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L^2-COHOMOLOGY AND INTERSECTION HOMOLOGY OF LOCALLY SYMMETRIC VARIETIES, III

Steven Zucker*

This article is the written account of, and elaboration on, the spontaneously organized lecture I gave on June 2, 1987 at Luminy. The title "The proofs of my conjecture" was designated for the talk, without contest, by Kashiwara. I must confess that, originally, I wanted to use this title for the present article, attributing it to Kashiwara, of course. However, his brief letter of permission wondrously tactfully steered me away from doing so.

The conjecture is the one that identifies the L^2-cohomology of a locally symmetric variety with the intersection homology of its Baily-Borel Satake compactification. It came as an outgrowth of attempts to understand the relation between the L^2 harmonic forms on these spaces — more generally, on arithmetic quotients of arbitrary symmetric spaces of non-compact type — and their ordinary cohomology (see [5],[43], and in particular their introductions). A fundamental, and modern, point of view that had been emerging is that the L^2 analysis of the Laplacian on non-compact Riemannian manifolds is not so different from that on compact manifolds (aside from the possibility of continuous spectrum); what is missing in general is the topological interpretation of the L^2 harmonic forms (Hodge theorem). The conjecture for locally symmetric varieties, proved recently by two very different lines of reasoning in [28] and [36], provides such a topological interpretation for these spaces. (Cases had been established, prior to 1987, in [7],[11], [43] and [47]. The results of [36] are announced in [35].)

The structure of this article is quite simple. In §1, we establish the notation and basic notions, culminating with the statement of the conjecture and its significance. We continue in §2 to discuss the basic issues involved in proving the conjecture, following [47]. (See [16] for some additional ideas.) In §3 and §4 respectively, we sketch the proofs of Looijenga [28] and Saper-Stern [36]. The main methods of the Saper-Stern proof date from the 1960's, but are manipulated in a very ingenious way. Looijenga's may be easier to grasp, yet is still quite

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clever, but it depends on a large amount of very elaborate, recently-developed machinery. Finally, §5 deals with Borel's 1983 extension of the conjecture to arithmetic quotients of equal-rank symmetric spaces (which includes the Hermitian ones), and covers some ideas and results that look useful, but are not needed in [28] and [36].

The natural setting for the conjecture seems to be the equal-rank symmetric spaces. Certainly, the ingredients in the conjecture require only a Riemannian manifold possessing a topological compactification that is stratified by even-dimensional strata. Thus, it had always been my goal to argue without mentioning the complex structure. The proofs of Looijenga and Saper-Stern both make use of the complex structure, the former in a more fundamental way than the latter. I continue to want and expect a unified proof of Borel's extended conjecture.

§1. PRELIMINARIES

(1.1) The setting of this article is the following:

a) Let $D = G/K$ be a symmetric space of non-compact type (here, G is a semi-simple Lie group, and K a maximal compact subgroup). Of course, there is a natural left action of G on D.

b) Let Γ be an arithmetically defined subgroup of G, i.e., there exists a finite-dimensional faithful representation of G for which Γ is commensurable with the subgroup of matrices with integer entries. (This likewise determines a unique algebraic group over $\mathbb{Q}$, for which G is the real-analytic space associated to the group of real points.)

c) Γ acts without fixed points on D (equivalently, Γ is torsion-free); then $X = \Gamma\backslash D$ is a manifold.

d) $\mathbb{E}$ is the local system on X associated to a finite dimensional representation E of G (a fortiori, of $\Gamma \simeq \pi_1(X)$).

e) Assume (until stated otherwise) that D has a G-invariant complex structure (i.e., is Hermitian). This is the case precisely when the intersection of K with every non-compact irreducible factor of G has a one-, as opposed to zero-dimensional, center. One then refers to X as an arithmetic (or locally symmetric) variety.

f) There is a compactification X^* of X (the Baily-Borel Satake compactification) that is a normal projective variety, with X contained as a Zariski-open subset [3], cf. [38]. It is not hard to give a rough description of how one obtains X^* from X: one puts in arithmetic varieties of lower dimension (and rank) at infinity. These are constructed from the so-called rational boundary components of D. They comprise the singular strata of X^*, the number of which is the **Q**-rank of G (an invariant of its rational structure).

(1.2) For an example of the above, let G be (the standard form of) the symplectic group $Sp(2r, \mathbb{R})$, the automorphisms of the standard non-degenerate skew-form on $\mathbb{R}^{2r}$. It is a group of **Q**-rank, and also absolute rank, r. Then:

a) D can be realized as the Siegel upper half-space of genus r, consisting of all $r \times r$ symmetric complex matrices with positive-definite imaginary part. Here G acts as fractional-linear transformations. ($r = 1$ gives the classical upper half-plane.)

b,c) Take, for instance, the principal congruence subgroups

$$\Gamma = \ker \{ Sp(2r,\mathbb{Z}) \to Sp(2r,\mathbb{Z}/\ell\mathbb{Z}) \},$$

with $\ell \geq 3$. Then X is the moduli space of r-dimensional abelian varieties with level ℓ structure.

f) X* is, as a topological space, the compactification constructed by Satake in [37], which was progressively proved to underlie a normal analytic space, then a projective algebraic variety [2] (see also [39]). To obtain X* from X, one adjoins arithmetic quotients of lower genus Siegel upper half-spaces.

(1.3) We return to the general setting. To X, qua Riemannian manifold, is associated its intrinsic L^2-cohomology with coefficients in $\mathbb{E}$, $H^{\bullet}_{(2)}(X,\mathbb{E})$, whose definition we recall briefly. Pick a G-invariant Hermitian metric on D (Bergman metric). It induces a complete Kähler metric on X. Likewise, a so-called admissible inner product (see [30:p.375]) on E determines a metrization of the local system $\mathbb{E}$, i.e., a (non-flat, whenever E is non-trivial) Hermitian metric on the associated vector bundle. Given a form ϕ on X with values in $\mathbb{E}$, its length $|\phi|$ defines a function on X, and thence its L^2 square-norm:

$$||\phi||^2 = \int |\phi|^2 dV_X. \tag{1.3.1}$$

The L^2-cohomology is the cohomology of the L^2 complex $L^{\bullet}_{(2)}(X,\mathbb{E})$, defined as

$$\{ \phi : \phi \text{ and } d\phi \text{ have finite } L^2 \text{ norm}\}. \tag{1.3.2}$$

Whether one uses C^∞ or just measurable forms in (1.3.2) is immaterial [18: §8]; the latter choice, which we take, gives the domain of the weakly defined Hilbert space exterior derivative.

