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by James A. Carlson and Richard M. Hain

1. Introduction.

In this paper we study the group $\mathrm{Ext}^n_{\mathcal{H}(X)}(B, A)$ of Yoneda n-extensions in the category $\mathcal{H}(X)$ of good unipotent variations of $\mathbb{Q}$-mixed Hodge structure over a smooth complex algebraic variety X. In particular, we show (Theorem (11.5)) that the continuous Deligne-Beilinson (absolute Hodge) cohomology

$$H^n_{\mathcal{H}}(\pi_1(X, x), \mathbb{Q}(p))$$

of the fundamental group of X is canonically isomorphic to the group

$$\mathrm{Ext}^n_{\mathcal{H}(X)}(\mathbb{Q}, \mathbb{Q}(p)).$$

To compute this object, one uses the exact sequence of [2] to write

(1.1)

$$0 \to \mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Q}, H^{n-1}_{cont}(\pi_1(X), \mathbb{Q}(p)) \to \mathrm{Ext}^n_{\mathcal{H}(X)}(\mathbb{Q}, \mathbb{Q}(p)) \to \mathrm{Ext}^0_{\mathcal{H}}(\mathbb{Q}, H^n_{cont}(\pi_1(X), \mathbb{Q}(p)) \to 0.$$

Here H_{cont} denotes the continuous cohomology of $\pi_1(X, x)$. These groups carry natural mixed Hodge structures, and so one can form the indicated groups of extensions in the category $\mathcal{H}$ of mixed Hodge structures.

There is also a natural map

$$H^\bullet_{\mathcal{H}}(\pi_1(X), \mathbb{Q}(p)) \longrightarrow H^\bullet_{\mathcal{H}}(X, \mathbb{Q}(p)) \qquad (1.2)$$

into absolute Hodge cohomology, Beilinson's refined version of Deligne cohomology, defined in [2]. This is an isomorphism when X is a rational $K(\pi, 1)$. Since every algebraic curve and every abelian variety is a rational $K(\pi, 1)$, we can interpret $H^n_{\mathcal{H}}(X, \mathbb{Q}(p))$ as the group of Yoneda n-extensions of $\mathbb{Q}$ by $\mathbb{Q}(p)$ in $\mathcal{H}(X)$ in these cases. In particular, the higher Ext groups of $\mathcal{H}(X)$ can be nontrivial.

The idea that extensions of $\mathbb{Z}$ by $\mathbb{Z}(p)$ in $\mathcal{H}(X)$ give elements of $H^\bullet_{\mathcal{H}}(X, \mathbb{Z}(p))$ appears in the beautiful example [7] of Deligne. There he shows, among other things, that, when X is a curve, the obstruction to the vanishing of the cup product $f \cup g \in H^2_{\mathcal{H}}(X, \mathbb{Z}(2))$ of two invertible functions

$$f, g \in H^0(X, \mathcal{O}^*_X) = H^1_{\mathcal{H}}(X, \mathbb{Z}(1)) = \mathrm{Ext}^1_{\mathcal{H}(X)}(\mathbb{Z}, \mathbb{Z}(1))$$

is the same as the obstruction to finding a variation of mixed Hodge structure $\mathcal{V}$ over X with weight graded quotients $\mathbb{Z}(0)$, $\mathbb{Z}(1)$, $\mathbb{Z}(2)$ such that the extension $\mathcal{V}/\mathbb{Z}(2)$ corresponds to the element g of $\mathrm{Ext}^1_{\mathcal{H}(X)}(\mathbb{Z}, \mathbb{Z}(1))$ and such that $W_{-2}\mathcal{V}$ corresponds to the element $f \otimes \mathbb{Z}(1)$ of $\mathrm{Ext}^1_{\mathcal{H}(X)}(\mathbb{Z}(1), \mathbb{Z}(2))$. Subsequently, Beilinson [2] related absolute Hodge cohomology to extensions in the category $\mathcal{H}$ of mixed Hodge structures. However, his construction does not appear to involve either variations of mixed Hodge structure or Griffiths transversality, both essential ingredients in Deligne's example.

The attempt to understand the relationship between Beilinson's general construction of absolute Hodge cohomology based on extensions, Deligne's example, and MacPherson's multivalued Deligne cohomology [12] led us to the constructions in this paper. The calculation of $\mathrm{Ext}^\bullet_{\mathcal{H}(X)}(B, A)$ is not completely trivial because there are never any projective or injective objects in $\mathcal{H}(X)$, even when X is a point. The actual calculation of $\mathrm{Ext}^\bullet_{\mathcal{H}(X)}(B, A)$ is analogous to Beilinson's calculation [1] of $\mathrm{Ext}^\bullet_{\mathcal{H}}(B, A)$ when X is a point, except that we work in the category of Hodge-theoretic representations of $\pi_1(X, x)$. This category, by [13], is equivalent to $\mathcal{H}(X)$, an essential ingredient of our calculations.

We now sketch the idea behind the proof. Since $\mathcal{H}(X)$ is the category of Hodge theoretic representations of $\pi_1(X, x)$, the completed group ring $\mathbb{Q}\pi_1(X, x)\hat{}$, viewed as a right $\pi_1(X, x)$-module via right multiplication, should play a distinguished role. Although $\mathbb{Q}\pi_1(X, x)\hat{}$ is not projective in $\mathcal{H}(X)$, or, more precisely, in a suitable completion $\mathcal{H}^{-\infty}(X)$ of $\mathcal{H}(X)$, it becomes projective if we forget either the rational structure of $\mathcal{H}(X)$, the Hodge filtration, or both. For this reason, we define the following three categories, each with the same objects as $\mathcal{H}(X)$.

(1.3) Definition.

i) $\mathcal{H}_{\mathbb{Q}}(X)$ *: objects are those of* $\mathcal{H}(X)$*, morphisms are those of the underlying* $\mathbb{Q}$*-local system which preserve the weight filtration,*

ii) $\mathcal{H}_F(X)$ *: objects are those of* $\mathcal{H}(X)$*, morphisms are those of the underlying* $\mathbb{C}$*-local system which preserve the Hodge and weight filtrations,*

iii) $\mathcal{H}_{\mathbb{C}}(X)$ *: objects are those of* $\mathcal{H}(X)$*, morphisms are those of the underlying* $\mathbb{C}$*-local system which preserve the weight filtration.*

To compute $\mathrm{Ext}^\bullet_{\mathcal{H}}(B, A)$, we first view B as a right $\pi_1(X, x)$-module. Next, we find a resolution $P_\bullet \longrightarrow B \longrightarrow 0$ of B in $\mathcal{H}(X)$ which is projective in each of the categories of (1.3). One can, for example, take $P_\bullet$ to be the bar resolution of B. Now, if

$$0 \longrightarrow A \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow B \longrightarrow 0$$

is an n-extension in $\mathcal{H}(X)$, we have chain maps $\phi^F : P_\bullet \longrightarrow E_\bullet$ and $\phi^{\mathbb{Q}} : P_\bullet \longrightarrow E_\bullet$ in $\mathcal{H}_F(X)$ and $\mathcal{H}_{\mathbb{Q}}(X)$, respectively, which become homotopic in $\mathcal{H}_{\mathbb{C}}(X)$, say, via a homotopy

$\phi^{\mathbb{C}} : P_\bullet \longrightarrow E_\bullet$ [1]. Schematically, we have

$$
\begin{array}{ccccccccccccc}
\cdots & P_{n+1} & \rightarrow & P_n & \longrightarrow & P_{n-1} & \rightarrow & \cdots & \rightarrow & P_0 & \rightarrow & B & \rightarrow & 0 \\
(1.4) & & & \phi^F_n, \phi^{\mathbb{Q}}_n \downarrow\downarrow & \phi^{\mathbb{C}}_{n-1} \swarrow & \downarrow\downarrow & \swarrow & & \swarrow & \downarrow\downarrow & \swarrow & \| & & \\
& 0 & \rightarrow & A & \longrightarrow & E_{n-1} & \rightarrow & \cdots & \rightarrow & E_0 & \rightarrow & B & \rightarrow & 0
\end{array}
$$

Now observe that

$$\phi = \begin{bmatrix} & \phi^{\mathbb{C}}_{n-1} & \\ \phi^{\mathbb{Q}}_n & & \phi^F_n \end{bmatrix}$$

satisfies $\delta\phi^{\mathbb{Q}}_n = \delta\phi^F_n = 0$ and $\delta\phi^{\mathbb{C}}_{n-1} = \phi^{\mathbb{Q}}_n - \phi^F_n$. Thus ϕ is an n-cocycle in the complex

(1.5)

$$\operatorname{cone}[W_0 \operatorname{Hom}_{\mathbb{Q}\pi_1(X,x)}(P_\bullet, A) \oplus F^0 W_0 \operatorname{Hom}_{\mathbb{C}\pi_1(X,x)}(P_\bullet, A) \longrightarrow W_0 \operatorname{Hom}_{\mathbb{C}\pi_1(X,x)}(P_\bullet, A)][-1].$$

When we take $B = \mathbb{Q}$ and $A = \mathbb{Q}(p)$, the above complex becomes

$$(1.6) \qquad \operatorname{cone}[W_{2p} \operatorname{Hom}_{\mathbb{Q}\pi_1}(P_\bullet, \mathbb{Q}) \oplus F^p W_{2p} \operatorname{Hom}_{\mathbb{C}\pi_1}(P_\bullet, \mathbb{C}) \longrightarrow W_{2p} \operatorname{Hom}_{\mathbb{C}\pi_1}(P_\bullet, \mathbb{C})][-1],$$

and this computes $H^\bullet_{\mathcal{H}}(\pi_1(X), \mathbb{Q}(p))$.

We note that there is a general procedure for computing extensions in any so-called mixed category [3]. This gives, in particular, a method for computing Ext in the category $\mathcal{H}(X)$. Nonetheless, we hope that the present approach will prove useful because of its directness.

One can construct a variant of the the absolute Hodge cohomology of an algebraic variety as follows. Associate to each open set U in X the cochains that compute $\operatorname{Ext}_{\mathcal{H}(U)}(\mathbb{Q}, \mathbb{Q}(p))$. By choosing suitable cochains, one can associate to each Zariski (hyper)-covering of X a double complex. The cohomology of X with coefficients in $\mathbb{Q}(p)$ is obtained as the limit of the cohomologies of these double complexes. This will be the subject of a future paper [11] by the second author.

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2. Yoneda Extensions.

