds. The result then follows from (6), (7) and from the fact that $d \circ d = 0$. □

Let $I_* : C_* \to R_*$ be the map, with values in the complex R_* of de Rham currents, which associates to each critical point x the current of integration over the oriented manifold $\overline{W}^u(x)$. By (7), I_* is a morphism of complexes. Of course, as the $W^u(x)$'s are the cells of a CW-complex, it is known that the homology of C_* is canonically isomorphic to the singular homology of M [M1, Appendix A]. But, in our context, the weaker result with real coefficients may be stated as follows.

Proposition 7. *$I_* : C_* \otimes \mathbb{R} \to R_*$ induces a homology isomorphism.*

Proof. The stable manifolds are naturally co-oriented and give rise to a complex $(\bar{C}_*, \bar{\partial})$, graded by the co-index of critical points : $\bar{i}(x) = n - i(x)$. The pairing $\langle x,x\rangle = 1, \langle x,y\rangle = 0$ when $x \neq y$, satisfies $\langle \bar{\partial}x, y\rangle = \pm\langle x, \partial y\rangle$ and creates a duality between $\bar{C}_{n-*}$ and C_*. Then $H_{n-k}(\overline{C}_*; \mathbb{R}) \cong \operatorname{Hom}(H_k(C_*); \mathbb{R})$.

Like the unstable manifolds, a co-oriented stable manifold of dimension $n-k$ defines a current, which can be paired with smooth $n-k$ forms twisted by the orientation bundle of TM. The de Rham regularization operator [Rh,§15] transforms such currents into smooth differential forms of degree k, and maps $\bar{\partial}$ to d.

Let $\sigma \in C_k$ be a cycle. If σ is not homologous to 0 in C_* , there exists a cycle $\bar{\sigma}$ in $\bar{C}_{n-k}$ such that $\langle\bar{\sigma},\sigma\rangle \neq 0$; then the de Rham regularization operator transforms $\bar{\sigma}$ into a closed $k-$ form ω such that $\langle\bar{\sigma},\sigma\rangle = \int_\sigma \omega$. Therefore σ is not homologous to 0 as a current, and so, I_* is injective in homology.

By duality, to show that I_* is surjective in homology, we only need to prove that if ω is a closed k-form on M such that $\int_\sigma \omega = 0$ for any $\sigma \in C_k$ with $\partial\sigma = 0$, then ω is exact. In fact, there exists $\xi \in \overline{C}_{n-k} \otimes \mathbb{R}$ such that for any critical point x, $\langle\xi, x\rangle = \int_{\overline{W^u(x)}} \omega$. Since $\langle\xi,\sigma\rangle = 0$ for any cycle σ, one has $\xi = \partial\eta, \eta \in \overline{C}_{n-k+1}$. By de Rham regularization, ξ is smoothed into a form ω', which is the differential of the de Rham regularized of η. Then, $\int_{\overline{W^u(x)}}(\omega-\omega') = 0$ for any x. The form $\omega - \omega'$ is shown to be exact by climbing the skeleton, and applying the Poincaré lemma to each cell ; this is detailed in [ST ; 6.2, Lemma 3]. In fact the structure of the closure of the unstable manifolds allows us to proceed in the same way as with the simplices of a triangulation. □

d) The Thom-Smale complex with local coefficients

Let F be a real flat vector bundle on M. Let $C_k(F)$ be the vector space generated by the $x \otimes f$, where x is a critical point of index k, and $f \in F_x$. Then if x, y are critical points of f such that $i(y) = i(x) - 1$, $W^u(x) \cap W^s(y)$ consists of a finite number of gradient lines. To each of these gradient lines, one can attach a sign ϵ and an identification $\alpha : F_x \to F_y$. Set $\partial = \Sigma\epsilon\alpha$. Then the obvious analogues of the results of $c)$ still hold.

e) Bifurcation of the Thom-Smale complex in a $1-$ parameter family

Now we consider a smooth path of pairs (f_t, X_t), $t \in [0,1]$, where X_t is the gradient of $-f_t$ with respect to a metric μ_t. We assume that f_0 and f_1 are Morse functions, and that X_0 and X_1 verify (T) and (SM). One may ask how the Thom-Smale complexes of X_0 and X_1 are related to each other. Observe that the given path can be modified into any other path having the same ends. We allow ourselves modifications which are based on classical tranversality arguments, as well as on a by-product of the universal unfolding of the x^3 singularity. So we assume that the following assumptions are verified:

a) Except on a finite set $\{t_1, \dots, t_k\}$ with $0 < t_1 < \dots < t_m < 1, f_t$ is a Morse function.