(1.4) The association

(1.4.1) $$U \text{ open in } X^* \longmapsto L^{\bullet}_{(2)}(U \cap X, \mathbb{E})$$

defines a presheaf on X^*, whose associated sheaf is denoted $\mathcal{L}^{\bullet}_{(2)}(X^*,\mathbb{E})$.

(1.4.2) PROPOSITION. For X^* as in (1.1,f)

i) $\mathcal{L}^{\bullet}_{(2)}(X^*,\mathbb{E})$ is a complex of fine sheaves,

ii) $L^{\bullet}_{(2)}(X,\mathbb{E}) = \Gamma(X^*, \mathcal{L}^{\bullet}_{(2)}(X^*,\mathbb{E}))$,

iii) $H^{\bullet}_{(2)}(X,\mathbb{E}) \simeq \mathbb{H}^{\bullet}(X^*, \mathcal{L}^{\bullet}_{(2)}(X^*,\mathbb{E}))$.

(see [47: (2.3),(3.6)],[43: (4.4)], etc.)

(1.5) On the other hand, the complex variety X^* is naturally stratified, so one may speak of its (middle perversity) intersection (co)homology with coefficients in $\mathbb{E}$, $IH^{\bullet}(X^*,\mathbb{E})$. The latter is the (hyper)cohomology of the complex of intersection chains, $\mathcal{IC}^{\bullet}(X^*,\mathbb{E})$, defined in [21: (2.1)]. Let it suffice for now to say that this complex is characterized up to quasi-isomorphism by certain axioms [21: (3.3),(4.1), or (6.1)], [9: V, §4]; some discussion of this will appear in the next section.

(1.6) With evidence supplied by some simple examples [43: §6], we had made the following conjecture:

(1.6.1) CONJECTURE (1980). Let X be an arithmetic variety, X^* its Baily-Borel Satake compactification, and $\mathbb{E}$ a metrized local system on X as above. Then $\mathcal{L}^{\bullet}_{(2)}(X^*,\mathbb{E})$ is quasi-isomorphic to $\mathcal{IC}^{\bullet}(X^*,\mathbb{E})$, so

$$H^{\bullet}_{(2)}(X,\mathbb{E}) \simeq IH^{\bullet}(X^*,\mathbb{E}).$$

Early in 1987, two proofs of the conjecture, quite different from each other, were announced [28],[35]. These will be outlined in §§3,4. Prior to that, cases of low $\mathbb{Q}$-rank had been proved [7],[11],[43],[47] (see also [16]).

(1.7) At this point, we give some indication of the significance of (1.6.1). According to [6], the L^2-cohomology admits a description as relative Lie algebra cohomology:

(1.7.1) $$H^{\bullet}_{(2)}(X,\mathbb{E}) \simeq H^{\bullet}(\underline{g},K;\ L^2(\Gamma\backslash G)^{\infty} \otimes E),$$

where $\underline{g}$ is the Lie algebra of G, and the superscript ∞ indicates C^{∞} vectors. This is given, as is often the case, by the L^2 harmonic forms with values in $\mathbb{E}$

(see [44: §2]) - the kernel of the Laplacian operator - which is here a finite dimensional space [12]. In terms of the isomorphism (1.7.1), they are in turn given, by Kuga's formula, in terms of the eigenfunctions of the Casimir element for $\underline{g}$ (see [13: II(2.5)] or [29: (6.8)]). These harmonic forms had been studied for a long time, first when X is compact (e.g., [29],[30],[32],[13]) and later in the non-compact case as well (e.g., [5],[10],[12]). From (1.6.1), we see that the L^2 harmonic forms have a topological interpretation, at least on X*. (It is too much to expect such on X itself, though the statement of (1.6.1) implies an isomorphism range for the natural mapping

(1.7.2) $$H^i_{(2)}(X,\mathbb{E}) \to H^i(X,\mathbb{E})$$

that one presumes is optimal. This was the immediate motivation behind [43].)

Moreover, there is a parallel construction of intersection étale cohomology, so one could say now that L^2-cohomology is part of a motive. As such, when one works in the adelic framework, in which X is a connected component of a Shimura variety over a number field, one is set up to define an intersection homology zeta function for X* à la Langlands (see [14], where the case of Hilbert modular varieties has been worked out), to which the L^2-cohomology contributes.

On the other hand, one can view (1.6.1) as providing a sort of de Rham theory for the intersection cohomology. Because X is metrically complete, and $H^{\bullet}_{(2)}(X,\mathbb{E})$ is finite dimensional, the isomorphism imparts a Hodge structure to $IH^{\bullet}(X^*,\mathbb{E})$ (see [44: §2],[48: §1]).[1] One would want, and even expect, this to coincide with the Hodge structure arising from the general Hodge theory for perverse sheaves given in [33] and [34], which is compatible with geometric constructions (see also [48]).

§2. THE GENERAL STRATEGY

The presentation here discusses the state of the problem prior to 1987, from the point of view of [47]. Though this reflects the biases of the author in his attempts at the conjecture, it does set up a common framework for the discussion of [28] and [36]. For another approach, the reader is referred to [16], which is also a reasonable survey.

(2.1) As we have already indicated, to prove Conjecture (1.6.1), one must

[1]From this angle, the conjecture resembles the one in [19: §4], though the two are independent of each other.

verify that the conditions that characterize intersection cohomology are satisfied by $\mathcal{L}^{\bullet}_{(2)}(X^*,\mathbb{E})$. The main ingredient in this characterization is that every point on the stratum of complex codimension j has a base of neighborhoods U with

(2.1.1) $$IH^i(U,\mathbb{E}) = 0 \quad \text{for} \quad i \geq j;$$

one eventually realizes that it is sufficient to establish this vanishing for local L^2-cohomology on X^*:

(2.1.2) $$H^i_{(2)}(U \cap X,\mathbb{E}) = 0 \quad \text{for} \quad i \geq j.$$

(2.2) We must know, then, what U and $U \cap X$ look like. Topologically, we have

(2.2.1) $$U \simeq V \times \text{Cone}(L),$$

where V is a contractible neighborhood within the stratum, and L is the link of that stratum. In terms of this, we give the precise version of (2.1.1):

(2.2.2) $$IH^i(U,\mathbb{E}) \simeq IH^i(\text{Cone}(L),\mathbb{E}) \simeq \begin{cases} IH^i(L,\mathbb{E}) & \text{if} \quad i < j, \\ 0 & \text{if} \quad i \geq j. \end{cases}$$

Of course,

$$U \cap X \simeq I \times (L \cap X),$$

where I is the interval $(0,1)$, so for ordinary cohomology, we have

(2.2.3) $$H^i(U \cap X,\mathbb{E}) \simeq H^i(L \cap X,\mathbb{E}).$$

(2.3) In the determination of $H^{\bullet}_{(2)}(U \cap X,\mathbb{E})$, one can be guided by:

(2.3.1) Claim. For certain simple situations, $H^{\bullet}_{(2)}(M,\mathbb{E})$ is the subset of $H^{\bullet}(M,\mathbb{E})$ of classes with L^2 representatives.

