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On Lusztig's parametrization of characters of finite groups of Lie type

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This paper has three parts. In the first part, we extend Lusztig's results of [11] about the parametrization of characters of finite reductive groups with a connected center, including [11, theorem 4.23] about multiplicities of irreducible characters in the Deligne-Lusztig characters, to the case of groups with non-connected center. We use mostly a method sketched in chapter 14 of [11] in the case of a cyclic center, based on Clifford theory and a result about the unicity of the parametrisation of characters constructed in [11] which we prove in section 6 (part II). This construction has been carried out by Lusztig in [13] but we need more information than he gets there, in order to get the results of section 5 and of part III.

In sections 1 and 2 we state the results we need from Clifford theory, from [11] and about non-connected groups. We also need a result about the commutation of Lusztig twisted induction with isogenies, whose proof is given in section 9 (part III) using Shintani descent. We then apply these results to the parametrization of characters in section 3, where we need the results of part II. Finally section 4 and section 5 describe the multiplicities of irreducible characters in Deligne-Lusztig characters using Lusztig "families", presented here from a simplified combinatorial viewpoint using the "Mellin transform".

Part II describes under which conditions Lusztig's parametrization of irreducible characters in [11] is unique; section 6 deals with families and Weyl groups, and section 7 gives the main theorem.

Part III studies Shintani descent in groups with non-connected center. We want to show how Shintani descent relates to the parametrization introduced in part I. Section 8 recalls facts about Shintani descent and "F'-twisted induction". In section 9 we prove a result about the commutation of F'-twisted induction with isogenies and deduce the analogous result for Lusztig's twisted induction. In section 10, we first extend to F-class functions the parametrization of section 5 (when the center is not connected we have to make assumptions that we cannot yet prove in all cases). Finally, we give a formula for Shintani descent of principal series characters using the Fourier transform on families (using section 5 of part I). For this last result we have to quote heavily from [7].

This paper * has been prompted by discussions with B. Srinivasan, and Shoji's papers [14] and [15] where he gets a complete description of Shintani descent $\mathrm{Sh}_{F^m/F}$ for m sufficiently divisible and for a group with connected center (Shoji himself uses results of Asai [1] which deal with the case $m=1$); this paper also has been prompted by the absence of a convenient written description dealing with groups with non-connected center.

0. Background.

In this section we recall some results from Clifford theory and the theory of F-class functions.

We denote by $\mathrm{Irr}(G)$ the set of irreducible characters of the finite group G (over an algebraically closed field of characteristic 0). We now give a general proposition which states basic (well known) results from Clifford theory. Most of these are easy consequences of Mackey formula and Frobenius reciprocity (see also [8, 2.1]).

0.1 Proposition (Clifford theory). *Let G be a normal subgroup of a finite group $\tilde G$ such that the quotient $\tilde G/G$ is abelian; let $\tilde Z$ be the center of $\tilde G$. For $\rho\in\mathrm{Irr}(\tilde G)$, we put $A(\rho)=\{\zeta\in\mathrm{Irr}(\tilde G/G\tilde Z)\mid \rho\otimes\zeta=\rho\}$ (note that $\tilde Z$ is in the kernel of any $\zeta\in\mathrm{Irr}(\tilde G/G)$ such that $\rho\otimes\zeta=\rho$). If $\mu\in\mathrm{Irr}(G)$ is a component of $\mathrm{Res}^{\tilde G}_G\rho$, we note $\tilde G(\rho)$ for the inertia group of μ in $\tilde G$ (it depends only on ρ (not on μ)). Then we have:*

(i) $\mathrm{Ker}(A(\rho))\subset\tilde G(\rho)$.

(ii) There exists $\tilde\mu\in\mathrm{Irr}(\mathrm{Ker}(A(\rho)))$, $\tilde\rho\in\mathrm{Irr}(\tilde G(\rho))$ and a positive integer e such that:

$$\mathrm{Ind}_G^{\mathrm{Ker}(A(\rho))}(\mu)=\sum_{\alpha\in\mathrm{Irr}(\mathrm{Ker}(A(\rho))/G)}\tilde\mu\otimes\alpha,\ \mathrm{Res}_G^{\mathrm{Ker}(A(\rho))}(\tilde\mu)=\mu,$$

$$\mathrm{Ind}_{\mathrm{Ker}(A(\rho))}^{\tilde G(\rho)}\tilde\mu=e\tilde\rho,\qquad \mathrm{Res}_{\mathrm{Ker}(A(\rho))}^{\tilde G(\rho)}\tilde\rho=e\tilde\mu,$$

$$\mathrm{Ind}_{\tilde G(\rho)}^{\tilde G}(\tilde\rho)=\rho,\qquad \mathrm{Res}_{\tilde G(\rho)}^{\tilde G}(\rho)=\sum_{x\in\tilde G/\tilde G(\rho)}{}^x\tilde\rho,$$

* part of this work was done during the authors visit at Essen university

(iii) The quotient group $\tilde{G}(\rho)/\mathrm{Ker}(A(\rho))$ *has cardinality* e^2 *and we have*

$$|A(\rho)| = \langle \mathrm{Res}_G^{\tilde{G}}(\rho), \mathrm{Res}_G^{\tilde{G}}(\rho) \rangle_G.$$

If $\tilde{G}(\rho)/\mathrm{Ker}(A(\rho))$ *is cyclic, then* $e = 1$.

The following result, which is proved in [7, 6.1] gives $e = 1$ in a general setting for Weyl groups:

0.2 LEMMA. *Assume that* G *is a Weyl group and* $\tilde{G}$ *is the semi-direct product of* G *by a group* A *of diagram automorphisms of* G*, then for any character* $\rho \in \mathrm{Irr}(\tilde{G})$ *we have* $e = 1$.

The proof of this lemma requires the following result (*cf.* [7, 6.2]) that we will need below in the proof of 5.5

0.3 LEMMA. *Let* G *be a finite group of the form* $G_1 \times \ldots \times G_l$ *and* A *be a finite group of automorphisms of* G *acting by permutation of the* G_i*. Let* $\mu = \mu_1 \otimes \ldots \otimes \mu_l$ *be an irreducible character of* G*. Let* A_i *be the subgroup of* $\mathrm{Stab}_A(\mu)$ *normalizing* G_i *(and so* μ_i*). If for each* i*, the character* μ_i *has an extension to* $G_i \rtimes A_i$*, then* μ *has an extension to* $G \rtimes \mathrm{Stab}_A(\mu)$ *(i.e.,* $e = 1$ *for the character* μ*).*

0.4 F-CLASS FUNCTIONS. If G is a finite group and if $\langle F \rangle$ is a group generated by an element F and acting on G, we denote by $\mathcal{C}(G/F)$ the space of complex valued F-class functions on G, i.e. functions φ which verify $\varphi(x.{}^F y) = \varphi(yx)$ for any x and y in G (note that the group $\langle F \rangle$ can be infinite). We may identify $\mathcal{C}(G/F)$ with the space of restrictions to the set $G.F$ of class functions on the semi-direct product $G \rtimes \langle F \rangle$. This space admits as a basis the set of restrictions to $G.F$ of an arbitrarily chosen extension to $G \rtimes \langle F \rangle$ of each F-invariant irreducible character of G. If φ_1 and φ_2 are elements of $\mathcal{C}(G/F)$, we put

$$\langle \varphi_1, \varphi_2 \rangle_{G.F} = |G|^{-1} \sum_{x \in G.F} \overline{\varphi_1(x)} \varphi_2(x).$$

We recall that if φ_1 and φ_2 are characters of $G \rtimes \langle F \rangle$ whose restrictions to G are irreducible, then

$$\langle \varphi_1, \varphi_2 \rangle_{G.F} = \begin{cases} 0, & \text{if } \mathrm{Res}_G^{G \rtimes \langle F \rangle} \varphi_1 \neq \mathrm{Res}_G^{G \rtimes \langle F \rangle} \varphi_2 \\ 1, & \text{if } \varphi_1 = \varphi_2 \end{cases}$$

(if φ_1 and φ_2 have equal restrictions to G, then they differ by multiplication by a linear character of $\langle F \rangle$).

If H is a subgroup of G stabilized by F, we denote by $\operatorname{Res}_{H.F}^{G.F}$ the restriction of F-class functions, and we define induction of F-class function by

$$\operatorname{Ind}_{H.F}^{G.F}(f)(gF) = |H|^{-1} \sum_{\{\gamma \in G \mid {}^{\gamma}(gF) \in H.F\}} f({}^{\gamma}(gF)).$$

Induction and restriction are adjoint with respect to the above scalar product.

I

1. Disconnected groups.

In this section we extend the definition of Deligne-Lusztig characters to non connected reductive groups. We begin with a proposition which gives the relation between the Weyl group of a reductive group and that of its connected component.

1.1 PROPOSITION. *Let* $\mathbf{H}$ *be a reductive algebraic group, and let* $\mathbf{T}$ *be a maximal torus of* $\mathbf{H}$*; then we may find representatives of* $\mathbf{H}/\mathbf{H}^\circ$ *in* $N_{\mathbf{H}}(\mathbf{T})$*. We set* $W = N_{\mathbf{H}}(\mathbf{T})/\mathbf{T}$ *and* $W^\circ = N_{\mathbf{H}^\circ}(\mathbf{T})/\mathbf{T}$*. Let* $\mathbf{B}$ *be a Borel subgroup containing* $\mathbf{T}$*. We put* $A = \{w \in W \mid {}^w\Phi^+ = \Phi^+\}$ *where* Φ *is the root system of* $\mathbf{H}^\circ$ *and* $+$ *denotes the order on* Φ *corresponding to* $\mathbf{B}$*. Then we have*

(i) $W = W^\circ \rtimes A$ *and* $A \simeq \mathbf{H}/\mathbf{H}^\circ$.

(ii) If $\mathbf{H}$ *is defined over* $\mathbb{F}_q$*, with corresponding Frobenius* F*, and* F *stabilizes* $\mathbf{T}$ *and* $\mathbf{B}$ *above, then* F *stabilizes* W*,* W° *and* A*.*

In the following we consider a (not necessarily connected) reductive algebraic group $\mathbf{H}$ defined over $\mathbb{F}_q$, and denote by F the corresponding Frobenius endomorphism. We fix a pair $\mathbf{T} \subset \mathbf{B}$ of an F-stable maximal torus in $\mathbf{H}$ included in an F-stable Borel subgroup. Let $\mathbf{H}^{\circ *}$ be a group dual to $\mathbf{H}^\circ$ containing a given torus $\mathbf{T}^*$ dual to $\mathbf{T}$. Finally we fix a Frobenius endomorphism F^* dual to F. We may identify W° with $N_{\mathbf{H}^{\circ *}}(\mathbf{T}^*)/\mathbf{T}^*$ by mapping w to the dual isogeny w^*, but note that this map is an anti-isomorphism. For any $v \in W^\circ$ we choose a representative $\dot{v}^* \in N_{\mathbf{H}^{\circ *}}(\mathbf{T}^*)$ of v^* and for any representative $\dot{a} \in N_{\mathbf{H}}(\mathbf{T})$ of an element $a \in A$, we choose an isogeny $(\dot{a}F)^*$ dual to $\dot{a}F$. For $w \in W$, if w is in the coset $W^\circ a$, we write $(\dot{w}\dot{a}F)^*$ for $(\dot{a}F)^*\dot{w}^*$.

1.2 DEFINITION. *For $s \in \mathbf{T}^*$, we put $W_F(s) = \{w \in W \mid {}^{(wF)^*}s = s\}$.* (Note that $W_F(s)$ depends only on F (not on the choices of F^* and of $(\dot{a}F)^*$) since two isogenies dual to the same one differ by $\operatorname{ad} t$ for some $t \in \mathbf{T}^*$).

We will assume given an isomorphism $\overline{\mathbb{F}}_q^{\times} \xrightarrow{\sim} (\mathbb{Q}/\mathbb{Z})_{p'}$ and an embedding
$(\mathbb{Q}/\mathbb{Z})_{p'} \hookrightarrow \overline{\mathbb{Q}}_l^{\times}$; these choices give for any $w \in W$ an identification $\operatorname{Irr}(\mathbf{T}^{wF}) \simeq (\mathbf{T}^*)^{(wF)^*}$.

We now generalize the definition of Deligne-Lusztig characters to non connected groups. The idea is to consider together the various rational forms of $\mathbf{H}$ that we get if we take as Frobenius endomorphisms hF, with $h \in \mathbf{H}$. If h and h' are in the same F-class of $\mathbf{H}$, the groups $\mathbf{H}^{hF}$ and $\mathbf{H}^{h'F}$ are isomorphic. As the set $H^1(F,\mathbf{H})$ of F-classes in $\mathbf{H}$ is isomorphic to $H^1(F,A)$, we may choose a set of representatives in $\mathbf{H}$ of $H^1(F,\mathbf{H})$ which are representatives of elements of A. Generalized Deligne-Lusztig characters will be constructed for each of the groups $\mathbf{H}^{\dot{a}F}$ with $\dot{a}$ such a representative.

Let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. For any $w \in W$, let us write $wF = b^{-1}vaFb$ with $a, b \in A$ and $v \in W^\circ$, where a represents the F-class of w as above. Let $\dot{v} \in N_{\mathbf{H}^\circ}(\mathbf{T})$ be a representative of v; we consider the variety $Y_{\dot{v},\dot{a}} = \{\, x \in \mathbf{H} \mid x^{-1}\,{}^{\dot{a}F}x \in \dot{v}\mathbf{U}\,\}$. Then $\mathbf{H}^{\dot{a}F}$ acts on $Y_{\dot{v},\dot{a}}$ by left multiplication, and $t \in \mathbf{T}^{wF}$ acts by right multiplication by ${}^b t$. These two actions commute, so taking the alternating sum of l-adic cohomology groups, we get a virtual representation of $\mathbf{H}^{\dot{a}F} \times \mathbf{T}^{wF}$, and for each character θ of $\mathbf{T}^{wF}$ we get a representation of $\mathbf{H}^{\dot{a}F}$ on the part of the cohomology on which $\mathbf{T}^{wF}$ acts by θ. If $s \in (\mathbf{T}^*)^{(wF)^*}$ corresponds to θ, we will denote by $R_{\mathbf{T}^{wF}}^{\mathbf{H}^{\dot{a}F}}(s)$ the (generalized) Deligne-Lusztig character that we have just defined. Note that we have defined Deligne-Lusztig characters of the groups $\mathbf{H}^{\dot{a}F}$ when $\dot{a}$ runs over a chosen set of representatives of $H^1(F,A)$. We have the following properties of (generalized) Deligne-Lusztig characters:

1.3 PROPOSITION.

(i) *Let $wF = b^{-1}vaFb$ as above, then $R_{\mathbf{T}^{wF}}^{\mathbf{H}^{\dot{a}F}}(s) = \operatorname{Ind}_{\mathbf{H}^{\circ\dot{a}F}}^{\mathbf{H}^{\dot{a}F}}(\operatorname{ad}\dot{b} \circ R_{\mathbf{T}^{wF}}^{\mathbf{H}^{\circ\dot{b}^{-1}\dot{a}F\dot{b}}}(s))$.*

(ii) *The union when $\dot{a}$ runs over representatives of $H^1(F,A)$ of the Deligne-Lusztig characters of $\mathbf{H}^{\dot{a}F}$ can be parametrized by pairs (s, wF) with $s \in \mathbf{T}^*$ and $w \in W_F(s)$, taken modulo W-conjugation.*

(iii) *Distinct Deligne-Lusztig characters are pairwise orthogonal.*

Note that $R_{\mathbf{T}_{wF}}^{\mathbf{H}^{\circ\dot{b}^{-1}\dot{a}F\dot{b}}}(s)$ where $b\in A$, becomes under conjugation by $\dot{b}$ the character $R_{\mathbf{T}_{vaF}}^{\mathbf{H}^{\circ\dot{a}F}}(s')$ where $s'={}^{b}s$, so (i) is equivalent to its particular case when $b=1$ which is $R_{\mathbf{T}_{wF}}^{\mathbf{H}^{\dot{a}F}}(s)=\operatorname{Ind}_{\mathbf{H}^{\circ\dot{a}F}}^{\mathbf{H}^{\dot{a}F}}R_{\mathbf{T}_{wF}}^{\mathbf{H}^{\circ\dot{a}F}}(s)$.

PROOF: As stated above, to prove (i) we may assume $b=1$, *i.e.*, $wF=vaF$ with $v\in W^{\circ}$. Let $Y_{\dot{v},\dot{a}}^{\circ}$ be the variety $\{\,y\in\mathbf{H}^{\circ}\mid y^{-1}\,{}^{\dot{a}F}y\in\dot{v}\mathbf{U}\,\}$. We claim that the map $(y,h)\mapsto hy$ from the variety $Y_{\dot{v},\dot{a}}^{\circ}\times\mathbf{H}^{\dot{a}F}$ to $Y_{\dot{v},\dot{a}}$ is an epimorphism whose fibers are the orbits of $\mathbf{H}^{\circ\dot{a}F}$ if the action of $h_0\in\mathbf{H}^{\circ\dot{a}F}$ is given by $(y,h)\mapsto(h_0y,hh_0^{-1})$. In fact, any element in $\dot{v}\mathbf{U}$ can be written, by Lang's theorem as $y^{-1}\,{}^{\dot{a}F}y$ with $y\in\mathbf{H}^{\circ}$, so any $x\in Y_{\dot{v},\dot{a}}$ differs by an element of $\mathbf{H}^{\dot{a}F}$ from an element of $Y_{\dot{v},\dot{a}}^{\circ}$. The assertion about fibers is clear. On $Y_{\dot{v},\dot{a}}^{\circ}\times\mathbf{H}^{\dot{a}F}$, we have an action of $\mathbf{H}^{\dot{a}F}\times\mathbf{T}^{wF}$ given by $(y,k)\mapsto(yt,hk)$. This action is compatible with the action of $\mathbf{H}^{\circ F}$. As the above epimorphism is compatible with the actions of $\mathbf{H}^{\dot{a}F}\times\mathbf{T}^{wF}$, we get an isomorphism of varieties with $\mathbf{H}^{\dot{a}F}\times\mathbf{T}^{wF}$-actions $(Y_{\dot{v},\dot{a}}^{\circ}\times\mathbf{H}^{\dot{a}F})/\mathbf{H}^{\circ F}\xrightarrow{\sim}Y_{\dot{v},\dot{a}}$. Using properties of l-adic cohomology, we get an isomorphism of $\mathbf{H}^{\dot{a}F}\times\mathbf{T}^{wF}$-modules $H_c^*(Y_{\dot{v},\dot{a}})\xrightarrow{\sim}H_c^*(Y_{\dot{v},\dot{a}}^{\circ})\otimes_{\overline{\mathbb{Q}}_l[\mathbf{H}^{\circ\dot{a}F}]}\overline{\mathbb{Q}}_l[\mathbf{H}^{\dot{a}F}]$, whence (i).

To prove (ii) and (iii), we have to show that generalized Deligne-Lusztig characters $R_{\mathbf{T}_{wF}}^{\mathbf{H}^{\dot{a}F}}(s)$ and $R_{\mathbf{T}_{w'F}}^{\mathbf{H}^{\dot{a}F}}(s')$ are equal if and only if (wF,s) and $(w'F,s')$ are conjugate under the action of W and are orthogonal otherwise. By conjugation by $\dot{b}$ with $b\in W$, we may assume that the elements w and w' are in the coset $W^{\circ}.a$. By (i), and Mackey formula, as distinct Deligne-Lusztig characters of $\mathbf{H}^{\circ\dot{a}F}$ are orthogonal, we see that the scalar product $\langle R_{\mathbf{T}_{wF}}^{\mathbf{H}^{\dot{a}F}}(s),R_{\mathbf{T}_{w'F}}^{\mathbf{H}^{\dot{a}F}}(s')\rangle_{\mathbf{H}^{\dot{a}F}}$ is non zero if and only if $\langle R_{\mathbf{T}_{wF}}^{\mathbf{H}^{\circ\dot{a}F}}(s),R_{\mathbf{T}_{w'F}}^{\mathbf{H}^{\circ\dot{a}F}}(s')\circ\operatorname{ad}\dot{x}\rangle_{\mathbf{H}^{\circ\dot{a}F}}$ is non zero for some representative $\dot{x}\in\mathbf{H}^{\dot{a}F}$ of some $x\in A^F$. As $R_{\mathbf{T}_{w'F}}^{\mathbf{H}^{\circ\dot{a}F}}(s')\circ\operatorname{ad}\dot{x}$ is equal to $R_{\mathbf{T}_{xw'Fx^{-1}}}^{\mathbf{H}^{\circ\dot{a}F}}(xs'x^{-1})$, the above condition is equivalent to the pairs (wF,s) and $(xw'Fx^{-1},xs'x^{-1})$ being W°-conjugate for some $x\in A^F$, which is equivalent to (wF,s) and $(w'F,s')$ being W-conjugate (note that as wF and $w'F$ are in the same coset $W^{\circ}.aF$, they are W-conjugate if and only if they are $W^{\circ}\rtimes A^F$-conjugate). ■

We then define (generalizing to non connected groups Lusztig's definition) series of characters in $\operatorname{Irr}(\mathbf{H}^{\dot{a}F})$ by

$$\mathcal{E}(\mathbf{H}^{\dot{a}F},s)=\{\,\chi\in\operatorname{Irr}(\mathbf{H}^{\dot{a}F})\mid\exists w\in W_F(s),\quad\langle\chi,R_{\mathbf{T}_{wF}}^{\mathbf{H}^{\dot{a}F}}(s)\rangle_{\mathbf{H}^{\dot{a}F}}\neq0\,\}.$$

Note that in the above definition wF has to be conjugate to an element of $W^{\circ}aF$.