An n-extension of B by A in $\mathcal{H}(X)$ is an exact sequence

$$E: \qquad 0 \longrightarrow A \longrightarrow E_{n-1} \longrightarrow E_{n-2} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow B \longrightarrow 0.$$

An *elementary equivalence* of extensions is a morphism $E \xrightarrow{\alpha} E'$ which is the identity on the extreme left and right-hand terms, and Yoneda equivalence is the relation generated by elementary equivalence. The set of all Yoneda classes of n-extension forms an abelian group, written $\operatorname{Ext}^n_{\mathcal{H}(X)}(B, A)$, relative to which there is a natural pairing,

$$\operatorname{Ext}^m_{\mathcal{H}(X)}(B, A) \otimes \operatorname{Ext}^n_{\mathcal{H}(X)}(C, B) \longrightarrow \operatorname{Ext}^{m+n}_{\mathcal{H}(X)}(C, A)$$

given by splicing extensions together at the common term B.

When X is a point, the group of 1-extensions has a particularly simple description [4,5], which we now recall. Let

$$E: \qquad 0 \longrightarrow A \xrightarrow{i} H \xrightarrow{\pi} B \longrightarrow 0$$

be a 1-extension. Then there are morphisms $s^{\mathbf{Q}} : B \longrightarrow H$ in $\mathcal{H}_{\mathbf{Q}}$ and $s^F : B \longrightarrow H$ in $\mathcal{H}_F$ which satisfy $\pi \circ s^{\mathbf{Q}} = id$ and $\pi \circ s^F = id$. Define a difference homomorphism $\psi = s^{\mathbf{Q}} - s^F \in \mathrm{Hom}_{\mathbf{C}}(B, A)$, and observe that the coset

$$\psi + \mathrm{Hom}_{\mathbf{Q}}(B, A) + \mathrm{Hom}_F(B, A)$$

is independent of the choice of the sections $s^{\mathbf{Q}}$ and s^F of the $\mathbf{Q}$- and F-structures. The correspondence $E \mapsto [\psi]$ then defines a natural isomorphism

$$\mathrm{Ext}^1_{\mathcal{H}}(B, A) \xrightarrow{\cong} \frac{\mathrm{Hom}_{\mathbf{C}}(B, A)}{\mathrm{Hom}_{\mathbf{Q}}(B, A) + \mathrm{Hom}_F(B, A)}. \tag{2.1}$$

From this identification, we see that any mixed Hodge structure has nontrivial extensions from the right, so that there are no projective objects. Indeed, if A is a mixed Hodge structure of highest weight k, and $2p > k$, then

$$\mathrm{Ext}^1_{\mathcal{H}}(\mathbf{Q}(-p), A) \cong A(p)_{\mathbf{C}} / A(p)_{\mathbf{Q}} \cong \mathbf{C}^{*m}.$$

Similarly, if A has lowest weight k and $2p < k$, then

$$\mathrm{Ext}^1_{\mathcal{H}}(A, \mathbf{Q}(-p)) \cong \check{A}(-p)_{\mathbf{C}} / \check{A}(-p)_{\mathbf{Q}} \cong \mathbf{C}^{*m},$$

so that there are no injective objects. By restricting to an arbitrary point $x \in X$, it follows that, other than 0, there are no projectives or injectives in $\mathcal{H}(X)$ whenever X is a smooth variety.

Extensions of degree one are easy to handle because every Yoneda equivalence is given by an elementary equivalence, in fact, one which induces an isomorphism on the middle term. For extensions of higher degree, equivalence is not the same as elementary equivalence, and elementary equivalences do not in general induce isomorphisms on the interior terms. The increased flexibility that this gives is so great that the higher Ext groups vanish when X is a point [1]. As we shall see, this is not generally true when X is nontrivial (e.g., $X = \mathbf{P}^1 - \{ 0, 1, \infty \}$).

3. Relative projectives.

As a substitute for the absence of projectives in $\mathcal{H}(X)$, we introduce the notion of relative projectives. To this end, let $\mathcal{B}$ be an abelian category and $\mathcal{A}$ an abelian subcategory with the same objects as $\mathcal{B}$. An object P of $\mathcal{A}$ is a *relative projective* for $\mathcal{A} \hookrightarrow \mathcal{B}$ if whenever $V \longrightarrow W \longrightarrow 0$ is an epimorphism in $\mathcal{A}$, each morphism $P \longrightarrow W$ in $\mathcal{B}$ lifts to V in $\mathcal{B}$:

$$\begin{array}{ccccc} & & P & & \\ & \swarrow & \downarrow & & \\ V & \longrightarrow & W & \longrightarrow & 0 \end{array}$$

A *relative projective resolution* of an object A in $\mathcal{A}$ with respect to $\mathcal{B}$ is an exact sequence

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$$

in $\mathcal{A}$ where each P_j is relatively projective with respect to $\mathcal{A} \hookrightarrow \mathcal{B}$. The category $\mathcal{B}$ has *enough projectives for* $\mathcal{A}$ if for each A in $\mathcal{A}$ there exists a relative projective P for $\mathcal{A} \hookrightarrow \mathcal{B}$ and an epimorphism $P \longrightarrow A \longrightarrow 0$ in $\mathcal{A}$. The standard argument (resolve kernels) gives the existence of resolutions:

(3.1) Proposition. *Every object of $\mathcal{A}$ has a relatively projective resolution in $\mathcal{B}$ if and only if $\mathcal{B}$ has enough projectives for $\mathcal{A}$.*

(3.2) Lemma. *When X is a point, each object of $\mathcal{H}$ is projective, relative to $\mathcal{H}_A$, where $A = \mathbb{Q}$, F, or $\mathbb{C}$.*

The proof is an easy exercise, given the following splitting lemma:

(3.3) Lemma . (Deligne) [6] *Let H be a mixed Hodge structure. Then there is a canonical functorial bigrading $H = \bigoplus_{p,q} H^{p,q}$ with the properties*

i) $F^p = \bigoplus_{a \geq p} H^{a,b}$,
ii) $W_l = \bigoplus_{a+b \leq l} H^{a,b}$.

(3.4) Proposition. *Suppose the $\mathcal{B}$ has enough projectives for $\mathcal{A}$. If $P_\bullet \longrightarrow A \longrightarrow 0$ and $Q_\bullet \longrightarrow A \longrightarrow 0$ are relatively projective resolutions of an object A of $\mathcal{A}$ with respect to $\mathcal{B}$, then there exists a third relatively projective resolution $R_\bullet \longrightarrow A \longrightarrow 0$, and a commutative diagram in $\mathcal{A}$ which relates them:*

$$\begin{array}{ccccc} P_\bullet & \longrightarrow & A & \longrightarrow & 0 \\ {\scriptstyle \pi_P}\uparrow & & \| & & \\ R_\bullet & \longrightarrow & A & \longrightarrow & 0 \\ {\scriptstyle \pi_Q}\downarrow & & \| & & \\ Q_\bullet & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Proof: Set $P_n = Q_n = R_n$ when $n < -1$. Set $P_{-1} = Q_{-1} = R_{-1} = A$. Suppose that $n \geq 0$ and that the complex $R_\bullet$ and the maps $\pi_P : R_\bullet \longrightarrow P_\bullet$ and $\pi_Q : R_\bullet \longrightarrow Q_\bullet$ have been constructed in dimensions $< n$. Let K_{n-1} be the kernel of $\partial : R_{n-1} \longrightarrow R_{n-2}$. Let L be the limit of the diagram

$$P_n \xrightarrow{\partial} P_{n-1} \xleftarrow{\pi_P} K_{n-1} \xrightarrow{\pi_Q} Q_{n-1} \xleftarrow{\partial} Q_n.$$

That is,

$$L = \{\ (p, k, q) \in P_n \oplus K_{n-1} \oplus Q_n \mid \partial p = \pi_P k, \partial q = \pi_Q k\ \}.$$

Since the restrictions of $\partial \circ \pi_P$ and $\partial \circ \pi_Q$ to K_{n-1} vanish, it follows tht the natural projection $L \longrightarrow K_{n-1}$ is surjective. Choose R_n to be any relative projective for $\mathcal{A} \hookrightarrow \mathcal{B}$ that surjects onto L. The projections $P_n \longleftarrow R_n \longrightarrow Q_n$ are then composites of $R_n \longrightarrow L$ with the canonical projections $P_n \longleftarrow L \longrightarrow Q_n$, as required.

Now observe that the categories $\mathcal{H}$, $\mathcal{H}_{\mathbf{Q}}$, $\mathcal{H}_F$,and $\mathcal{H}_{\mathbf{C}}$ fit together via the natural forgetful functors to give a 'Mayer-Vietoris' diagram

$$\mathcal{MH} = \begin{bmatrix} & & \mathcal{H}_{\mathbf{C}} & & \\ & \nearrow & & \nwarrow & \\ \mathcal{H}_{\mathbf{Q}} & & & & \mathcal{H}_F \\ & \nwarrow & & \nearrow & \\ & & \mathcal{H} & & \end{bmatrix} \tag{3.5}$$

We shall often write

(3.6) $\quad \mathrm{Hom}(B, A)$ for $\mathrm{Hom}_{\mathcal{H}}(B, A)$, $\quad \mathrm{Hom}_{\mathbf{Q}}(B, A)$ for $\mathrm{Hom}_{\mathcal{H}_{\mathbf{Q}}}(B, A)$, $\quad$ etc.

Let us make the notion just used precise:

(3.7) Definition. *A commutative diagram of categories*

$$\mathcal{MA} = \begin{bmatrix} & & \mathcal{A}_{01} & & \\ & \nearrow & & \nwarrow & \\ \mathcal{A}_0 & & & & \mathcal{A}_1 \\ & \nwarrow & & \nearrow & \\ & & \mathcal{A} & & \end{bmatrix}$$

is Mayer-Vietoris if

i) each arrow is the identity on the level of objects,

ii) each arrow is an inclusion on the level of morphisms,

iii) $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ *is the intersection of* $Hom_{\mathcal{A}_0}(X, Y)$ *and* $\mathrm{Hom}_{\mathcal{A}_1}(X, Y)$ *in* $\mathrm{Hom}_{\mathcal{A}_{01}}(X, Y)$.

The diagram $\mathcal{MH}(X)$ of (2.1) is, by definition, the *canonical Mayer-Vietoris diagram* for $\mathcal{H}(X)$.

(3.8) Proposition. *The diagram* $\mathcal{H}(X)$ *is Mayer-Vietoris.*

(3.9) Definition. *An object in* $\mathcal{A}$ *is a relative projective for a Mayer-Vietoris diagram* $\mathcal{MA}$ *if it is relatively projective for* $\mathcal{A} \hookrightarrow \mathcal{A}_I$, *with* $I = 0, 1, 01$. $\mathcal{MA}$ *has enough projectives if for every object* A *in* $\mathcal{A}$ *there is an object* P *in* $\mathcal{A}$ *which is relative projective in* $\mathcal{MA}$.

According to Lemma (3.2), the diagram $\mathcal{H}$(point) has enough relative projectives. One of our main concerns is to show that for each X, a suitable completion of $\mathcal{H}(X)$ has enough relative projectives. This we do in section 7.