(2.3.2) WARNING. It will turn out that the neighborhoods U are among the "certain simple situations" mentioned above. In general, however, the canonical mapping

$$H^{\bullet}_{(2)}(M,\mathbb{E}) \to H^{\bullet}(M,\mathbb{E})$$

can have a kernel.

(2.4) We next describe $L \cap X$ and $U \cap X$ in detail. We tacitly assume in what follows that Γ is sufficiently small. This is no loss of generality, for (1.7.1) is hereditary for quotients by finite groups. We also may assume that G is irreducible over $\mathbb{Q}$.

The boundary components correspond to the maximal rational parabolic subgroups P of G, with Langlands decomposition (over $\mathbb{Q}$)

(2.4.1) $$P = M_P \cdot A_P \cdot N_P,$$

in which N_P is the unipotent radical, and $M_P \cdot A_P$ is a Levi subgroup. The split component A_P is one-dimensional, diffeomorphic to $(0,\infty)$ - a half-line in its Lie algebra - as P is, up to conjugacy, determined by the deletion of a single simple $\mathbb{Q}$-root β. This, in turn, decomposes M_P into

(2.4.2) $$M_P = M_P^A \cdot M_P^B,$$

for its $\mathbb{Q}$-root system has two components, one of type A, and the other of type B (which includes the classification type BC, and its degenerate form C). Here M_P^B is (essentially) the automorphism group of the boundary component, here denoted D^B.

From this, we can see that $L \cap X$ is the image of $M_P^A \times N_P$ in X. It admits a fibration over X^A, an arithmetic quotient of the symmetric space[2] D^A (usually not Hermitian) of M_P^A, with fiber a compact nilmanifold associated to N_P, To describe $U \cap X$ in similar terms, one must be a little careful (see [47: (1.3)]).

(2.4.3) LEMMA. There is a positive, real-valued function f on the symmetric space D^A of M_P^A, such that open sets of the form (2.2.1) have $U \cap X$ given as the image of

$$\{(x,y,s,n) \in D^A \times X^B \times (0,\infty) \times N_P : y \in V,\ s > \lambda + f(x)\}$$

for $\lambda >> 0$. The singular stratum lies at $s = \infty$.

(2.5) It is well-known that the Leray spectral sequence of $(L \cap X) \to X^A$ degenerates at E_2. This gives a canonical isomorphism

(2.5.1) $$H^\bullet(U \cap X, \mathbb{E}) \simeq H^\bullet((0,\infty), \mathbb{C}) \otimes_{\mathbb{C}} H^\bullet(X^A, \mathbb{H}^\bullet(\underline{n}, E)),$$

where $\underline{n}$ is the Lie algebra of N_P, and $\mathbb{H}^\bullet(\underline{n},E)$ is the local system on X^A associated to the representation of M^A on the Lie algebra cohomology $H^\bullet(\underline{n},E)$.

[2] D^A may have Euclidean factors.

We have written in the extraneous "$H^{\bullet}((0,\infty),\mathbb{C})\otimes_{\mathbb{C}}$" for comparison with the following result (whose notation will soon get explained):

(2.5.2) PROPOSITION. ("L^2-Künneth theorem") [47: (3.19)]

$$H^{\bullet}_{(2)}(U \cap X,\mathbb{E}) \simeq \bigoplus_{\alpha} [H^{\bullet}_{(2)}((0,\infty),\mathbb{C};\ w_1^{\alpha}) \otimes_{\mathbb{C}} H^{\bullet}_{(2)}(X^A,\mathbb{H}^{\bullet}_{\alpha}(\underline{n},E);\ w_2^{\alpha})].^3$$

In terms of the variables (x,y,s,n) of (2.4.3), the metric on $U \cap X$ is, according to [5: (4.3)],

$$dx^2 + dy^2 + ds^2 + e^{-2s}\, dn_1^2(x) + e^{-4s}\, dn_2^2(x), \tag{2.5.3}$$

where we have decomposed $\underline{n}$ into $\underline{n}_1 \oplus \underline{n}_2$ according to weights $\beta, 2\beta$ with respect to A_P. (Then $\underline{n}_2$ is the center of $\underline{n}$.) When we change variables by

$$s = r + f(x)$$

to recover the product structure, we get, up to quasi-isometry:

$$dx^2 + dy^2 + dr^2 + e^{-2r}\, e^{-2f(x)} dn_1^2(x) + e^{-4r}\, e^{-4f(x)} dn_2^2(x). \tag{2.5.4}$$

Similarly, an element $e \in E$ on which A_P acts by $e^{\ell s}$ (write $e \in E_{\ell}$) determines a (multi-valued) section of $\mathbb{E}$ with

$$|e|^2 \sim e^{-2\ell r}\, e^{-2\ell f(x)} \tag{2.5.5}$$

In view of (2.5.3) or (2.5.4), the factor V of $U \cap X$ (recall (2.2.1)) can be omitted, as a quasi-isometrically trivial factor, in the discussion of calculating L^2-cohomology. With this understanding, the volume density becomes

$$dV \sim (e^{-2jr}dr)(e^{-2jf(x)}dV(x))dV(n), \tag{2.5.6}$$

as

$$2j = \dim \underline{n}_1 + 2 \dim \underline{n}_2. \tag{2.5.7}$$

Then, for $i = i_1 + i_2$ correspondingly decomposed, an i-form on N_P with bidegree (i_1,i_2) and values in E_{ℓ} has an action of A_P by

$$\alpha(s) = e^{(\ell - i_1 - 2i_2)s}, \tag{2.5.8}$$

and the integral defining its L^2 square-norm on $U \cap X$ is weighted by

[3] This formula requires an additional hypothesis, which is satisfied via the last assertion in (2.6). See [47: (3.19(8))].

(2.5.9) $$w_1^{\alpha}(r) = e^{(2i_1+4i_2-2\ell-2j)r}$$

and by $w_2^{\alpha} = w_1^{\alpha} \circ f$. Moreover, the decomposition of

$$\Lambda^{\bullet}\underline{n}^* \otimes E$$

into weight spaces is compatible with passage to cohomology. The precise weights that occur non-trivially in $H^{\bullet}(\underline{n},E)$ are determined by a theorem of Kostant [27: (5.14)] (see [47: (3.4)]); the weight spaces are indicated by the subscript α.

