EMMANUEL DROR FARJOUN
J. SMITH

A geometric interpretation of Lannes' functor T

Astérisque, tome 191 (1990), p. 87-95
<http://www.numdam.org/item?id=AST_1990__191__87_0>

A Geometric Interpretation of Lannes' Functor T.

E. DROR FARJOUN AND J. SMITH

1. Introduction. In this note we are concerned with a question raised by [Lannes 2.3]. In what follows R will denote a finite field of the form $\mathbf{Z}/p\mathbf{Z}$, homology and cohomology are always taken with coefficient in R and denoted by H_*X etc. For a space X let $\{R_sX\}_s$ denote the Bousfield-Kan localization tower. We denote by $B\tau$ the classifying space of the underlying abelian group of R. Let P_sX denote the s-Postnikov section of X. By a "space" we mean a Kan complex or a C.W. complex.

1.1 Theorem: *If $H^iX < \infty$ for all $i \geq 0$, then $TH^*X \cong \varinjlim_s H^*(P_sR_sX)^{B\tau}$, where T is Lannes' functor* (see below). *If, in addition, X is nilpotent then $TH^*X \cong \varinjlim$ $H^*(P_sX)^{B\tau} \cong \varinjlim H^*(P_sR_\infty X)^{B\tau}$*

The proof of this theorem yields a new proof for Lannes theorem 1.5 below that essentially asserts 1.1 for dimension zero and was a the motivation for his question [Lannes 2.3]. The proof of theorem is based on the following technical proposition:

1.2 Proposition: *Let $G \to E \to B$ be a principal fibration where G is a (topological or simplicial) group. Assume that in each dimension the R-cohomology of the mapping spaces E^{B_t} and B^{B_t} is finite. Then if the relation $TH^*W \cong H^*W^{B_t}$ is satisfied by $W = E$ and $W = B$ then it is also satisfied by $W = G$.*

Remark: The finiteness assumption, noted by the referee, is necessary in order to use cohomological Eilenberg-Moore spectral sequence.

Corollary: *If W is a nilpotent space of finite type with $\pi_iW = 0$ for $i >> 0$ or a R-localization thereof then*

$$TH^*W \cong H^*W^{B\tau}.$$

Further, as a direct corollary of 1.1 and 9.3 of [Bousfield] one gets the following interesting special case due to Lannes [4].

1.3 Corollary. *Let $H^iX < \infty$ for all $i \geq 0$ for nilpotent space X of finite type. Assume that a given algebraic component T_cH^*X of TH^*X is finite in all dimensions and vanishes in dimension one. Then* $T_cH^*X \cong H^*X_c^{B\tau} \cong H^*(R_\infty X)_c^{B\tau}$

where $X_c^{B\tau}$ is the corresponding component.

Another example where the main result (1.1) implies a result on $H^*map(B\tau, X)$ is when the latter has a finite homotopy group in each dimension.

1.4 Corollary: *Let X be nilpotent space of finite type with $\pi_1 X$ finite. Assume that for a given $f : B\tau \to X$ the groups $\pi_i map(B\tau, P_nX)_{f_n}$ are finite for all $i, n \geq 0$. Then $H^*map(B\tau, X) \cong T_cH^*X$ where T_c is algebraic component of T that corresponds to f.*

The referee also notes that theorem 1.1 gives a new proof of the following result [Lannes, 0.4].

1.5 Corollary: *If Y is a nilpotent space with $H^n(Y, R)$ finite for all n, then the natural map*

$$[Bt, Y] \cong [Bt, R_\infty Y] \to Hom_K(H^*Y, H^*B_t)$$

is an isomorphism of profinite sets.

Proof: This follows directly from 1.1 above in light of the algebraic fact [Lannes 3.5] and the old result of [Dror] about the tower R_sY.

The authors would like to thank W. Dwyer for his suggestion to consider the tower R_sX as a starting point for a geometric interpretation of T, and to H. Miller for several useful discussions. The authors would also like to thank the referee for his careful reading and for correcting a non-fatal error in an earlier version of this paper. The referee notes that if one considers the fibration $\Omega X \to * \to X$ for X being the infinite wedge of $\mathbb{R}P^\infty/\mathbb{R}P^{2n+2}$ over the integers, the formula in 1.2 holds for $W = \Omega X$ but not for X itself. Therefore one cannot turn around 1.2 to conclude that either E or B satisfy the property $TH^*W = H^*W^{B_t}$, assuming the other two spaces do.