2. Rational series.

From now on we will consider a connected reductive algebraic group $\mathbf{G}$ and will keep the notations of the previous section with $\mathbf{H} = \mathbf{H}^\circ = \mathbf{G}$ and $\mathbf{H}^{\circ *} = \mathbf{G}^*$.

Given $x \in \mathbf{G}$, we denote by $A(x)$ the group $C_\mathbf{G}(x)/C_\mathbf{G}(x)^\circ$. In the sequel we fix an element $s \in \mathbf{T}^*$ such that $W_F(s)$ is not empty, and we write simply A for $A(s)$. We will write $W(s)$ for the Weyl group of $C_{\mathbf{G}^*}(s)$; we have $W(s) = \{w \in W \mid {}^{w^*}s = s\}$, so $W_F(s)$ is a single coset in $W(s)\backslash W$. We will denote by $W^\circ(s)$ the Weyl group of $C_{\mathbf{G}^*}(s)^\circ$, and Φ_s its root system. According to 1.1, (i), if we identify A with $\{w \in W(s) \mid {}^w\Phi_s^+ = \Phi_s^+\}$, then we have $W(s) = W^\circ(s) \rtimes A$; and if we choose $w_1 \in W_F(s)$ such that ${}^{(w_1F)^*}\Phi_s^+ = \Phi_s^+$, then by 1.1, (ii) w_1F induces an automorphism of $W(s)$ which stabilizes $W^\circ(s)$ and A.

As we want to use results from the theory of groups with a connected center, we will embedd $\mathbf{G}$ in such a group: Let $i : \mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$ be an embedding defined over $\mathbb{F}_q$ of $\mathbf{G}$ in a group $\tilde{\mathbf{G}}$ such that the center $\tilde{\mathbf{Z}}$ of $\tilde{\mathbf{G}}$ is connected, and $\mathbf{G}$ and $\tilde{\mathbf{G}}$ have the same derived group (we may identify $\tilde{\mathbf{G}}$ with $\mathbf{G} \times_\mathbf{Z} \tilde{\mathbf{Z}}$, where $\mathbf{Z}$ is the center of $\mathbf{G}$). We denote again by F the Frobenius endomorphism on $\tilde{\mathbf{G}}$, so we have $F \circ i = i \circ F$.

We denote by $\tilde{\mathbf{T}}$ the maximal torus of $\tilde{\mathbf{G}}$ containing $\mathbf{T}$ ($\tilde{\mathbf{T}}$ is $\mathbf{T} \times_\mathbf{Z} \tilde{\mathbf{Z}}$ in the above identification). Let $\tilde{\mathbf{G}}^*$ be a group dual to $\tilde{\mathbf{G}}$ and $\tilde{\mathbf{T}}^* \subset \tilde{\mathbf{G}}^*$ be a torus dual to $\tilde{\mathbf{T}}$. We fix $i^* : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$ dual to i. It is an epimorphism mapping $\tilde{\mathbf{T}}^*$ onto $\mathbf{T}^*$ and $\operatorname{Ker} i^*$ is a central torus of $\tilde{\mathbf{G}}^*$. The embedding i gives a natural identification of $W = N_\mathbf{G}(\mathbf{T})/\mathbf{T}$ with $N_{\tilde{\mathbf{G}}}(\tilde{\mathbf{T}})/\tilde{\mathbf{T}}$.

2.1 DEFINITION.

(i) *For $a \in A$ we denote by $[[a, s]] \in \operatorname{Ker} i^*$ the commutator of any two preimages in $\tilde{\mathbf{G}}^*$ of a and s* (it clearly does not depend on the chosen preimages).

(ii) *We denote by $[[A, s]]$ the set $\{ [[a, s]] \mid a \in A \}$.*

With these notations, a straightforward computation shows that:

2.2 PROPOSITION. *$[[A, s]]$ is a group and the map $a \mapsto [[a, s]]$ is a group isomorphism $A \to [[A, s]]$ which maps the action of w_1F on A to the action of F on $[[A, s]] \hookrightarrow \operatorname{Ker} i^*$.*

We want to study how Lusztig series restrict from $\tilde{\mathbf{G}}^F$ to $\mathbf{G}^F$. First we choose a suitable preimage of s.

2.3 LEMMA. *There exists a preimage* $\tilde{s}$ *of* s *such that* $W_F(\tilde{s}) = W^\circ(s)w_1$.

PROOF: It is a straightforward computation, using Lang's theorem in the connected group $\operatorname{Ker} i^*$. ■

In the sequel we fix $\tilde{s}$ as in 2.3. It is well known that for any $t \in \tilde{\mathbf{T}}^*$, the restriction of $i^* : C_{\tilde{\mathbf{G}}^*}(t) \to C_{\mathbf{G}^*}(i^*(t))^\circ$ is an isogeny of the derived groups; so the Weyl group $W(t)$ is identified to $W^\circ(i^*(t))$. Next proposition relates series in $\operatorname{Irr}(\tilde{\mathbf{G}}^F)$ corresponding to various preimages of s. Before stating this proposition we need the following definition:

2.4 DEFINITION. *We will call Lang's map in an algebraic group defined over* $\mathbb{F}_q$, *with Frobenius endomorphism* F, *the map* $\mathcal{L} : g \mapsto g^{-1}\,{}^F g$.

In this section, we will use only Lang's map from $\operatorname{Ker} i^*$ into itself, and denote it again by $\mathcal{L}$. Recall that, as $\operatorname{Ker} i^*$ is connected, Lang's map is onto.

2.5 PROPOSITION.

(i) *For any pair* (t, w) *with* $t \in \tilde{\mathbf{T}}^*$ *and* $w \in W_F(t)$, *we have*

$$\operatorname{Res}_{\mathbf{G}^F}^{\tilde{\mathbf{G}}^F} R_{\tilde{\mathbf{T}}_{wF}}^{\tilde{\mathbf{G}}^F}(t) = R_{\mathbf{T}_{wF}}^{\mathbf{G}^F}(i^*(t)).$$

(ii) *Let* $\tilde{s}z$ *be a preimage of* s *(where* $z \in \operatorname{Ker} i^*$*); then* $W_F(\tilde{s}z)$ *is not empty if and only if* $z \in \mathcal{L}^{-1}([[A,s]])$, *and then* $W_F(\tilde{s}z) = W^\circ(s)aw_1$, *where* $\mathcal{L}(z) = [[a,s]]$.

(iii) *Two series* $\mathcal{E}(\tilde{\mathbf{G}}^F, \tilde{s}z)$ *and* $\mathcal{E}(\tilde{\mathbf{G}}^F, \tilde{s}z')$ *are equal if and only if* z *and* z' *differ by an element of* $[[A,s]]$.

PROOF: We will show in part III that (i) is actually true for any morphism i inducing an isogeny of derived groups and with connected kernel. Statement (ii) results from a straightforward computation. Statement (iii) just reflects the fact that W-action by conjugation on elements $\tilde{s}z$ is the same as $[[A,s]]$-action by translation. ■

By (ii) and (iii) above we see that the series $\mathcal{E}(\tilde{\mathbf{G}}^F, \tilde{s}z)$ are parametrized by

$$\mathcal{L}^{-1}([[A,s]])/[[A,s]].$$

For any $t \in \tilde{\mathbf{T}}^*$, let $\overline{\mathcal{E}}(\mathbf{G}^F, t)$ be the subset of $\operatorname{Irr}(\mathbf{G}^F)$ whose elements occur in the restriction of some $\chi \in \mathcal{E}(\tilde{\mathbf{G}}^F, t)$. By (i) above, $\overline{\mathcal{E}}(\mathbf{G}^F, \tilde{s}z)$ is a subset of $\mathcal{E}(\mathbf{G}^F, s)$ and the series $\mathcal{E}(\tilde{\mathbf{G}}^F, \tilde{s}z)$ are all the series $\mathcal{E}(\tilde{\mathbf{G}}^F, t)$ such that $\overline{\mathcal{E}}(\mathbf{G}^F, t)$ intersects $\mathcal{E}(\mathbf{G}^F, s)$.

The following properties are well known.

2.6 PROPOSITION.

(i) The characters of $\tilde{\mathbf{G}}^F/\mathbf{G}^F$ correspond by duality to elements of $(\operatorname{Ker} i^*)^F$.

(ii) The characters of $\tilde{\mathbf{G}}^F/\mathbf{G}^F$ whose kernel contains $\tilde{\mathbf{Z}}^F.\mathbf{G}^F$ correspond by duality to elements of $(\operatorname{Ker} i^*)^F$ *which are in the derived group of $\tilde{\mathbf{G}}^*$.*

Using the fact that, if $\hat{z} \in \operatorname{Irr}(\tilde{\mathbf{G}}^F/\mathbf{G}^F)$ corresponds to $z \in (\operatorname{Ker} i^*)^F$, then we have $R^{\tilde{\mathbf{G}}^F}_{\tilde{\mathbf{T}}_{wF}}(\tilde{s}z'z) = R^{\tilde{\mathbf{G}}^F}_{\tilde{\mathbf{T}}_{wF}}(\tilde{s}z') \otimes \hat{z}$ and proposition 0.1, we get:

2.7 PROPOSITION.

(i) The action of $\otimes\hat{z}$, for $z \in (\operatorname{Ker} i^*)^F$, maps $\mathcal{E}(\tilde{\mathbf{G}}^F, \tilde{s}z')$ to $\mathcal{E}(\tilde{\mathbf{G}}^F, \tilde{s}z'z)$.

(ii) Two sets $\overline{\mathcal{E}}(\mathbf{G}^F, \tilde{s}z)$ and $\overline{\mathcal{E}}(\mathbf{G}^F, \tilde{s}z')$ are equal or disjoint, and are equal if and only if z and z' differ by an element of $(\operatorname{Ker} i^*)^F$.

As a consequence of 2.5, (iii) and of 2.7, we see that the sets $\overline{\mathcal{E}}(\mathbf{G}^F, \tilde{s}z)$ are parametrized by the set $\mathcal{L}^{-1}([[A,s]])/[[A,s]].(\operatorname{Ker} i^*)^F \simeq H^1(w_1F, A)$. So, for a representative $a \in A$ of an element of $H^1(w_1F, A)$, we will denote by $\mathcal{E}(\mathbf{G}^F, s, a)$ the set $\overline{\mathcal{E}}(\mathbf{G}^F, \tilde{s}z)$ when z is such that $\mathcal{L}(z) = [[a, s]]$. We will call such sets **rational** series. We have

$$\mathcal{E}(\mathbf{G}^F, s) = \coprod_{a \in H^1(w_1F, A)} \mathcal{E}(\mathbf{G}^F, s, a)$$

(note that

$$\mathcal{E}(\mathbf{G}^F, s, a) = \{\chi \in \operatorname{Irr}(\mathbf{G}^F) \mid \exists v \in W^\circ(s), \langle \chi, R^{\mathbf{G}^F}_{\mathbf{T}_{avw_1F}}(s) \rangle_{\mathbf{G}^F} \neq 0\},$$

and that $H^1(w_1F, A)$ also parametrizes the rational classes in $\mathbf{G}^{*F}$ which are geometrically conjugate to s).

3. Parametrization.

Lusztig has shown that, if $\mathbf{Z}$ is connected, there is a bijection $\pi_s : \mathcal{E}(\mathbf{G}^F, s) \to \mathcal{E}(C_{\mathbf{G}^*}(s)^{(\dot{w}_1F)^*}, 1)$ (where $\dot{w}_1$ is any representative of w_1 in $N_{\mathbf{G}^*}(\mathbf{T}^*)$) such that (extending π_s by linearity to the $\mathbb{Z}$-span of $\mathcal{E}(\mathbf{G}^F, s)$)

$$\pi_s(R^{\mathbf{G}^F}_{\mathbf{T}_{wF}}(s)) = (-1)^{l(w_1)} R^{C_{\mathbf{G}^*}(s)^{(\dot{w}_1F)^*}}_{\mathbf{T}^*(wF)^*}(1) \tag{3.1}$$

for any $w \in W_F(s)$ (it is an immediate consequence of [11, 4.23] applied once in the group $\mathbf{G}^F$ and once in the group $C_{\mathbf{G}^*}(s)^{(\dot{w}_1F)^*}$).

We want to show that a similar result holds in general. In fact, as we will show in part II, condition 3.1 above determines uniquely π_s for classical groups (with connected centre), but there is some ambiguity for other groups; the main result of part II will be to give additional conditions which define uniquely π_s in all cases. In what follows, we will assume that for the group $\tilde{\mathbf{G}}$ (*cf.* notations of section 2), the map π_s is defined as in part II. Using results from part II we prove the following:

3.2 THEOREM. *Let $b \in A^{w_1 F}$; we denote by $\hat{b} \in \operatorname{Irr}(\tilde{\mathbf{G}}^F/\mathbf{G}^F)$ the character corresponding to $[[b,s]] \in (\operatorname{Ker} i^*)^F$ by 2.6. Then, if the representative $\dot{b}^*$ is chosen $(\dot{a}\dot{w}_1 F)^*$-fixed, the action of $\otimes \hat{b}$ on $\mathcal{E}(\tilde{\mathbf{G}}^F, \tilde{s}z)$ corresponds by π_s to that of $\operatorname{ad} \dot{b}^*$ on $\mathcal{E}(C_{\tilde{\mathbf{G}}^*}(\tilde{s}z)^{(\dot{a}\dot{w}_1 F)^*}, 1)$.*

PROOF: By 7.1 (iii), if $\zeta = [[b,s]]$, we have

$$\pi_{\tilde{s}z\zeta}(\chi \otimes \hat{b}) = \pi_{\tilde{s}z}(\chi) \in \mathcal{E}(C_{\tilde{\mathbf{G}}^*}(\tilde{s}z)^{(\dot{a}\dot{w}_1 F)^*}, 1).$$

By 7.3, we have the equality $\operatorname{ad}(\dot{b}^*) \circ \pi_{\tilde{s}z\zeta} = \pi_{\tilde{s}z}$, whence the result. ∎

We need the following assumption:

3.3 HYPOTHESIS. *Restriction of representations from $\tilde{\mathbf{G}}^F$ to $\mathbf{G}^F$ are multiplicity-free, as well as restrictions of unipotent representations from $C_{\mathbf{G}^*}(s)^{(\dot{a}\dot{w}_1 F)^*}$ to $C_{\mathbf{G}^*}(s)^{\circ(\dot{a}\dot{w}_1 F)^*}$, for any $a \in A$.*

By 0.1, (iii), both parts of this hypothesis are true if $\mathbf{G}$ has no components of type D_{2n}, since both $\tilde{\mathbf{G}}^F/\mathbf{G}^F$ and $A^{w_1 F}$ identify to subquotients of the fundamental group of $\mathbf{G}$, which is then cyclic; Lusztig has proved the first part for type D_{2n} (*cf.* [13]). In the sequel we will assume that 3.3 is true.

By theorem 3.2, we see that, for $\rho \in \mathcal{E}(\tilde{\mathbf{G}}^F, \tilde{s}z)$, the map $a \mapsto \hat{a}$ is a group isomorphism from $A(\rho)$ (*cf.* notations of 0.1) to $\operatorname{Stab}_{A^{w_1 F}}(\pi_{\tilde{s}z}(\rho))$; so, using 0.1 and assumption 3.3, we see that the number of irreducible components of $\operatorname{Res}^{\tilde{\mathbf{G}}^F}_{\mathbf{G}^F}(\rho)$ is $|\operatorname{Stab}_{A^{w_1 F}}(\pi_{\tilde{s}z}(\rho))|$.

Since i^* defines an isogeny of the derived groups of $C_{\tilde{\mathbf{G}}^*}(\tilde{s}z)$ and of $C_{\mathbf{G}^*}(s)^\circ$ and since unipotent characters factorize through isogenies, we may consider $\pi_{\tilde{s}z}(\rho)$ as an element of $\operatorname{Irr}(C_{\mathbf{G}^*}(s)^{\circ(\dot{a}\dot{w}_1 F)^*})$ (where $\mathcal{L}(z) = [[a,s]]$). By 0.1 and 3.3, the number of irreducible components of $\operatorname{Ind}^{C_{\mathbf{G}^*}(s)^{(\dot{a}\dot{w}_1 F)^*}}_{C_{\mathbf{G}^*}(s)^{\circ(\dot{a}\dot{w}_1 F)^*}} \pi_{\tilde{s}z}(\rho)$ is $|\operatorname{Stab}_{A^{w_1 F}}(\pi_{\tilde{s}z}(\rho))|$. So we get:

3.4 PROPOSITION. *The sets $\mathcal{E}(\mathbf{G}^F, s, a)$ and $\mathcal{E}(C_{\mathbf{G}^*}(s)^{(\dot{a}\dot{w}_1 F)^*}, 1)$ are in bijection.*

We define π_s as being the collection over $a \in H^1(w_1F, A)$ of these bijections. Note that the parametrization of individual components of a restriction is not completely defined, but, once a component μ of $\mathrm{Res}_{\mathbf{G}^F}^{\tilde{\mathbf{G}}^F} \rho$ and a component $\pi_s(\mu)$ of $\mathrm{Ind}_{C_{\mathbf{G}^*}(s)^\circ(\dot{a}\dot{w}_1F)^*}^{C_{\mathbf{G}^*}(s)(\dot{a}\dot{w}_1F)^*} \pi_{\tilde{s}z}(\rho)$ are chosen, we can define π_s uniquely for other components of $\mathrm{Res}_{\mathbf{G}^F}^{\tilde{\mathbf{G}}^F} \rho$: first note that by 0.1 $\mathrm{Ker}\, A(\rho) = \tilde{\mathbf{G}}^F(\rho)$ under assumption 3.3; so $A(\rho) \xrightarrow{\sim} \mathrm{Irr}(\tilde{\mathbf{G}}^F/\tilde{\mathbf{G}}^F(\rho))$, and we get a canonical isomorphism

$$\tilde{\mathbf{G}}^F/\tilde{\mathbf{G}}^F(\rho) \xrightarrow{\sim} \mathrm{Irr}(\mathrm{Stab}_{A^{w_1F}}(\pi_{\tilde{s}z}(\rho)));$$

if we demand that the action of $\tilde{\mathbf{G}}^F/\tilde{\mathbf{G}}^F(\rho)$ on components of $\mathrm{Res}_{\mathbf{G}^F}^{\tilde{\mathbf{G}}^F} \rho$ be mapped by π_s on that of tensorization by linear characters of the group $\mathrm{Stab}_{A^{w_1F}}(\pi_{\tilde{s}z}(\rho))$, we define canonically π_s. We now clearly have the analogue of 3.1 for groups with a non connected center, that is, for $a \in A$ and $v \in W^\circ(s)$,

$$\pi_s(R_{\mathbf{T}_{avF}}^{\mathbf{G}^F}(s)) = (-1)^{l(aw_1)} R_{\mathbf{T}^*(aw_1F)^*}^{C_{\mathbf{G}^*}(s)(\dot{a}\dot{w}_1F)^*}. \tag{3.5}$$

4. Families.

From section 3 it follows that, in order to know the multiplicities of the irreducible characters in the Deligne-Lusztig characters, it is enough to solve the problem for unipotent characters. We will write R_{wF} for $R_{\mathbf{T}_{wF}}^{\mathbf{G}^F}(1)$ to abbreviate the notations. Lusztig [11] has shown in the connected-centre case how the multiplicities are given by a "Fourier transform" over "families" in the Weyl group. He defines a partition $\Xi(W)$ of $\mathrm{Irr}(W)$ in "families", and to each $\mathcal{F} \in \Xi(W)$ associates a finite group $\Gamma_{\mathcal{F}}$, the set $\mathcal{M}(\Gamma_{\mathcal{F}}) = \{(x, \chi) \mid x \in \Gamma_{\mathcal{F}}, \chi \in \mathrm{Irr}(C_{\Gamma_{\mathcal{F}}}(x))\}/\Gamma_{\mathcal{F}}$ (the action of $\Gamma_{\mathcal{F}}$ being by conjugation), and an embedding $\mathcal{F} \hookrightarrow \mathcal{M}(\Gamma_{\mathcal{F}})$.