(2.6) We recall the (elementary) computation of the L^2-cohomology of a half-line with exponential weights:

(2.6.1) $$H^0_{(2)}((0,\infty),\mathbb{C};\ e^{-kr}) = \begin{cases} \mathbb{C} & \text{if } k > 0, \\ 0 & \text{if } k \le 0; \end{cases}$$

(2.6.2) $$H^1_{(2)}((0,\infty),\ \mathbb{C};\ e^{-kr}) = \begin{cases} 0 & \text{if } k \neq 0, \\ \text{infinite dimensional} & \text{if } k = 0. \end{cases}$$

(Lest there be any confusion, the imposition of the weight w in (1.3.1) would change the square-norm to

(2.6.3) $$\int |\phi|^2\ w\ dV_X.)$$

The infinite dimensionality in (2.6.2) results from the fact that $dL^0_{(2)}((0,\infty),\ \mathbb{C})$ is a proper dense subspace of $L^1_{(2)}((0,\infty),\ \mathbb{C})$, and is the range of an operator. This does not contribute in (2.5.2), for one ultimately sees that the right-hand factor vanishes if $\alpha(s) = e^{-js}$.

(2.7) At this point, we can see that the proof of Conjecture (1.6.1) comes down to a comparison of truncations. Intersection cohomology effects a truncation by <u>degree</u> (recall (2.2.2)), whereas L^2-cohomology does so by <u>weight</u> (compare (2.5.1) and (2.5.2)). One must show that the two coincide.

The problem can be expressed in explicit "combinatorial" terms. Suppose that the A_p weight α occurs in $H^q(\underline{n},E)$. Show that either

i) the coefficient of r in (2.5.9) is positive,

ii) $H^p_{(2)}(X^A, H^q_{\alpha}(\underline{n},E);\ w_2^{\alpha}) = 0$ for $p \ge j - q$.

(Here, we are assuming that we know that (2.6.2) does not contribute in (2.5.2).)

§3. LOOIJENGA'S PROOF

(3.1) The first feature of the proof in [28] is that it proceeds by induction on j, the codimension of the stratum, or equivalently, the $\mathbb{Q}$-rank of M^A. Thus, given j, we may assume that $\mathcal{L}^{\bullet}_{(2)}(X^*,\mathbb{E})$ and $\mathcal{IC}^{\bullet}(X^*,\mathbb{E})$ are quasi-isomorphic outside the closure of the codimension j stratum. It follows (cf. (2.5.2)) that in fact

(3.1.1) $$\bigoplus_{\alpha} [H^0((0,\infty), \mathbb{C}) \otimes H^{\bullet}_{(2)}(X^A, \mathbb{H}^{\bullet}_{\alpha}(\underline{n},E);\ w_2^{\alpha})] \simeq IH^{\bullet}(L,\mathbb{E}).$$

(3.2) The next step is the elimination of explicit mention of L^2-cohomology from the statement of what must be proved. This is achieved by converting the weights α into the eigenvalues of geometrically defined endomorphisms, as follows:

(3.2.1) PROPOSITION. i) _There exists_ $a \in A_P$ _for which the conjugation of_ P _by_ a _induces a proper, finite-to-one, endomorphism of the stratified space_ U. _Call it_ Ψ_a.

ii) Ψ_a _respects the nilmanifold fibration of_ $U \cap X$ (_implicit in_ (2.4), [47: (3.21)]).

iii) _The induced actions of_ Ψ_a (_together with the action of_ a _on_ E) _on_

$$IH^{\bullet}(U,\mathbb{E}) \quad \underline{\text{and}} \quad IH^{\bullet}(L,\mathbb{E}) \simeq H^{\bullet}_{(2)}(L \cap X,\mathbb{E})$$

are compatible and semi-simple. On the right-hand group, it is induced by the action by $\alpha(\log a)$ _on_ $H^{\bullet}_{\alpha}(\underline{n},E)$.

From this, it follows that the summands of (3.1.1) are detectable by the action of Ψ_a on $IH^{\bullet}(L,\mathbb{E})$.

(3.2.2) _Remark_. It is observed in [28: (3.7)] that the simple left-translation Φ_a by a also induces an endomorphism of U, with the same action on the cohomology groups in (3.2.1, iii).

(3.3) Thus, (2.1.2) — and indeed the sharper

(3.3.1) $$H^i_{(2)}(U,\mathbb{E}) \simeq \begin{cases} IH^i(L,\mathbb{E}) & \text{if } i < j \\ 0 & \text{if } i \geq j \end{cases}$$

— is verified if one shows:

(3.3.2) Claim. For $i \geq j$, the eigenvalues of Φ_a on $IH^i(L,\mathbb{E})$ are of the form a^k with $k > j$. (We will reword this from now on as: the weights of Φ_a are $> j$.)

(See also (2.6)). By duality, - E is isomorphic to its conjugate contragredient; see ([10: (1.6)]) - this is equivalent to

(3.3.3) Claim. For $i < j$, the weights of Φ_a on $IH^i(L,\mathbb{E})$ are $< j$.

Given the truncation property (2.2.2) of intersection cohomology, (3.3.3) is established immediately by the following "purity theorem" (akin to [17: (1.13)], [25: (4.0.1)]):

(3.3.4) PROPOSITION. The weights of Φ_a on $IH^i(U,\mathbb{E})$ are $\leq i$.

(3.4) It is time to bring in the complex structure. The Hermitian symmetric spaces are, in a sense, characterized by the relation

(3.4.1) $$\dim X^A + \dim A_p = \dim \underline{n}_2.$$

Moreover, $\dim \underline{n}_1$ is even. The complex structure of U can be seen in terms of

(3.4.2) $$V \times \underline{n}_1 \times (\underline{n}_2 \otimes_{\mathbb{R}} \mathbb{C})$$

(not complex-analytically a product), and is described as a Siegel domain in terms of an open convex cone C in $\underline{n}_2$ (see [28: §2] and also our (4.15)).

The introduction of C also permits the construction of a resolution of singularities

(3.4.3) $$\pi : \tilde{\mathcal{U}} \to U$$

by the method of toroidal embeddings [1], in which the singular locus of U is replaced in $\tilde{U}$ by a divisor with normal crossings. It can be arranged that the morphism π is projective. When we forget the contractible V again, the construction gives

(3.4.4) PROPOSITION. i) $\tilde{U}$ is covered by open sets $\tilde{U}(\sigma)$ that are Δ^{ν}-bundles over nilmanifolds, such that $\tilde{U}(\sigma) \cap X$ is the corresponding $(\Delta^*)^{\nu}$-bundle. (Here, $\nu = \dim \underline{n}_2$.)

ii) A non-empty intersection of k distinct $\tilde{U}(\sigma)$'s is a $\Delta^{\nu-\ell}$-bundle over a nilmanifold for some $\ell \geq k-1$, and its intersection with X is the corresponding $(\Delta^*)^{\nu-\ell}$-bundle.

iii) Φ_a lifts to a (stratified) endomorphism of $\tilde{U}$.