2 First examples.

Let $\mathcal{U}$ denote the category of unstable modules over the algebra $\mathcal{A}$ of Steenrod reduced powers relative to a prime $p = char\ R$. Let $\mathcal{K}$ denote the category of unstable $\mathcal{A}$-algebra. In [Lannes] a left adjoint T is defined to the functor $- \otimes H^*B\tau$, where the latter is taken either as a functor from $\mathcal{U}$ to itself or from $\mathcal{K}$ to itself. If one regards an element $A \in \mathcal{K}$ as an element of $\mathcal{U}$, the value of T does not change. Thus the defining property of T is $hom_C(TM, N) = hom_C(M, N \otimes H^*B\tau)$ where C is either $\mathcal{U}$ or $\mathcal{K}$.

2.1 Three basic properties [Lannes]: (i) T is exact. (ii) T commutes with tensor products. (iii) T commutes with direct limits.

2.2 Motivation: It can be seen from 1.1, 1.2, 1.3 that the construction of T is motivated by attempts to describe the cohomology of $X^{B\tau} \equiv map(B\tau, X)$ in terms of H^*X, when the latter is given as an object in $\mathcal{K}$. Lannes proves the relation between the homotopy class $[B\tau, X]$ and $(TH^*X)^0$ and X as in 1.3, see [Lannes 7.1.1]. [Miller] proves it for $dimX < \infty$.

2.3 Example. It is easy to calculate directly from the adjointness relation that if V is a finite dimensional vector space over R then

$$TH^*K(V,n) \cong \bigotimes_{n \geq i \geq 0} H^*K(V,i) \cong H^*map(B\tau, K(V,n)).$$

Here $map(X,Y)$ denotes the space of maps $X \to Y$ otherwise denoted by Y^X. Similar calculation holds for a finite products of $K(V_i, n_i)$ with $dim_R V_i < \infty$. However it turns out that for homotopically large space one cannot, in general, interpret TH^*X as the cohomology of $map(B\tau, X)$, (see 2.5 below).

2.4. Example. An important class of spaces on which T behaves nicely are finite Postnikov pieces of nilpotent spaces. The prime examples of such spaces are $K(\mathbf{Z}, n)$ spaces for $n \geq 0$.

Proposition: *For any $n \geq 0$ there is an isomorphism $TH^*K(\mathbf{Z},n) \cong H^*map(B\tau, K(\mathbf{Z},n))$.*

Proof: For $p = 2$ we show by a direct computation that $TH^*K(\mathbf{Z},n) \cong H^* \prod_{i=1}^{[n/2]} K(\mathbf{Z}/2\mathbf{Z},\ 2i)$. For $p > 2$ the argument is similar. Now since $H^*K(\mathbf{Z},n) \cong P(S_q^I \mid I$ admissible with $e_1(I) \geq 2$ and $e\ (I) \leq n-1)$ a map of the algebra $H^*K(\mathbf{Z},n)$ over A is given by the image of the generator in dimension n. Thus

$$hom_{\mathcal{K}}(K(\mathbf{Z},n), K) = ker\beta : K_n \to K_{n+1}$$

where K is any object in $\mathcal{K}$ and β is The Bockstein operation. Now compute:

$$\begin{aligned}
hom_K(TH^*K(\mathbf{Z},n),K) &\cong hom_K(H^*K(\mathbf{Z},n),K\otimes H^*B\tau)\\
&\cong ker\beta : K\otimes H^*B\tau)_n \to (K\otimes H^*B\tau)_{n+1}\\
&\cong ker\beta : \bigoplus_{i+j=n} K_i\otimes H^jB\tau \to \bigoplus_{i+j=n+1} K_i\otimes H^jB\tau\\
&\cong \bigoplus_{\substack{i+j=n\\ j \ \text{even}}} K_i =\\
&\cong hom_K(H^*\prod_{i=1}^{[n/2]} K(\mathbf{Z}(2\mathbf{Z},i),K).
\end{aligned}$$

This together with the adjointness property of T completes the proof. Similarly let $\mathbf{Z}_p$ denotes the p-adic numbers $\mathbf{Z}_p = \text{invlim } \mathbf{Z}/p^k\mathbf{Z}$. Then [B - K VI 6.4] one has an R homology equivalence $K(\mathbf{Z},n) \to K(\mathbf{Z}_p,n)$ for all $n \geq 0$. There is a pro-isomorphism on R-homology of $K(\mathbf{Z},n) \to (K(\mathbf{Z}/p^k\mathbf{Z},n))_n$. Therefore

$$H^*K(\mathbf{Z},n) \cong H^*K(\mathbf{Z}_p,n) = \varinjlim_k H^*K(\mathbf{Z}/p^k\mathbf{Z},n)$$