4. Construction of $\mathrm{Ext}_{\mathcal{MA}}$.

Let $\mathcal{A}$ be an abelian category, $\mathcal{MA}$ a Mayer-Vietoris diagram on $\mathcal{A}$ which admits enough projectives. Given an object A in $\mathcal{A}$, let $P_\bullet(A)$ be a relative projective resolution. Define, by analogy with Deligne-Beilinson cohomology [1], the complex

(4.1)

$$\mathrm{RHom}_{\mathcal{MA}}(B,A) = \begin{bmatrix} & & \mathrm{Hom}_{\mathcal{A}_{01}}(P(B),A) & & \\ & i_{\mathbf{Q}} \nearrow & & \nwarrow i_F & \\ \mathrm{Hom}_{\mathcal{A}_0}(P(B),A) & & & & \mathrm{Hom}_{\mathcal{A}_1}(P(B),A) \end{bmatrix},$$

where $i_{\mathbf{Q}}$ and i_F are the forgetful functors. Following Beilinson [1] we use the notation

$$D = \begin{bmatrix} & & C & & \\ & \alpha \nearrow & & \nwarrow \beta & \\ A & & & & B \end{bmatrix}$$

to denote the complex

$$D = [\ A \oplus B \xrightarrow{\delta} C\ ][-1],$$

where $\delta = \alpha - \beta$. We shall sometimes use the more compact notation

$$\text{(4.2)} \qquad \mathrm{RHom}_{\mathcal{MA}}(B,A) = \mathrm{cone}[\ \mathrm{Hom}_{\mathcal{A}[0]}(P(B),A) \xrightarrow{\delta} \mathrm{Hom}_{\mathcal{A}[1]}(P(B),A)\ ][-1],$$

where $\delta = i_{\mathbf{Q}} - i_F$. As an immediate consequence of (3.4), we have

(4.3) Proposition. *If $\mathcal{MA}$ has enough relative projectives, then $\mathrm{RHom}_{\mathcal{MA}}(B,A)$ is well defined in the derived category of abelian groups.*

Define

$$\text{(4.4)} \qquad \mathrm{Ext}^n_{\mathcal{MA}}(B,A) = H^n\,\mathrm{RHom}_{\mathcal{MA}}(B,A).$$

The relation between this and the usual functor is given by the following

(4.5) Theorem. *There is a natural isomorphism $c:\ \mathrm{Ext}^n_{\mathcal{A}}(B,\ A) \longrightarrow \mathrm{Ext}^n_{\mathcal{MA}}(B,\ A)$.*

We shall give a proof of this result for $\mathcal{MH}$ in the next section. Let us see, however, what we have for $\mathcal{A} = \mathcal{H} = \mathcal{H}(\text{point})$. According to Lemma (3.2), every object in $\mathcal{H}$ is projective, so that we may take for $P(B)$ the complex B concentrated in degree 0. Then

$$\text{(4.6)} \qquad \mathrm{RHom}_{\mathcal{MH}}(B,\ A) = [\ \mathrm{Hom}_{\mathbf{Q}}(B,A) \oplus \mathrm{Hom}_F(B,A) \longrightarrow \mathrm{Hom}_{\mathbf{C}}(B,A)\ ][-1]$$

is a complex concentrated in degrees 0 and 1. Then 2.3.iii gives

$$\text{(4.7)} \qquad \mathrm{Ext}^0_{\mathcal{MH}}(B,\ A) \cong \mathrm{Hom}_{\mathcal{H}}(B,A)$$

and 3.1 yields

$$\text{(4.8)} \qquad \mathrm{Ext}^1_{\mathcal{MH}}(B,\ A) = \frac{\mathrm{Hom}_{\mathbf{C}}(B,A)}{\mathrm{Hom}_{\mathbf{Q}}(B,A) + \mathrm{Hom}_F(B,A)} \cong \mathrm{Ext}^1_{\mathcal{H}}(B,A).$$

Since RHom has no cochains in degrees greater than 1, we find

$$\text{(4.9)} \qquad \mathrm{Ext}^n_{\mathcal{MH}}(B,\ A) = 0 \qquad \text{for } n > 1.$$

Note that this agrees with Beilinson's result [1] for the vanishing of $\mathrm{Ext}_{\mathcal{H}}$.

5. Complete variations.

To have enough relative projectives to compute $\mathrm{Ext}_{\mathcal{H}}(B, A)$, we need to adjoin certain inverse limits of unipotent variations of mixed Hodge structure to $\mathcal{H}(X)$.

(5.1) Definition. *A complete variation $\mathcal{V}$ of mixed Hodge structure over a smooth variety X consists of*

- *i) a local system $\mathbf{V}_{\mathbb{Q}}$ of rational vector spaces over X,*
- *ii) a filtration $W_\bullet$ of $\mathbf{V}_{\mathbb{Q}}$ by sublocal systems,*
- *iii) a filtration $F^\bullet$ of $\mathcal{V}_{\mathbb{C}} = \mathbf{V}_{\mathbb{Q}} \otimes \mathcal{O}_X$ satisfying*
- *iv) for all r, s, $W_r\mathcal{V}/W_s\mathcal{V}$ is a good unipotent variation of mixed Hodge structure [13] relative to the induced rational structure and Hodge and weight filtrations,*
- *iv.a) $\mathbf{V}_{\mathbb{Q}} = \bigcup W_r\mathbf{V}_{\mathbb{Q}}$, and*
- *iv.b) $W_r\mathbf{V}_{\mathbb{Q}} = \varprojlim W_r\mathbf{V}_{\mathbb{Q}}/W_s\mathbf{V}_{\mathbb{Q}}$ for all r.*

The weight filtration defines a natural topology on $\mathcal{V}$ that we shall call the W-adic topology. Morphisms between complete variations over X are maps that preserve all structures (local system, $\mathbb{Q}$-structure, and all filtrations). Let $\mathcal{H}_X\hat{}$ denote the category of complete variations over X, define $\mathcal{H}^{+\infty}(X)$ to be the full subcategory of $\mathcal{H}_X\hat{}$ whose objects satisfy $W_l\mathcal{V} = 0$ for some l, and define $\mathcal{H}^{-\infty}(X)$ to be the full category whose objects satisfy $\mathcal{V} = W_l\mathcal{V}$ for some l. Objects of $\mathcal{H}^{+\infty}(X)$ will be called *upper complete* variations and those of $\mathcal{H}^{-\infty}(X)$ will be called *lower complete.*

(5.2) Proposition. *The category of lower complete (respectively, upper complete) unipotent variations of mixed Hodge structures over X is abelian.*

(5.3) Remark. *We have bifunctors*

$$\mathcal{H}^{-\infty} \otimes \mathcal{H}^{-\infty} \longrightarrow \mathcal{H} \qquad \text{etc.}$$

The faithful inclusion $\mathcal{H} \hookrightarrow \mathcal{H}^{\pm\infty}$ induces a natural transformation

$$\mathrm{Ext}^{\bullet}_{\mathcal{H}(X)} \longrightarrow \mathrm{Ext}^{\bullet}_{\mathcal{H}^{\pm\infty}(X)}$$

(5.4) Proposition. *If A and B are in $\mathcal{H}(X)$, then the natural maps $\mu :$ $\mathrm{Ext}^n_{\mathcal{H}(X)}(B, A) \longrightarrow$ $\mathrm{Ext}^n_{\mathcal{H}^{\pm\infty}(X)}(B, A)$ are isomorphisms.*

Proof: We give a proof for the case of $\mathcal{H}^{-\infty}$ by constructing an inverse for μ. To this end, choose an integer s such that

$$\mathrm{Gr}_k^W A = \mathrm{Gr}_k^W B = 0 \qquad \text{if } k \leq s.$$

If

$$E: \quad 0 \longrightarrow A \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow B \longrightarrow 0$$

is an n-extension in $\mathcal{H}^{-\infty}$ then so is

$$E/W_s: \quad 0 \longrightarrow A \longrightarrow E_{n-1}/W_s \longrightarrow \cdots \longrightarrow E_0/W_s \longrightarrow B \longrightarrow 0,$$

since $* \longrightarrow */W_s$ is an exact functor. Because the diagram

$$E/W_s \longrightarrow E$$

is a Yoneda equivalence, we may define the required inverse for μ by $\nu(E) = E/W_s$.

6. Classification of complete variations.

Throughout this section, X will denote a fixed smooth variety. Let $\mathbb{Q}\pi_1(X,x)$ be the rational group ring of the fundamental group, let J denote the augmentation ideal, and let $\mathbb{Q}\pi_1(X,x)\hat{}$ be the J-adic completion of $\mathbb{Q}\pi_1(X,x)$. According to [9], $\mathbb{Q}\pi_1(X,x)\hat{}$ carries a natural complete mixed Hodge structure. The next two theorems [13] give a classification of unipotent variations.

(6.1) Theorem. *If $\mathcal{V} \longrightarrow X$ is a complete unipotent variation of mixed Hodge structure, then the monodromy representation*

$$\rho: \mathbb{Q}\pi_1(X,x)\hat{} \longrightarrow \mathrm{End}(V_x)$$

is a continuous morphism of complete mixed Hodge structures.

Let HRep(X, x) be the category of morphisms

$$\rho: \mathbb{Q}\pi_1(X,x)\hat{} \longrightarrow \mathrm{End}(V).$$

where V is a lower complete mixed Hodge structure and ρ is a morphism in $\mathcal{H}^{-\infty}$.

(6.2) Theorem. *The functor $\mathcal{H}^{-\infty}(X) \longrightarrow$ HRep(X, x) that takes a variation to the monodromy representation at the base point is an equivalence of categories.*

Note that if $\mathcal{V}$ and $\mathcal{W}$ are objects in $\mathcal{H}^{-\infty}(X)$, then $Hom(\mathcal{V}, \mathcal{W})$ is the object of $\mathcal{H}(X)$ which corresponds to the $\mathbb{Q}\pi_1(X,x)\hat{}$-module $\mathrm{Hom}(V_x, W_x)$ with the π_1 action given by

$$(\phi \cdot g)(v) = \phi(vg^{-1})g,$$

where $\phi \in \mathrm{Hom}(V_x, W_x)$, $g \in \pi_1((X,x),)v \in V_x$.

Let $\mathbb{Q}$ be the constant variation of weight 0, and note that the global sections of type $(0,0)$ may be written

$$\Gamma\mathcal{V} = \mathrm{Hom}_{\mathcal{H}(X)}(\mathbb{Q}, \mathcal{V}).$$

(6.3) Proposition. $\mathrm{Hom}_{\mathcal{H}(X)}(\mathcal{V}, \mathcal{W}) = \Gamma Hom(\mathcal{V}, \mathcal{W}) = \Gamma H^0(X, Hom(\mathcal{V}, \mathcal{W}))$.