(3.5) It was observed in [42: §4] that the coefficient system $\mathbb{E}$ on X underlies a polarized variation of complex Hodge structure. Then $\mathbb{E} \oplus \overline{\mathbb{E}}$ underlies a real variation, placing us in the realm of the decomposition theorem of [34], which is the complex analytic analogue of the one in [4: (5.4.6)], that itself could be applied only when $\mathbb{E}$ is known to be of geometric origin. As a result, we have the existence of embeddings

(3.5.1) $$IH^{\bullet}(U,\mathbb{E}) \hookrightarrow IH^{\bullet}(\tilde{U},\mathbb{E}),$$

so it is sufficient for (3.3.4) to prove

(3.5.2) PROPOSITION. The weights of Φ_a on $IH^i(\tilde{U},\mathbb{E})$ are $\leq i$.

(3.6) From here, the discussion involves a fairly crude estimation of weights to reduce to the "purity" theorem for variations of Hodge structure on products of punctured discs, as follows.

We first retreat a step, and explain the notation in (3.4.4). The parameter σ is an equivalence class of top-dimensional simplicial cones in $\overline{C}$ that occurs in the construction of $\tilde{U}$. Given $k = p + 1$ sets $\tilde{U}(\sigma_i)$ $(i = 0,\dots,p)$, one has that

(3.6.1) $$\bigcap_{i=0}^{p} \tilde{U}(\sigma_i) = \tilde{U}(\tau),$$

the part of $\tilde{U}$ associated to $\tau = \bigcap_{i=0}^{p} \sigma_i$; the number ℓ occurring in (3.4.4,ii) is precisely the dimension of τ. From the spectral sequence of a covering (see (3.8.3)), we see that (3.5.2) would follow from

(3.6.2) <u>Claim</u>. With notation as above, the weights of Φ_a on $IH^i(\hat{U}(\tau),\mathbb{E})$ are $\leq i - p$.

(3.7) Very roughly, the fibration described in (3.4.4,ii), arises with the punctured discs produced from a subspace of $\underline{n}_2$, which is duly divided out of $\underline{n}$ to yield the base, here denoted $B(\tau)$. Consider, then, the Leray spectral sequence (please pardon the abuse of notation):

(3.7.1) $$E_2^{r,s} = H^r(B(\tau),\ IH^s(\Delta^{\nu-\ell},\mathbb{E})) \Longrightarrow IH^{r+s}(\hat{U}(\tau),\mathbb{E}).$$

By [17: (1.13)] or [25: (4.0.1)], the weights of Φ_a on $IH^s(\Delta^{\nu-\ell},\mathbb{E})$ are $\leq s$. Those on forms on $B(\tau)$ are of the form $i_1 + 2i_2$ (cf.(2.5.8)), with $i_1 + i_2 = r$ and

$$0 \leq i_2 \leq \min\{r,\ \mathrm{codim}(\tau)\},$$

so are at most $r + \mathrm{codim}(\tau)$. From (3.7.1), one sees

(3.7.2) PROPOSITION. <u>The weights of</u> Φ_a <u>on</u> $IH^i(\hat{U}(\tau),\mathbb{E})$ <u>are</u> $\leq i + \mathrm{codim}(\tau)$.

(3.8) Unfortunately, (3.7.2) is strong enough for our purposes ((3.6.2)) only when $\mathrm{codim}(\tau) \leq p$, i.e. (for non-degenerate intersections) when the intersection of the simplicial cones is of maximal dimension. However, these are the only ones that matter, as is seen by the following argument.

Let Σ denote the set of cones. For any contravariant functor

(3.8.1) $$F : \Sigma \to \{\text{Cochain complexes}\},$$

one defines the associated alternating Cech double complex, determined by

(3.8.2) $$\underline{\sigma} = (\sigma_0,\ldots,\sigma_p) \longmapsto F(\tau) \qquad (\tau = \bigcap_{i=0}^{p} \sigma_i).$$

The spectral sequence for the cohomology of the associated single complex, $C^\bullet(F)$, and its filtration by simplicial degree, begins:

(3.8.3) $$E_1^{p,q}(C^\bullet(F)) = \bigoplus_{\underline{\sigma}\in\Sigma^{p+1}} H^q(F(\tau)).$$

From this, one sees:

(3.8.4) $$E_2^{p,q}(C^\bullet(F)) = H^p(C^\bullet[H^q(F)]).$$

Let $C^\bullet_k(F)$ denote the subcomplex of $C^\bullet$ consisting of those elements that are trivial on all simplices of codimension $\leq k$; equivalently,

$$C^\bullet_k(F) = C^\bullet(F_k),$$

where

$$F_k(\tau) = \begin{cases} F(\tau) & \text{if} \quad (\operatorname{codim} \tau) > k, \\ 0 & \text{if} \quad (\operatorname{codim} \tau) \leq k. \end{cases}$$

(3.8.5) PROPOSITION. *For any functor (compare* (3.8.1))

$$L : \Sigma \to \{\text{Abelian groups}\},$$

$H^p(C^\bullet_k(L)) = 0$ *whenever* $k \leq p$.

(3.8.6) COROLLARY. *The mapping (compatible with* Φ_a)

$$E_2^{p,q}(C^\bullet(F)) \to H^p(C^\bullet[H^q(F)]/C^\bullet_p[H^q(F)])$$

is injective.

(3.8.7) COROLLARY. *The weights of* $E_2^{p,q}(C^\bullet(F))$ *are among those of the summands of* $E_1^{p,q}$ *with* $\operatorname{codim} \tau = p$.

Of course, the above is applied in the case where F is given by

$$F(\tau) = IC^\bullet(\tilde{U}(\tau), \mathbb{E}),$$

so the spectral sequence (3.8.3) abuts to $IH^\bullet(\tilde{U}, \mathbb{E})$.

(3.9) It remains to prove (3.8.5). The main point is that the simplicial complex of cones is topologically a manifold.

Consider $\{C^\bullet_k(L)\}$ as a (decreasing) filtration of $C^\bullet(L)$. There is a spectral sequence

(3.9.1) $$E_1^{p,q}(C^\bullet(L)) = H^{p+q}(C^\bullet_{p-1}(L)/C^\bullet_p(L)) \Longrightarrow H^{p+q}(C^\bullet(L))$$

(restrict to $p > k$, and the abutment becomes the cohomology of $C^\bullet_k(L)$).