Moreover it follows by a spectral sequence argument that the tower $\{map(B\tau,K(\mathbf{Z}/p^k\mathbf{Z},k))\}_*$ is an R completion tower for the function complex $map(B\tau,K(\mathbf{Z},n))$, since all function complexes involved here are R-nilpotent. Again using comparison of spectral sequences converging to homology we see that there is an R-homology (hence R-cohomology) equivalence $map(B\tau,K(\mathbf{Z},n)) \to map(B\tau,K(\mathbf{Z}_p,n))$. Therefore the R-cohomology of the last range is isomorphic to the limit of the R-cohomologies $\varinjlim_k H^*map(B\tau,K(\mathbf{Z}/p^k\mathbf{Z},n))$. But since the functor T commutes with direct limits we get the desired example:

$$TH^*K(\mathbf{Z}_p,n) \cong H^*map(B\tau,K(\mathbf{Z}_p,n)).$$

The second example of $K(\mathbf{Z}_p,n)$ is in reality equivalent to the first using the isomorphism of cohomologies $H^*(B\tau,\mathbf{Z}) \cong H^*(B\tau,\mathbf{Z}_p)$. Since the function complexes $hom(B\tau,K(\mathbf{Z},n))$ and $hom(B\tau,K(\mathbf{Z}_p,n))$ are built out of these cohomology groups, the map $\mathbf{Z} \to \mathbf{Z}_p$ induces a homotopy equivalence between them. Now since $TH^*K(\mathbf{Z},n) \cong TH^*K(\mathbf{Z}_p,n)$ one satisfies 2.4 if and only if the other does.

2.5 **Example.** It is not hard to check that if V is an infinite dimensional vector space over R then 2.3 fails to hold.

Similarly, let $RB\mathcal{T}$ be the free (simplicial) R-module generated by $B\mathcal{T}$, then R has non-trivial homotopy groups in all dimensions and $H^0 map(B\mathcal{T}, RB\mathcal{T})$ is larger then $T^0 H^* B\mathcal{T}$ which is countable.

3. **Proof of 1.2.** The main observation of this note is (1.2) from which (1.1) and (1.3) follow. We use the Eilenberg Moore spectral sequence (EMSS) to gain information on H^*W as an object in $\mathcal{U}$, i.e. as an unstable module over the Steenrod algebra $\mathcal{A}$. D. Rector, L. Smith, A. Heller and others showed that there is a natural action on the Eilenberg-Moore spectral sequence $E^r(W \to E \xrightarrow{u} B)$ by $\mathcal{A}$ making the differentials $\mathcal{A}$-linears and such that E_∞ is a graded $\mathcal{A}$-module associated to a filtration:

3.1. $H^*W \supseteq \cdots \supseteq F^{-2} \supseteq F^{-1} \supseteq F^0 \supseteq 0 \supseteq 0 \supseteq \cdots$ of H^*W by $\mathcal{A}$-submodules. We shall need the following result of [Dwyer] that gives a necessary and sufficient condition for a strong convergences of the spectral sequence: For every n the above filtration of H^nW is *finite* iff $\pi_1 B$ operates nilpotently on H^i (fibre) for all i.

3.2 Observe that if $p : E \to B$ is a fibre map with B not necessarily connected and with $\pi_1(B, *)$ operates nilpotently on $H^i(p^{-1}(*))$, then EMSS of $(* \to B \leftarrow E)$ will be identical to the one associated to the connected component of $* \in B$ and therefore will likewise converge strongly to $H^*(p^{-1}(*))$. This is because the functor Tor_{H^*B} appearing in E_2 'eliminates' all the components of H^*E not hitting the component of $* \in B$ in H^*B, due to the trivial action of off base point elements in H^*B on $H^*(b) = R$.

Claim: If L is any space of then the Eilenberg-Moore spectral sequence for the fibre square

$$\begin{array}{ccc} map(L, W) & \to & map(L, E) \\ \downarrow & & \downarrow \bar{u} \\ map(L, *) & \to & map(L, B) \end{array} \tag{3.3}$$

induced by the fibration in (3.1) converges strongly.

Since map$(L, *) = *$ the above pull back square is, in fact, a fibre map $\bar{u}$ with a non-connected space map(L, b) as the base and with map(L, G) as the fibre are the component of the null homotopic maps in the base.