By an extension [13, (8.6)] of a theorem of Deligne (cf [16, 15]), the right-hand group carries a natural mixed Hodge structure.

By analogy with the definitions of §2, we may define categories

$$\mathrm{HRep}_{\mathbb{Q}}(\mathrm{X},\mathrm{x}) \qquad \mathrm{HRep}_{\mathrm{F}}(\mathrm{X},\mathrm{x}) \qquad \mathrm{HRep}_{\mathbb{C}}(\mathrm{X},\mathrm{x})$$

of $\mathbb{Q}\pi_1(X,x)\hat{}$-modules. In each case the objects are the same as those of HRep(X, x), but the structural requirements on morphisms are relaxed in the obvious way. Consequently we get a Mayer-Vietoris diagram

(6.4) $$\mathcal{M}\mathrm{HRep} = \left[\begin{array}{ccccc} & & \mathrm{HRep}_{\mathbb{C}} & & \\ & \nearrow & & \nwarrow & \\ \mathrm{HRep}_{\mathbb{Q}} & & & & \mathrm{HRep}_{\mathrm{F}} \\ & \nwarrow & & \nearrow & \\ & & \mathrm{HRep} & & \end{array}\right]$$

(6.5) Proposition. *There are equivalences of categories*

$$\mathcal{H}_{\mathbb{Q}} \cong \mathrm{HRep}_{\mathbb{Q}}$$
$$\mathcal{H}_F \cong \mathrm{HRep}_{\mathrm{F}}$$
$$\mathcal{H}_{\mathbb{C}} \cong \mathrm{HRep}_{\mathbb{C}}.$$

Proof: To prove the first equivalence, consider two objects $\mathcal{V}$ and $\mathcal{W}$ of $\mathcal{H}^{-\infty}(X)$, and observe that

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Q}}(\mathcal{V},\mathcal{W}) &\cong W_0 H^0(Hom_{\mathbb{Q}}(\mathcal{V}, \mathcal{W}) \\ &\cong W_0 H^0(\mathbb{Q}\pi_1(X,x)\hat{}, \mathrm{Hom}_{\mathbb{Q}}(V_x, W_x)) \\ &\cong W_0 \,\mathrm{Hom}_{\mathbb{Q}}(V_x, W_x)^{\mathbb{Q}\pi_1(X,x)\hat{}}\end{aligned}$$

To prove the second equivalence, observe that

$$\begin{aligned}\mathrm{Hom}_F(\mathcal{V}, \mathcal{W}) &= F^0 \cap W_0 H^0(X, Hom(\mathcal{V},\mathcal{W})) \\ &\cong F^0 \cap W_0 H^0(\mathbb{C}\pi_1(X,x)\hat{}, \mathrm{Hom}_{\mathbb{C}}(V_x, W_x)) \\ &= F^0 \cap W_0 \,\mathrm{Hom}(V_x, W_x)^{\mathbb{C}\pi_1(X,x)\hat{}}\end{aligned}$$

The verification for the third equivalence is similar to that of the first two.

7. Existence of relative projectives

We now sketch a proof of the existence of relative projectives in $\mathcal{H}^{-\infty}(X)$. To begin, we recall that if X is a smooth variety with basepoint x, then the *canonical variation* associated with (X, x) is the lower complete variation of mixed Hodge structure whose fiber over $z \in X$ is

$$H_0(P_{x,z}X)^\wedge,$$

where $P_{x,z}X$ is the space of homotopy classes of paths from x to z [14].

Now suppose that A is a lower complete variation of mixed Hodge structure over X, with A_x as fiber over the basepoint. Define $H_0(P_{x,-}X; A_x)^\wedge$ to be the lower complete variation over X whose fiber over z is $A_x \otimes H_0(P_{x,z}X)^\wedge$.

(7.1) Proposition. *There is a surjective morphism*

$$H_0(P_{x,-}X; A_x)^\wedge \xrightarrow{\epsilon} A$$

in $\mathcal{H}(X)$.

Proof: By the classification theorem (6.1) it suffices to construct a homomorphism

$$H_0(P_{x,x}X, A_x)^\wedge \longrightarrow A_x$$

that preserves the filtrations. First, observe that

$$H_0(P_{x,x}X, \mathbb{Q})^\wedge = \mathbb{Q}\pi_1(X, x)^\wedge,$$

and that the augmentation

$$H_0(P_{x,x}X, \mathbb{Q})^\wedge \longrightarrow \mathbb{Q} \longrightarrow 0$$

is a homomorphism of $\mathbb{Q}\pi_1(X, x)^\wedge$-modules that preserves filtrations. Now tensor with A_x (a constant mixed Hodge structure) and apply (6.1) to obtain the map.

The final step is to show

(7.2) Proposition. *If A is a lower complete variation and $x \in X$, then $H_0(P_{x,-}X; A_x)^\wedge$ is a relatively projective object of $\mathcal{H}(X)$.*

Proof: Suppose that $B \longrightarrow C$ is an epimorphism in $\mathcal{H}(X)$, and that $H_0(P_{x,-}X; A) \longrightarrow C$ is a morphism in (say) $\mathcal{H}_{\mathbb{Q}}(X)$. By the classification theorem, the problem of constructing a lift $H_0(P_{x,-}X; A) \longrightarrow B$ such that the diagram

$$\begin{array}{ccccc} & & H_0(P_{x,-}X; A) & & \\ & \swarrow & \downarrow \phi & & \\ B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

is equivalent to constructing the commutative diagram of $\mathbf{Q}\pi_1(X,x)\hat{}$-modules below, where the base and altitude are given:

$$\begin{array}{ccccc} & & A_x \otimes \mathbf{Q}\pi_1(X,x)\hat{} & & \\ & \swarrow & \downarrow \phi & & \\ B_x & \longrightarrow & C_x & \longrightarrow & 0 \end{array}$$

Using the existence of relative projective for the simplest case, $X = \{x\}$, we obtain a lift $A_x \longrightarrow B_x$ in $\mathcal{H}(x)$ of the composition $A_x \hookrightarrow A_x \otimes \mathbf{Q}\pi_1(X,x)\hat{} \longrightarrow C_x$:

$$\begin{array}{ccccc} & & A_x & & \\ & \swarrow & \downarrow \phi & & \\ B_x & \longrightarrow & C_x & \longrightarrow & 0. \end{array}$$

Now define $\tilde{\phi} : A \otimes \mathbf{Q}\pi_1(X,x)\hat{} \longrightarrow B_x$ by

$$a \otimes g \mapsto \hat{\phi}(a) \cdot g.$$

This clearly (i) is a $\mathbf{Q}\pi_1(X,x)\hat{}$-module homomorphism, (ii) lifts ϕ. In addition, $\tilde{\phi}$ preserves the filtrations. If $a \in W_l A$, $u \in W_m \mathbf{Q}\pi_1(X,x)\hat{}$, then $\phi(a) \in W_l C_x$ and $\phi(a \otimes u) \in W_{l+m} H_0(P_{x,x}X; A)$, and so $\hat{\phi}(a) \in W_l B_x$. Since the module map $B_x \otimes \mathbf{Q}\pi_1(X,x)\hat{} \longrightarrow B_x$ preserves the filtrations,

$$\tilde{\phi}(a \otimes u) = \hat{\phi}(a) \cdot u \in W_{l+m} B_x.$$

Since

$$W_k(A_x \otimes \mathbf{Q}\pi_1(X,x)\hat{}) = \sum_{l+m=k} W_l A_x \otimes W_m \mathbf{Q}\pi_1(X,x)\hat{},$$

this shows that $\tilde{\phi}$ is a morphism in $\mathcal{H}_{\mathbf{Q}}(X)$. Lifts in $\mathcal{H}_F$ and $\mathcal{H}_{\mathbf{C}}$ are treated in the same way.

(7.3) Remark. *Similarly, we can define relative injectives. A dual argument shows that the dual canonical variation, $H^0(P_{x,-}X; A)$ is injective and that*

$$0 \longrightarrow A \longrightarrow H^0(P_{x,-}X; A)$$

is a morphism in $\mathcal{H}(X)$. The fiber can be identified with the space of closed iterated integrals, $H^0B(A^\bullet X)$.

Sometimes it is convenient to use the bar resolution, which gives a relatively projective resolution in $\mathcal{H}(X)$: Fix $x \in X$ and consider

$$B_\bullet(A_x, \mathbf{Q}\pi_1(X,x)\hat{}, H_0(P_{x,-}X\hat{}) \longrightarrow A.$$

The piece of degree n is

$$A_x \otimes (\bigotimes^n J(X,x)) \otimes H_0(P_{x,-}X)\hat{}.$$

8. From $\mathrm{Ext}_{\mathcal{H}}$ to $\mathrm{Ext}_{\mathcal{MH}}$.

We shall now construct a natural transformation $c: \mathrm{Ext}^n_{\mathcal{H}} \longrightarrow \mathrm{Ext}^n_{\mathcal{MH}}$. Throughout this section X is a fixed variety and $\mathcal{H} = \mathcal{H}^{-\infty}(X)$. To begin, let $P(B)$ be a relative projective resolution for B, let

$$E: \qquad 0 \longrightarrow A \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_0 \longrightarrow B \longrightarrow 0$$

be an n-extension of B by A, and let $K(E)$ be the complex

$$A \longrightarrow E_{n-1} \cdots \longrightarrow E_0$$

with canonical augmentation $K(E) \longrightarrow B$. Construct chain maps $\phi^{\mathbf{Q}}: P(B) \longrightarrow K(E)$ and $\phi^F: P(B) \longrightarrow K(E)$ which lift $id: B \longrightarrow B$ in $\mathcal{H}_{\mathbf{Q}}$ and $\mathcal{H}_F$, respectively. Since both maps lift the identity in $\mathcal{H}_{\mathbf{C}}$, they must be homotopic there. Let $\phi^{\mathbf{C}}: P(B) \longrightarrow E[1]$ be such a homotopy, and define a characteristic class

$$c(E) = (\ \phi^{\mathbf{Z}}_n,\ \phi^F_n,\ \phi^{\mathbf{C}}_{n-1}\ )$$

We claim that $c(E)$ is a well-defined cocycle in $\mathrm{RHom}_{\mathcal{MH}}(B,A)$, and that c is a homomorphism of groups. For the first verification, observe that

$$D(\ \phi^{\mathbf{Z}}_n,\ \phi^F_n,\ \phi^{\mathbf{C}}_{n-1}\ ) = (\ \phi^{\mathbf{Z}}_n\partial,\ \phi^F_n\partial,\ \phi^{\mathbf{Z}}_n - \phi^F_n - \phi^{\mathbf{C}}_{n-1}\partial\ )$$

where D is the differential in $\mathrm{RHom}_{\mathcal{MH}}$. The fact that $E_{n+1} = 0$, that $\phi^{\mathbf{Z}}$ and ϕ^F are chain maps, and that $\phi^{\mathbf{C}}$ is a homotopy imply that $Dc(E) = 0$, as required.