(3.9.2) LEMMA. $C^\bullet_{p-1}(L)/C^\bullet_p(L) \simeq \bigoplus_{\operatorname{codim} \upsilon = p} (K^\bullet(\upsilon) \otimes_{\mathbb{Z}} L(\upsilon))$,

with

$$H^{\bullet}(K^{\bullet}(\upsilon)) \simeq H^{\bullet}(\mathrm{Star}(\upsilon), \mathrm{Star}(\upsilon) - \upsilon).$$

Proof. Because

$$\mathrm{codim}\ (\tau \cap \sigma) < \mathrm{codim}\ \tau$$

unless $\tau \subseteq \sigma$, the direct sum decomposition follows at once (recall (3.8.2)). To understand $K^{\bullet}(\upsilon)$, one observes that it is the integral Cech complex for the set of simplices σ_i containing υ, i.e., for Star(υ), modulo the terms associated to those $\underline{\sigma}$ whose intersection τ strictly contains υ. By retracting such $\underline{\sigma}$ away from υ, one identifies the latter with the Cech complex for Star(υ) $-$ υ.

Because our simplicial complex is a manifold, there is a homotopy equivalence

(3.9.3) $$(\mathrm{Star}(\upsilon), \mathrm{Star}(\upsilon) - \upsilon) \sim (D^p, S^{p-1})$$

whenever codim $\upsilon = p$. Thus,

(3.9.4) $$H^i(C^{\bullet}_{p-1}(L)/C^{\bullet}_p(L)) = 0 \quad \text{for} \quad i \neq p.$$

It follows that in (3.9.1),

(3.9.5) $$E_1^{p,q}(C^{\bullet}_k(L)) = 0, \quad \text{unless} \quad p > k \text{ and } q = 0,$$

which implies (3.8.5).

§4. THE PROOF OF SAPER AND STERN

(4.1) In contrast with Looijenga's proof, the first feature of the Saper-Stern argument is that it goes directly, not inductively, towards the required vanishing of local L^2-cohomology that is described in (2.7).

(4.2) The main principle of the proof goes back to [24] and [26], which (in retrospect) reduces the vanishing of L^2-cohomology to an a priori estimate. As in (1.3), let d denote the weakly defined exterior derivative (a densely-defined unbounded operator on the Hilbert space of L^2 forms), and d^* its Hilbert space adjoint. The following is well-known (see [24: §1],[26: (8.10)]):

(4.2.1) PROPOSITION. Let U be a Riemannian manifold, $\mathbb{E}$ a metrized local system on U. The following are equivalent:

i) $H^i_{(2)}(U,\mathbb{E}) = 0$, and $dL^i_{(2)}(U,\mathbb{E})$ is closed in the $L^2(i+1)$-forms.

ii) *There is a constant* $\kappa > 0$ *such that*

$$||d\phi||^2 \geq \kappa||\phi||^2$$

whenever $\phi \in L^i_{(2)}(U,\mathbb{E})$ *is orthogonal to* $dL^{i-1}_{(2)}(U,\mathbb{E})$, *and whenever* $\phi \in L^{i-1}_{(2)}(U,\mathbb{E})$ *is orthogonal to* ker d.

iii) *There is a constant* $\kappa > 0$ *such that*

$$||d\phi||^2 + ||d^*\phi||^2 \geq \kappa||\phi||^2$$

whenever $\phi \in L^i_{(2)}(U,\mathbb{E}) \cap (\mathrm{Dom}\ d^*)$.

iv) *There is a constant* $\kappa > 0$ *such that*

$$||\Delta\phi|| \geq \kappa||\phi||$$

whenever ϕ *is an* i-*form in the domain of* Δ ($\Delta = dd^* + d^*d$, *interpreted under the conventions of functional analysis; see* [20: §2]).

If, moreover, U *is the interior of* $\overline{U}$, *a complete manifold with boundary* (*corners allowed*), *it is enough to verify the estimate of* (ii), (iii) *or* (iv) *for forms* ϕ *that are smooth in* $\overline{U}$, *and have compact support therein*.

(4.3) According to [47: (3.6)-(3.7)], the open set U from §2 admits so-called *distinguished coverings*, whose members are indexed by conjugacy classes of parabolic subgroups Q of M^A (recall (2.4)). We write

$$U \cap X = \bigcup_Q U(Q). \tag{4.3.1}$$

For a suitably chosen such covering, it suffices to verify any of the estimates of (4.2.1) for smooth forms of compact support on each U(Q) — here, U(Q) is considered to have as boundary $\partial U \cap U(Q)$, so is in particular incomplete whenever the $\mathbb{Q}$-rank of M^A is positive; the estimate for U(Q) then has no L^2-cohomological significance. This is by no means a tautology, but rather it depends on the ability to find partitions of unity subordinate to (4.3.1) with certain properties [35: Prop. 2]; it is stronger than a Mayer-Vietoris argument.

The breaking up of U into these pieces allows one to avoid explicit discussion of the weighting of L^2-cohomology on X^A (2.5.2), at the expense of having to consider all parabolic subgroups of M^A. (Similar issues occur in [43: §4],[47: (3.9),(3.23)].)

(4.4) Fix a rational parabolic subgroup Q of M^A, with rational Langlands decomposition

(4.4.1) $$Q = L_Q \cdot A_Q \cdot N_Q.$$

Then

(4.4.2) $$Q' = L_Q(A_Q \cdot A_P)(N_Q \cdot N_P) =: \ LAN$$

is parabolic in the centralizer of the boundary component associated to P, and $Q' \cdot M^B$ is among the rational parabolic subgroups of P. Here,

(4.4.3) $$A_Q \simeq (0,\infty)^{\nu},$$

where ν is the parabolic $\mathbb{Q}$-rank of Q. The set $U(Q)$ can be taken to be the full nilmanifold fibration (with fiber a compact quotient of N) over $\tilde{X}_Q \times \tilde{A}$, where $\tilde{X}_Q$ is a relatively compact deformation retract of an arithmetic quotient X_Q associated to L, and $\tilde{A} \subset A$ is

(4.4.4) $$\{(a,s) \in (\mu,\infty)^{\nu} \times (\lambda,\infty)\},$$

though the product decomposition in (4.4.4) does not come from the one in (4.4.2), but rather from simple roots (see [47: (1.3)]); this discrepancy is also the source of the w_2^{α}'s in (2.5.2). The boundary of $U(Q)$, for the purposes described in (4.3), is given by $s = \lambda$.

(4.5) It is necessary now to get more technical.