3.4 **Lemma:** *If $G \to E \to B$ is a principal fibration of spaces where G is a group, then for any space L the map* $\mathrm{map}(L, E) \to \mathrm{map}(L, B; E)$, *where the range is the space of maps $L \to B$ liftable to E, is a principal fibration with the group being* $\mathrm{map}(L, G)$ *and the action is pointwise.*

Proof: One checks directly that the pointwise action is a transitive action of $\mathrm{map}(L, G)$ on the fibres of the above maps.

It follows from the above that the EMSS of (3.3) converges strongly, and as argued above the E^s-terms are the same for the fibrations $\mathrm{map}(L, E;$ null homolopic on $B) \to \mathrm{map}(L, B;$ null homotopic$)$.

Now we wish to compare the Eilenberg-Moore spectral sequence of (3.4) to the spectral sequence gotten by applying T to the Eilenberg-Moore sequence of the given fibration $W \to E \xrightarrow{u} B$. Let $E_r(u)$ be the spectral sequence of the fibration u.

For each $r \geq 1$ and $s \leq 0$ the $\mathbf{Z}$ - graded objects $E_1^{s,*}, E_2^{s,*}$ are unstable modules over the Steenrod algebra since the first one is, being the cohomology of the space

$B \times B \times \cdots \times B \times E$ (product taken s times). (Notice, however, that if we grade $\{E_r^{p,q}\}$ by the total degree $p+q$, we do not get an unstable module, but rather a stable one - e.g. Sq^i can operate non-trivially on elements of bi-degree $(-s, s)$, for any $s \geq i > 0$.). Therefore, we can form a spectral sequence $\{TE_r;\ Td_r\}$ by applying T to each $E_r^{-s,*}(u)$ as an object in U, to get another object in U namely $TE_r^{-s,*}$.

3.5. **Claim.** $TE_r(u)$ converges to TH^*W in the sense that $TE_\infty^{s,*}(u)$ is associated graded $\mathcal{A}$-module to the $\mathcal{A}$-filtration TF^i with $\varinjlim TF^i \cong TH^*W$.

To see why notice (2.1) that T is exact so it converts an exact couple to an exact couple, and since all the terms in spectral sequence $E_r^{s,*}(u)$ are $\mathcal{A}$- modules in $\mathcal{U}$ and all derivations $\mathcal{A}$-maps one can apply the functor T to get another spectral sequence. Since T is a left adjoint it commutes with direct limits so that $H^*W = \varinjlim F^i$ implies what we need.

Let $E_r(\bar{u})$ be the spectral sequence of the fibre-square (3.4) for $L = B\tau$. One can construct a comparison map $TE_r(u) \to E_r(\bar{u})$ using the adjointness properties of T: the evaluation map $B\tau \times map(B\tau, X) \to X$ induces [Lannes] a map

$$TH^*X \to H^*map(B\tau, X).$$

Therefore there is a natural map of $\mathcal{A}$-modules

$$TE_r(u) \to E_r(\bar{u}).$$

Claim: Under the assumption of lemma 1.2, this map is an isomorphism.

Proof: First notice that if $K \in \mathcal{K}$ and $M, N \in \mathcal{U}$ are K-modules then $T(A \underset{K}{\otimes} B) = TA \underset{TK}{\otimes} TB$. This is because $A \underset{K}{\otimes} B$ is the cokernel of a difference map $A \otimes K \otimes B \to A \otimes B$ of the two operation of K. Now T commutes with $\otimes$ in $\mathcal{U}$ so TN, TM are TK-modules and again by commutation and (right) exactness of T (2.3) we get the tensor product. Next notice that since the unstable $\mathcal{A}$-model $Tor_s(M, N)$ is the s - homology group of a chain complex consisting in degree s of $M \otimes K \otimes K \otimes \ldots \otimes K \otimes N$ and since T preserves tensor products one has for all $s \geq 0$.

$$T(Tor^s_K(M, N)) = Tor^s_{TK}(TM, TN).$$

By assumption on the space E and by (2.3) we get the desired result. Thus we have an isomorphism for $r = 2$ thus for all r.

It follows that one has an isomorphism $TE_\infty(u) \cong E_\infty(\bar{u})$. Now we get for each submodule in the filtration an isomorphism:

$$TF^i(u) \cong F^i(\bar{u})$$

and taking direct limits, noting again that T commutes with direct limits, we get the desired result by comparison of spectral sequence, namely

$$TH^*W \cong H^*map(B\tau, W).$$

Thus using the bar construction of the EMSS we saw that $E_1^{s,*}$ is an unstable module over A. This proves 1.2.