These data are functorial, *i.e.*,

4.1 PROPOSITION. *Given an isomorphism $W \xrightarrow{\pi} W'$ of Weyl groups (i.e., an isomorphism of groups coming from an isomorphism of root systems), we have $\pi(\Xi(W)) = \Xi(W')$, and for any $\mathcal{F} \in \Xi(W)$, there is a well-defined isomorphism $\pi_{\mathcal{F}} : \Gamma_{\mathcal{F}} \xrightarrow{\sim} \Gamma_{\pi(\mathcal{F})}$ (and $\pi \mapsto \pi_{\mathcal{F}}$ is functorial); furthermore, if we again denote by $\pi_{\mathcal{F}}$ the induced isomorphism from $\mathcal{M}(\Gamma_{\mathcal{F}})$ to $\mathcal{M}(\Gamma_{\pi(\mathcal{F})})$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{M}(\Gamma_{\mathcal{F}}) & \xrightarrow{\pi_{\mathcal{F}}} & \mathcal{M}(\Gamma_{\pi(\mathcal{F})}) \\ \uparrow & & \uparrow \\ \mathcal{F} & \xrightarrow{\pi} & \pi(\mathcal{F}) \end{array}$$

Note that in the case of classical groups we get $\pi_{\mathcal{F}}$ using the facts that the groups $\Gamma_{\mathcal{F}}$ are commutative and that the image of $\mathcal{F}$ in $\mathcal{M}(\Gamma_{\mathcal{F}})$ contains the elements of the form (x, Id).

Proposition 4.1 implies that, given $\mathcal{F} \in \Xi^F$, a well-defined automorphism of $\Gamma_{\mathcal{F}}$ is associated to F; we will again denote it by F. Lusztig shows that $\mathcal{E}(\mathbf{G}^F, 1)$ is in bijection with the union over $\mathcal{F} \in \Xi^F$ of the sets

$$\overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes\!\!<\!F\!>) = \{(xF, \chi) \mid x \in \Gamma_{\mathcal{F}},\ \chi \in \mathrm{Irr}(C_{\Gamma_{\mathcal{F}}}(xF))\}/\Gamma_{\mathcal{F}} \rtimes\!\!<\!F\!>$$

We will denote by $\rho_{(xF,\chi)}$ the irreducible unipotent character parametrized by (xF, χ). To describe the multiplicities $\langle \rho_{(xF,\chi)}, R_{wF} \rangle_{\mathbf{G}^F}$ we first define another basis of the space spanned by the $\{R_{wF}\}$. For any F-class function f on W, we define $R_f = \sum_{w\in W} f(w) R_{wF}$. We can take as a basis of the space of F-class functions on W those obtained by choosing one extension to $W \rtimes\!\!<\!F\!>$ of each element of $\mathrm{Irr}(W)^F$ (the corresponding R_f are called the "almost-characters").

4.2 Notations.

(i) For $\mathcal{F} \in \Xi$, we will denote by $\mathcal{E}(\mathbf{G}^F, 1, \mathcal{F})$ the set of unipotent characters parametrized by $\overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes\!\!<\!F\!>)$.

(ii) For $\mathcal{F} \in \Xi^F$ we will denote by $\tilde{\mathcal{F}}$ the subspace of F-class functions spanned by extensions of elements of $\mathcal{F}^F$.

With this notation, Lusztig defines an embedding of $\tilde{\mathcal{F}}$ in the vector space

$$\underline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes\!\!<\!F\!>) = \oplus_{y\in\Gamma_{\mathcal{F}}} V_y,$$

where V_y is the set of functions on $C_{\Gamma_{\mathcal{F}}.F}(y)$ (the set of elements in $\Gamma_{\mathcal{F}}.F$ which centralize y) invariant under conjugation by $C_{\Gamma_{\mathcal{F}}}(y)$ (note that only elements y such that $C_{\Gamma_{\mathcal{F}}.F}(y)$ is not empty are relevant).

4.3 Then, Lusztig shows that there is a pairing between $\underline{\mathcal{M}}$ and $\overline{\mathbb{Q}}_l\overline{\mathcal{M}}$ given by

$$\{(y,\tau),(xF,\chi)\} = |C_{\Gamma_{\mathcal{F}}}(y)|^{-1}|C_{\Gamma_{\mathcal{F}}}(xF)|^{-1}\Delta_{(xF,\chi)} \sum_{\{g\in\Gamma_{\mathcal{F}} \mid [{}^g y, xF]=1\}} \overline{\chi({}^g y)}\,\tau({}^{g^{-1}}(xF)),$$

(where $\Delta_{(xF,\chi)}$ is a sign which is constant over each family except for so-called exceptional families *i.e.*, families containing a character such that the corresponding representation of the Hecke algebra is not defined over $\mathbb{Q}[q, q^{-1}]$, which occurs only in types E_7, E_8) such that, if we consider all the above class functions as taking their values in $\overline{\mathbb{Q}}_l$, the scalar product $\langle \rho_{(xF,\chi)}, R_f \rangle_{\mathbf{G}^F}$ is given by $\{(y,\tau),(xF,\chi)\}$ where $f \in \tilde{\mathcal{F}}$ corresponds to (y,τ).

Let us remark that the pairing formula depends only on the action of F on $\Gamma_{\mathcal{F}}$ (not on the order of F). We also remark that this formula depends only on the restriction of τ to the centralizer of y in the coset $\Gamma_{\mathcal{F}}.F$. We now introduce new bases in order to get simpler formulas for the pairing.

4.4 For $x, y \in \Gamma_{\mathcal{F}}$ such that $[xF, y] = 1$ we define "Mellin transforms"

$$(xF, y) = \sum_{\chi \in \mathrm{Irr}(C_{\Gamma_{\mathcal{F}}}(xF))} \chi(y)\Delta_{(xF,\chi)}(xF,\chi) \in \overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F \rangle)$$

and

$$(y, xF) = \sum_{\tau} \tau(xF)(y,\tau) \in \underline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F \rangle),$$

where the sum runs over a basis of the $C_{\Gamma_{\mathcal{F}}}(y)$-class functions on $C_{\Gamma_{\mathcal{F}}.F}(y)$ which consists of the restrictions to the coset $C_{\Gamma_{\mathcal{F}}.F}(y)$ of one extension to $C_{\Gamma_{\mathcal{F}} \rtimes \langle F \rangle}(y)$ of each xF-invariant irreducible character of $C_{\Gamma_{\mathcal{F}}}(y)$. The pairs (xF, y) and (y, xF) are taken up to $\Gamma_{\mathcal{F}}$-conjugacy. The following proposition gives a formula for the pairing using Mellin transforms.

4.5 Proposition. *The pairing between $\overline{\mathbb{Q}}_l\overline{\mathcal{M}}$ and $\underline{\mathcal{M}}$ is given by*

$$\{(xF,y),(y',x'F)\} = \delta_{(xF,y),(x'F,y')}|C_{\Gamma_{\mathcal{F}}}((xF,y))|.$$

Proof: By definition, we have

$$\{(xF,y),(y',x'F)\} = \sum_{\chi,\tau} \chi(y)\Delta_{(xF,\chi)}\overline{\tau(x'F)}\{(xF,\chi),(y',\tau)\}.$$

If we replace $\{(xF,\chi),(y',\tau)\}$ by its value given above, and if we use the fact that $\Delta^2_{(xF,\chi)} = 1$, we get

$$\sum_{\{g \in \Gamma_{\mathcal{F}} \mid [{}^g y', xF]=1\}} |C_{\Gamma_{\mathcal{F}}}(y')|^{-1}|C_{\Gamma_{\mathcal{F}}}(xF)|^{-1} \sum_{\chi} \overline{\chi({}^g y')}\chi(y) \sum_{\tau} \overline{\tau(x'F)}\tau({}^{g^{-1}}(xF)).$$

By orthogonality formulas for characters and for the chosen basis of $C_{\Gamma_{\mathcal{F}}}(y)$-class functions on $C_{\Gamma_{\mathcal{F}}.F}(y)$, we get the result. ∎

5. Fourier coefficients.

In this section we show that the results stated in the preceding section hold in the non-connected centre case. For this we have to give the appropriate definition for families of a group of the form $W \rtimes A$, and to generalize Lusztig's parametrization of unipotent characters and of almost characters to non connected groups (the non connected groups we have to consider are the centralizers of semi-simple elements in $\mathbf{G}^*$, so we must keep in mind that the results of this section are to be applied when $\mathbf{H}$, A, and F stand respectively for $C_{\mathbf{G}^*}(s)$, $A(s)$, and $(w_1F)^*$). First we give the appropriate definition of families (*cf.* also [12]).

5.1 DEFINITION. *In the situation of 1.1, we define $\Xi(W)$ to be the set of A-orbits in $\Xi(W^\circ)$, and, for $\mathcal{F} \in \Xi(W)$, we put $\Gamma_{\mathcal{F}} = \Gamma_{\mathcal{F}_0} \rtimes \mathrm{Stab}_A(\mathcal{F}_0)$, where $\mathcal{F}_0$ is in the A-orbit $\mathcal{F}$. If $\mathcal{F}$ is F-stable, we define the action of F on $\Gamma_{\mathcal{F}}$ to be that induced by aF on $\Gamma_{\mathcal{F}_0} \rtimes \mathrm{Stab}_A(\mathcal{F}_0)$ where a is such that ${}^{aF}\mathcal{F}_0 = \mathcal{F}_0$.*

Note that the action of F is defined only up to an inner automorphism, but that this does not matter since all the constructions we made are invariant by $\Gamma_{\mathcal{F}}$-conjugacy.

We can then consider objects such as $\overline{\mathcal{M}}$, $\underline{\mathcal{M}}$, *etc...* for non connected groups. We extend the definition of Δ (*cf.* 4.3) to our situation: given $(xF, \chi) \in \overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F\rangle)$ we define $\Delta_{(xF,\chi)} = \Delta_{(xaF,\psi)}$ for any a such that ${}^{aF}\mathcal{F}_0 = \mathcal{F}_0$ and any component ψ of $\mathrm{Res}^{C_{\Gamma_{\mathcal{F}}}(xF)}_{C_{\Gamma_{\mathcal{F}_0}}(xaF)} \chi$. This makes sense since Δ is invariant by any automorphism of W°. Next proposition will show that, with these definitions, $\coprod_{a \in H^1(F,A)} \mathcal{E}(\mathbf{H}^{\dot{a}F}, 1)$ is again parametrized by $\coprod_{\mathcal{F} \in \Xi(W)} \overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F\rangle)$.

5.2 PROPOSITION. *Let $\mathbf{H}$ be a non connected group such that $A = \mathbf{H}/\mathbf{H}^\circ$ is abelian. We keep the notation of 1.1 and of 1.3. We assume that restrictions of irreducible representations from $\mathbf{H}^{\dot{a}F}$ to $\mathbf{H}^{\circ\dot{a}F}$ are multiplicity free for any $a \in A$. Then there is an isomorphism*

$$\bigoplus_{a \in H^1(F,A)} \overline{\mathbb{Q}}_l \mathcal{E}(\mathbf{H}^{\dot{a}F}, 1) \xrightarrow{\sim} \bigoplus_{\mathcal{F} \in \Xi(W)^F} \overline{\mathbb{Q}}_l \overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F\rangle)$$

such that for any family $\mathcal{F}_0 \in \Xi(W^\circ)$, the map

$$\mathrm{Ind}^{\mathbf{H}^{\dot{a}F}}_{\mathbf{H}^{\circ\dot{a}F}} : \mathcal{E}(\mathbf{H}^{\circ\dot{a}F}, 1, \mathcal{F}_0) \to \mathcal{E}(\mathbf{H}^{\dot{a}F}, 1, \mathcal{F})$$

(with the notations of definition 5.1) corresponds to the linear map

$$\overline{\mathbb{Q}}_l\overline{\mathcal{M}}(\Gamma_{\mathcal{F}_0} \subset \Gamma_{\mathcal{F}_0} \rtimes <aF>) \to \overline{\mathbb{Q}}_l\overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes <F>)$$

defined on the basis of Mellin transforms by $((xaF, y) \bmod \Gamma_{\mathcal{F}_0}) \mapsto ((xF, y) \bmod \Gamma_{\mathcal{F}})$.

PROOF: Irreducible characters in the left hand side are by definition extensions to $\mathbf{H}^{\dot{a}F}$ of the sums of A^F-orbits of characters in $\mathcal{E}(\mathbf{H}^{\circ\dot{a}F}, 1)$. By part II (*cf.* 6.4) we have a well defined bijection

$$\mathcal{E}(\mathbf{H}^{\circ\dot{a}F}, 1) \xrightarrow{\sim} \coprod_{\mathcal{F}\in\Xi(W^\circ)^{aF}} \overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes <aF>).$$

We denote by $\mathcal{F}_0, \ldots, \mathcal{F}_r$ the A-orbit of $\mathcal{F}_0 \in \Xi(W^\circ)$; by definition it is the element $\mathcal{F}$ of $\Xi(W)$. It is clear from proposition 4.1 that there is a bijection from the set of A^F-orbits in $\coprod_{\{\, i \mid {}^{aF}\mathcal{F}_i = \mathcal{F}_i \,\}} \overline{\mathcal{M}}(\Gamma_{\mathcal{F}_i} \subset \Gamma_{\mathcal{F}_i} \rtimes <aF>)$ onto $\{\, (ya'F, \psi) \mid y \in \Gamma_{\mathcal{F}_0},\ a'F \underset{A}{\sim} aF,\ {}^{a'F}\mathcal{F}_0 = \mathcal{F}_0, \psi \in \operatorname{Irr}(C_{\Gamma_{\mathcal{F}_0}}(ya'F)) \,\}/\Gamma_{\mathcal{F}}$, given by

$$(xaF, \chi) \bmod \Gamma_{\mathcal{F}_i} \mapsto ({}^{\tau_i}xa_i a^F a_i^{-1} F, {}^{\tau_i}\chi) \bmod \Gamma_{\mathcal{F}},$$

where $\Gamma_{\mathcal{F}}$ is identified to $\Gamma_{\mathcal{F}_0} \rtimes \operatorname{Stab}_A(\mathcal{F}_0)$, where a_i is an element of A such that ${}^{a_i}\mathcal{F}_i = \mathcal{F}_0$, and where $\tau_i : \Gamma_{\mathcal{F}i} \xrightarrow{\sim} \Gamma_{\mathcal{F}_0}$ is the isomorphism associated to a_i as in 4.1.

We now claim that the number of extensions to $\mathbf{H}^{\dot{a}F}$ of the sum of an A^F-orbit in $\mathcal{E}(\mathbf{H}^{\circ\dot{a}F}, 1)$ corresponding via the parametrization of proposition II, 6.4 and the above bijection to the class of $(ya'F, \psi)$ modulo $\Gamma_{\mathcal{F}_0} \rtimes \operatorname{Stab}_A(\mathcal{F}_0)$ is equal to the number of irreducible components of the induced representation of ψ to $C_{\Gamma_{\mathcal{F}_0} \rtimes \operatorname{Stab}_C(\mathcal{F}_0)}(ya'F)$. By assumption, restrictions of irreducible characters from $\mathbf{H}^{\dot{a}'F}$ to $\mathbf{H}^{\circ\dot{a}'F}$ are multiplicity free. We need the same property for restrictions of irreducible characters from $C_{\Gamma_{\mathcal{F}_0} \rtimes \operatorname{Stab}_A(\mathcal{F}_0)}(ya'F)$ to $C_{\Gamma_{\mathcal{F}_0}}(ya'F)$. This can be proved in the following way: by 0.3 we can reduce ourselves to the case where W° is irreducible; then both $a'F$ and $\operatorname{Stab}_A(\mathcal{F}_0)$ act trivially on $\Gamma_{\mathcal{F}_0}$; so $C_{\Gamma_{\mathcal{F}_0} \rtimes \operatorname{Stab}_A(\mathcal{F}_0)}(ya'F)$ is the direct product of $C_{\Gamma_{\mathcal{F}_0}}(ya'F) = C_{\Gamma_{\mathcal{F}_0}}(y)$ by the abelian group $\operatorname{Stab}_A(\mathcal{F}_0, F)$ and the result is clear.

Now, proving our claim is equivalent to show that the stabilizer in $\mathbf{H}^{\dot{a}F}/\mathbf{H}^{\circ\dot{a}F}$ of the character ρ of $\mathbf{H}^{\circ\dot{a}F}$ parametrized by $(xaF, \chi) \in$

$\overline{\mathcal{M}}(\Gamma_{\mathcal{F}_i} \subset \Gamma_{\mathcal{F}_i} \rtimes <aF>)$ (a preimage of $(ya'F, \psi)$ by the above bijection) has the same cardinality as
$\mathrm{Stab}_{\Gamma_\mathcal{F}}(ya'F, \psi)/C_{\Gamma_{\mathcal{F}_0}}(ya'F)$. But this quotient group is isomorphic to the stabilizer in $\mathrm{Stab}_A(\mathcal{F}_0)$ of $((ya'F, \psi) \bmod \Gamma_{\mathcal{F}_0}))$, the isomorphism being induced by the projection of $\Gamma_\mathcal{F}$ onto $\mathrm{Stab}_A(\mathcal{F}_0)$. On the other hand, the stabilizer of ρ in A^F is equal by II, 6.5 and II, 6.6 to the stabilizer of (xaF, χ) in $\mathrm{Stab}_{A^F}(\mathcal{F}_i)$. As this latter group is isomorphic to $\mathrm{Stab}_{A^F}(\mathcal{F}_0, (ya'F, \psi)) = \mathrm{Stab}_A(\mathcal{F}_0, (ya'F, \psi))$, we get our claim. So we have a bijection

$$\mathcal{E}(\mathbf{H}^{\dot{a}F}, 1, \mathcal{F}) \xrightarrow{\sim}$$
$$\{(ya'F, \chi) \mid y \in \Gamma_{\mathcal{F}_0}, a'F \underset{A}{\sim} aF,\ {}^{a'F}\mathcal{F}_0 = \mathcal{F}_0,\ \chi \in \mathrm{Irr}(C_{\Gamma_\mathcal{F}}(ya'F))\}/\Gamma_\mathcal{F}$$

such that if $\rho \in \mathcal{E}(\mathbf{H}^{\circ\dot{a}F}, 1)$ corresponds to $(xaF, \psi) \in \overline{\mathcal{M}}(\Gamma_{\mathcal{F}_0} \subset \Gamma_{\mathcal{F}_0} \rtimes <aF>)$, the irreducible components of $\mathrm{Ind}_{\mathbf{H}^{\circ\dot{a}F}}^{\mathbf{H}^{\dot{a}F}} \rho$ correspond to the elements (xaF, χ) in the right hand side set, where χ runs over the irreducible components of $\mathrm{Ind}_{C_{\Gamma_{\mathcal{F}_0}}(xaF)}^{C_{\Gamma_\mathcal{F}}(xaF)} \psi$. If we take the union over $H^1(F, A)$, we then get

$$\coprod_{a \in H^1(F,A)} \mathcal{E}(\mathbf{H}^{\dot{a}F}, 1, \mathcal{F}) \xrightarrow{\sim}$$
$$\{(ya'F, \chi) \mid y \in \Gamma_{\mathcal{F}_0}, a' \in A,\ {}^{a'F}\mathcal{F}_0 = \mathcal{F}_0,\ \chi \in \mathrm{Irr}(C_{\Gamma_\mathcal{F}}(ya'F))\}/\Gamma_\mathcal{F}.$$

The set in the above right hand side is isomorphic to

$$\{\, (xF, \chi) \mid x \in \Gamma_\mathcal{F},\ \chi \in \mathrm{Irr}(C_{\Gamma_\mathcal{F}}(xF)) \,\}$$

by definition of $\Gamma_\mathcal{F}$. So, taking the union over all F-invariant families $\mathcal{F}$, we get the isomorphism of the proposition.

We now show the statement on Mellin transforms. It will be an immediate consequence of what we have shown above and of the following lemma.

5.3 LEMMA. *The map*

$$\mathrm{Ind}_{\overline{\mathcal{M}}} : \overline{\mathbb{Q}}_l\overline{\mathcal{M}}(\Gamma_{\mathcal{F}_0} \subset \Gamma_{\mathcal{F}_0} \rtimes <aF>) \to \overline{\mathbb{Q}}_l\overline{\mathcal{M}}(\Gamma_\mathcal{F} \subset \Gamma_\mathcal{F} \rtimes <F>)$$

defined by $(xaF, \psi) \mapsto (xF, \mathrm{Ind}_{C_{\Gamma_{\mathcal{F}_0}}(xaF)}^{C_{\Gamma_\mathcal{F}}(xF)} \psi)$ *maps* $((xaF, y) \bmod \Gamma_{\mathcal{F}_0})$ *on* $((xF, y) \bmod \Gamma_\mathcal{F})$.