Let

(4.5.1) $$K_Q = K \cap Q = K \cap L.$$

Because $U(Q)$ is a principal K_Q-bundle over $\tilde{X}_Q \times \tilde{A} \times N$, the L^2 $\mathbb{E}$-valued forms on $U(Q)$ admit a description as the K_Q-invariant elements of

(4.5.2) $$L^2(\Gamma_Q \backslash (\tilde{L} \times \tilde{A} \times N)) \otimes \Lambda^{\bullet}\underline{p}_Q^* \otimes (\Lambda^{\bullet}\underline{u}_Q^* \otimes E),$$

where $\tilde{L}$ is the full inverse image of $\tilde{X}_Q$ in L, $\Gamma_Q = \Gamma \cap Q$, $\underline{p}_Q$ is the orthogonal complement (with respect to the Killing form of $\underline{g}$) of the Lie algebra $\underline{k}_Q$ of K_Q in that of LA, and (sorry) $\underline{u}_Q$ is the Lie algebra of N. (The analogous assertion for G and X underlies (1.7.1).) The exterior derivative d

decomposes as

(4.5.3) $$d = D + d_\rho,$$

where D is given by differentiating the L^2 functions in (4.5.2), and d_ρ is a 0-th order operator, determined by the natural representations of Q on its Lie algebra and on E.

According to [27: (5.7)], there is a splitting

(4.5.4) $$\Lambda^\bullet \underline{u}_Q^* \otimes E \simeq H^\bullet(\underline{u}_Q, E) \oplus R^\bullet,$$

equivariant with respect to the action of LA, with $R^\bullet$ d_ρ'-acyclic. By an averaging argument [36: §8], it suffices to get the estimate(s) of (4.2.1) on the N-invariant forms, i.e. for

(4.5.5) $$\phi \in L^\bullet_{(2)}(\tilde{X}_Q \times \tilde{A}, \mathbb{H}^\bullet(\underline{u}_Q, E))^{K_Q},$$

on which the derivatives and Laplacians become the natural ones on $\tilde{X}_Q \times \tilde{A}$ (cf. [22: (2.6),(2.7)]). This brings us close to the analogue of (2.5.2) for the pieces of $U \cap X$.

(4.6) At this point, we depart a little from what appears in [36] by emphasizing the estimate for the Laplacian (4.2.1, iv), rather than (iii), and we rearrange the arguments.

Let δ denote the formal adjoint of d, and also its weakly defined extension to a closed operator on L^2 differential forms. By [24: (1.2)], d and δ are the closures of their restriction to C^∞ forms smooth to the full boundary of $U(Q)$. Write d_0 and δ_0 for the closures on forms of compact support (in the interior). One has Hilbert space adjoint relations

(4.6.1) i) $d^* = \delta_0$ (Neumann boundary conditions),

ii) $(d_0)^* = \delta$ (Dirichlet boundary conditions).

Let $\tilde{d}_0$ denote the closure of d on smooth forms of compact support <u>in the sense of</u> (4.3). The estimates sought in (4.3) imply that the kernel of $(\tilde{d}_0\delta_0 + \delta_0\tilde{d}_0)$ is trivial, and its range is closed. As was remarked previously, this has, by itself, no cohomological interpretation. Of course, these estimates would be a consequence of the same for any extension to a larger domain; e.g. we can take $\Delta = d\delta_0 + \delta_0 d$, or mix in Dirichlet boundary conditions on any of the first ν factors of $\tilde{A}$ (see (4.4.4)) and on $\tilde{X}_Q$. Indicate the boundary conditions chosen by a left-subscript b; these may be chosen independently on the different summands:

(4.6.2) PROPOSITION. (Modulo an abuse of notation on the left-hand side)

$$_bH^\bullet_{(2)}(U(Q),\mathbb{E}) \simeq \bigoplus_\beta [{}_bH^\bullet_{(2)}((0,\infty)^{\nu+1}, \mathbb{C}; w^\beta) \otimes {}_bH^\bullet_{(2)}(\tilde{X}_Q,\mathbb{H}^\bullet_\beta(\underline{u}_Q,E))].$$

In the above, the notation is parallel to that in (2.5.2): for instance, β is a weight of A, and w^β a multivariate exponential of the form

$$w^\beta = \exp(-\sum_{k=0}^{\nu} n_k r_k). \tag{4.6.3}$$

From the direct calculation of the weighted L^2-cohomology of half-lines and their products (cf. [43: (4.51)]), one gets

(4.6.4) LEMMA. i) $H^\bullet_{(2)}((0,\infty), \mathbb{C}; e^{-nr}) = 0$ if $n < 0$.

ii) ${}_0H^\bullet_{(2)}((0,\infty), \mathbb{C}; e^{-nr}) = 0$ if $n > 0$.

Using the Künneth formula (essentially [43: (2.36)]), we get

(4.6.5) PROPOSITION. If in (4.6.3) $n_k \neq 0$ for some $k < \nu$, or if $n_\nu < 0$, then there is a choice of boundary conditions for which ${}_bH^\bullet_{(2)}((0,\infty)^{\nu+1}, \mathbb{C}; w^\beta) = 0$.

We see now that to verify the conditions given in (2.7), it suffices to show the following for each parabolic subgroup Q of M^A. Suppose that the weight β occurs in $H^q(\underline{u}_Q,E)$. Then one of the following holds:

(4.6.6)

i) $n_\nu < 0$,

ii) $n_k \neq 0$ for some $k < \nu$,

iii) $n_\nu > 0$, and $H^p(\tilde{X}_Q,\mathbb{H}^q_\beta(\underline{u}_Q,E)) = 0$ for $p \geq j - q - \nu$.

(4.6.7) Remark. Although the L^2-cohomology in (4.6.5) is infinite dimensional when $n_\nu \geq 0$ and $n_k = 0$ for $k < \nu$, it could enter only in low degrees, hence not at all (see [9: V, (4.3)]).

(4.7) Recall the formula $d = D + d_\rho$ (4.5.3), which we now consider on the space $\tilde{X}_Q$. Since d_ρ is the extension of scalars of the differential in the finite dimensional complex

(4.7.1) $$\wedge^{\bullet}\underline{p}_Q^* \otimes H^{\bullet}(\underline{u},E),$$

it is a bounded operator of order zero. Likewise are its adjoint, $(d_\rho)^*$, and Laplacian

(4.7.2) $$\Delta_\rho = d_\rho(d_\rho)^* + (d_\rho)^* d_\rho.$$

It follows that d and D, and likewise d^* and D^*, have the same domain. Moreover, the identity

(4.7.3) $$\Delta = \Delta_D + \Delta_\rho,$$

first in the formal sense (see [32: p. 244]), and then in the sense of operators, decomposes the Laplacian Δ into two (semi)positive operators. It follows now by an argument of Bochner type, that

(4.7.4) PROPOSITION (cf. [32: §1, Prop. 2]). <u>For any</u> Δ-<u>invariant summand of</u>

$$L^p_{(2)}(\tilde{X}_Q, H^q(\underline{u}_Q, E)),$$

<u>a sufficient condition for the vanishing of its contribution to</u>

$$H^p_{(2)}(\tilde{X}_Q, H^q(\underline{u}_Q, E))$$

<u>is that the eigenvalues of</u> Δ_ρ <u>thereon are strictly positive.</u>

That this will be strong enough to establish (4.6.6) may come as a big surprise. In other words, the more precise vanishing theorems for $(\underline{g},K)$-cohomology (see [40]), which in effect take into account Δ_D as well as Δ_ρ, on the whole space X^A, are not needed here.