4. Proof of 1.1 and 1.3. If $H^iX < \infty$ for all i, then $\pi_i P_s R_s X < \infty$ for all $s, i \geq 0$ where P_s is the Postnikov section. This means that the space $P_s R_s X$ satisfies the conditions of (1.2) and we have $TH^* P_s R_s X = H^* map(B\tau, P_s R_s X)$. But since $H_i X \to \{H_i R_s X\}$ is a pro-isomorphism of towers [Dror] of finite groups, we have $H^* X \cong \varinjlim H^* P_s R_s X$ therefore (2.3)(iii) $TH^*X \cong \varinjlim TH^* P_s R_s X = \varinjlim H^* map(B\tau, H^* P_s R_s X)$, This gives 1.1.

Now one uses the following lemma of [Bousfield 9.3]. Consider a tower of fibrations of pointed R-nilpotent spaces $\{X_m\}$

4.1. Lemma. *If $\{\tilde{H}_i(X_m, R)\}_m$ are pro-trivial for $i \leq 1$ and pro-constant for all i, then $\varprojlim X_m = X_\infty$ is simply connected and the map $H_i(X_\infty, R) \to \{H_i(X_m, R)\}$ is a pro-isomorphism for all i.*

We use (4.1) with $X_m = P_m R_m X$.

Notice that if $\{H_m\}$ is an inverse tower of finite groups with $A_\infty = \varprojlim A_m$ a finite group then the map $A_\infty \to \{A_m\}_m$ is a pro-isomorphism, because $\varprojlim^1 (-)$ vanishes on tower of finite groups and $\varprojlim$ is left exact. Consider the tower $H_i(X_s^{B\tau})_c = (H^i(X_s^{B\tau})_c^*$ where $(-)^*$ denotes the R-dual. By 1.2 this is a tower of finite groups since one considers only a component $(X_s^{B\tau})_c$ for which T_c is finite in all dimension. Therefore $H_0(X_s^{B\tau})_c \cong R$ the tower $H_1(X_s^{B\tau})_c$ is pro-trivial, being pro-isomorphic to $(T_c^1 H^*(X))^*$. Therefore by lemma 4.1 the tower $\{H_i(X_s^{B\tau})_c\}_{s\geq 0}$ is pro-isomorphic to $H_i(\varprojlim (X_s^{B\tau})_c)$.

Since for any tower $\cdots \to Y_{m+1} \to Y_m \to \cdots Y_0$ of fibrations taking inverse limit commutes with taking function complex $map(L, -)$ the desired conclusion follow from $\varprojlim (X_s)^{B\tau} \sim (\varprojlim X_s)^{B\tau} = X^{B\tau}$, since X is assumed to be R-nilpotent.

4.2. Proof of 1.4. By [Lannes 7.1.1] we have again $H^0 map(B\tau, X) \cong T^0 H^* X$ so that as in 1.3. $T_c H^* X$ is a well defined component corresponding to $[f] \in [B\tau, X]$. We have $X_f^{B\tau} = \varprojlim map(B\tau, P_n X)_{f_n}$ where f_n is the obvious composition $B\tau \xrightarrow{f} X \to P_n X$. Since all the relevant homotopy groups are finite, one gets vanishing lim^1 - term and thus $\pi_i X_f^{B\tau} \cong \varprojlim \pi_i(map \quad (B\tau, P_n X), \quad f_n)$. But, again this means that there is a pro-isomorphism $\pi_i X_f^{B\tau} \cong \{\pi_i(map(B\tau, P_n X), f_n)\}_n$ for each i, so that the constant tower $X_f^{B\tau}$ is pro-equivalent to the tower $\{P_n X)_{f_n}^{B\tau}\}_n$. Thus, they have the same R - cohomology.

But the R - cohomology of the latter is pro-isomorphic to TH^*X as needed.

REFERENCES

1. A. K. Bousfield, *One the homology spectral sequence of a cosimplicial space.* American J. of Math., vol. 109 (1987), pp. 361-394.

2. E. Dror, *The pro-nilpotent completion of a space.* Proc. AMS, vol. 38 (1973).

3. W. G. Dwyer, *Strong convergence of the Eilenberg-Moore Spectral Sequence.* Topology, vol. 13 (1974),

4. J. Lannes, *Sur la cohomologie modulo p des p-groupes abéliens élémentaires,* Homotopy theory Ed. E. Rees and J.D.S. Jones, London Math. Series #117 p. 112. (1987).

5. H. Miller, *The Sullivan Conjecture on maps from classifying spaces.* Ann. of Math. 120 (1984) p. 39-87, Corrigendum Annals of Math. 121.

Emmanuel Dror Farjoun - Hebrew University

Jerusalem, Israel.

Jeffrey H. Smith - Purdue University,

Indiana, USA