b) Near t_k, the path f_t is an "elementary" path of birth or death of a pair of critical points. The word "elementary" means the path is described as in [C, p. 244 - 246] : near the degenerate critical point the path of functions is given by,

$$f_t(x) = \frac{1}{3}x_1^3 - (t - t_k)x_1 \pm x_2^2 \dots \pm x_n^2 + \text{const.} \tag{8}$$

for $t \in [t_k - \epsilon, t_k + \epsilon]$, when the birth happens for increasing t.

c) For $t \in [t_k - \epsilon, t_k + \epsilon]$, the metric μ_t is constant. In the chart where (8) holds, μ_t is a small C^0-perturbation of the canonical Euclidean metric, so that (SM) holds at the two new critical points $(\pm\sqrt{\epsilon}, 0, \dots, 0)$ of $f_{t_k+\epsilon}$.

d) The stable and unstable manifolds of X_{t_k} are transversal ; at the cubical singularities, they are manifolds with boundary.

e) For any t and any critical point x of f_t, distinct from the critical points which appear in the birth/death process when $t \in]t_k - \epsilon, t_k + \epsilon[$, the condition (SM) is satisfied at x with respect to the metric μ_t.

f) At the end points $t = t_k \pm \epsilon$, assumption (T) is verified.

To describe the modification of the Thom-Smale complex along such a path, we consider in succession the following two problems: how does the complex change when one passes a birth-death point, and how does it vary along a path of Morse gradient fields, at the points where (T) is not satisfied.

f) Modification of the Thom-Smale complex near a birth-death point

Change the orientation of the t-axis if necessary and assume that t_k is the birth point of a pair of critical points of index $i, i+1$.

Set $g_- = f_{t_k-\epsilon}, \quad g_0 = f_{t_k}, \quad g_+ = f_{t_k+\epsilon}$. Let x be the cubic singularity of g_0 ; let x' (resp. x'') be the index i (resp. $i+1$)-critical point of g_+ just created from x. The point x is a degenerate critical point with index i. Its local unstable manifold is a half-disc of dimension $i+1$ and its local unstable manifold is a half-disc of dimension $n-i$; they meet only at x which lies in their boundaries. The kernel of the Hessian at x is the unique direction tangent to the stable and unstable manifolds. The singularities of g_+, all quadratic, are those of g_- plus x' and x''.

If y (resp. z) is a critical point of index $i+1$ (resp. i) of g_0, the integer $n(y,x)$ (resp. $n(x,z)$) is well defined because the transversality condition is assumed for $g_0 = f_{t_k}$: it is the algebraic number of gradient lines descending from y to x (resp. from x to z).

The formulae which calculate the complex (C_+, ∂_+) associated to $(g_+, \mathrm{grad} g_+)$ from the complex (C_-, ∂_-) associated to $(g_-, \mathrm{grad} g_-)$ are the following :

$$\partial_+ p = \partial_- p \text{ for any critical point } p \text{ of } g_+ \quad \text{with } i(p) \neq i+1, i+2 \text{ and } p \neq x', \tag{9}$$

$$\partial_+ x'' = x' + \sum_{i(z)=i} n(x,z) z, \tag{10}$$

$$\partial_+ x' = -\sum_{i(z)=i} n(x,z) \partial_- z, \tag{11}$$

$$\partial_+ y = \partial_- y + \ n(y,x)[x' + \sum_{i(z)=i} n(x,z) z], \tag{12}$$

for any critical point y of g_+, $i(y) = i+1$ and $y \neq x''$;

$$\partial_+ y = \partial_- y - \ n(\partial_- y, x) x'' \tag{13}$$

for any critical point y of $g_+, i(y) = i+2$.

In (13), $n(\alpha_1 y_1 + \cdots + \alpha_k y_k, x) = \alpha_1 n(y_1, x_1) + \cdots + \alpha_k n(y_k, x)$, where the α_j 's are integers and the y_j 's are critical points of index $i+1$. These formulae are complicated, but, except when i is 0, $n-1$ or $n-2$, one can easily make all the $n(x,z)$ and $n(y,x)$ zero, in which case they become trivial. This is the case when the box where the new pair of critical points of index $i, i+1$ is far from the unstable manifolds of points of index $i+1$ and from the stable manifolds of points of index i.

All these formulae are consequences of the following geometrical fact : if L is a level set of g_+ just below x', then $L \cap W^u(x')$ is the boundary of $L \cap W^u(x'')$ which is a small deformation of $L \cap W^u(x)$; if L is a level set just above x'', then $L \cap W^s(x'')$ is the boundary of $L \cap W^s(x')$ which is a small deformation of $L \cap W^s(x)$.

Now we put these formulae in a more concentrated form. For this, we introduce the split extension (C_-^e, ∂_-^e) of (C_-, ∂_-) by the acyclic complex $0 \to \mathbb{Z}x'' \xrightarrow{\times 1} \mathbb{Z}x' \to 0$.