Now suppose that $(\ \phi^{\mathbf{Z}}_n,\ \phi^F_n,\ \phi^{\mathbf{C}}_{n-1}\ )$ and $(\ \overline{\phi}^{\mathbf{Q}}_n,\ \overline{\phi}^F_n,\ \overline{\phi}^{\mathbf{C}}_{n-1}\ )$ are two representatives for $c(E)$. Because $\phi^{\mathbf{Q}}$ and $\overline{\phi}^{\mathbf{Q}}$ are both liftings of the identity in $\mathcal{H}_{\mathbf{Q}}$, there is a homotopy $\eta^{\mathbf{Q}}$ from one to the other. We may express this as $D\eta^{\mathbf{Q}} = \phi^{\mathbf{Q}} - \overline{\phi}^{\mathbf{Q}}$, where D is the natural differential in $\mathrm{Hom}(P(B),E)$. Construct also η^F with $D\eta^F = \phi^F - \overline{\phi}^F$. Since $\phi^{\mathbf{Q}}$ is homotopic to ϕ^F, we have $D\phi^{\mathbf{C}} = \phi^{\mathbf{Q}} - \phi^F$, and, for the same reasons, $D\overline{\phi}^{\mathbf{C}} = \overline{\phi}^{\mathbf{Q}} - \overline{\phi}^F$. Then

$$D(\phi^{\mathbf{C}} - \overline{\phi}^{\mathbf{C}}) = (\phi^{\mathbf{Q}} - \overline{\phi}^{\mathbf{Q}}) - (\phi^F - \overline{\phi}^F) = D(\eta^{\mathbf{Q}} - \eta^F),$$

so that $D\lambda = 0$ in $\mathcal{H}_{\mathbf{C}}$, where $\lambda = \phi^{\mathbf{C}} - \overline{\phi}^{\mathbf{C}} - \eta^{\mathbf{Q}} + \overline{\eta}^F$. We claim that there is a solution to the equation $D\mu = \lambda$. If so, we have

$$D(\eta^{\mathbf{Q}}, \eta^F, \mu^{\mathbf{C}}) = (D\eta^{\mathbf{Q}}, D\eta^F, \eta^{\mathbf{Q}} - \eta^F + D\mu^{\mathbf{C}}) = (\phi^{\mathbf{Q}}, \phi^F, \phi^C) - (\overline{\phi}^{\mathbf{Q}}, \overline{\phi}^F, \overline{\phi}^C),$$

as required. To justify the claim, we note that $(D\lambda)_0 = \partial\lambda_0$, where the subscript denotes the degree of the domain of the map. Consequently the diagram below exists and commutes:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & B \\ & & \downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \lambda_0} & & \downarrow{\scriptstyle 0} \\ \cdots & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & E_0 \end{array}$$

Since λ and the zero map both lift $0: B \longrightarrow E_0$, and since P is projective in $\mathcal{H}_{\mathbf{C}}$, the equation $D\mu = \lambda$ is solvable, as claimed.

We leave as an exercise the verification that $\mathrm{Ext}_{\mathcal{MH}}$ and c are independent of the resolution; for the former, see (4.3).

9. From $\mathrm{Ext}_{\mathcal{MH}}$ to $\mathrm{Ext}_{\mathcal{H}}$.

We shall now construct a natural transformation from $\mathrm{Ext}_{\mathcal{MH}}$ to $\mathrm{Ext}_{\mathcal{H}}$.

To begin, we define the *twisted direct sum*

$$M = T \oplus_\eta S, \tag{9.1}$$

where S and T are in $\mathcal{H}$ and $\eta : S \longrightarrow T$ is in $\mathcal{H}_{\mathbf{C}}$. For the complex local system, set $M = T \oplus S$, and define $F^p\mathcal{M} = F^p\mathcal{T} \oplus F^p\mathcal{S}$. For the rational structure, set

$$M_{\mathbf{Q}} = T_{\mathbf{Q}} + (\eta, 1)S_{\mathbf{Q}}.$$

and define the weight filtration by

$$W_l M = W_l T \oplus (\eta, 1) W_l S.$$

(9.2) Lemma. *$T \oplus_\eta S$ is in $\mathcal{H}$, as is the extension*

$$0 \longrightarrow T \xrightarrow{\ i\ } T \oplus_\eta S \xrightarrow{\ \pi\ } S \longrightarrow 0.$$

Proof: Because η is a map of $\mathbf{C}$-local systems, it commutes with the connection operators on S and T. Therefore the lattice and weight structure just defined are flat. Moreover, since both the Hodge filtration and the connection are given by direct sum, Griffiths transversality holds. To see that the Hodge and weight filtrations give a mixed Hodge structure on fibers, observe that the maps i and π of the lemma are strictly compatible with the Hodge and weight filtration. But if the end terms of an such exact sequence are mixed Hodge structures, then so is the middle term [8].

Next, we define *twisted pushouts*, an extension of the usual notion of pushout.

(9.3) Definition. *Consider the diagram D below,*

$$\begin{array}{ccc} R & \xrightarrow{\ \partial\ } & S \\ {\scriptstyle \rho^F,\, \rho^{\mathbf{Q}}}\big\downarrow & \swarrow {\scriptstyle \eta^{\mathbf{C}}} & \\ T & & \end{array}$$

where the superscript indicates the category to which the morphism belongs; if there is no superscript, then the arrow is in $\mathcal{H}$. If

$$\rho^{\mathbf{Q}} - \rho^F = \eta^{\mathbf{C}} \circ \partial,$$

then $D = (T, R, S)$ *is a set of* pushout data.

(9.4) Definition. *A* pushout diagram $\epsilon : (T, R, S) \longrightarrow X$ *is a diagram*

$$\begin{array}{ccc} R & \xrightarrow{\partial} & S \\ {\scriptstyle \rho^F,\, \rho^{\mathbf{Q}}}\downarrow & \swarrow \eta^{\mathbf{C}} & \downarrow{\scriptstyle \sigma^F,\sigma^{\mathbf{Q}}} \\ T & \xrightarrow{\partial'} & X \end{array}$$

where

a) $\sigma^F \circ \partial = \partial' \circ \rho^F$,

b) $\sigma^{\mathbf{Q}} \circ \partial = \partial' \circ \rho^{\mathbf{Q}}$,

c) $\partial' \circ \eta^{\mathbf{C}} = \sigma^{\mathbf{Q}} - \sigma^F$.

(9.5) Definition. *A* morphism *of pushout diagrams,* $\lambda : \epsilon \longrightarrow \epsilon_1$ *is an* $\mathcal{H}$*-morphism* $\lambda : X \longrightarrow X_1$ *such that*

a) $\lambda \circ \sigma^F = {\sigma_1}^F$,

b) $\lambda \circ \sigma^{\mathbf{Q}} = {\sigma_1}^{\mathbf{Q}}$,

c) $\lambda \circ \partial' = \partial'_1$

(9.6) Definition. *A* pushout *for* (T, R, S) *is a pushout diagram* $\epsilon_U : (T, R, S) \longrightarrow U$ *for which the following hold:*

a) *Structure :* σ^F *is an* $\mathcal{H}$*-morphism.*

b) *Factorization: If* $\epsilon_V : (T, R, S) \longrightarrow V$ *is a pushout diagram, then there is an* $\mathcal{H}$*-morphism* $\lambda : \epsilon_U \longrightarrow \epsilon_V$.

c) *Functoriality: If* $\alpha : \epsilon_1 \longrightarrow \epsilon_2$ *is a morphism of pushout diagrams, then there is the canonical diagram*

$$\begin{array}{ccc} \epsilon_U & \longrightarrow & \epsilon_1 \\ \downarrow & \swarrow & \\ \epsilon_2 & & \end{array}$$

commutes.

Thus, a pushout is a universal pushout diagram.

(9.7) Proposition. *For every set of pushout data* (T, R, S) *there is a pushout* $\epsilon_U : (T, R, S) \longrightarrow U$.

Proof:

Construction of U

Form $\tilde{U} = T \oplus_\eta S$, and define a map $i : R \longrightarrow \tilde{U}$ by

$$x \mapsto (\rho^F x, \, -\partial x).$$

One verifies that i is a morphism of variations. Let us check only that i preserves the rational structure. Take a local section ζ of $L_{\mathbf{Q}}$, and note that

$$\begin{aligned} i(\zeta) &= (\rho^F \zeta, -\partial\zeta) \\ &= (\rho^F \zeta - \rho^{\mathbf{Q}}\zeta + \rho^{\mathbf{Q}}\zeta, -\partial\zeta) \\ &= (-\eta^{\mathbf{C}}\partial\zeta + \rho^{\mathbf{Q}}\zeta, -\partial\zeta) \\ &= (\rho^{\mathbf{Q}}\zeta, \, 0) - (\eta^{\mathbf{C}}, \, 1)(\partial\zeta) \; \in \; \tilde{P}_{\mathbf{Q}} \end{aligned}$$

Then

$$U = \tilde{U}/iR.$$

is a variation of mixed Hodge structures. Note that a variation of mixed Hodge structure is unipotent if and only if the induced connection on the graded quotients of the weight filtration are zero. Therefore this property is satisfied by U if it is satisfied by S and T.

Construction of the maps

Let us denote the class of $(x, y) \in \tilde{U}$ in U by $[x, y]$. Define

$$\begin{aligned} \sigma^{\mathbf{Q}} y &= [\eta^{\mathbf{C}} y, y] \\ \sigma^F y &= [0, y] \\ \partial x &= [x, 0] \end{aligned}$$

By construction, part (a) of definition (9.4)is satisfied. For part (b), we find

$$\begin{aligned} \sigma^{\mathbf{Q}} \circ \partial w &= [\eta^{\mathbf{C}} \partial w, \partial w] \\ &= [\rho^{\mathbf{Q}} w - \rho^F w, \partial w] \\ &= [\rho^Z w, 0] \\ &= \partial \rho^{\mathbf{Q}} w, \end{aligned}$$

as required.