PROOF: By definition we have

$$(xaF, y) = \sum_{\psi \in \operatorname{Irr}(C_{\Gamma_{\mathcal{F}_0}}(xaF))} \psi(y)\Delta_{(xaF,\psi)}(xaF, \psi).$$

So its image is

$$\sum_{\psi \in \operatorname{Irr}(C_{\Gamma_{\mathcal{F}_0}}(xaF))} \psi(y)\Delta_{(xaF,\psi)}(xaF, \operatorname{Ind}\psi) =$$

$$\sum_{\psi \in \operatorname{Irr}(C_{\Gamma_{\mathcal{F}_0}}(xaF))} \psi(y)\Delta_{(xaF,\psi)} \sum_{\chi \in \operatorname{Irr}(C_{\Gamma_{\mathcal{F}}}(xF))} \langle \operatorname{Ind}\psi, \chi \rangle_{C_{\Gamma_{\mathcal{F}}}(xF)}(xF, \chi).$$

By definition we have $\Delta_{(xaF,\psi)} = \Delta_{(xF,\chi)}$ for any χ appearing in $\operatorname{Ind}\psi$. So we can exchange the summations to get

$$\sum_{\chi \in \operatorname{Irr}(C_{\Gamma_{\mathcal{F}}}(xF))} \Delta_{(xF,\chi)}(xF, \chi) \sum_{\psi \in \operatorname{Irr}(C_{\Gamma_{\mathcal{F}_0}}(xaF))} \langle \operatorname{Ind}\psi, \chi \rangle_{C_{\Gamma_{\mathcal{F}}}(xF)}\psi(y).$$

As the second sum is equal to $\chi(y)$, we get

$$\sum_{\chi \in \operatorname{Irr}(C_{\Gamma_{\mathcal{F}}}(xF))} \Delta_{(xF,\chi)}(xF, \chi)\chi(y),$$

which is by definition the element (xF, y) of $\overline{\mathbb{Q}}_l\overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes\!\!\!\lhd\, <F>)$, whence the lemma, and the proposition. ∎

We will now prove an analogous statement for F-class functions on W and spaces $\underline{\mathcal{M}}$. For this we generalize to W Lusztig's embedding of F-class functions on W° mapping $\mathcal{F}_0$ into $\underline{\mathcal{M}}(\Gamma_{\mathcal{F}_0} \subset \Gamma_{\mathcal{F}_0} \rtimes\!\!\!\lhd\, <aF>$.

5.4 PROPOSITION. *Let $\mathcal{F} \in \Xi(W)^F$ be the orbit $\{\mathcal{F}_0, \ldots, \mathcal{F}_r\}$. There is an embedding*

$$\tilde{\mathcal{F}} \subset \underline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes\!\!\!\lhd\, <F>)$$

such that the map $\operatorname{Ind}_{W^\circ.\dot{a}F}^{W.F} : \tilde{\mathcal{F}}_0 \to \tilde{\mathcal{F}}$ is the restriction of the linear map $\underline{\mathcal{M}}(\Gamma_{\mathcal{F}_0} \subset \Gamma_{\mathcal{F}_0} \rtimes\!\!\!\lhd\, <aF>) \to \underline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes\!\!\!\lhd\, <F>)$ defined on the basis of Mellin transforms by $((y, xaF) \bmod \Gamma_{\mathcal{F}_0}) \mapsto ((y, xF) \bmod \Gamma_{\mathcal{F}})$.

PROOF: The elements of $\mathcal{F}$ are the components of $\operatorname{Ind}_{W^\circ}^{W} \varphi$ for $\varphi \in \mathcal{F}_0$. Let $A_0 = \operatorname{Stab}_A(\Gamma_{\mathcal{F}_0})$ so that $\Gamma_{\mathcal{F}} = \Gamma_{\mathcal{F}_0} \rtimes\!\!\!\lhd\, A_0$; if φ corresponds to $(x, \sigma) \in \mathcal{M}(\Gamma_{\mathcal{F}_0})$, by 4.1 we have $\operatorname{Stab}_A(\varphi) = \operatorname{Stab}_{A_0}(x, \sigma)$, so the components of $\mathrm{Ind}_{W^\circ}^W(\varphi)$ are in bijection with those of $\mathrm{Ind}_{C_{\Gamma_{\mathcal{F}_0}}(x)}^{C_{\Gamma_\mathcal{F}}(x)}\sigma$, whence an embedding $\mathcal{F}\subset\mathcal{M}(\Gamma_\mathcal{F})$. Let us prove that the action of F on $\mathcal{F}$ corresponds via this embedding to the action of F on $\mathcal{M}(\Gamma_\mathcal{F})$: since components of $\mathrm{Ind}_{W^\circ}^W\varphi$ (resp. components of $\mathrm{Ind}_{C_{\Gamma_{\mathcal{F}_0}}(x)}^{C_{\Gamma_\mathcal{F}}(x)}\sigma$) differ from each other by characters of A, if we prove that there exists an F-stable component in each of these sets, then the action of F on these components will be in either case given by that of F on $\mathrm{Irr}(A)$. The following lemma proves that there exists an F-stable component in each of these sets.

5.5 LEMMA.
(i) *Let $\varphi\in\mathrm{Irr}(W^\circ)$ have an F-invariant A-orbit. Then $\mathrm{Ind}_{W^\circ}^W\varphi$ has an F-invariant irreducible component.*
(ii) *Let $a\in A^F$ and let $\mathcal{F}\in\Xi(W)^F$ be the F-stable A-orbit $\{\mathcal{F}_0,\ldots,\mathcal{F}_r\}$ in $\Xi(W^\circ)^a$. For any $(xa,\chi)\in\overline{\mathcal{M}}(\Gamma_{\mathcal{F}_0}\subset\Gamma_{\mathcal{F}_0}\rtimes\langle a\rangle)$, there exists a component $\tilde\chi$ of $\mathrm{Ind}_{C_{\Gamma_{\mathcal{F}_0}}(x)}^{C_{\Gamma_\mathcal{F}}(x)}\chi$, such that $(x,\tilde\chi)$ is an F-stable element of $\mathcal{M}(\Gamma_\mathcal{F})$.*

PROOF: We first prove (i). By 0.2 φ extends to some $\chi\in\mathrm{Irr}(\tilde I)$ where $\tilde I=\mathrm{Stab}_{W\rtimes\langle F\rangle}\varphi$. Let $I=\mathrm{Stab}_W(\varphi)$; by hypothesis there exists $a\in A$ such that $aF\in\tilde I$, so $\mathrm{Res}_I^{\tilde I}\chi$ is an (irreducible) aF-stable character, and so $\mathrm{Ind}_I^W\mathrm{Res}_I^{\tilde I}\chi$ is F-stable and is an irreducible component of $\mathrm{Ind}_{W^\circ}^W\varphi$, whence the result.

The proof of (ii) is similar: the preceding argument shows that it suffices to show that χ extends to $\mathrm{Stab}_{\Gamma_{\mathcal{F}'}\rtimes\langle F\rangle}(xa,\chi)$. By using 0.3 we can reduce ourselves to the case where W is irreducible, and then the result is trivial since in this case $\mathrm{Stab}_{A^F}(\mathcal{F}_0)$ acts trivially on $\Gamma_{\mathcal{F}_0}$. ■

We now define the embedding of the proposition in the following way: for each F-invariant component τ of $\mathrm{Ind}_{W^\circ}^W\varphi$ mapped onto (x,χ) (where χ is a component of $\mathrm{Ind}_{C_{\Gamma_{\mathcal{F}_0}}(x)}^{C_{\Gamma_\mathcal{F}}(x)}\psi$) we choose a mapping $\tilde\tau\mapsto(x,\tilde\chi)$ compatible with tensorization by characters of A. We then get $x_{\mathrm{Ind}_{W^\circ\rtimes\langle aF\rangle}^{W\rtimes\langle F\rangle}\tilde\tau}=(x,\mathrm{Ind}_{C_{\Gamma_{\mathcal{F}_0}\rtimes\langle aF\rangle}(x)}^{C_{\Gamma_\mathcal{F}\rtimes\langle F\rangle}(x)}\tilde\psi)$. The statement about Mellin transforms is then an immediate consequence of the following lemma:

5.6 LEMMA. *The map $\mathrm{Ind}_{\underline{\mathcal{M}}}:\underline{\mathcal{M}}(\Gamma_{\mathcal{F}_0}\subset\Gamma_{\mathcal{F}_0}\rtimes\langle aF\rangle)\to\underline{\mathcal{M}}(\Gamma_\mathcal{F}\subset\Gamma_\mathcal{F}\rtimes\langle F\rangle)$ defined by $(x,\tilde\psi)\mapsto(x,\mathrm{Ind}_{C_{\Gamma_{\mathcal{F}_0}\rtimes\langle aF\rangle}(x)}^{C_{\Gamma_\mathcal{F}\rtimes\langle F\rangle}(x)}\tilde\psi)$ maps the element $(y,xaF)\bmod\Gamma_{\mathcal{F}_0}$ on $(y,xF)\bmod\Gamma_\mathcal{F}$.*

PROOF: By definition we have $(y, xaF) = \sum_{\tilde{\psi}} \tilde{\psi}(xaF)(y, \tilde{\psi})$, where $\tilde{\psi}$ runs over a set consisting of one extension of each xaF-invariant irreducible character of $C_{\Gamma_{\mathcal{F}_0}}(y)$. So its image is

$$\sum_{\tilde{\psi}} \tilde{\psi}(xaF)(y, \operatorname{Ind}\psi) = \sum_{\tilde{\psi}} \tilde{\psi}(xaF) \sum_{\chi} \langle \operatorname{Ind}\tilde{\psi}, \tilde{\chi} \rangle (x, \tilde{\chi}),$$

where $\tilde{\chi}$ runs over a set consisting of one extension of each xF-invariant irreducible character of $C_{\Gamma_{\mathcal{F}}}(y)$. If we exchange the summations, we get

$$\sum_{\tilde{\chi}} (y, \tilde{\chi}) \sum_{\tilde{\psi}} \langle \operatorname{Ind}\tilde{\psi}, \tilde{\chi} \rangle \tilde{\psi}(xaF).$$

As the second sum is equal to $\tilde{\chi}(xF)$, we get

$$\sum_{\tilde{\chi}} (y, \tilde{\chi}) \tilde{\chi}(xF),$$

which is by definition the element (xF, y) of $\underline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes <F>)$, whence the lemma, and the proposition. ∎

We will denote by $f \mapsto x_f$ the embedding defined by the previous proposition.

We want now to show that the scalar products $\langle \rho_{(xF,\chi)}, R_f \rangle_{\mathbf{H}^F}$ (where R_f will be defined in an analogous way to the connected case, using generalized Deligne-Lusztig characters) are still given by 4.5. We first generalize the notation R_f for non connected groups.

5.7 DEFINITION. *For any F-class function f on W, we define*

$$R_f = |W|^{-1} \sum_{w \in W} f(w) R_{\mathbf{T}_{wF}}^{\mathbf{H}^{\dot{a}F}}(1)$$

(an element of the direct sum of the spaces of class functions on $\mathbf{H}^{\dot{a}F}$ when $\dot{a}$ runs over representatives of $H^1(F, A)$).

Note that the linear span considered above has a natural scalar product, which is the orthogonal sum of the scalar products of class functions on each group $\mathbf{H}^{\dot{a}F}$, and that similarly we can extend the pairing 4.3 to a pairing between $\oplus_{\mathcal{F} \in \Xi(W)^F} \overline{\mathbb{Q}}_l \overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes <F>)$ and $\oplus_{\mathcal{F} \in \Xi(W)^F} \underline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes <F>)$. With these notations, we can now state:

5.8 THEOREM. *Let $\rho \in \mathcal{E}(\mathbf{H}^{\dot{a}F}, 1)$ be parametrized by*

$$x_\rho \in \coprod_{\mathcal{F} \in \Xi(W)^F} \overline{\mathcal{M}}(\Gamma_\mathcal{F} \subset \Gamma_\mathcal{F} \rtimes \langle F \rangle)$$

and let f be an F-class function on W; then the scalar product $\langle \rho, R_f \rangle$ is given by $\{x_\rho, x_f\}$.

PROOF: It is sufficient to prove the result for a basis of the space of F-class functions on W. We get such a basis, if we consider the functions γ_{wF}, where, for $w \in W$, we denote by γ_{wF} the function whose value is the cardinality of the centralizer $C_W(wF)$ on the class of wF under W and zero outside. We have $R_{\gamma_{wF}} = R_{\mathbf{T}_{wF}}^{\mathbf{H}^{\dot{b}F}}(1)$, if w is of the form vbF, with b in the chosen set of representatives of $H^1(F, A)$ and $v \in W^\circ$. If $b \neq a$, both the scalar product that we want to compute and the pairing $\{x_\rho, x_{\gamma_{wF}}\}$ are zero, so we may assume $b = a$. By 1.3, (i), we have $R_{\gamma_{wF}} = \operatorname{Ind}_{\mathbf{H}^{\circ\dot{a}F}}^{\mathbf{H}^{\dot{a}F}}(R_{\gamma^\circ_{wF}})$, where γ°_{wF} is the function defined similarly to γ_{wF} for the class of wF under the group W°. We have clearly $\gamma_{wF} = \operatorname{Ind}_{W^\circ}^{W}(\gamma^\circ_{wF})$.

We want to compute $\langle \rho, R_{\gamma_{wF}} \rangle$. By Frobenius reciprocity and the preceding remarks, this is equal to $\langle \operatorname{Res}_{\mathbf{H}^{\circ\dot{a}F}}^{\mathbf{H}^{\dot{a}F}} \rho, R_{\gamma^\circ_{wF}} \rangle_{\mathbf{H}^{\circ\dot{a}F}}$. By 4.3, this scalar product is equal to $\{x_{\operatorname{Res}\rho}, x_{\gamma^\circ_{wF}}\}$, if $x_{\operatorname{Res}\rho} \in \overline{\mathbb{Q}}_l\overline{\mathcal{M}}(\Gamma_{\mathcal{F}_0} \subset \Gamma_{\mathcal{F}_0} \rtimes \langle aF \rangle)$ parametrizes $\operatorname{Res}_{\mathbf{H}^{\circ\dot{a}F}}^{\mathbf{H}^{\dot{a}F}} \rho$. So we are reduced to proving $\{x_{\operatorname{Res}\rho}, x_{\gamma^\circ_{wF}}\} = \{x_\rho, x_{\operatorname{Ind}\gamma^\circ_{wF}}\}$.

With the notations of 5.3 and 5.6, we will show that in general $\{\operatorname{Res}_{\overline{\mathcal{M}}} \overline{m}, \underline{m}\} = \{\overline{m}, \operatorname{Ind}_{\underline{\mathcal{M}}} \underline{m}\}$ for any $\overline{m} \in \overline{\mathbb{Q}}_l\overline{\mathcal{M}}(\Gamma_\mathcal{F} \subset \Gamma_\mathcal{F} \rtimes \langle F \rangle)$ and any $\underline{m} \in \underline{\mathcal{M}}(\Gamma_\mathcal{F} \subset \Gamma_\mathcal{F} \rtimes \langle F \rangle)$, where $\operatorname{Res}_{\overline{\mathcal{M}}}$ is the adjoint of $\operatorname{Ind}_{\overline{\mathcal{M}}}$ for the scalar product $\langle\, , \rangle$ on $\overline{\mathbb{Q}}_l\overline{\mathcal{M}}(\Gamma_\mathcal{F} \subset \Gamma_\mathcal{F} \rtimes \langle F \rangle)$ deduced from the scalar product on class functions via the isomorphism of 5.2 (*i.e.*, $\operatorname{Res} x_\rho = x_{\operatorname{Res}\rho}$). For this scalar product the elements (xF, χ) form an orthonormal basis, so Mellin transforms are pairwise orthogonal and we have $\langle (xF, y), (xF, y) \rangle = |C_{\Gamma_\mathcal{F}}((xF, y))|$. So $\langle (xF, y), (x'F, y') \rangle = \{(xF, y), (y', x'F)\}$. Whence

$$\{\operatorname{Res}(xF, y), (x_\circ F, y_\circ)\} = \langle \operatorname{Res}(xF, y), (x_\circ F, y_\circ) \rangle = \\ \langle (xF, y), \operatorname{Ind}(x_\circ F, y_\circ) \rangle = \{(xF, y), \operatorname{Ind}(x_\circ F, y_\circ)\};$$

whence the result. ∎

II

6. Unicity of the parametrization.

The goal of this part is to give conditions which unambiguously determine the map π_s in the case of a group $\mathbf{G}$ connected and with a connected center. Our strategy is as follows:

We first determine (*cf.* 6.3) under which conditions 4.3 determines the map

$$\rho \in \mathcal{E}(\mathbf{G}^F, 1) \to (xF, \chi) \in \coprod_{\mathcal{F} \in \Xi^F} \overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes <F>)$$

We need to add two conditions to define completely the map (*cf.* 6.4).

We then consider the general situation of $\mathcal{E}(\mathbf{G}^F, s)$, and we have to deal with all the cases where 4.3 is not sufficient and 6.4 does not apply.

6.1 Since the families we consider depend on F, we will consider couples (W, F) where W is a Coxeter group and F a diagram automorphism of W. We say that (W, F) is irreducible if W is a product of irreducible Coxeter groups, permuted transitively by F.

With the above definition we have $(W, F) = \prod_i (W_i, F_i)$ where the (W_i, F_i) are irreducible; each $\mathcal{F} \in \Xi(W)^F$ identifies to a product $\prod_i \mathcal{F}_i \in \prod_i \Xi(W_i)^{F_i}$, and we have $\Gamma_{\mathcal{F}} = \prod_i \Gamma_{\mathcal{F}_i}$,

$$\overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes <F>) = \prod_i \overline{\mathcal{M}}(\Gamma_{\mathcal{F}_i} \subset \Gamma_{\mathcal{F}_i} \rtimes <F_i>),$$

$$\underline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes <F>) = \bigotimes_i \underline{\mathcal{M}}(\Gamma_{\mathcal{F}_i} \subset \Gamma_{\mathcal{F}_i} \rtimes <F_i>)$$

and $\tilde{\mathcal{F}} = \bigotimes_i \tilde{\mathcal{F}}_i$, and the pairing between the spaces

$$\overline{\mathbb{Q}}_l \overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes <F>) \text{ and } \underline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes <F>)$$

is the product of the pairings between

$$\overline{\mathbb{Q}}_l \overline{\mathcal{M}}(\Gamma_{\mathcal{F}_i} \subset \Gamma_{\mathcal{F}_i} \rtimes <F_i>) \text{ and } \underline{\mathcal{M}}(\Gamma_{\mathcal{F}_i} \subset \Gamma_{\mathcal{F}_i} \rtimes <F_i>).$$

When (W,F) is irreducible where $W = \prod_{i=1}^r W_i$ with W_i irreducible, then (*cf.* [11, 4.20]) $\Xi(W)^F$ identifies to $\Xi(W_1)^{F^r}$, $\Gamma_{\mathcal{F}}$ identifies to a product $\Gamma_{\mathcal{F}_1} \times \ldots \times \Gamma_{\mathcal{F}_r}$ permuted transitively by F and $\overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F\rangle)$ identifies to $\overline{\mathcal{M}}(\Gamma_{\mathcal{F}_1} \subset \Gamma_{\mathcal{F}_1} \rtimes \langle F^r\rangle)$: any $\overline{x} \in \overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F\rangle)$ has a representative of the form $((a,1,\ldots,1)F,\chi)$ where χ is a character of $C_{\Gamma_{\mathcal{F}}}((a,1,\ldots,1)F) \simeq C_{\Gamma_{\mathcal{F}_r}}(a)$. Similarly there exists isomorphisms $\underline{\mathcal{M}}(\Gamma_{\mathcal{F}_1} \subset \Gamma_{\mathcal{F}_1} \rtimes \langle F^r\rangle) \simeq \underline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F\rangle)$ and $\tilde{\mathcal{F}} \simeq \tilde{\mathcal{F}}_1$ (relative to F^r), and the pairing is compatible with these isomorphisms.

Finally, when W is irreducible, for any automorphism F of W, any $\mathcal{F}$ with more than one element is in Ξ^F and is fixed pointwise by F (*cf.* [11, 4.19] for A_n, E_6 and D_4 (this last case with an automorphism of order 3), and for D_n the only characters not fixed by the diagram automorphism are those corresponding to symbols with identical parts, which are in a one-element family). So for any element $\mathcal{F} \in \Xi^F$, we have $\underline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F\rangle) \simeq \overline{\mathbb{Q}}_l\overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F\rangle) \simeq \overline{\mathbb{Q}}_l\mathcal{M}(\Gamma_{\mathcal{F}})$ and the pairing between $\underline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F\rangle)$ and $\overline{\mathbb{Q}}_l\overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F\rangle)$ has the same value as that between $\mathcal{M}(\Gamma_{\mathcal{F}})$ and itself, and $\tilde{\mathcal{F}} \simeq \overline{\mathbb{Q}}_l\mathcal{F}$.

6.2 So, in general, for any $\mathcal{F} \in \Xi(W)^F$ there is a set $\{\mathcal{F}_i \in \Xi(W_i)^{F_i}\}_i$ where W_i is irreducible, such that $\overline{\mathbb{Q}}_l\overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F\rangle)$ and $\mathcal{M}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F\rangle)$ both identify to a product $\bigotimes_i \mathcal{M}(\Gamma_{\mathcal{F}_i})$, compatibly with the pairing, and $\tilde{\mathcal{F}}$ identifies to $\bigotimes_i \overline{\mathbb{Q}}_l\mathcal{F}_i$. This will allow us to reduce the problem to the case of an irreducible W.

PROPOSITION 6.3. *Suppose that no component $\mathcal{F}_i$ of $\mathcal{F}$ is exceptional (cf. 4.4), and that $\overline{x} \in \overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F\rangle)$ has no component x_j of the following types (using Lusztig's [11, chap. 4] notations):*

$x_j = (g_4, i)$ or $(g_4, -i)$ ($\Gamma_{\mathcal{F}_j}$ isomorphic to $\mathbf{S}_4$).