Consider the following automorphism A of C_-^e : in degree distinct from i, $i+1$, it is the identity. For $i(y) = i+1, y \neq x''$, put $A(y) = y + n(y,x)x''$ and $A(x') = x' + \Sigma_{i(z)=i}\, n(x,z)z$. This automorphism is "elementary" in the sense of algebraic K-theory. We get

Proposition 8. *(C_+, ∂_+) is obtained from (C_-, ∂_-) by setting $C_+ = C_-^e$ and $\partial_+ = A^{-1} \circ \partial_-^e \circ A$.*

g) The Thom-Smale complex near points where (T) is not satisfied

After the above discussion, we are reduced to consider a path of Morse functions f_t, $t \in [0,1]$, where both ends f_i, $i = 0,1$, are equipped with gradient vector fields X_i satisfying (T) and (SM). The Morse lemma holds with parameters and the space of Morse charts of a given Morse function, near one fixed critical point, is connected, up to the Euclidean symmetries of the model (Alexander trick). Then it is easy to construct a path of metrics μ_t such that $X_t = -\mathrm{grad}_{\mu_t} f_t$ satisfies (SM) for every $t \in [0,1]$ and coincides with the given vector fields for $t = 0,1$.

Now, by approximation, we can suppose that X_t satisfies the transversality condition (T) except for $0 < t_1' < \ldots < t_p' < 1$; moreover, the $f_{t_k'}$'s have distinct critical values. The lack of transversality in a $1-$ parameter family can be described generically as follows : let L be a regular level of $f = f_{t_k'}$, $L = f^{-1}(a)$, just above a critical point x of index i. In $f^{-1}([a, +\infty[)$ and in $f^{-1}(]-\infty, a])$, the transversality condition (T) is valid for the stable and unstable manifolds of each cobordism considered alone. The unstable manifolds of critical points of f, with critical values $> a$, induce on L some stratification St with conical singularities. Let $S \subset L$ be the trace of the stable manifold of x: S is non transversal to exactly one stratum Σ of St; there is a unique point p where Σ and S meet non transversally and the tangency at p is a "codimension 1" singularity.

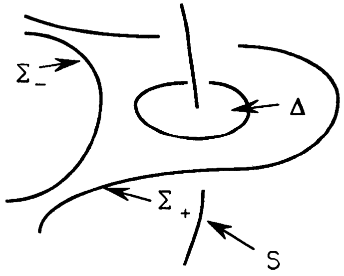
The stratification of the space of embeddings $S \to L$ induce by Σ is described in [C, p.123]. When going from $f_{t'_k-\epsilon}$ to $f_{t'_k+\epsilon}$, the picture of the stable-unstable manifolds is itself stable above L and below L. But the glueing of both pictures in L is not stable; it crosses a codimension 1 stratum in the space of embeddings mentionned above.

In the following, we only consider failures of transversality which a priori generate modifications of the associated algebraic complex. They are of two types:

First type. $\dim\Sigma + \dim S = \dim L = n-1$. In this case $\Sigma = W^u(y) \cap L$, where y is a critical point of index $i+1$; during the transition, some pair of gradient lines descending from y to x is created or cancelled. But the integer $n(x,y)$ is preserved and the algebraic complex does not change.

Second type. $\dim\Sigma + \dim S = \dim L - 1$.

In this case y is a critical point of index i. The transition is pictured in L : we have a small disc Δ cutting S in one point and one moves from Σ_- to Σ_+ through Δ.



As unstable manifolds are oriented, Σ is oriented and S is transversally oriented; therefore, the above operation comes with a sign ϵ. The boundary morphism changes from ∂_- to ∂_+ according to the following formulae :

(14) $$\partial_+(z) = \partial_-(z) - \epsilon\, n(z,y)x \text{ if } \operatorname{ind}(z) = i+1 \ ,$$

(15) $$\partial_+(z) = \partial_-(z) \text{ if } \operatorname{ind}(z) = i \text{ and } z \neq y,$$

$$\partial_+(y) = \partial_-(y) + \epsilon\partial_-(x), \tag{16}$$

and, for the other critical points, $\partial_+ = \partial_-$.

Here is a sketch of proof for (16). Let L' be a level set of f just below $f(x)$; as Δ is a small meridian disc of S, the gradient lines descending from $\partial\Delta$ intersect L' along a sphere parallel to $L' \cap W^u(x)$. If Σ'_+ (resp. Σ'_-) denotes the trace in L' of the gradient lines descending from Σ_+ (resp. Σ_-) then Σ'_+ is the connected sum of Σ'_- with a sphere parallel to $L' \cap W^u(x)$. Formula (16) follows.