For (c) we have

$$\begin{aligned} \partial \eta^{\mathbf{C}} y &= [\eta^{\mathbf{C}} y, 0] \\ &= [\eta^{\mathbf{C}} y, y] - [0, y] \\ &= (\sigma^{\mathbf{Q}} - \sigma^F)(y). \end{aligned}$$

Factorization

Let $\epsilon_V : (T, R, S) \longrightarrow V$ be a pushout diagram for (T, R, S). Define $\lambda : U \longrightarrow V$ by

$$\lambda(x, y) = \partial x + \sigma^F y.$$

Note that λ preserves the rational structure:

$$\lambda(\phi^{\mathbf{C}}y, y) = \partial \circ \phi^{\mathbf{C}}y + \sigma^F y = \phi^{\mathbf{Q}}y.$$

In addition, $\lambda \circ i = 0$, since

$$\lambda \circ iw = \lambda(\rho^F w, -\partial w) = \partial\rho^F w - \sigma^F \partial w.$$

One easily verifies that λ is part of a morphism of pushout diagrams:

i) $\lambda\sigma^F y = \lambda[0, y] = \sigma^F y$,

ii) $\lambda\sigma^{\mathbf{Q}} y = \lambda[\eta^{\mathbf{C}}y, y] = \partial\eta^{\mathbf{C}}y + \sigma^F y = \sigma^{\mathbf{Q}}y$, and

iii) $\lambda\partial x = \lambda[x, 0] = \partial x$.

Functoriality

Suppose given pushout diagrams ϵ_1 and ϵ_2 and a morphism $\phi : \epsilon_1 \longrightarrow \epsilon_2$. Let V_1 and V_2 be the targets of the pushout diagram, and let $\phi : V_1 \longrightarrow V_2$ be the natural $\mathcal{H}$-morphism. Let $\lambda_1 : U \longrightarrow V_1$ and $\lambda_2 : U \longrightarrow V_2$ be canonical factors. Then

$$\begin{aligned} \phi\lambda_1[x, y] &= \phi(\partial_1 x + \sigma_1^F y) \\ &= \phi\partial_1 x + \phi\sigma_1^F y \\ &= \partial_2 x + \sigma_2^F y \\ &= \lambda_2[x, y], \end{aligned}$$

as required.

This completes the proof of Proposition (9.7).

(9.8) Proposition. *If $\epsilon_U : (T, R, S) \longrightarrow U$ is a pushout then* coker $\partial \cong$ coker ∂'.

We omit the proof.

We can now construct a natural transformation $p : \mathrm{Ext}^n_{\mathcal{MH}} \longrightarrow \mathrm{Ext}^n_{\mathcal{H}}$. Let $\phi_n = (\phi_n^{\mathbf{Q}}, \phi_n^F, \phi_{n-1}^{\mathbf{C}})$ be an n-cocycle in $R\,\mathrm{Hom}_{\mathcal{MH}}(B, A)$, calculated with respect to a relative projective resolution $P_\bullet \longrightarrow B$. Apply the above result to the triple $(T, R, S) = (A, P_n, P_{n-1})$ to form a twisted pushout U, and then use this to construct an exact commutative diagram

$$\begin{array}{ccccccccccc} P_n & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & B \\ \downarrow{\scriptstyle \phi_n^{F,\mathbf{Q}}} & & \downarrow{\scriptstyle \phi_{n-1}^{F,\mathbf{Q}}} & & \downarrow{\scriptstyle id} & & & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\ A & \longrightarrow & U & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & B \end{array} \tag{9.9}$$

in which the rows are in $\mathcal{H}$. The bottom row is $p(\phi_n)$.

As usual there are a number of verifications:

i) If ϕ_n and ϕ'_n are cohomologous, then $p(\phi_n)$ and $p(\phi'_n)$ are Yoneda equivalent.

ii) Independence of the projective resolution.

We leave the verification of item (i) to the reader.

For item (ii), suppose that $P_\bullet$ and $Q_\bullet$ are two relative projective resolutions for A, with a commutative diagram

$$\begin{array}{ccc} P_\bullet & \xrightarrow{\mu} & Q_\bullet \\ \downarrow & & \downarrow \\ A & \xrightarrow{id} & A \end{array}$$

Let ϕ_n^P and ϕ_n^Q be the respective cocycles. Then $\mu^*(\phi_n^Q)$ and ϕ_n^P are (we claim) cohomologous, so that, by (i), $p(\phi_n^P)$ and $p(\phi_n^Q)$ are Yoneda equivalent.

(9.10) Lemma. $c \circ p = 1$

(9.11) Corollary. $p : \mathrm{Ext}^n_{\mathcal{MH}} \longrightarrow \mathrm{Ext}^n_{\mathcal{H}}$ *is injective.*

Proof: Begin with a cocycle $\phi_n = (\phi_n^{\mathbf{Q}}, \phi_n^F, \phi_{n-1}^{\mathbf{C}})$ and construct the pushout for $p(\phi_n)$ as in (9.9). Let $\phi_{n-1} = (\sigma^{\mathbf{Q}}, \sigma^F, 0)$, and set $\phi_k = (id, id, 0)$ for $k < n-1$. Using Proposition (9.7), we verify that the triple $\phi = (\phi^{\mathbf{Q}}, \phi^F, \phi^{\mathbf{C}})$ consists of two liftings of $id : B \longrightarrow B$ in $\mathcal{H}_{\mathbf{Q}}$ and $\mathcal{H}_F$, respectively, and a homotopy $\phi^{\mathbf{C}}$ between them. Consequently ϕ_n is the n-cocycle canonically associated to the extension $p(\phi_n)$. This concludes the proof.

We now pass to the proof of theorem (4.5), for which we must show that $p \circ c = 1$. The key step is to construct a relative projective resolution for which the chain map ϕ has the form $(\phi^{\mathcal{H}}, \phi^{\mathcal{H}}, 0)$ with $\phi^{\mathcal{H}}$ a morphism in the category of complexes on $\mathcal{H}$. Such a resolution, which we shall call *adapted* to the n-extension, yields a Deligne-Beilinson cocycle

$$\phi_n = \begin{bmatrix} & 0 & \\ \phi_n^{\mathcal{H}} & & \phi_n^{\mathcal{H}} \end{bmatrix}$$

Since $H^\bullet(\mathrm{RHom}(B, A))$, p, and c are independent of the resolution, we can use an adapted resolution for the proof, assuming that such exists.

Suppose for a moment that this is the case. Then the pushout construction gives a commutative diagram

$$(9.12)\qquad \begin{array}{ccccccccccc} P_n & \longrightarrow & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ A & \longrightarrow & U & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & B \end{array}$$

with all morphisms in $\mathcal{H}$. From the universal property of the pushout one obtains a further diagram

$$(9.13)\qquad \begin{array}{ccccccccccc} A & \longrightarrow & U & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ A & \longrightarrow & E_{n-1} & \longrightarrow & E_{n-2} & \longrightarrow & \cdots & \longrightarrow & E_0 & \longrightarrow & B \end{array}$$

But this is an elementary equivalence, as required. This shows that, for extension classes containing an adapted resolution, $p \circ c = id$.

To show that adapted resolutions exist, we first observe the following.

(9.14) Lemma. *Let R, S and T be objects in $\mathcal{A}$, and suppose given $\mathcal{A}$-morphisms $R \longrightarrow S$ and $T \longrightarrow S$. Then there exists a relative projective P and $\mathcal{A}$-morphisms $P \longrightarrow R$, $P \longrightarrow T$ such that the diagram below commutes:*

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & T \\ \downarrow & & \downarrow \\ R & \xrightarrow{\beta} & S \end{array}$$

Moreover, if β is surjective, then so is α.

Proof: First construct the pullback diagram

$$\begin{array}{ccc} P' & \longrightarrow & T \\ \downarrow & & \downarrow \\ R & \longrightarrow & S \end{array}$$

Next, construct $P \longrightarrow P'$ with P' relatively projective. Finally, substitute P for P' in the pullback diagram. The surjectivity assumption is easily verified, thus completing the proof.

We complete the existence proof by an induction argument. Begin with an n-extension

$$\cdots \longrightarrow E_2 \longrightarrow E_1 \longrightarrow E_0 \xrightarrow{\epsilon} B \longrightarrow 0,$$

choose a relative projective resolution $\phi_0 : P_0 \longrightarrow E_0 \longrightarrow 0$, and construct the commutative diagram

$$\begin{array}{ccccc} P_0 & \xrightarrow{\epsilon\circ\phi} & B & \longrightarrow & 0 \\ {\scriptstyle \phi_0}\downarrow & & \| & & \\ E_0 & \xrightarrow{\epsilon} & B & \longrightarrow & 0 \end{array}$$

This gives stage 0 of an adapted resolution. Next, suppose that an adapted resolution has been constructed through stage $l-1$, so that we have the diagram below, commutative in $\mathcal{H}(X)$, and with exact rows:

$$(9.15)\qquad \begin{array}{ccccccccc} P_{l-1} & \longrightarrow & P_{l-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & B \\ \downarrow & & \downarrow & & & & \downarrow & & \| \\ E_{l-1} & \longrightarrow & E_{l-2} & \longrightarrow & \cdots & \longrightarrow & E_0 & \longrightarrow & B \end{array}$$

Set

$$K_{l-1}(P) = \ker\{ P_{l-1} \longrightarrow P_{l-2} \}$$

$$K_{l-1}(E) = \ker\{ E_{l-1} \longrightarrow E_{l-2} \}$$

These are objects of $\mathcal{H}(X)$, and the morphism $P_{l-1} \longrightarrow E_{l-1}$ induces a morphism $K_{l-1}(P) \longrightarrow K_{l-1}(E)$ in $\mathcal{H}(X)$. Thus we can apply the lemma above to form a commutative square

$$(9.16)\qquad \begin{array}{ccc} P_l & \xrightarrow{\alpha} & K_{l-1}(P) \\ \downarrow & & \downarrow \\ E_l & \xrightarrow{\beta} & K_{l-1}(E) \end{array}$$

Adjoin this square to the diagram (9.15) to complete the inductive step, and, with it, the proof.

10. Products and Functoriality.

Suppose that X and Y are smooth varieties. Let $p : X \times Y \longrightarrow X$ and $q : X \times Y \longrightarrow Y$ be the canonical projections. There is a natural functor

$$\mathcal{H}^{-\infty}(X) \times \mathcal{H}^{-\infty}(Y) \longrightarrow \mathcal{H}^{-\infty}(X \times Y)$$

that takes $(A,\ B)$ to $p^*A \otimes q^*B$. This induces a pairing

$$\text{(10.1)} \qquad \mathrm{Ext}^n_{\mathcal{H}(X)}(A, B) \otimes \mathrm{Ext}^n_{\mathcal{H}(Y)}(C, D) \longrightarrow \mathrm{Ext}^n_{\mathcal{H}(X\times Y)}(p^*A \otimes q^*C,\ p^*B \otimes q^*D).$$

which takes two extensions to their tensor product, which, by the Kunneth theorem, is again an extension.