$x_j = (g_3, \theta)$ or (g_3, θ^2) ($\Gamma_{\mathcal{F}_j}$ isomorphic to $\mathbf{S}_3$ or $\mathbf{S}_4$).

Then the set of coefficients $\{\overline{x}, x_{\tilde{E}}\}$, when $\tilde{E}$ runs over $\tilde{\mathcal{F}}$, determine the element $\overline{x} \in \overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F\rangle)$.

PROOF: In the identification of 6.2, we have $\tilde{E} = (E_1,\ldots,E_r)$ where $E_i \in \mathcal{F}_i$, $\overline{x} = (x_1,\ldots,x_r)$ with $x_i \in \mathcal{M}(\Gamma_{\mathcal{F}_i})$ and $\{x_{\tilde{E}}, \overline{x}\} = \prod_i \{x_{E_i}, x_i\}$. Let $V(x_i)$ be the vector $\{x_{E_i}, x_i\}_{E_i \in \mathcal{F}_i}$; to show the proposition we have to prove that the vectors $V(\overline{x}) = V(x_1) \otimes \ldots \otimes V(x_r)$ are all distinct when $\overline{x}$ runs over $\overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F\rangle)$ (unless an x_i is one of the exceptions of the proposition). If this were not true, then there would exist $x_i \neq x_i'$ and $\lambda \in \mathbb{C}^\times$ such that $V(x_i) = \lambda V(x_i')$.

We can do a case-by-case check that this is impossible for an irreducible W: when $\Gamma_{\mathcal{F}} \simeq \mathbf{S}_3$, $\mathbf{S}_4$ or $\mathbf{S}_5$ it is an easy check on the table of coefficients. In the case where $\Gamma_{\mathcal{F}} \simeq (\mathbb{Z}/2\mathbb{Z})^d$ we first note that $\{x_E, x\}$ are equal for all x when E is the special character in the family. So any λ as above has to be 1, and it is enough to check that the vector $\{x_E, x\}_{E\in\mathcal{F}}$ determines x.

The types of W to consider are B_n, D_n and ${}^2D_n$.

- Case B_n. With Lusztig's [11] notations, x is parametrized by a subset X of even cardinality of a set Z_1 with $2d+1$ elements in which a fixed subset M_0 of cardinal d has been distinguished. The elements x_E correspond to subsets $Y \subset Z_1$ such that $|Y \Delta M_0| = d$, and with these parametrizations, we have $\{x, x_E\} = (-1)^{|X\cap Y|}$. Given $a \in Z_1$ we can find two subsets S_1 and S_2 of cardinal d such that $S_1 \cup S_2 = Z_1 - \{a\}$. If we put $Y_i = S_i \Delta M_0$ (for $i = 1, 2$), then Y_i corresponds to some x_{E_i} and:
$$(-1)^{|X\cap Y_1|+|X\cap Y_2|} = 1 \Longleftrightarrow a \notin X$$
so X is clearly determined by the $\{x, x_{E_i}\}$.
- Case D_n or ${}^2D_n$. This time x corresponds to a subset X of odd cardinality of a set Z_1 with $2d$ elements, taken modulo the equivalence relation $X \sim Z_1 - X$, and elements x_E correspond to subsets Y of even cardinality of Z_1 taken modulo the same equivalence relation and such that $|Y \Delta M_0| = d$ where M_0 is a fixed subset of cardinality d; and we have:
$$\{x, x_E\} = (-1)^{|X\cap Y|}.$$
Given two elements $a, b \in Z_1$ there exists two subsets S_1 and S_2 of cardinal d such that $S_1 \Delta S_2 = \{a, b\}$; if we set $Y_i = S_i \Delta M_0$ (for $i = 1, 2$) then:
$$(-1)^{|X\cap Y_1|}(-1)^{|X\cap Y_2|} = (-1)^{|X\cap\{a,b\}|}$$
Modulo the equivalence $X \sim Z_1 - X$ we may assume that $a \notin X$, and then the above equality clearly determines X.

This concludes the proof of proposition 6.3. ∎

We now give sufficient conditions for Lusztig's parametrization $\rho \mapsto (xF, \chi)$ to be uniquely determined in the case of unipotent characters.

For $\overline{x} \in \overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F\rangle)$, we define a root of unity $\lambda(\overline{x})$ as follows: if $\overline{x}$ identifies to $(x_1, \ldots, x_r)$ (*cf.* 6.2) then $\lambda(\overline{x}) = \lambda(x_1)\ldots\lambda(x_r)$; and if $x_1 = (a, \chi)$ where $a \in \Gamma_{\mathcal{F}_1}$ and χ is a character of $C_{\Gamma_{\mathcal{F}_1}}(a)$, we put $\lambda(x_1) = \chi(a)/\chi(1)$ unless $\mathcal{F}_1$ is exceptional and (x_1, χ) is not in the image of $\mathcal{F}_1$ by the embedding $\mathcal{F}_1 \hookrightarrow \mathcal{M}(\Gamma_{\mathcal{F}})$, when we put $\lambda(x_1) = i\chi(a)/\chi(1)$.

REMARK. In the next proposition we do not know a reference for the proof of (ii) when (W, F) is of type ${}^2D_n$ though we believe it is true. The same restriction will consequently apply in (ii) of theorem 7.1 about the unicity of Lusztig's parametrization, but we will not use that part of the hypotheses in the proof of the unicity.

PROPOSITION 6.4. *Suppose (W, F) irreducible. Then, for $\mathcal{F} \in \Xi(W)^F$, there is a unique bijection*

$$\rho \mapsto \overline{x}_\rho : \mathcal{E}(\mathbf{G}^F, 1, \mathcal{F}) \to \overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes <F>)$$

such that:

(i) For any $\tilde{E} \in \tilde{\mathcal{F}}$ and $\rho \in \mathcal{E}(\mathbf{G}^F, 1, \mathcal{F})$ we have

$$\langle \rho, R_{\tilde{E}} \rangle = \Delta(\overline{x}_\rho)\{\overline{x}_\rho, x_{\tilde{E}}\}.$$

(ii) If F^δ is the smallest split power of F then the eigenvalues of F^δ associated to $\rho \in \mathcal{E}(\mathbf{G}^F, 1, \mathcal{F})$ are equal, up to a power of $q^{\delta/2}$ to $\lambda(\overline{x}_\rho)$.

(iii) The character ρ of the principal series associated (via Lusztig's explicit isomorphism of W with the Hecke $H(\mathbf{G}, \mathbf{B})$) to the special character of $\mathcal{F}$ is such that $\overline{x}_\rho = (F, \mathrm{Id}) \in \overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes <F>)$.

PROOF: Lusztig has shown the existence of a bijection compatible with (i)–(iii): indeed (i) is [11, 4.23]; to check that we may have a bijection satisfying (ii), if $W = \prod_{i=1}^r W_i$ where the W_i are irreducible and cyclically permuted by F, the elements $w = (w_1, 1, .., 1)$ span the F-classes of W and we have an isomorphism of the Deligne-Lusztig varieties: $X_{w,\mathbf{G},F} \simeq X_{w_1,\mathbf{G}_1,F^r}$; moreover, if F^δ is split the eigenvalues of F^δ on $H_c^*(X_{w,\mathbf{G},F})$ are equal to the eigenvalues of $F^{r\delta}$ on $H_c^*(X_{w_1,\mathbf{G}_1,F^r})$. So we may assume that W is irreducible. For irreducible groups, (ii) for untwisted groups is [11, 11.2], and for groups of type ${}^2E_6$ and ${}^3D_4$ can be deduced from [9, 7.3] (we can restrict ourselves to the case where ρ is cuspidal, and in these cases all cuspidal unipotent representations occur in $H^*(X_w, \overline{\mathbb{Q}}_l)$ where w is the Coxeter element of W).

We note that at this stage we have already resolved all ambiguities left by proposition 6.3 (whence the unicity in the proposition) except for exceptional families, since the $\lambda(x_i)$ differ in the listed cases.

Checking compatibility with (iii) can also be reduced to the case W irreducible, where it is a property of the embedding $\mathcal{F} \subset \mathcal{M}(\Gamma_{\mathcal{F}})$ which can be easily checked case by case, except that in the case of exceptional families there is some ambiguity on the embedding that (iii) is precisely designed to resolve. ∎

PROPOSITION 6.5. *Assume that (W, F) is irreducible and that we are given an automorphism φ of $W \rtimes <F>$ which stabilizes W, fixes F and induces a diagram automorphism of W. Then for any $\mathcal{F} \in \Xi^F$ we have either $|\mathcal{F}| = 1$ or φ fixes $\tilde{\mathcal{F}}$ pointwise.*

PROOF: Assume first that W is irreducible. If $\mathcal{F}$ has more than one element, then it is pointwise fixed by F and φ and the condition $\varphi(F) = F$ ensures that an extension in $\tilde{\mathcal{F}}$ of an element of $\mathcal{F}$ will be φ-invariant.

Suppose now that $W = W_1 \times \ldots \times W_r$ permuted cyclically by F. Since φ and F commute, there exists i such that φF^{-i} stabilizes W_j for any j. Recall that we have an isomorphism: $(\tilde{\mathcal{F}}_1, W_1, F^r) \simeq (\tilde{\mathcal{F}}, W, F)$. Write $\varphi = \varphi F^{-i}.F^i$; φF^{-i} stabilizes all W_j and is clearly compatible with the above isomorphism, and from the first part it fixes pointwise $\tilde{\mathcal{F}}_1$ and so its image. We conclude by observing that F^i acts trivially on elements of $\tilde{\mathcal{F}}$ since they are class functions on $W \rtimes <F>$. ■

PROPOSITION 6.6. *Assume that (W, F) is irreducible and let φ be an isogeny of $\mathbf{G}$ commuting with F. Then for any $\mathcal{F} \in \Xi^F$ the set $\mathcal{E}(\mathbf{G}^F, 1, \mathcal{F})$ is pointwise fixed by φ if it has more than one element.*

PROOF: Let $\rho \in \mathcal{E}(\mathbf{G}^F, 1, \mathcal{F})$. For any $\tilde{E} \in \tilde{\mathcal{F}}$ we have:

$$\langle \rho, R_{\tilde{E}} \rangle = \langle \rho \circ \varphi, R_{\tilde{E}} \circ \varphi \rangle = \langle \rho \circ \varphi, R_{\tilde{E} \circ \varphi} \rangle$$

(the last equality by corollary 9.2). This means that:

$$\{\overline{x}_\rho, x_{\tilde{E}}\} = \{\overline{x}_{\rho\circ\varphi}, x_{\tilde{E}\circ\varphi}\} = \{\overline{x}_{\rho\circ\varphi}, x_{\tilde{E}}\}$$

(the last equality since $\tilde{E} = \tilde{E} \circ \varphi$ by proposition 6.5). By proposition 6.3 this gives $\rho = \rho \circ \varphi$ unless $\mathcal{F}$ is one of the exceptions of the proposition. To deal with the remaining cases, note that φ induces an isomorphism: $H_c^*(X_w, \overline{\mathbb{Q}}_l) \xrightarrow{\sim} H_c^*(X_{\varphi(w)}, \overline{\mathbb{Q}}_l)$ which commutes with F, so the eigenvalues of F associated to ρ are equal to those associated to $\rho \circ \varphi$; this gives the result by proposition 6.4 (ii) except if $\mathcal{F}$ is exceptional. In this last case we use the fact that a diagram isomorphism is compatible with Lusztig's isomorphism from W to $H(\mathbf{G}, \mathbf{B})$. ■

7. Unicity of Lusztig's parametrization.

We now study Lusztig's parametrization of non-unipotent characters. If we fix $s \in \mathbf{T}^*$ such that $Z_F(s)$ is non empty, Lusztig's parametrization of characters in $\mathcal{E}(\mathbf{G}^F, s)$ may be precised as follows.

THEOREM 7.1. *Given s, there exists a unique bijection:*

$$\pi_s : \mathcal{E}(\mathbf{G}^F, s) \xrightarrow{\sim} \mathcal{E}(C_{\mathbf{G}^*}(s)^{(\dot{w}_1 F)^*}, 1)$$

satisfying the following conditions:

(i) For any $\chi \in \mathcal{E}(\mathbf{G}^F, s)$ and any $w \in Z_F(s)$ we have:

$$\langle \chi, R_{\mathbf{T}^{wF}}^{\mathbf{G}^F}(\hat{s}) \rangle = \langle \pi_s(\chi), (-1)^{l(w_1)} R_{\mathbf{T}^{*(wF)^*}}^{C_{\mathbf{G}^*}(s)^{(\dot{w}_1 F)^*}}(1) \rangle.$$

(ii) If $s = 1$ then:

(a) The eigenvalues of F^δ associated to χ are equal, up to a power of $q^{\delta/2}$ to the eigenvalues of $F^{*\delta}$ associated to $\pi_1(\chi)$.

(b) If χ is in the principal series then $\pi_1(\chi)$ and χ correspond to the same character of the Hecke algebra.

(iii) If $\zeta \in \mathbf{G}^{*F^*}$ is central and $\chi \in \mathcal{E}(\mathbf{G}^F, s)$ then $\pi_{s\zeta}(\chi \otimes \hat{\zeta}) = \pi_s(\chi)$, where $\hat{\zeta}$ is the character of $\mathbf{G}^F$ corresponding to ζ (see for instance [6, 6.7 and 6.8]).

(iv) If $\mathbf{L}$ is a standard Levi subgroup of $\mathbf{G}$ such that $\mathbf{L}^*$ contains $C_{\mathbf{G}^*}(s)$ and such that $\mathbf{L}$ is $\dot{w}F$-stable (where $\dot{w}$ is a representative of the element w reduced with respect to $W_{\mathbf{L}}$ in the class of w_1); then the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{E}(\mathbf{G}^F, s) & \xrightarrow{\pi_s} & \mathcal{E}(C_{\mathbf{G}^*}(s)^{(\dot{w}_1 F)^*}, 1) \\ \uparrow R_{\mathbf{L}^{\dot{w}F}}^{\mathbf{G}^F} & & \| \\ \mathcal{E}(\mathbf{L}^{\dot{w}F}, s) & \xrightarrow{\pi_s} & \mathcal{E}(C_{\mathbf{L}^*}(s)^{(\dot{v}_1 \dot{w} F)^*}, 1) \end{array}$$

(where $\dot{v}_1$ is defined by $\dot{v}_1 \dot{w} = \dot{w}_1$).

(v) Assume that (W, F) is irreducible, that $(\mathbf{G}, F)$ is of type E_8 and that $(C_{\mathbf{G}^*}(s), (\dot{w}_1 F)^*)$ is of type $E_7 \times A_1$ (resp. $E_6 \times A_2$, resp. ${}^2E_6 \times {}^2A_2$) then we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C}(\mathbf{G}^F, s) & \xrightarrow{\pi_s} & \mathcal{C}(C_{\mathbf{G}^*}(s)^{(\dot{w}_1 F)^*}, 1) \\ \uparrow R_{\mathbf{L}^{\dot{w}_2 F}}^{\mathbf{G}^F} & & \uparrow R_{\mathbf{L}^{*(\dot{w}_2 F)^*}}^{C_{\mathbf{G}^*}(s)^{(\dot{w}_1 F)^*}} \\ \mathcal{E}(\mathbf{L}^{\dot{w}_2 F}, s)^\bullet & \xrightarrow{\pi_s} & \mathcal{E}(\mathbf{L}^{*(\dot{w}_2 F)^*}, 1)^\bullet \end{array}$$

with $\mathbf{L}$ a Levi subgroup of $\mathbf{G}$ of type E_7 (resp. E_6, resp. E_6) containing the corresponding component of $C_{\mathbf{G}^*}(s)$, where the superscript $\bullet$ means the cuspidal part of the series, and where $w_2 = 1$

(resp. 1, resp. the $W_{\mathbf{L}}$-reduced element of $Z_F(s)$ which is in a parabolic subgroup of type E_7 of W).

(vi) *Given an epimorphism with kernel a central torus:*

$$(\mathbf{G}, F) \xrightarrow{\varphi} (\mathbf{G}_1, F_1)$$

and elements $s_1 \in \mathbf{G}_1^*$ and $s = \varphi^*(s_1) \in \mathbf{G}^*$, then the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{E}(\mathbf{G}^F, s) & \xrightarrow{\pi_s} & \mathcal{E}(C_{\mathbf{G}^*}(s)^{(\varphi(\dot{w}_1)F)^*}, 1) \\ \uparrow {}^t\varphi & & \downarrow {}^t\varphi^* \\ \mathcal{E}(\mathbf{G}_1^{F_1}, s_1) & \xrightarrow{\pi_{s_1}} & \mathcal{E}(C_{\mathbf{G}_1^*}(s_1)^{(\dot{w}_1 F_1)^*}, 1) \end{array}$$

(vii) *If $\mathbf{G}$ is a direct product $\prod \mathbf{G}_i$, then $\pi_{\prod s_i} = \prod \pi_{s_i}$.*

PROOF: We have to show that conditions (i)–(vi) completely specify the map π_s, and that they are compatible with each other. Our strategy will be to consider each condition in turn, and each time to show that (together with the previous conditions) it specifies π_s for a larger set of situations, and the previous conditions hold for the new situations (as well as the new condition for the previous situations).

We first suppose that (W, F) is irreducible and that $s = 1$, and remark that by applying 6.4 in $\mathbf{G}$ and $\mathbf{G}^*$, we obtain that (i) and (ii) completely specify π_s in this case (and are compatible).

If now s is central, (iii) clearly specifies π_s from its value when $s = 1$. (ii) is not relevant unless $s = 1$, and (i) holds, since $R^{\mathbf{G}^F}_{\mathbf{T}_{wF}}(\hat{s} \otimes \hat{\zeta}) = R^{\mathbf{G}^F}_{\mathbf{T}_{wF}}(\hat{s}) \otimes \hat{\zeta}$ (*cf.* [3, 1.67]).

Suppose now that $C_{\mathbf{G}^*}(s)$ is contained in a proper Levi subgroup $\mathbf{L}^*$ of $\mathbf{G}^*$; then (iv) specifies π_s completely in $\mathbf{G}$ if π_s is uniquely defined in $\mathbf{L}$, which we may assume by induction on the rank of $\mathbf{G}$ (we assume that the whole of 4.1 holds for groups of smaller rank); indeed in this case by [11, 6.21], $R^{\mathbf{G}^F}_{\mathbf{L}_{\dot{w}F}}$ is an isometry between $\mathcal{E}(\mathbf{L}^{\dot{w}F}, s)$ and $\mathcal{E}(\mathbf{G}^F, s)$ which is such that (i) is compatible with (iv); (ii) is not relevant here, and (ii) holds since $R^{\mathbf{G}^F}_{\mathbf{L}_{\dot{w}F}}(\chi \otimes \hat{\zeta}) = R^{\mathbf{G}^F}_{\mathbf{L}_{\dot{w}F}}(\chi) \otimes \hat{\zeta}$ (*cf.* par exemple [6, 3.8]).

When (W, F) is irreducible, it only remains the case when s is exceptional (*i.e.*, , $C_{\mathbf{G}^*}(s)$ is not included in any proper Levi subgroup, but s is not central). Suppose first that W has no family which contains one of the exceptions of proposition 6.3; then (i) completely specifies

π_s; (ii) and (iv) are not relevant here, and (iii) holds again because of $R^{\mathbf{G}^F}_{\mathbf{T}^{wF}}(\hat{s}\otimes\hat{\zeta}) = R^{\mathbf{G}^F}_{\mathbf{T}^{wF}}(\hat{s})\otimes\hat{\zeta}$.

From the explicit knowledge of root subsystems of maximal rank in root systems (which parameterize centralizers of semi-simple elements in $\mathbf{G}^*$), we see that $W(s)$ has a family which contains one of the exceptions in proposition 6.3 only if $\mathbf{G}$ is of type E_8 and $(C_{\mathbf{G}^*}(s),(\dot{w}_1F)^*)$ is of type $E_7\times A_1$, $E_6\times A_2$ or ${}^2E_6\times{}^2A_2$ (we assume that W is irreducible; it is easy to extend the arguments below to the case (W,F) irreducible by "descent of scalars"). Furthermore, in the above cases, the exceptions to proposition 6.3 are each time two characters occuring in a family where $\Gamma_{\mathcal{F}}\simeq\mathcal{S}_3$ and coincide with the list of cuspidal unipotent characters of $\mathbf{L}^{\dot{w}_2F}$ (excepted that in the case ${}^2E_6$ the character parametrized by $(1,1)\in\mathcal{M}(\mathcal{S}_3)$ is also cuspidal). We will show that in this case, (v) can be assumed to hold, is compatible with (i)-(iv), and together with them completely specifies π_s.