If A is the "elementary" automorphism of the module C_* defined by $A(p) = p$ for any generator $p \neq y$ and by $A(y) = y + \epsilon x$, then we get :

Proposition 9. *(C_+, ∂_+) is obtained from (C_-, ∂_-) by setting $C_+ = C_-$ and $\partial_+ = A^{-1} \circ \partial_- \circ A$.*

The formulas from (9) to (16) still make sense with local coefficients. Then, if for some adhoc system of coefficients the complex becomes acyclic, its torsion (Franz-Reidemeister or Whitehead) does not depend on the pair - function, gradient vector field - chosen at the beginning. Of course, this fact is well known (compare Milnor [M2, §9]).

h) Final comments and complements

The only new fact proved in this appendix is that the pair (f, X) of a function and a gradient vector field (with some conditions) produces an embedding I_* of the Thom-Smale complex C_* into the complex R_* of de Rham currents, because the unstable manifolds of critical points are currents. Then, by Proposition 7, we have a canonical isomorphism between the Thom-Smale homology (homology of the Thom-Smale complex) and the de Rham homology. In this Section, we will verify directly that the identifications of complexes of Proposition 8 and 9 induce the corresponding canonical identifications of their homology groups.

When we need to specify the pair (f, X) which is used, $C_*(f, X)$ and $I_*(f, X)$ will denote the Thom-Smale complex and the corresponding embedding into the de Rham complex.

First, let us consider a one-parameter family (f_t, X_t), $t \in [0, 1]$, of Morse functions and gradient vector fields satisfying both conditions (T) and (SM) on the whole interval. In this case, $C_*(f_0, X_0)$ and $C_*(f_1, X_1)$ are the same as the critical points of both functions are in canonical correspondance and we have two embeddings of the same Thom-Smale complex. We claim that $I_*(f_0, X_0)$ and $I_*(f_1, X_1)$ are homotopic; this means that there exists a morphism K of degree $+1$ from C_* to R_* such that

$$I_*(f_1, X_1) - I_*(f_0, X_0) = \partial \circ K + K \circ \partial$$

This equation is satisfied if for each generator x of $C_k(f_0, X_0)$, we set $K(x) = \bigcup_t W^u(x_t)$. Here x_t is the critical point of f_t corresponding to x and $K(x)$ is of course a $(k+1)$-dimensional current; it is the direct image by the projection $M \times [0,1]$ to M of the obvious current $\bigcup_t W^u(x_t) \times \{t\}$ in $M \times [0,1]$. As a consequence, one has the following result.

Proposition 10. *$I_*(f_0, X_0)$ and $I_*(f_1, X_1)$ induce the same isomorphism in homology.*

The crossing of an "accident" along the path (f_t, X_t) - failure of transversality or birth-death point - involves a little bit more technicality. But with the notation of Propositions 8 and 9, and using homotopies like above, one can prove the following.

Proposition 11. *1) Near a generic no-transversality point, the morphisms $I_*(f_+, X_+)$ and $I_*(f_-, X_-) \circ A$ induce the same isomorphism in homology.*

2) Near a birth point, $I_*(f_+, X_+)$ and $I_*(f_-, X_-) \circ p \circ A$ induce the same isomorphism in homology, where p is the natural projection of C_-^e onto C_-.

To conclude this Appendix, we give a Fubini formula which only makes sense by our use of currents. Here (f, X) is a pair satifying the (T) and (SM) conditions, ω is a closed k-form, Ω is a closed orientation-twisted $(n-k)$-form; the k-dimensional unstable manifolds are oriented and the $(n-k)$-dimensional stable manifolds are co-oriented.

Proposition 12.

$$\int_M \omega \wedge \Omega = \sum_x \int_{\overline{W^u(x)}} \omega \int_{\overline{W^s(x)}} \Omega, \tag{17}$$

where the sum is taken over all the critical points of index k.

Proof. The transpose of I_k maps the cocycle ω to a cycle of $\overline{C}_{n-k}$, given by

$$\sum_x \left(\int_{\overline{W}^u(x)} \omega\right) x,$$

which itself gives rise to the twisted closed current

$$\sigma = \sum_x \left(\int_{\overline{W}^u(x)} \omega\right) \int_{\overline{W}^s(x)} .$$

Thus the homology class of σ only depends on the cohomology class of ω. Therefore, the right-hand side of (17) only depends on the cohomology classes of ω and Ω. The same is obviously true for the left-hand side. So we are reduced to the case where ω vanishes near the $(k-1)$-skeleton of the stratification by unstable manifolds and Ω vanishes near the $(n-k-1)$-skeleton of the stratification by stable manifolds. Then $\omega \wedge \Omega$ vanishes everywhere except on blocks $D^k \times D^{n-k}$, usually called handlebodies. On each handlebody the formula reduces more or less to Fubini. $\square$

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Ecole Normale Supérieure
46, Avenue d'Italie
69364 - Lyon cédex 07 - France
et
Ecole Polytechnique
Centre de Mathématiques
91128 Palaiseau - France