On the other hand we can choose a relatively projective resolutions $P_\bullet A$ and $Q_\bullet C$ in $\mathcal{H}^{-\infty}(X)$ whose tensor product $p^*P_\bullet \otimes q^*Q_\bullet$ is a a resolution of $p^*A \otimes q^*C$ in $\mathcal{H}^{-\infty}(X \times Y)$. By lemma (9.14) there is a relatively projective resolution $R_\bullet \longrightarrow p^*A \otimes q^*C$ in $\mathcal{H}^{-\infty}(X \times Y)$ with a morphism $R_\bullet \longrightarrow p^*P_\bullet \otimes q^*Q_\bullet$ in $\mathcal{H}^{-\infty}(X \times Y)$. Using $P_\bullet$ to compute $\mathrm{RHom}_{\mathcal{H}(X)}(A, B)$, $Q_\bullet$ to compute $\mathrm{RHom}_{\mathcal{H}(Y)}(C,\ D)$, and $R_\bullet$ to compute $\mathrm{RHom}_{\mathcal{H}(Y)}(p^*A \otimes q^*C,\ p^*B \otimes q^*D)$, any one of the usual formulae for cup product in Deligne cohomology [1] then yields a pairing

$$\text{(10.2)} \quad \cup_\alpha : \mathrm{RHom}_{\mathcal{H}(X)}(A, B) \times \mathrm{RHom}_{\mathcal{H}(Y)}(C,\ D) \longrightarrow \mathrm{RHom}_{\mathcal{H}(X\times Y)}(p^*A \otimes q^*C,\ p^*B \otimes q^*D)$$

where $0 \leq \alpha \leq 1$.

(10.3) Proposition. *For each $\alpha \in [0,1]$, the pairing* (10.2) *induces the pairing* (10.1) *via the isomorphism* (4.5).

The proof is straightforward.

Now suppose that $f : X \longrightarrow Y$ is a morphism of smooth varieties. The functor

$$f^* : \mathcal{H}^{-\infty}(Y) \longrightarrow \mathcal{H}^{-\infty}(X)$$

is exact and thus induces a functor

$$\text{(10.4)} \qquad f^* : \mathrm{Ext}_{\mathcal{H}(Y)}(A, B) \longrightarrow \mathrm{Ext}_{\mathcal{H}(X)}(f^*A, f^*B).$$

Choose a relatively projective resolution $P_\bullet \longrightarrow A$ in $\mathcal{H}^{-\infty}(Y)$. By (9.14) we can find a relatively projective resolution $Q_\bullet \longrightarrow f^*A$ in $\mathcal{H}^{-\infty}(X)$ and a morphism $Q_\bullet \longrightarrow f^*P_\bullet$ in $\mathcal{H}^{-\infty}(Y)$. One obtains a chain map

$$\text{(10.5)} \qquad f^* : \mathrm{RHom}_{\mathcal{H}(Y)}(A,\ B) \longrightarrow \mathrm{RHom}_{\mathcal{H}(X)}(f^*A,\ f^*B)$$

which one can easily show is well-defined in the derived category of abelian groups.

(10.6) Proposition. *The chain map* (10.5) *induces* (10.4) *via the isomorphism* (4.5).

We omit the proof.

If X is a smooth variety we obtain products

$$\mathrm{Ext}^m_{\mathcal{H}(X)}(A,\,B)\otimes \mathrm{Ext}^n_{\mathcal{H}(X)}(C,\,D) \longrightarrow \mathrm{Ext}^{m+n}_{\mathcal{H}(X)}(A\otimes C,\,B\otimes D) \tag{10.7}$$

by composing the natural map (10.1) with the map

$$\Delta^* : \mathrm{Ext}^\bullet_{\mathcal{H}(X\times X}(p^*A\otimes q^*C,\, p^*B\otimes q^*D) \longrightarrow \mathrm{Ext}^\bullet_{\mathcal{H}(X)}(A\otimes C,\,B\otimes D)$$

induced by the diagonal. Combining (10.3) and (10.6) we obtain

(10.8) Proposition. *The natural product* (10.7) *is induced by*

$$\cup_\alpha : \mathrm{RHom}_{\mathcal{H}(X)}(A,\,B)\otimes \mathrm{RHom}_{\mathcal{H}(X)}(C,\,D) \longrightarrow \mathrm{RHom}_{\mathcal{H}(X)}(A\otimes C,\,B\otimes D)$$

for each $\alpha\in[0,1]$.

11. Absolute Hodge Cohomology of π_1.

The material of this section is described in more detail in [11]. Suppose that X is a finite complex and $k = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{M}_k(X)$ be the category of k-local systems $\mathbb{M}$ of finitely generated torsion-free k-modules over X that have a finite filtration $W_\bullet$ by sublocal systems satisfying

a) $0 = W_{l-1}\mathbb{M}\subset W_l\mathbb{M}\subset\cdots\subset W_m\mathbb{M} = \mathbb{M}$.

b) $\mathrm{Gr}^W_r\,\mathbb{M}$ is a trivial local system for each r, finite dimensional over k.

As in (5.1) we enlarge the category $\mathcal{M}_k^{-\infty}(X)$ to obtain categories $\mathcal{M}_k^{+\infty}$, $\mathcal{M}_k^{-\infty}$, $\mathcal{M}\hat{}_k$. When X is a point we shall generally omit the 'X'.

Now let $\mathrm{Rep}_k^{-\infty}(X,x)$ be the category of lower complete continuous $k\pi_1(X,x)$-modules. One easily verifies the following.

(11.1) Proposition. a) *The correspondence* $\mathbb{M}\mapsto\mathbb{M}_x$ *defines a functor*

$$\mathcal{M}_k^{-\infty}(X) \longrightarrow \mathrm{Rep}_k^{-\infty}(X,x)$$

which is an equivalence of categories. b) [11 §3] The categories $\mathcal{M}_k^{-\infty}(X)$ *and* $\mathrm{Rep}_k^{-\infty}(X,x)$ *are abelian.*

If X is a smooth variety, we have forgetful functors $\mathcal{H}(X)\longrightarrow\mathcal{M}_k(X)$ for $k=\mathbb{Q}$, $\mathbb{C}$, etc.

(11.2) Proposition. *If* X *is a smooth variety and* A, B *are objects of* $\mathcal{H}(X)$, *then* $\mathrm{Ext}^\bullet_{\mathcal{M}_{\mathbb{Q}}^{-\infty}(X)}(A,B)$ *has a natural mixed Hodge structure (in* $\mathcal{H}^{+\infty}$*).*

Proof: If A is an object of $\mathcal{H}^{-\infty}(X)$, then, by (7.1), (7.2), and (4.3), we can find a resolution $P_\bullet \longrightarrow A$ in $\mathcal{H}^{-\infty}$ that is projective in $\mathcal{M}_k^{-\infty}(X)$. Consequently, if B is an object of $\mathcal{H}^{+\infty}(X)$, then

$$\mathrm{RHom}_{\mathcal{M}_{\mathbb{Q}}^{-\infty}(X)}(A,B) := \mathrm{Hom}_{\mathcal{M}_{\mathbb{Q}}^{-\infty}(X)}(P_\bullet,\, B) \in \mathrm{Ob}\,\mathcal{H}^{+\infty}.$$

From (4.3) it follows already that this complex is well defined in the derived category of $\mathcal{H}$. Consequently

$$\mathrm{Ext}^n_{\mathcal{M}_{\mathbb{Q}}^{-\infty}(X)}(A,\, B) = H^n(\mathrm{RHom}_{\mathcal{M}_{\mathbb{Q}}^{-\infty}(X)}(A,\, B))$$

has a well-defined mixed Hodge structure in $\mathcal{H}$.

(11.3) Theorem. *If X is a smooth variety and A, B are objects in $\mathcal{H}(X)$, then there is a natural short exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathcal{H}}(\mathbb{Q}, \mathrm{Ext}^{n-1}_{\mathcal{M}_{\mathbb{Q}}^{-\infty}(X)}(A,B)) \to \mathrm{Ext}^n_{\mathcal{H}(X)}(A,B) \to \mathrm{Ext}^0_{\mathcal{H}}(\mathbb{Q}, \mathrm{Ext}^n_{\mathcal{M}_{\mathbb{Q}}^{-\infty}(X)}(A,B)) \to 0$$

Proof: Since $\mathrm{RHom}_{\mathcal{H}(X)}(A,B)$ is a cone, there is a long exact sequence of cohomology groups which calculates $\mathrm{Ext}^\bullet_{\mathcal{H}(X)}(A,B)$. Apply proposition (11.2) to the remaining terms to get the sequence

$$\cdots \longrightarrow W_0\,\mathrm{Ext}^{n-1}_{\mathcal{M}_{\mathbb{Q}}^{-\infty}}(A,B) \oplus F^0 W_0\,\mathrm{Ext}^{n-1}_{\mathcal{M}_{\mathbb{Q}}^{-\infty}}(A,B) \longrightarrow W_0\,\mathrm{Ext}^{n-1}_{\mathcal{M}_{\mathbb{C}}^{-\infty}}(A,B) \longrightarrow \mathrm{Ext}^n_{\mathcal{H}}(A,B) \longrightarrow$$
$$\longrightarrow W_0\,\mathrm{Ext}^{n}_{\mathcal{M}_{\mathbb{Q}}^{-\infty}}(A,B) \oplus F^0 W_0\,\mathrm{Ext}^{n}_{\mathcal{M}_{\mathbb{Q}}^{-\infty}}(A,B) \longrightarrow W_0\,\mathrm{Ext}^{n}_{\mathcal{M}_{\mathbb{C}}^{-\infty}}(A,B) \longrightarrow \cdots.$$

From this long exact sequence form the natural short exact sequences and apply (2.1) to obtain the required result.

Give $\pi_1(X,x)$ the topology induced by the natural map $\pi_1(X,x) \longrightarrow \mathbb{Q}\pi_1(X,x)\hat{}$, where the group ring is, as usual, given the J-adic topology. This topology, which may not be separated, is equivalent to the W-adic topology, since $J^{-2l} \subset W_l \subset J^{-l}$.

(11.4) Corollary. *The continuous cohomology $H^\bullet_{cont}(\pi_1(X,x),\,\mathbb{Q})$ has a natural mixed Hodge structure (in $\mathcal{H}^{+\infty}$) which is independent of x. In other words, each canonical isomorphism $\pi_1(X,y) \longrightarrow \pi_1(X,x)$ given by a path from x to y in X induces an isomorphism of mixed Hodge structures on cohomology.*

Proof: This follows from the canonical isomorphism

$$H^\bullet_{cont}(\pi_1(X,x),\,\mathbb{Q}) \cong \mathrm{Ext}^\bullet_{\mathcal{M}_{\mathbb{Q}}^{-\infty}(X)}(\mathbb{Q},\,\mathbb{Q}).$$

In view of the corollary, there is no ambiguity when we write $H^\bullet_{cont}(\pi_1(X),\,\mathbb{Q})$. Now define the absolute Hodge cohomology $H_{\mathcal{H}}(\pi_1(X),\mathbb{Q}(p))$ of $\pi_1(X)$ to be the homology of the complex

$$\mathrm{cone}[\, W_{2p}\,\mathrm{Hom}_{\mathcal{M}_{\mathbb{Q}}^{-\infty}(X)}(P_\bullet,\mathbb{Q}) \oplus F^p W_{2p}\,\mathrm{Hom}_{\mathcal{M}_{\mathbb{C}}^{-\infty}(X)}(P_\bullet,\mathbb{C}) \xrightarrow{\delta} W_{2p}\,\mathrm{Hom}_{\mathcal{M}_{\mathbb{C}}^{-\infty}(X)}(P_\bullet,\mathbb{C})\,][-1],$$

where δ is the difference map.