We first prove:

LEMMA 7.2. *Suppose that s is central in $\mathbf{L}^*$, a Levi subgroup of $\mathbf{G}^*$, and that $W(s)$ has a w_1F-stable direct product decomposition $W(s)=W_{\mathbf{L}}\times W'$. Let $w_2\in W_F(s)$, let $\rho\in\mathcal{E}(\mathbf{L}^{\dot{w}_2F},1)$, and let $\tilde{E}$ be a w_2F-class function on W; write $\tilde{E}=\tilde{E}_{\mathbf{L}}\otimes\tilde{E}'$ where $\tilde{E}_{\mathbf{L}}$ is a w_2F-class function on $W_{\mathbf{L}}$ and $\tilde{E}'$ a w_2F-class function on W'. Then:*

$$\langle R_{\tilde{E}}(s), R^{\mathbf{G}^F}_{\mathbf{L}^{\dot{w}_2F}}(\rho\otimes\hat{s})\rangle_{\mathbf{G}^F} = \tilde{E}'(w_2F)\langle R_{\tilde{E}_{\mathbf{L}}},\rho\rangle_{\mathbf{L}^{\dot{w}_2F}}$$

PROOF: We first compute the scalar product $\langle R^{\mathbf{G}^F}_{\mathbf{T}^{ww_2F}}(s), R^{\mathbf{G}^F}_{\mathbf{L}^{\dot{w}_2F}}(\rho\otimes\hat{s})\rangle$ for $w\in W(s)$. By Mackey's formula [6, 2.1(b)] it is equal to:

$$\sum_{\mathbf{L}^{\dot{w}_2F}\backslash\{v\in\mathbf{G}^F\mid {}^v\mathbf{T}\subset\mathbf{L}\}/\mathbf{T}^{ww_2F}} \langle R^{\mathbf{L}^{\dot{w}_2F}}_{v(\mathbf{T}^{ww_2F})}({}^vs),\rho\otimes\hat{s}\rangle_{\mathbf{L}^{\dot{w}_2F}} = |W_{\mathbf{L}}|^{-1}\sum_{\substack{v\in W_{\mathbf{G}}\\ vw^{w_2F}v^{-1}\in W_{\mathbf{L}}}} \langle R^{\mathbf{L}^{\dot{w}_2F}}_{\mathbf{T}^{vww_2Fv^{-1}}}({}^vs),\rho\otimes\hat{s}\rangle_{\mathbf{L}^{\dot{w}_2F}}$$

Since $\rho\otimes\hat{s}\in\mathcal{E}(\mathbf{L}^{\dot{w}_2F},s)$, the only non-zero summands are those where ${}^vs$ and s are conjugate in $\mathbf{L}^*$, i.e. equal since s is central in $\mathbf{L}^*$, and the above sum reduces to:

$$|W_{\mathbf{L}}|^{-1}\sum_{\substack{v\in W(s)\\ vw^{w_2F}v^{-1}\in W_{\mathbf{L}}}} \langle R^{\mathbf{L}^{\dot{w}_2F}}_{\mathbf{T}^{vww_2Fv^{-1}}}(1),\rho\rangle_{\mathbf{L}^{\dot{w}_2F}}.$$

Write $w = w'w_{\mathbf{L}}$ where $w' \in W'$ and $w_{\mathbf{L}} \in W_{\mathbf{L}}$. Then $vw^{w_2F}v^{-1}$ is w_2F-conjugate to $w_{\mathbf{L}}$ in $W_{\mathbf{L}}$, so $R^{\mathbf{L}^{\dot{w}_2F}}_{\mathbf{T}_{vww_2Fv^{-1}}}(1) = R^{\mathbf{L}^{\dot{w}_2F}}_{\mathbf{T}_{w_{\mathbf{L}}w_2F}}(1)$ and the condition under the sum is that w' is w_2F-conjugate to 1 in W'. So we get for the sum:

$$|W'^{w_2F}|\langle R^{\mathbf{L}^{\dot{w}_2F}}_{\mathbf{T}_{w_{\mathbf{L}}w_2F}}(1), \rho\rangle_{\mathbf{L}^{\dot{w}_2F}}.$$

From that we get:

$$\begin{aligned}
&\langle R_{\tilde{E}}(s), R^{\mathbf{G}^F}_{\mathbf{L}^{\dot{w}_2F}}(\rho\otimes\hat{s})\rangle_{\mathbf{G}^F}\\
&= |W(s)|^{-1}\sum_{w\in W(s)} \tilde{E}(ww_2F)\langle R^{\mathbf{G}^F}_{\mathbf{T}_{ww_2F}}(\hat{s}), R^{\mathbf{G}^F}_{\mathbf{L}^{\dot{w}_2F}}(\rho\otimes\hat{s})\rangle_{\mathbf{G}^F}\\
&= |W(s)|^{-1}|W_{\mathbf{L}}|^{-1}\sum_{\{w\in W(s),w_{\mathbf{L}}\in W_{\mathbf{L}}\mid w\overset{w_2F}{\sim} w_{\mathbf{L}}\in W_{\mathbf{L}}\}} \tilde{E}(w_{\mathbf{L}}w_2F)\\
&\qquad\qquad |W'^{w_2F}|\langle R^{\mathbf{L}^{\dot{w}_2F}}_{\mathbf{T}_{w_{\mathbf{L}}w_2F}}(1), \rho\rangle_{\mathbf{L}^{\dot{w}_2F}}\\
&= |W_{\mathbf{L}}|^{-1}\tilde{E}'(w_2F)\sum_{w_{\mathbf{L}}\in W_{\mathbf{L}}} \tilde{E}_{\mathbf{L}}(w_{\mathbf{L}}w_2F)\langle R^{\mathbf{L}^{\dot{w}_2F}}_{\mathbf{T}_{w_{\mathbf{L}}w_2F}}(1), \rho\rangle_{\mathbf{L}^{\dot{w}_2F}}\\
&= \tilde{E}'(w_2F)\langle R_{\tilde{E}_{\mathbf{L}}}, \rho\rangle_{\mathbf{L}^{\dot{w}_2F}}
\end{aligned}$$

■

By the general structure theorem for reductive groups, in the cases we consider we may write a $(w_1F)^*$-stable decomposition of $C_{\mathbf{G}^*}(s)$ as $\mathbf{L}^* \times_{\mathbf{T}} \mathbf{H}$ where $\mathbf{H}$ is a reductive group of type A_n with Weyl group W' and $\mathbf{T}$ is a maximal torus of $\mathbf{H}$ central in $\mathbf{L}^*$. Let ρ be a unipotent representation of $\mathbf{L}^{*(\dot{w}_2F)^*}$. Since ρ is trivial on the center of $\mathbf{L}^*$, it factors through a representation of the form $\rho'\otimes\mathrm{Id}$ of $\mathbf{L}^{*(\dot{w}_2F)^*}_{ad} \times \mathbf{T}^{(\dot{w}_2F)^*}_{ad}$ and we have:

$$R^{C_{\mathbf{G}^*}(s)^{(\dot{w}_1F)^*}_{ad}}_{\mathbf{L}^{*(\dot{w}_2F)^*}_{ad}\times\mathbf{T}^{(\dot{w}_2F)^*}_{ad}}\rho'\otimes\mathrm{Id} = \rho'\otimes R^{H^{(\dot{w}_2F)^*}_{ad}}_{\mathbf{T}^{(\dot{w}_2F)^*}_{ad}}\mathrm{Id} = \sum_{E'\in\mathrm{Irr}(W')^{w_2F}} \tilde{E}'(w_2F)\rho'\otimes\rho_{E'}$$

where $\rho_{E'}$ is the irreducible unipotent character of $\mathbf{H}$ parameterized by E' (when $\mathbf{H}$ is split we take $\tilde{E}'(w_1F) = 1$ and when $\mathbf{H}$ is a unitary group the extension $\tilde{E}'$ is determined by the above formula).

Since $\mathbf{H}$ is of type A_n, the families of W' are reduced to single characters, and the families in $\Xi(W(s))^{w_1F}$ are of the form $\mathcal{F}\otimes E'$ with $\mathcal{F}\in\Xi(W_{\mathbf{L}})^{w_1F}$ and $E'\in\mathrm{Irr}(W')^{w_1F}$, and there is a bijection that we will denote $x\mapsto x\otimes E' : \overline{\mathcal{M}}(\Gamma_{\mathcal{F}}\subset\Gamma_{\mathcal{F}}\rtimes{<}w_1F{>})\to\overline{\mathcal{M}}(\Gamma_{\mathcal{F}\otimes E'}\subset\Gamma_{\mathcal{F}\otimes E'}\rtimes{<}w_1F{>})$. If we denote by ρ'_x the unipotent character in the set $\mathcal{E}(\mathbf{L}_{ad}^{*(\dot w_2F)^*},1,\mathcal{F})$ corresponding to $x\in\overline{\mathcal{M}}(\Gamma_{\mathcal{F}}\subset\Gamma_{\mathcal{F}}\rtimes{<}w_1F{>})$ we get:

$$R^{C_{\mathbf{G}^*}(s)_{ad}^{(\dot w_1F)^*}}_{\mathbf{L}_{ad}^{*(\dot w_2F)^*}\times\mathbf{T}_{ad}^{(\dot w_2F)^*}}\rho'_x\otimes\mathrm{Id}=\sum_{E'\in\mathrm{Irr}(W')^{w_2F}}\tilde E'(w_2F)\rho'_{x\otimes E'}$$

And using (vi) of the theorem (which holds in $C_{\mathbf{G}^*}(s)$ by induction) we get the analogous result in $C_{\mathbf{G}^*}(s)^{(\dot w_1F)^*}$:

$$R^{C_{\mathbf{G}^*}(s)^{(\dot w_1F)^*}}_{\mathbf{L}^{*(\dot w_2F)^*}}(\rho_x)=\sum_{E'\in\mathrm{Irr}(W')^{w_1F}}\tilde E'(w_2F)\rho_{x\otimes E'}$$

To prove (v) we will show that when ρ_x is cuspidal, $R^{\mathbf{G}^F}_{\mathbf{L}^{\dot w_2F}}(\rho_x\otimes\hat s)$ is of the form $\sum_{E'\in\mathrm{Irr}(W')^{w_1F}}\tilde E'(w_2F)\rho^s_{x\otimes E'}$ where $\rho^s_{x\otimes E'}\in\mathcal{E}(\mathbf{G}^F,s,\mathcal{F}\otimes E')$ and

$$\langle R_{\tilde E}(s),\rho^s_{x\otimes E'}\rangle_{\mathbf{G}^F}=(-1)^{l(w_1)}\langle R_{\tilde E},\rho_{x\otimes E'}\rangle_{C_{\mathbf{G}^*}(s)^{(\dot w_1F)^*}}$$

for any $E\in(\mathcal{F}\otimes E')\check{}$ (then if we define π_s by $\pi_s(\rho^s_{x\otimes E'})=\rho_{x\otimes E'}$ (i) and (v) are satisfied; (iii) is clearly satisfied also and (ii) and (iv) do not apply here, so we will have finished with the case (W,F) irreducible).

In the cases where $(C_{\mathbf{G}^*}(s),(\dot w_1F)^*)$ is of type $E_6\times A_2$ or $E_7\times A_1$, we have $w_1=1$ and $\mathbf{L}$ is a Levi subgroup of a rational parabolic subgroup of $\mathbf{G}$ so, by Howlett-Lehrer's theory, we know that the decomposition of $R^{\mathbf{G}^F}_{\mathbf{L}^F}(\rho\otimes\hat s)$ for a cuspidal $\rho\in\mathcal{E}(\mathbf{L}^{*F^*},1)$ (which corresponds to a cuspidal $\rho\otimes\hat s\in\mathcal{E}(\mathbf{L}^F,s))$ is of the form $\sum_{\chi\in\mathrm{Irr}(W(\rho))}\chi(1)\rho_\chi$. When $C_{\mathbf{G}^*}(s)$ is of type $E_6\times A_2$, $W(\rho)$ is equal to W' since $N_W(W_{\mathbf{L}})=W(s)\rtimes{<}w{>}$ where w is an element of order 2 acting on $\mathbf{L}$ by the non-trivial diagram automorphism of E_6 and w cannot fix ρ since it maps s to s^{-1}. Similarly $W(\rho)$ is W' if $C_{\mathbf{G}^*}(s)$ is of type $E_7\times A_1$ since in that case $N_W(W_{\mathbf{L}})=W(s)$. So we know that $R^{\mathbf{G}^F}_{\mathbf{L}^F}(\rho\otimes\hat s)$ has a decomposition of the form $\sum_{E'\in\mathrm{Irr}(W')}E'(1)\rho^s_{x\otimes E'}$; it remains to show that $\langle R_{\tilde E}(s),\rho^s_{x\otimes E'}\rangle_{\mathbf{G}^F}=\langle R_{\tilde E},\rho_{x\otimes E'}\rangle_{C_{\mathbf{G}^*}(s)^{F^*}}(=\langle R_{\tilde E_{\mathbf{L}}},\rho_x\rangle_{\mathbf{L}^{*F^*}})$.

Consider the projection $\rho^{E'}$ of $R_{\mathbf{L}^F}^{\mathbf{G}^F}(\rho_x \otimes \hat{s})$ on $\overline{\mathbb{Q}}_l\mathcal{E}(\mathbf{G}^F, s, \mathcal{F} \otimes E')$; by lemma 7.2, we have $\langle \rho^{E'}, R_{\tilde{E}}(s) \rangle_{\mathbf{G}^F} = E'(1)\langle \rho_x, R_{\tilde{E}_{\mathbf{L}}} \rangle_{\mathbf{L}^F}$ so $\rho^{E'}$ is not 0; furthermore, since $R_{\mathbf{L}^F}^{\mathbf{G}^F}(\rho_x \otimes \hat{s})$ has $|\operatorname{Irr}(W')|$ isotypic components, $\rho^{E'}$ must be one of them. Looking at the table of Fourier coefficients for $\mathcal{M}(\mathbf{S}_3)$ which gives us $\langle \rho_x, R_{\tilde{E}_{\mathbf{L}}} \rangle_{\mathbf{L}^F}$ we see that $|\rho^{E'}| \geq 3/4E'(1)$ which implies $|\rho^{E'}| \geq E'(1)$ since both are integers and $E'(1) \leq 2$; but $\sum_{E'} |\rho^{E'}|^2 = |R_{\mathbf{L}^F}^{\mathbf{G}^F}(\rho_x \otimes \hat{s})|^2 = \sum_{E'} E'(1)^2$ so $\rho^{E'}$ must be of the form $E'(1)\rho^s_{x\otimes E'}$ with $\rho^s_{x\otimes E'}$ irreducible.

Similarly, in the case where $(C_{\mathbf{G}^*}(s), (\dot{w}_1F)^*)$ is of type ${}^2E_6 \times {}^2A_2$, it is enough to show that

$$\langle R_{\mathbf{L}_{\dot{w}_2 F}}^{\mathbf{G}^F}(\rho_x \otimes \hat{s}), R_{\mathbf{L}_{\dot{w}_2 F}}^{\mathbf{G}^F}(\rho_x \otimes \hat{s}) \rangle_{\mathbf{G}^F} = 2$$

(then for each of the two elements $E' \in \operatorname{Irr}(W')^{w_2 F}$ we can similarly deduce by using 7.2 and the table of fourier coeffcients for $\mathcal{M}(\mathcal{S}_3)$ that the projection $\rho^{E'}$ of $R_{\mathbf{L}_{\dot{w}_2 F}}^{\mathbf{G}^F}(\rho_x \otimes \hat{s})$ on $\overline{\mathbb{Q}}_l\mathcal{E}(\mathbf{G}^F, s, \mathcal{F} \otimes E')$ is an irreducible character with sign $\tilde{E}'(w_2F)$).

We have $w_2 = w_0^{\mathbf{L}} w_0^{\mathbf{M}}$ where $w_0^{\mathbf{L}}$ is the longest element of $W_{\mathbf{L}}$ and $w_0^{\mathbf{M}}$ is the longest element of the standard Levi subgroup $\mathbf{M}$ of $\mathbf{G}$ of type E_7 which contains $\mathbf{L}$. We will consider $R_{\mathbf{L}_{\dot{w}_2 F}}^{\mathbf{M}^F}(\rho_x \otimes \hat{s})$, which is an irreducible representation since $C_{\mathbf{M}^*}(s) = C_{\mathbf{L}^*}(s)$ (in that case $R_{\mathbf{L}_{\dot{w}_2 F}}^{\mathbf{M}^F}$ is an isometry, *cf.* [10, proof of 7.9]) and we will conclude by proving that for any $\chi \in \mathcal{E}(\mathbf{M}^F, s)$, we have $\langle R_{\mathbf{M}^F}^{\mathbf{G}^F}\chi, R_{\mathbf{M}^F}^{\mathbf{G}^F}\chi \rangle_{\mathbf{G}^F} = 2$. Indeed, $\langle R_{\mathbf{M}^F}^{\mathbf{G}^F}\chi, R_{\mathbf{M}^F}^{\mathbf{G}^F}\chi \rangle_{\mathbf{G}^F} = \langle \chi, {}^*R_{\mathbf{M}^F}^{\mathbf{G}^F} R_{\mathbf{M}^F}^{\mathbf{G}^F}\chi \rangle_{\mathbf{M}^F}$ and by Mackey's formula (which we may apply since $\mathbf{M}$ is included in a rational parabolic subgroup):

$${}^*R_{\mathbf{M}^F}^{\mathbf{G}^F} R_{\mathbf{M}^F}^{\mathbf{G}^F}\chi = \sum_{w \in W_{\mathbf{M}} \backslash W / W_{\mathbf{M}}} R_{({}^{w^{-1}}\mathbf{M} \cap \mathbf{M})^F}^{\mathbf{M}^F} \circ \operatorname{ad} w^{-1} \circ {}^*R_{(\mathbf{M} \cap {}^w\mathbf{M})^F}^{\mathbf{M}^F}\chi$$

Here is a list of reduced representatives $w \in W_{\mathbf{M}} \backslash W / W_{\mathbf{M}}$, and, below each, the value of $\mathbf{M} \cap {}^w\mathbf{M}$:

$w =$	1	$w_0^{\mathbf{M}} w_0^{\mathbf{G}}$	s_8	$w_0^{\mathbf{G}} w_0^{\mathbf{M}} s_8 w_0^{\mathbf{M}} w_0^{\mathbf{L}}$	$w_0^{\mathbf{G}} w_0^{\mathbf{M}} s_8 w_0^{\mathbf{M}} w_0^{\mathbf{D}}$
$\mathbf{M} \cap {}^w\mathbf{M} =$	$\mathbf{M}$	$\mathbf{M}$	$\mathbf{L}$	$\mathbf{L}$	$\mathbf{D}$

where s_8 is the generating reflection in $W - W_{\mathbf{M}}$ and where $\mathbf{D}$ is the standard Levi subgroup of $\mathbf{G}$ of type D_6. Since $w_0^{\mathbf{M}} w_0^{\mathbf{G}}$ acts like the identity on $\mathbf{M}$, the first two values of w give a contribution of 2χ to ${}^*R_{\mathbf{M}^F}^{\mathbf{G}^F} R_{\mathbf{M}^F}^{\mathbf{G}^F}\chi$. We have to prove that the other terms are 0; for ${}^*R_{(\mathbf{M}\cap {}^w\mathbf{M})^F}^{\mathbf{M}^F}\chi$ to be non-zero, s must have a $W_{\mathbf{M}}$-conjugate $s' \in (\mathbf{M}\cap {}^w\mathbf{M})$ such that $Z_F(s')$ meets $W_{\mathbf{M}} \cap {}^wW_{\mathbf{M}}$. There is no conjugate of s in $\mathbf{D}$, so ${}^*R_{\mathbf{D}^F}^{\mathbf{M}^F}\chi = 0$. Now, $Z_F(s) \cap W_{\mathbf{M}} = W_{\mathbf{L}} w_0^{\mathbf{M}}$; when $\mathbf{M}\cap {}^w\mathbf{M} = \mathbf{L}$ we must find $v \in W_{\mathbf{M}}$ such that $W_{\mathbf{L}} \cap {}^v(W_{\mathbf{L}} w_0^{\mathbf{M}}) \neq \emptyset \iff W_{\mathbf{L}} v W_{\mathbf{L}} = W_{\mathbf{L}} v W_{\mathbf{L}} w_0^{\mathbf{M}}$ but it can be checked that there is no element of $W_{\mathbf{L}}\backslash W_{\mathbf{M}}/W_{\mathbf{L}}$ invariant by multiplication by $w_0^{\mathbf{M}}$.

We now consider the general case of $\mathbf{G}$ connected reductive with connected center and use (vi) and (vii).

First we check that (vi) and (vii) hold in the previous situations: (vii) is irrelevant in the case (W, F) irreducible; (vi) holds when (i) specifies π_s by 6.3; (vi) holds when (ii) specifies π_s since φ induces an isomorphism of the Deligne-Lusztig varietes X_w for $\mathbf{G}$ et $\mathbf{G}_1$, and of the Hecke algebras; when (iii) specifies π_s, (vi) clearly holds since $\hat{\zeta}$ factors through $\mathbf{G}_1$; and when (iv) or (v) specify π_s, (vi) holds since φ commutes with a Lusztig functor $R_{\mathbf{L}}^{\mathbf{G}}$ (*cf.* 9.2).