(11.5) **Theorem.** *There is a natural isomorphism*

$$H^\bullet_{\mathcal{H}}(\pi_1(X), \mathbb{Q}(p)) \cong \mathrm{Ext}^\bullet_{\mathcal{H}(X)}(\mathbb{Q}, \mathbb{Q}(p))$$

which is compatible with the natural product structures on

$$H^\bullet_{\mathcal{H}}(\pi_1(X), \mathbb{Q}(\bullet)) \qquad \text{and} \qquad \mathrm{Ext}^\bullet_{\mathcal{H}(X)}(\mathbb{Q}, \mathbb{Q}(\bullet)).$$

Proof: This follows from (4.5) and the definition of $H_{\mathcal{H}}(\pi_1(X), \mathbb{Q}(p))$.

We conclude this section by relating the absolute Hodge cohomology of a smooth variety to the absolute Hodge cohomology of its fundamental group. Composing the canonical homomorphisms

$$H^\bullet_{cont}(\pi_1(X)) \longrightarrow H^\bullet(\pi_1(X))$$
$$H^\bullet(\pi_1(X)) \longrightarrow H^\bullet(X)$$

we obtain, for each smooth variety X, a canonical homomorphism

$$\theta^* : H^\bullet_{cont}(\pi_1(X)) \longrightarrow H^\bullet(X). \tag{11.6}$$

(11.7) **Theorem.** *The homomorphism* (11.6) *is a morphism of mixed Hodge structure.*

Proof: Choose a relatively projective resolution $P_\bullet \longrightarrow \mathbb{Q} \longrightarrow 0$ of $\mathbb{Q}$ in $\mathcal{H}^{-\infty}(X)$ Denote the continuous dual

$$\varinjlim \mathrm{Hom}_{\mathbb{Q}}(P_\bullet/W_l, \mathbb{Q})$$

of $P_\bullet$ by $P^\bullet$. The canonical inclusion $\mathbb{Q} \longrightarrow P^\bullet$ is a quasi-isomorphism in $\mathcal{H}^{+\infty}(X)$. Let $\mathbf{E}^\bullet(X, P^\bullet)$ be the mixed Hodge complex defined in [13, 8.6] that computes the mixed Hodge structure on the cohomology of X with with coefficients in the complex $P^\bullet$ of variations of mixed Hodge structure over X. The differential of $P^\bullet$ makes $\mathbf{E}^\bullet(X, P^\bullet)$ into a double complex, and the inclusion

$$\mathbf{E}^\bullet(X) \longrightarrow \mathbf{E}^\bullet(X, P^\bullet)$$

of a standard mixed Hodge complex for X into $\mathbf{E}^\bullet(X, P^\bullet)$ induced by $\mathbb{Q} \longrightarrow P^\bullet$ is a quasi-isomorphism. Thus we can compute the mixed Hodge structure on $H^\bullet(X)$ using $\mathbf{E}^\bullet(X, P^\bullet)$. On the other hand, the inclusion

$$\mathrm{RHom}_{\mathcal{M}(X)}(\mathbb{Q}, \mathbb{Q}) = \mathrm{Hom}_{\pi_1^\cdot}(P_\bullet, \mathbb{Q}) \hookrightarrow P^\bullet$$

induces a morphism

$$\theta : \mathrm{RHom}_{\mathcal{M}(X)}(\mathbb{Q}, \mathbb{Q}) \longrightarrow \mathbf{E}(X, P^\bullet)$$

of mixed Hodge complexes. The result follows, since θ induces (11.6) on homology.

As a corollary of the proof, we obtain

(11.8) **Theorem.** *For each $p \geq 0$, there is a natural ring homomorphism*

$$H^\bullet_{\mathcal{H}}(\pi(X), \mathbb{Q}(p)) \longrightarrow H^\bullet_{\mathcal{H}}(X, \mathbb{Q}(p)).$$

12. Applications.

A topological space is a *rational* $K(\pi,1)$ if the natural homomorphism (11.6) is an isomorphism with $\mathbb{Q}$ coefficients. (This is easily seen to be equivalent to the more familiar definition: X is a rational $K(\pi,1)$ if its Sullivan minimal model is generated by elements of degree 1.) As an immediate consequence of (11.3), (11.5), and (11.8), we have:

(12.1) Theorem. *Suppose that $p \geq 0$. If the smooth variety X is a rational $K(\pi,1)$, then there are canonical ring isomorphisms*

$$H^\bullet_{\mathcal{H}}(X, \mathbb{Q}(p)) \cong H^\bullet_{\mathcal{H}}(\pi_1(X), \mathbb{Q}(p)) \cong \mathrm{Ext}^\bullet_{\mathcal{H}(X)}(\mathbb{Q}, \mathbb{Q}(p)).$$

Since $(\mathbb{C}^*)^p$ is a rational $K(\pi,1)$, we have

(12.2) Corollary. *For all p and all $0 \leq k \leq p$, there exists a smooth variety X with*

$$\mathrm{Ext}^k_{\mathcal{H}(X)}(\mathbb{Q}, \mathbb{Q}(p)) \neq 0.$$

Since every incomplete smooth curve is a rational $K(\pi,1)$ [12, §8], we obtain

(12.3) Theorem. *If X is an incomplete smooth curve, then there is a natural ring isomorphism*

$$H^\bullet_{\mathcal{H}}(X, \mathbb{Q}(p)) \cong \mathrm{Ext}^\bullet_{\mathcal{H}(X)}(\mathbb{Q}, \mathbb{Q}(p)).$$

From the bar resolution (7.3) one obtains vanishing criteria:

(12.4) Theorem. *If X is a smooth variety, then*

$$\mathrm{Ext}^k_{\mathcal{H}(X)}(\mathbb{Q}, \mathbb{Q}(p)) = 0 \quad when \quad k > 2p.$$

If in addtion $W_1 H^1(X) = 0$ then

$$\mathrm{Ext}^k_{\mathcal{H}(X)}(\mathbb{Q}, \mathbb{Q}(p)) = 0 \quad when \quad k > p.$$

From a variation of Hodge structure $\mathcal{V}$ one can construct n-extensions in a standard way. For $n = 2$, suppose a, b are two weights, with $a < b$, and form the exact sequence:

$$0 \longrightarrow W_a \longrightarrow W_b \longrightarrow \mathcal{V}/W_a \longrightarrow \mathcal{V}/W_b \longrightarrow 0$$

We shall say that such an extension is *deduced from a variation.* The diagram below gives an equivalence of this extension with a representative of the zero class:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & W_a & \longrightarrow & W_b & \longrightarrow & \mathcal{V}/W_a & \longrightarrow & \mathcal{V}/W_b & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & W_a & \longrightarrow & W_a \oplus W_b & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{V}/W_b & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & W_a & \xrightarrow{id} & W_a & \xrightarrow{0} & \mathcal{V}/W_b & \xrightarrow{id} & \mathcal{V}/W_b & \longrightarrow & 0 \end{array}$$

We have therefore established the following vanishing criterion:

(12.5) Proposition. *The Yoneda class of a 2-extension which is deduced from a variation vanishes.*

For general $n > 2$ one finds (for example) that an extension deduced from a variation is Yoneda equivalent to zero.

We conclude by considering once again Deligne's example [7] (cf. Introduction). A variation of mixed Hodge structure $\mathcal{V}$ over X satisfying

$$(12.6) \qquad \mathrm{Gr}_l^W \mathcal{V} = \left\{ \begin{array}{ll} \mathbb{Q} & l = 0 \\ \mathbb{Q}(1)^N & l = -2 \\ \mathbb{Q}(2) & l = -4 \\ 0 & \text{otherwise} \end{array} \right\}$$

determines N pairs $\{ (f_j, g_j) \mid j = 1, \cdots N \}$ of elements of $\mathcal{O}^*(X) \otimes \mathbb{Q}$ as follows: Let $(f_1, \cdots, f_N)$ be the element of $\mathcal{O}^*(X) \otimes \mathbb{Q}$ corresponding to

$$W_{-2}\mathcal{V} \in \mathrm{Ext}^1_{\mathcal{H}(X)}(\mathbb{Q}(1)^N, \mathbb{Q}(2)) \cong \bigoplus_{j=1}^{N} \mathrm{Ext}^1_{\mathcal{H}(X)}(\mathbb{Q}(1), \mathbb{Q}(2)) \cong \mathcal{O}^*(X)^N \otimes \mathbb{Q},$$

and let $(g_1, \cdots, g_N)$ be the element corresponding to

$$\mathcal{V}/W_{-4} \in \mathrm{Ext}^1_{\mathcal{H}(X)}(\mathbb{Q}, \mathbb{Q}(1)^N) \cong \bigoplus_{j=1}^{N} \mathrm{Ext}^1_{\mathcal{H}(X)}(\mathbb{Q}, \mathbb{Q}(1)) \cong \mathcal{O}^*(X)^N \otimes \mathbb{Q}.$$

Let

$$e(\mathcal{V}) \in \mathrm{Ext}^2_{\mathcal{H}(X)}(\mathbb{Q}, \mathbb{Q}(2))$$

be the extension

$$0 \longrightarrow \mathbb{Q}(2) \longrightarrow W_{-2} \longrightarrow \mathcal{V}/W_{-4} \longrightarrow \mathbb{Q} \longrightarrow 0.$$

Then we have:

(12.7) **Proposition.** *The class of* $\epsilon(\mathcal{V})$ *in* $\mathrm{Ext}^2_{\mathcal{H}(X)}(\mathbb{Q},\mathbb{Q}(2))$ *is*

$$\epsilon(\mathcal{V}) = \sum_{j=1}^{N} f_j \cup g_j.$$

Combining this with (11.5), (11.8), and (12.5), we obtain Deligne's result:

(12.8) **Theorem** (Deligne [7]). *Suppose that* $\mathcal{V}$ *is a variation of mixed Hodge structure over a smooth variety* X *that satisfies* (12.8). *If* $\{ (f_j, g_j) \mid j = 1, \cdots, N \}$ *are the* N *pairs of elements of* $\mathcal{O}^*(X) \otimes \mathbb{Q}$ *associated to* $\mathcal{V}$ *as above, then*

$$\sum_{j=1}^{N} f_j \cup g_j = 0 \quad in \quad H^2_{\mathcal{D}}(X,\mathbb{Q}(2)).$$

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