Now, in general, we can find an epimorphism with kernel a central torus $\prod \mathbf{G}_i \to \mathbf{G}$, where each $\mathbf{G}_i$ is a connected reductive algebraic group with a connected center, with a Weyl group W_i such that (W_i, F) is irreducible; so applying (vi) and (vii) π_s is completely determined from its value in the case (W_i, F) irreducible. We next show that the π_s thus constructed does not depend on the particular choice of $\prod \mathbf{G}_i \to \mathbf{G}$. If we have another such morphism $\prod \mathbf{H}_i \to \mathbf{G}$, then $\mathbf{H}_i$ and $\mathbf{G}_i$ have the same derived group and we can embed both in some $\mathbf{K}_i$ with still the same derived group such that the maps $\prod \mathbf{G}_i \to \mathbf{G}$ and $\prod \mathbf{H}_i \to \mathbf{G}$ factor through $\prod \mathbf{K}_i$. It is enough to show that the parametrization given through $\prod \mathbf{G}_i$ factors through $\prod \mathbf{K}_i$, and that comes from the unicity in the case (W, F) irreducible and the fact that (vi) holds in that case so the embedding $\mathbf{G}_i \to \mathbf{K}_i$ is compatible with π_{s_i}.

The last thing we have to check is that the parametrization π_s defined above satisfies (i) to (v). It is clear for the parametrization in $\prod \mathbf{G}_i$, and the remarks above proving the compatibility of (vi) with (i)-(v) in the case (W, F) ireducible can be applied here too to prove that (i)-(v) are preserved via $\prod \mathbf{G}_i \to \mathbf{G}$. ■

COROLLARY 7.3. *Let $w \in W$ be an element such that ${}^w\Phi_s^+ \subset \Phi^+$.*

Then the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{E}(\mathbf{G}^F, s) & \xrightarrow{\pi_s} & \mathcal{E}(C_{\mathbf{G}^*}(s)^{(\dot{w}_1 F)^*}, 1) \\
\| & & \downarrow \scriptstyle{\mathrm{ad}\,\dot{w}^*} \\
\mathcal{E}(\mathbf{G}^F, {}^w s) & \xrightarrow{\pi_{{}^w s}} & \mathcal{E}(C_{\mathbf{G}^*}({}^w s)^{(\dot{w}\dot{w}_1 F \dot{w}^{-1})^*}, 1)
\end{array}$$

PROOF: Note first that this corollary is a special case of 7.1(vi) when $w \in W^F$ (taking then $\varphi = \mathrm{ad}\,\dot{w}^*$; the proof will be similar to that of 7.1(vi). We have to check that $\mathrm{ad}\,\dot{w} \circ \pi_s$ satisfies (i)–(vii) of 7.1 applied replacing s by ${}^w s$.

(i) results from 9.2 applied in $\mathbf{G}^*$ with $\varphi = \mathrm{ad}\,\dot{w}^*$ which gives

$$R_{\mathbf{T}_{(v w_1 F)^*}}^{C_{\mathbf{G}^*}(s)^{(\dot{w}_1 F)^*}} \circ \mathrm{ad}\,\dot{w}(1) = R_{\mathbf{T}_{(w v w_1 F w^{-1})^*}}^{C_{\mathbf{G}^*}({}^w s)^{(\dot{w}\dot{w}_1 F \dot{w}^{-1})^*}} (1)$$

for any $v \in W(s)$ and from $R_{\mathbf{T}_{w v w_1 F w^{-1}}}^{\mathbf{G}^F}({}^w s) = R_{\mathbf{T}_{v w_1 F}}^{\mathbf{G}^F}(\hat{s})$.

(ii) and (iii) are trivial here (if s is central then $w = 1$).

To prove that (iv) holds we have to prove that the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{E}(\mathbf{G}^F, {}^w s) & \xrightarrow{\mathrm{ad}\,\dot{w}^* \circ \pi_s} & \mathcal{E}(C_{\mathbf{G}^*}({}^w s)^{(\dot{w}\dot{w}_1 F \dot{w}^{-1})^*}, 1) \\
\uparrow \scriptstyle{R_{{}^w\mathbf{L}_{w v F w^{-1}}}^{\mathbf{G}^F}} & & \| \\
\mathcal{E}({}^{\dot{w}}\mathbf{L}^{w v F w^{-1}}, {}^w s) & \xrightarrow{\pi_{{}^{w^*} s}} & \mathcal{E}(C_{{}^{\dot{w}}\mathbf{L}^*}({}^w s)^{(\dot{w}\dot{w}_1 F \dot{w}^{-1})^*}, 1)
\end{array}$$

where $\mathbf{L}$ is a standard, vF-stable Levi subgroup of $\mathbf{G}$ such that $\mathbf{L}^* \supset C_{\mathbf{G}^*}(s)$ and where v is the $W_{\mathbf{L}}$-reduced element in $W_{\mathbf{L}} w_1$. But since $R_{\mathbf{L}_{\dot{v}F}}^{\mathbf{G}^F} \circ \mathrm{ad}\,\dot{w}^{-1} = R_{{}^w\mathbf{L}_{w v F w^{-1}}}^{\mathbf{G}^F}$ the diagram above is the composed of the diagram given by 7.1(iv) (with v for w) and of the following diagram given by 7.1(vi) with $\varphi = \mathrm{ad}\,\dot{w}$:

$$\begin{array}{ccc}
\mathcal{E}(\mathbf{L}^{\dot{v}F}, s) & \xrightarrow{\pi_s} & \mathcal{E}(C_{\mathbf{L}^*}(s)^{(\dot{w}_1 F)^*}, 1) \\
\uparrow \scriptstyle{\mathrm{ad}\,\dot{w}^{-1}} & & \downarrow \scriptstyle{\mathrm{ad}\,\dot{w}^{-1}} \\
\mathcal{E}({}^w\mathbf{L}^{\dot{w}\dot{v}F\dot{w}^{-1}}, {}^w s) & \xrightarrow{\pi_{{}^w s}} & \mathcal{E}(C_{{}^w\mathbf{L}^*}({}^w s)^{(\dot{w}\dot{w}_1 F \dot{w}^{-1})^*}, 1)
\end{array}$$

The proof for (v) is entirely similar; and (vi) and (vii) are quite obvious. ■

III

The aim of this part is to give a general conjecture about the action of Shintani descent on the parametrizations of parts I and II and to prove some results in the direction of that conjecture. We begin by recalling some facts about Shintani descent and the definition of "F-twisted induction" (*cf.* [4], [5] and [7]).

8. Shintani descent, F'-twisted induction. Let $\mathbf{G}$ be a connected reductive algebraic group defined over an algebraically closed field $\overline{\mathbb{F}}_q$. We consider two rational structures on $\mathbf{G}$ defined over some finite subfields of $\overline{\mathbb{F}}_q$ and given by two Frobenius endomorphisms F and F' which commute. In this background, we recall the definition of Shintani descent.

8.1 DEFINITION.

(i) *We denote by* $\mathrm{N}_{F'/F}$ *the map from F-classes on* $\mathbf{G}^{F'}$ *to F'-classes on* $\mathbf{G}^F$ *which maps the F-class of* $g \in \mathbf{G}^{F'}$ *to the F'-class in* $\mathbf{G}^F$ *of* $h^{-1}\,{}^{F'}h \in \mathbf{G}^F$ *where* $h \in \mathbf{G}$ *is such that* $h^{-1}\,{}^{F}h = g$.

(ii) *We call Shintani descent the map* $\mathrm{Sh}_{F'/F}$ *from F-class functions on* $\mathbf{G}^{F'}$ *to F'-class functions on* $\mathbf{G}^F$ *defined by* $\mathrm{Sh}_{F'/F} = {}^t\mathrm{N}_{F'/F}^{-1}$.

It is easy to show that Shintani descent is an isometry for the usual scalar products on $\mathcal{C}(\mathbf{G}^F/F')$ and $\mathcal{C}(\mathbf{G}^{F'}/F)$.

Let us now assume that we are given a pair $\mathbf{T} \subset \mathbf{B}$ of a maximal torus and a Borel subgroup both F and F'-stable. Let $\mathbf{L}$ be a standard Levi subgroup stable under F' and let w be an element of $W^{F'}$ such that $\mathbf{L}$ is stabilized by $\dot{w}F$ where $\dot{w}$ is a fixed representative of w in the normalizer $N_{\mathbf{G}}(\mathbf{T})^{F'}$ (note that $\dot{w}F$ and F' commute). With this background we want to define the F'-twisted induction which will be denoted by $R^{\mathbf{G}^F/F'}_{\mathbf{L}^{\dot{w}F}/F'}$ from the space $\mathcal{C}(\mathbf{L}^{\dot{w}F}/F')$ to the space $\mathcal{C}(\mathbf{G}^F/F')$. We first define it for the elements of a basis of $\mathcal{C}(\mathbf{L}^{\dot{w}F}/F')$: let π be an F'-invariant irreducible character of $\mathbf{L}^{\dot{w}F}$ and $\tilde{\pi}$ be an extension of π to the semi-direct product $\mathbf{L}^{\dot{w}F} \rtimes \langle F' \rangle$; let $[\tilde{\pi}]$ be a complex vector space affording a representation of $\mathbf{L}^{\dot{w}F}$ with character $\tilde{\pi}$; let S be the algebraic variety

$$S = \{x \in \mathbf{G} \mid x^{-1}.{}^{F}x \in \mathbf{U}.\dot{w}.{}^{F}\mathbf{U}\}$$

(the endomorphism F' acts naturally on S). We define the F'-twisted induction of $\tilde{\pi}$ (actually of the restriction of $\tilde{\pi}$ to $\mathbf{L}^{\dot{w}F}.F'$; we shall not mention the restriction when the context is unambiguous) as being the restriction to $\mathbf{G}^F.F'$ of the character of the representation of $\mathbf{G}^F \rtimes \langle F' \rangle$ on the space $(H_c^*(S) \otimes [\tilde{\pi}])^{\mathbf{L}^{\dot{w}F}}$, where F' acts naturally on $H_c^*(S)$ and acts as defined on $[\tilde{\pi}]$. We immediately get:

8.2 PROPOSITION. *With the preceding notations we have:*

$$\mathrm{Trace}(gF' \mid R_{\mathbf{L}^{\dot{w}F}/F'}^{\mathbf{G}^F/F'}(\tilde{\pi})) = |\mathbf{L}^{\dot{w}F}|^{-1} \sum_{l \in \mathbf{L}^{\dot{w}F}} \tilde{\pi}(lF')\,\mathrm{Trace}((g,l)F' \mid H_c^*(S)).$$

This allows us to extend the definition to the whole space of F'-class functions:

8.3 DEFINITION. *We call F'-twisted induction the map*

$$R_{\mathbf{L}^{\dot{w}F}/F'}^{\mathbf{G}^F/F'} : \mathcal{C}(\mathbf{L}^{\dot{w}F}/F') \to \mathcal{C}(\mathbf{G}^F/F')$$

which extends by linearity the formula of proposition 8.2.

9. Isogenies.

In this section we show that induction and F'-twisted induction from a Levi subroup behave well with respect to isogenies. We will consider a morphism $\varphi : \mathbf{G} \to \mathbf{G}_1$ of connected reductive algebraic groups such that φ induces an isogeny from the derived group of $\mathbf{G}$ to that of $\mathbf{G}_1$. In this situation, if $\mathbf{T}$ is a given maximal torus of $\mathbf{G}$, there is a unique maximal torus $\mathbf{T}_1$ of $\mathbf{G}_1$ such that $\varphi(\mathbf{T}) \subset \mathbf{T}_1$. In fact, we have $\mathbf{T}_1 = \varphi(\mathbf{T}).Z\mathbf{G}_1$ since $\mathbf{G}_1 = \varphi(\mathbf{G}).Z\mathbf{G}_1$; we have also $\mathbf{T}_1 \cap \varphi(\mathbf{G}) = \varphi(\mathbf{T})$. In addition, we have: $\operatorname{Ker}\varphi \subset Z\mathbf{G}$ and $N_{\mathbf{G}_1}(\mathbf{T}_1) = \varphi(N_{\mathbf{G}}(\mathbf{T})).Z\mathbf{G}_1$ so φ defines an isomorphism of Weyl groups: $W \to W_1$.

Let F and F' (resp. F_1 and F_1') be two commuting Frobenius endomorphisms on $\mathbf{G}$ (resp. on $\mathbf{G}_1$) with F' (resp. F_1') split such that:

$$\varphi \circ F' = F_1' \circ \varphi \text{ and } \varphi \circ F = F_1 \circ \varphi.$$

Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ such that the pair $\mathbf{T} \subset \mathbf{B}$ is F and F'-stable; then it is clear that the pair $\mathbf{T}_1 \subset \mathbf{B}_1$ is also F' and F_1'-stable, where $\mathbf{B}_1 = \varphi(\mathbf{B}).Z\mathbf{G}_1$.

9.1 THEOREM. *In the above situation, assume in addition that* $\operatorname{Ker}\varphi$ *is connected. Let* $\dot w \in N_{\mathbf G}(\mathbf T)^{F'}$, *let* $\dot w_1 = \varphi(\dot w)$, *and let* $\mathbf L_1$ *be a standard Levi subgroup of* $\mathbf G_1$ *which is* w_1F_1 *and* F_1'*-stable. Then for any* F_1'*-class function* $\tilde\pi_1$ *on* $\mathbf L_1^{\dot w_1F_1}$ *we have:*

$$R^{\mathbf G_1^{F_1}/F_1'}_{\mathbf L_1^{\dot w_1F_1}/F_1'}(\tilde\pi_1)\circ\varphi = R^{\mathbf G^F/F'}_{\mathbf L^{\dot wF}/F'}(\tilde\pi_1\circ\varphi)$$

where $\mathbf L = \varphi^{-1}(\mathbf L_1)$.

PROOF: In view of 8.2 and the Lefschetz formula we have to show the equality of:

$$|\mathbf L^{\dot wF}|^{-1}\sum_{l\in\mathbf L^{\dot wF}}\tilde\pi(l)\,|\{x\in\mathbf G\mid x^{-1}.{}^Fx\in\mathbf U\dot w^F\mathbf U \text{ and } g^{F'}x = xl\}| \quad (1)$$

(where $\tilde\pi = \tilde\pi_1\circ\varphi$, $g\in\mathbf G^F$ and $\mathbf U$ is the unipotent radical of a parabolic subgroup containing $\mathbf L$) with the analogous expression for $\mathbf G_1$:

$$|\mathbf L_1^{\dot w_1F_1}|^{-1}\sum_{l_1\in\mathbf L_1^{\dot w_1F_1}}\tilde\pi_1(l_1)$$
$$|\{x\in\mathbf G_1\mid x^{-1}.{}^{F_1}x\in\mathbf U_1\dot w_1{}^{F_1}\mathbf U_1 \text{ and } \varphi(g)^{F_1'}x = xl_1\}| \quad (2)$$

The equalities $\mathbf G_1 = \mathbf L_1\varphi(\mathbf G)$ (since $Z\mathbf G_1\subset\mathbf L_1$) and $\mathbf L_1\cap\varphi(\mathbf G) = \varphi(\mathbf L)$ show that $\mathbf G_1/\varphi(\mathbf G)\simeq\mathbf L_1/(\mathbf L_1\cap\varphi(\mathbf G))\simeq\mathbf L_1/\varphi(\mathbf L)$ and this isomorphism is compatible with w_1F_1 since $\varphi(\mathbf L)$ is w_1F_1-stable (because $\mathbf L$ is wF-stable).

So we get $(\mathbf G_1/\varphi(\mathbf G))^{w_1F_1}\simeq(\mathbf L_1/\varphi(\mathbf L))^{w_1F_1}$ and the first group is also isomorphic to $(\mathbf G_1/\varphi(\mathbf G))^{F_1}$ because $\mathbf G_1$ is central modulo $\varphi(\mathbf G)$. As $\varphi(\mathbf L)$ is connected we have: $(\mathbf L_1/\varphi(\mathbf L))^{w_1F_1}\simeq\mathbf L_1^{\dot w_1F_1}/\varphi(\mathbf L)^{\dot w_1F_1}$. Now $\mathbf U_1 = \varphi(\mathbf U)$ so that $\mathbf U_1\dot w_1{}^{F_1}\mathbf U_1\subset\varphi(\mathbf G)$; so any $x_1\in\mathbf G_1$ such that $x_1^{-1}.{}^{F_1}x_1\in\mathbf U_1\dot w_1{}^{F_1}\mathbf U_1$ is F_1-stable modulo $\varphi(\mathbf G)$ and since we have $(\mathbf G_1/\varphi(\mathbf G))^{F_1} = \mathbf L_1^{\dot w_1F_1}/\varphi(\mathbf L)^{\dot w_1F_1}$ we can find an element $\lambda\in\mathbf L_1^{\dot w_1F_1}$ such that $y_1 = x_1\lambda$ is in $\varphi(\mathbf G)$. Then $y_1^{-1}.{}^{F_1}y_1\in\mathbf U_1\dot w_1{}^{F_1}\mathbf U_1$ and there are $|\varphi(\mathbf L)^{\dot w_1F_1}|$ elements y_1 corresponding to $|\mathbf L_1^{\dot w_1F_1}|$ elements x_1. Moreover x_1 satisfies $x_1^{-1}\varphi(g)^{F_1'}x_1 = l_1\in\mathbf L_1^{\dot w_1F_1}$ if and only if y_1 satisfies $y_1^{-1}\varphi(g)^{F_1'}y_1 = \lambda^{-1}l_1{}^{F_1'}\lambda\in\mathbf L_1^{\dot w_1F_1}$ and, by assumption, $\tilde\pi_1(l_1) = \tilde\pi_1(\lambda^{-1}l_1{}^{F_1'}\lambda)$. So expression (2) is equal to:

$$|\varphi(\mathbf L)^{\dot w_1F_1}|^{-1}\sum_{l_1\in\mathbf L_1^{\dot w_1F_1}}\tilde\pi_1(l_1)$$
$$|\{y_1\in\varphi(\mathbf G)\mid y_1^{-1}.{}^{F_1}y_1\in\mathbf U_1\dot w_1{}^{F_1}\mathbf U_1 \text{ and } \varphi(g)^{F_1}y_1 = y_1l_1\}|. \quad (3)$$

We show now that the map $(x,l)\mapsto(\varphi(x),\varphi(l))$:

$$\{(x,l)\in\mathbf{G}\times\mathbf{L}^{\dot w F}\mid x^{-1}.{}^Fx\in\mathbf{U}\dot w^F\mathbf{U}\text{ and }g^{F'}x=xl\}\to$$
$$\{(y_1,l_1)\in\varphi(\mathbf{G})\times\mathbf{L}_1^{\dot w_1F_1}\mid y_1^{-1}.{}^{F_1}y_1\in\mathbf{U}_1\dot w_1{}^{F_1}\mathbf{U}_1\text{ and }\varphi(g)^{F_1'}y_1=y_1l_1\}$$

is surjective. If (y_1,l_1) is in the right-hand side set then $y_1=\varphi(y)$ with $y^{-1}.{}^Fy\in\mathbf{U}\dot w^F\mathbf{U}z$ for some $z\in\operatorname{Ker}\varphi$. As $\operatorname{Ker}\varphi$ is connected (so that Lang's theorem holds in $\operatorname{Ker}\varphi$) we can modify y to get another element such that $\varphi(y)=y_1$ and $y^{-1}.{}^Fy\in\mathbf{U}\dot w^F\mathbf{U}$. Then we have $\varphi(y^{-1}g^{F'}y)=l_1\in\varphi(\mathbf{L})^{\dot w_1F_1}$ which implies $y^{-1}g^{F'}y\in\mathbf{L}^{\dot wF}\operatorname{Ker}\varphi$ since $\varphi(\mathbf{L})^{w_1F_1}\simeq(\mathbf{L}/\operatorname{Ker}\varphi)^{wF}\simeq\mathbf{L}^{\dot wF}/(\operatorname{Ker}\varphi)^{wF}$ as $\operatorname{Ker}\varphi$ is connected. Let us write $y^{-1}g^{F'}y=lz$ with $l\in\mathbf{L}^{\dot wF}$ and $z\in\operatorname{Ker}\varphi$; then:

$$\begin{aligned}{}^Fz={}^Fl^{-1}\,{}^Fy^{-1}\,g^{F'}\,{}^Fy&={}^Fl^{-1}\,{}^Fy^{-1}y(y^{-1}g^{F'}y)^{F'}(y^{-1}\,{}^Fy)\\&={}^Fl^{-1}({}^Fy^{-1}.y)lz^{F'}(y^{-1}\,{}^Fy).\end{aligned}$$

Setting $a=y^{-1}\,{}^Fy\in\mathbf{U}\dot w^F\mathbf{U}$ we get

$$z^{-1}\,{}^Fz={}^Fl^{-1}a^{-1}l^{F'}a=b^{-1}\,{}^{wF}l^{-1}l^{F'}a$$

with $b\in\mathbf{U}\dot w^F\mathbf{U}$. But ${}^{wF}l=l$, so $z^{-1}\,{}^Fzb={}^{F'}a\in\mathbf{U}\dot w^F\mathbf{U}$, which implies $z^{-1}\,{}^Fz=1$ (using the Bruhat decomposition), so the element (y,lz) is a preimage of (y_1,l_1). The fibers of the above map are clearly all of cardinality $|(\operatorname{Ker}\varphi)^F|$. So expression (3) becomes:

$$|\varphi(\mathbf{L})^{\dot w_1F_1}|^{-1}|(\operatorname{Ker}\varphi)^F|^{-1}\sum_{l\in\mathbf{L}^{\dot wF}}\tilde\pi(l)|\{y\in\mathbf{G}\mid y^{-1}.{}^Fy\in\mathbf{U}w^F\mathbf{U}\text{ and }g^{F'}y=yl\}$$

which is the desired result since $|\mathbf{L}^{\dot wF}|=|(\operatorname{Ker}\varphi)^F||\varphi(\mathbf{L})^{\dot w_1F_1}|$. ∎

9.2 Corollary. *Under the assumptions of theorem 9.1 (except the existence of F' and F_1'), for any $\pi_1\in\operatorname{Irr}(\mathbf{L}_1^{\dot w_1F_1})$ we have:*

$$R^{\mathbf{G}_1^{F_1}}_{\mathbf{L}_1^{\dot w_1F_1}}(\pi_1)\circ\varphi=R^{\mathbf{G}^F}_{\mathbf{L}^{\dot wF}}(\pi_1\circ\varphi).$$

Proof: Let d be such that F_1^d and F^d are split and such that $\mathbf{L}_1$ and π_1 are fixed by F_1^d. Then, for any $m\in\mathbb{N}$, we can apply the theorem with $F'=F^{dm}$ and $F_1'=F_1^{dm}$. Applying the standard process of "taking the limit for $m=0$" (*cf.* [3, 3.3 and 4.1.2] and [5, 3.2]) gives the result. ∎

If $\mathbf{G}^*$ (resp. $\mathbf{G}_1^*$) is a group dual to $\mathbf{G}$ (resp. $\mathbf{G}_1$), a morphism φ as above induces a morphism $\varphi^*:\mathbf{G}_1^*\to\mathbf{G}^*$ with a central kernel, which also induces an isogeny for the derived groups, and an isomorphism of the Weyl groups.

9.3 COROLLARY. *Under the above hypotheses, assume in addition that* $Z\mathbf{G}$ *and* $Z\mathbf{G}_1$ *are connected. Then for any* $s_1 \in \mathbf{T}_1^*$ *such that* $W_{F_1}(s_1)$ *is non empty we have:*

(i) Let $s = \varphi^*(s_1)$*; then* φ^* *induces a morphism* $C_{\mathbf{G}_1^*}(s_1) \to C_{\mathbf{G}^*}(s)$ *which is an isogeny on the derived groups.*

We denote $\mathcal{F} \mapsto \varphi^*(\mathcal{F})$ the induced bijection from $\Xi(W(s_1))$ to $\Xi(W(s))$. Then

(ii) $\chi \mapsto \chi \circ \varphi$ *defines a bijection:* $\chi \in \mathcal{E}(\mathbf{G}_1^{F_1}, s_1, \mathcal{F}) \to \mathcal{E}(\mathbf{G}^F, s, \varphi^*(\mathcal{F}))$.

PROOF: Since $Z\mathbf{G}_1$ (resp. $Z\mathbf{G}$) is connected, the group $W(s_1)$ (resp. $W(s)$) is generated by reflections corresponding to the roots in Φ_{s_1} (resp. Φ_s), so to prove (i) we have to prove that $\alpha_1 \in \Phi_{s_1}$ (i.e. $\alpha_1(s_1) = 1$) is equivalent to $\alpha(s) = 1$ where α corresponds to α_1 by φ^*. This is true since (by definition of isogenies) $\alpha \circ \varphi^* = q(\alpha)\alpha_1$ for some power $q(\alpha)$ of the characteristic.

We now identify $W(s)$ to $W(s_1)$ by φ^*. To prove (ii), since by 9.2 we have $R_{\mathbf{T}_1^{wF_1}}^{\mathbf{G}_1^{F_1}}(\hat{s}_1) \circ \varphi = R_{\mathbf{T}^{wF}}^{\mathbf{G}^F}(\hat{s})$ for any $w \in Z_F(s)$, it is enough to prove that $\chi \circ \varphi$ is irreducible for any $\chi \in \mathcal{E}(\mathbf{G}_1^{F_1}, s_1)$.

If χ and ψ are characters in $\operatorname{Irr}(\mathbf{G}_1^{F_1})$, by Clifford theory $\chi \circ \varphi$ and $\psi \circ \varphi$ are either disjoint or equal, and the latter occurs only if they differ by a (linear) character of $\mathbf{G}_1^{F_1}/\varphi(\mathbf{G})^{F_1}$, and $\langle \chi \circ \varphi, \chi \circ \varphi \rangle_{\mathbf{G}^F} = |\{\theta \in \operatorname{Irr}(\mathbf{G}_1^{F_1}/\varphi(\mathbf{G})^{F_1}) \mid \theta \otimes \chi = \chi\}|$. Such a θ corresponds to an element z in $\operatorname{Ker} \varphi^*$. If $z \neq 1$, the elements zs_1 and s_1 cannot be conjugate under G_1^*: they would then be conjugate by some $w \in W$, but we would have ${}^w s = s$ so $w \in W(s) = W(s_1)$ so ${}^w s_1 = s_1$, a contradiction. So if $\chi \in \mathcal{E}(\mathbf{G}_1^{F_1}, s_1)$ then $\theta \otimes \chi \in \mathcal{E}(\mathbf{G}_1^{F_1}, s_1 z)$ which is disjoint from $\mathcal{E}(\mathbf{G}_1^{F_1}, s_1)$, so we cannot have $\theta \circ \chi = \chi$, so $\chi \circ \varphi$ is irreducible. ∎

10. Shintani descent.

In this section, we want to express the action of Shintani descent in a connected reductive algebraic group $\mathbf{G}$ in terms of the parametrization of section 5. As in section 8, we assume that $\mathbf{G}$ has two rationnal structures given by Frobenius endomorphisms F and F', both stabilizing a pair $\mathbf{T} \subset \mathbf{B}$. Furthermore, we assume that F' is a split Frobenius endomorphism. We fix a semi-simple element $s \in \mathbf{T}^*$ assumed to be F'-fixed, and such that $W_F(s)$ is not empty and we choose $w_1 \in W_F(s)$ as in section 2. We keep all notations of part I. With these notations, we will give some evidence for the existence and commutativity of the following diagram (which properties we conjecture to hold in general):

10.1

$$\begin{array}{ccc}
\mathcal{C}(\mathbf{G}^{F'}/F,s) & \xrightarrow{\Phi\circ \mathrm{Sh}_{F'/F}} & \mathcal{C}(\mathbf{G}^{F}/F',s) \\
\downarrow \mathbf{a} & & \downarrow \mathbf{b} \\
\coprod_{a\in A^{w_1F}} \mathcal{C}(C_{\mathbf{G}^*}(s)^{(\dot a F')^*}/(\dot w_1 F)^*,1) & & \coprod_{a\in H^1(w_1F,A)} \mathcal{C}(C_{\mathbf{G}^*}(s)^{(\dot a\dot w_1 F)^*}/(F')^*,1) \\
\downarrow \mathbf{c} & & \downarrow \mathbf{d} \\
\coprod_{\mathcal{F}\in\Xi(W(s))^{w_1F}} \underline{\mathcal{M}}(\Gamma_{\mathcal{F}}\subset\Gamma_{\mathcal{F}}\rtimes<w_1F>) & \xrightarrow{\mathrm{FT}} & \coprod_{\mathcal{F}\in\Xi(W(s))^{w_1F}} \overline{\mathbb{Q}}_l\overline{\mathcal{M}}(\Gamma_{\mathcal{F}}\subset\Gamma_{\mathcal{F}}\rtimes<w_1F>)
\end{array}$$

where map **a** (resp. **b**) is a bijection generalizing to F-class functions (resp F'-class function) the map π_s of section 3, and maps **c** and **d** are generalizations of the parametrization of section 5. We have denoted by FT the Fourier transform associated to the pairing between spaces $\overline{\mathbb{Q}}_l\overline{\mathcal{M}}$ and $\underline{\mathcal{M}}$ and by Φ the endomorphism of $\mathcal{C}(\mathbf{G}^F/F',s)$ defined on the basis of extensions of F'-invariant irreducible characters by multiplication by the associated eigenvalue of F'. We will construct vertical maps under suitable assumptions: we keep the assumption 3.3 that restrictions of irreducible characters are multiplicity free and we will assume some degree of compatibility of π_s with the action of the Frobenius endomorphisms (*cf.* 10.3, 10.4 below). Then we will show that Shintani descent maps the subspace of $\mathcal{C}(\mathbf{G}^{F'}/F,s)$ corresponding to principal series characters of $C_{\mathbf{G}^*}(s)^{(\dot a F')^*}$ into $\mathcal{C}(\mathbf{G}^F/F',s)$ and that when $F' = F_0^m$, where F_0 a split Frobenius and m sufficiently divisible, the restriction of the above diagram to the corresponding subspaces is commutative. We do not know in general if Shintani descent maps $\mathcal{C}(\mathbf{G}^{F'}/F,s)$ on $\mathcal{C}(\mathbf{G}^F/F',s)$. Shoji and Asai ([14], [15] and [2]) proved this result for groups with connected center and for the special linear group. The proof of the commutativity of the restricted diagram relies heavily on the results of [7]. The idea is to show that the "limit for m=0" of this diagram ([7] shows the existence of the limit of Shintani descent) is known to be commutative by theorem 5.8, and to apply results of [7] which relate the value of Shintani descent to the value of its limit.

We need also the following result of [7] which is an easy generalization of [11, 2.20]:

10.2 PROPOSITION. *Let F' be a split Frobenius endomorphism over the field $\mathbb{F}_{q_{F'}}$ such that ${}^{F'}s = s$; then*
(i) Any $\rho \in \mathcal{E}(\mathbf{G}^F, s)$ is fixed by F'.

(ii) For any extension $\tilde\rho$ of ρ to $\mathbf{G}^F \rtimes \langle F'\rangle$, such that $\langle F'\rangle$ acts on $\tilde\rho$ via a finite quotient, there exists an algebraic number $\lambda_{\tilde\rho} \in \overline{\mathbb{Q}}_l$ all of which complex conjugates have absolute value 1 such that if ρ is a component of $H^*_c(Y_{\dot w}, \overline{\mathbb{Q}}_l)$ (where $Y_{\dot w}$ denotes a Deligne-Lusztig variety) then the isotypic component of type ρ in this module has a $(\mathbf{G}^F, F')$-stable filtration whose quotients are isomorphic to $\tilde\rho$, if we multiply the action of F' in each quotient by $\lambda_{\tilde\rho}^{-1} q_{F'}^{-i}$ for some integer i.

So the right hand side of diagram 10.1 is given by 3.4 and 5.2 provided that F' acts trivially on $\mathcal{C}(C_{\mathbf{G}^*}(s)^{(\dot a\dot w_1 F)^*}, 1)$. By 10.2 above, it acts trivially on $\mathcal{C}(C^\circ_{\mathbf{G}^*}(s)^{\dot a\dot w_1 F^*}, 1)$. We assume that it acts again trivially on unipotent characters of the full centralizer (it may be possible to check case by case that this assumption holds):

10.3 ASSUMPTION. *The action of F' on $\mathcal{E}(C_{\mathbf{G}^*}(s)^{(\dot w_1 F)^*}, 1)$ is trivial.*

We now look at the left part of diagram 10.1. Here the Frobenius endomorphism is F'. It is assumed to fix s, so $W_{F'}(s) = W(s)$, *i.e.*, the corresponding element w_1 is equal to 1. As F' is split, it acts trivially on the Weyl group and we have $H^1(F', A) = A$. We will assume that all representatives of elements of W we consider are chosen in $\mathbf{G}^{F'}$. To construct map **a** we have to show that the map $\pi_s : \mathcal{E}(\mathbf{G}^{F'}, s) \to \coprod_{a\in A} \mathcal{E}(C_{\mathbf{G}^*}(s)^{(\dot a F')^*}, 1)$ restricts to a bijection from the set of F-fixed characters in the left hand side onto the set of $(\dot w_1 F)^*$-fixed characters in the right hand side. Recall that π_s was constructed using the embedding $\mathbf{G} \hookrightarrow \tilde{\mathbf{G}}$, and that it maps the elements of the $\tilde{\mathbf{G}}^{F'}/\operatorname{Stab}_{\tilde{\mathbf{G}}^{F'}}(\rho)$-orbit of $\rho \in \mathcal{E}(\mathbf{G}^{F'}, s)$ onto the elements $\pi_s(\rho) \otimes \zeta$ where ζ runs over $\operatorname{Irr}(C_{\mathbf{G}^*}(s)^{\dot a F'}(\pi_s(\rho))/C^\circ_{\mathbf{G}^*}(s)^{\dot a F'})$. Moreover, as stated after 3.4, there is a canonical isomorphism

$$\tilde{\mathbf{G}}^{F'}/\operatorname{Stab}_{\tilde{\mathbf{G}}^{F'}}(\rho) \xrightarrow{\sim} \operatorname{Irr}(C_{\mathbf{G}^*}(s)^{\dot a F'}(\pi_s(\rho))/C^\circ_{\mathbf{G}^*}(s)^{\dot a F'})$$

induced by the map $a \mapsto \hat a$ of 3.2. So if both ρ is F-fixed and $\pi_s(\rho)$ is $\dot w_1 F$-fixed, the map π_s is compatible with the actions of F and $\dot w_1 F$. To ensure this compatibility in the sequel, we will make the following assumption:

10.4 ASSUMPTION.

(i) If the A-orbit of $\rho \in \mathcal{E}(\mathbf{G}^{F'}, s)$ is F-fixed, then it contains an F-fixed character.

(ii) *For any $a \in A$ and each $(\dot w_1 F)^*$-fixed A-orbit in $\mathcal{E}(C^\circ_{\mathbf{G}^*}(s)^{\dot aF'},1)$, there exists a $(\dot w_1 F)^*$-fixed extension to $C_{\mathbf{G}^*}(s)^{\dot cF'}$ of the sum of the characters in that orbit.*

This assumption gives us the desired existence of bijection $\mathbf{a}$ because a character ρ can be F-fixed only if its orbit is F-fixed, and a character in $\mathcal{E}(C_{\mathbf{G}^*}(s)^{F'},1)$ can be $\dot w_1 F$-fixed only if its restriction to $C^\circ_{\mathbf{G}^*}(s)^{\dot aF'}$ is $\dot w_1 F$-fixed.

Let us remark that in the case where $a = 1$, and ρ is in the principal series of $C_{\mathbf{G}^*}(s)^{F'}$, property 10.4, (ii) holds by the lemma 5.5, (i) applied to $W(s)$ with the automorphism $w_1 F$.

10.5 We now construct map $\mathbf{c}$. We simplify the notations by writing $\mathbf{H}$ for $C_{\mathbf{G}^*}(s)$. Note that for any family $\mathcal{F}$ of $W(s)$, as F' acts trivially on $\Gamma_{\mathcal{F}}$, we have

$$\overline{\mathcal{M}}(\Gamma_{\mathcal{F}} \subset \Gamma_{\mathcal{F}} \rtimes \langle F\rangle) = \mathcal{M}(\Gamma_{\mathcal{F}}),$$

so the parametrization of 5.2 in this case is a bijection

$$\bigoplus_{a\in A} \overline{\mathbb{Q}}_l \mathcal{E}(\mathbf{H}^{aF'},1) \xrightarrow{\sim} \bigoplus_{\mathcal{F}\in\Xi(W)} \overline{\mathbb{Q}}_l \mathcal{M}(\Gamma_{\mathcal{F}}).$$

We now show that this parametrization maps $\dot w_1 F$-invariant irreducible characters on $w_1 F$-invariant elements in the right hand side, so we can restrict ourselves to $w_1 F$-invariant families. Let $\mathcal{F}$ be such a family. It is defined as the A-orbit of a family $\mathcal{F}_0 \in \Xi(W^\circ(s))^a$ with $a \in A^{w_1 F}$. The action of $\dot w_1 F$ on the orbit of $\mathcal{E}(\mathbf{H}^{\circ aF'},1,\mathcal{F}_0)$ corresponds to the action of $w_1 F$ on the A-orbit of $\overline{\mathcal{M}}(\Gamma_{\mathcal{F}_0} \subset \Gamma_{\mathcal{F}_0} \rtimes \langle a\rangle)$, which induces the action of $w_1 F$ on $\mathcal{M}(\Gamma_{\mathcal{F}})$. The parametrization maps the extensions to $\mathbf{H}^{aF'}$ of the sum of the A-orbit of $\psi_0 \in \mathcal{E}(\mathbf{H}^{\circ aF'}, s, \mathcal{F}_0)$ onto elements $(x,\chi) \in \mathcal{M}(\Gamma_{\mathcal{F}})$ where χ runs over irreducible components of $\mathrm{Ind}_{C_{\Gamma_{\mathcal{F}_0}}(x)}^{C_{\Gamma_{\mathcal{F}}}(x)}(\chi_0)$ where $(x,\chi_0) \in \overline{\mathcal{M}}(\Gamma_{\mathcal{F}_0} \subset \Gamma_{\mathcal{F}_0} \rtimes \langle a\rangle)$ parametrizes ψ_0. Extensions of the sum of the orbit of ψ_0 differ by tensorization by linear characters of $\mathrm{Stab}_A(\psi_0)$. Irreducible components of $\mathrm{Ind}_{C_{\Gamma_{\mathcal{F}_0}}(x)}^{C_{\Gamma_{\mathcal{F}}}(x)}(\chi_0)$ differ by tensorization by linear characters of $\mathrm{Stab}_A(\mathcal{F}_0,(x,\chi_0))$ which is equal to $\mathrm{Stab}_A(\psi_0)$. By assumption 10.4, (ii), the sum of any $\dot w_1 F$-invariant orbit in $\mathcal{E}(\mathbf{H}^{\circ aF'},1)$ has a $\dot w_1 F$-invariant irreducible extension to $\mathbf{H}^{aF'}$. So showing that $\dot w_1 F$-invariant characters in $\mathcal{E}(\mathbf{H}^{aF'},1,\mathcal{F})$ corresponds to $w_1 F$-invariant elements of $\mathcal{M}(\Gamma_{\mathcal{F}})$ is equivalent to showing that for any $(x,\chi_0)\in\overline{\mathcal{M}}(\Gamma_{\mathcal{F}_0}\subset\Gamma_{\mathcal{F}_0}\rtimes\langle a\rangle)$ there exists an irreducible component χ of $\operatorname{Ind}_{C_{\Gamma_{\mathcal{F}_0}}(x)}^{C_{\Gamma_{\mathcal{F}}}(x)}(\chi_0)$ such that (x,χ) is invariant as an element of $\mathcal{M}(\Gamma_{\mathcal{F}})$. But this is exactly the statement of 5.5, (ii) applied for the automorphism w_1F. So under assumptions 3.3 and 10.4 we have constructed map **c**.

Note that assumptions 10.3 and 10.4 are clearly true if the center of **G** is connected. So in this case, maps **a**, **b**, **c** and **d** are proved to exist.

Before giving the announced result about Shintani descent, we recall some of the main results of [7] (*cf.* [7] 4.2, 7.4, 8 and 9.7). We keep the same notations.

10.6 THEOREM. *We assume that $F'=F_0^{2m}$ where F_0 is a split Frobenius endomorphism on* **G** *and m is such that ${}^{F_0^m}s=s$. We denote by $\psi_{\mathbf{G},F'}$ the linear operator from $\mathcal{C}(\mathbf{G}^F/F',s)$ to $\mathcal{C}(\mathbf{G}^F,s)$ which maps any $\tilde\rho$ on $\lambda_{\tilde\rho}^{-1}\rho$, where $\lambda_{\tilde\rho}$ is the eigenvalue of F' associated to the extension $\tilde\rho$ of $\rho\in\mathcal{E}(\mathbf{G}^F,s)$ (cf. 10.2). Then, for any extension $\tilde E$ to $W(s)\rtimes\langle w_1F\rangle$ of $E\in(\operatorname{Irr}(W(s)))^{w_1F}$, there exists an extension $\rho_{\tilde E}$ of the character ρ_E (inverse image by $\pi_{s^{-1}}$ of the principal series character corresponding to E) such that*

$$\psi_{\mathbf{G},F'}\circ\operatorname{Sh}_{F'/F}\rho_{\tilde E}=\pi_{s^{-1}}(R_{\tilde E})$$

(and so is independent of m).

We can now state:

10.7 PROPOSITION. *As above, we assume that $F'=F_0^m$ where F_0 is a split Frobenius endomorphism on* **G** *and m is such that ${}^{F_0^m}s=s$; then for any $\rho\in\mathcal{E}(\mathbf{G}^F,s^{-1})$, any extension $\tilde\rho$ of ρ to $\mathbf{G}^F\rtimes\langle F'\rangle$ (where the action of F' is of finite order) and any $E\in(\operatorname{Irr}(W(s)))^{w_1F}$, if $\lambda_{\tilde\rho}$ is the eigenvalue of F' associated to $\tilde\rho$, we have*

$$\langle\tilde\rho,\operatorname{Sh}_{F'/F}\rho_{\tilde E}\rangle_{\mathbf{G}^F.F'}=\lambda_{\tilde\rho}^{-1}(-1)^{l(w_1)}\{x_{\pi_{s^{-1}}(\rho)},x_{\tilde E}\}.$$

PROOF: It is an immediate consequence of 10.6, of 3.5 and of 5.8. ■

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