(4.8) The Δ-invariant subspaces in (4.7.4) are given by the K_Q-invariant subspaces of (4.7.1), as one can see from Kuga's formula (compare [45]). Using the compatible normalizations of Casimir elements $C_{\underline{\ell}}$ and $C_{\underline{k}}$, for the reductive algebras $\underline{\ell}_Q$ and $\underline{k}_Q$, coming from the Killing form of $\underline{g}$ (see [31: p. 109]), one gets

(4.8.1) PROPOSITION ([32: p. 246],[35: Lemma 5]). <u>If</u> σ <u>is an irreducible</u> $\underline{k}_Q$-<u>constituent of</u> $\wedge^{\bullet}\underline{p}_Q^*$, and ρ <u>is an irreducible</u> $\underline{\ell}_Q$-<u>constituent of</u> $H^q(\underline{u}_Q,E)$, <u>then on</u> $\sigma \otimes \rho$, Δ_ρ <u>is the semi-simple operator</u>

$$\rho(C_{\underline{\ell}}) - \frac{1}{2}\rho(C_{\underline{k}}) + \frac{1}{2}\sigma(C_{\underline{k}}) - \frac{1}{2}(\sigma\otimes\rho)(C_{\underline{k}}).$$

The above expression admits a nice description in terms of highest weights. One makes use of a fundamental Cartan subalgebra $\underline{h}_Q$ of $\underline{\ell}_Q$; i.e., one starts with a Cartan subalgebra $\underline{h}_k$ of $\underline{k}_Q$, and enlarges it to one for $\underline{\ell}_Q$. One also uses compatible systems of positive roots; what is meant by this will be explained in (4.10). Let δ' and δ denote the respective half-sums of the positive roots. Finally, let τ be an irreducible constituent of $\rho|_{\underline{k}_Q}$, and ξ of $\sigma\otimes\tau$. In what follows, we always adopt the convention of identifying representations with their highest weights. Then, as

(4.8.2) $$\rho(C_{\underline{\ell}}) = |\rho+\delta|^2 - |\delta|^2$$

(see [31: §2, Lemma 4]), etc., one obtains

(4.8.3) PROPOSITION. *On ξ, Δ_ρ is the scalar operator*

$$|\rho+\delta|^2 - |\delta|^2 - \frac{1}{2}(|\tau+\delta'|^2 - |\sigma+\delta'|^2 + |\xi+\delta'|^2 - |\delta'|^2).$$

(4.8.4) COROLLARY 1. *For fixed σ and τ, with ξ as variable, the eigenvalue is minimized (strictly) by* $\xi = \sigma + \tau$.

(*Proof*: Write $\xi = \sigma + \tau - \varepsilon$, where ε is a non-negative linear combination of simple roots, etc.)

A little manipulation gives:

(4.8.5) COROLLARY 2. *For $\xi = \sigma + \tau$, the eigenvalue of Δ_ρ on ξ is*

$$|\rho|^2 - |\tau|^2 + 2(\rho-\tau,\delta) + (\tau, 2\delta-2\delta'-\sigma).$$

For the purposes described in (4.7.4), we need to be able to decide whether the above expression is zero. This can happen only when ρ and τ are equal, i.e., ρ vanishes on

(4.8.6) $$\underline{h}_0 = \underline{h}_Q \cap \underline{p}_Q,$$

the orthogonal complement of $\underline{h}_k$ in $\underline{h}_Q$, in which case one must also have

(4.8.7) $$(\tau,\mu) = 0 \quad [\mu = 2\delta-2\delta'-\sigma].$$

In other words, τ must be orthogonal to the restrictions of certain positive roots. In summary:

(4.8.8) PROPOSITION. The Δ_ρ eigenvalue on ξ is zero if and only if

$$\tau = \rho, \text{ and } (\rho,\tilde{\mu}) = 0$$

for any $\tilde{\mu} \in \underline{h}_Q^*$ whose projection onto $\underline{h}_k$ is μ.

(4.9) If λ denotes the highest weight of E as a representation of G, the highest weights ρ occurring in $H^q(\underline{u}_Q,E)$, as a representation of $\underline{\ell}_Q \oplus \underline{a}_Q$ are given by

(4.9.1) $$\rho = w(\lambda+\delta_G) - \delta_G,$$

where δ_G is the half-sum ... for $\underline{g}$, and w is an element of length q in the Weyl group for G, with the property that whenever β is a positive root with $w^{-1}(\beta)$ negative, then β occurs in $\underline{u}_Q$ [27: (5.14)]. In the preceding, we have extended $\underline{h}_Q \oplus \underline{a}_Q$ to a Cartan subalgebra of $\underline{g}_{\mathbb{C}}$ by adding on a subalgebra of $\underline{k}$, which is possible since M^B is equal-rank (see (5.1), and recall (2.4)).[4]

In [35], the authors argue instead with the highest weight vectors in $H^q(\underline{u}_Q,E)$, though this has been reformulated in [36] in terms of the weights themselves.

(4.10) We recall now the structure of root systems with respect to a fundamental Cartan subalgebra, such as our $\underline{h}_Q$. Suppose that

(4.10.1) $$\underline{h} = \underline{h}_k \oplus \underline{h}_0$$

is fundamental for $\underline{g} \supset \underline{k}$, and $\underline{h}$ and $\underline{h}_k$ are given compatible systems of positive roots. Then the positive roots of $\underline{g}_{\mathbb{C}}$ fall into three classes (see [31: p. 123]):

a) Roots of $\underline{g}_{\mathbb{C}}$ whose root spaces lie in $\underline{k}_{\mathbb{C}}$, hence restrict to (positive) roots of $\underline{k}_{\mathbb{C}}$. These are fixed by the Cartan involution θ of $\underline{g}$ with respect to $\underline{k}$.

b) Pairs of positive roots, interchanged by θ, with common (positive) restriction to $\underline{h}_k$. The sum of the two root spaces, being θ-invariant, has a one-dimensional intersection with each of $\underline{k}_{\mathbb{C}}$ and $\underline{p}_{\mathbb{C}}$, both of which are

[4] Thus, each Q determines its own Cartan subalgebra of $\underline{g}_{\mathbb{C}}$.

invariant under $\underline{h}_k$.

c) Roots of $\underline{g}_{\mathbb{C}}$ whose root spaces lie in $\underline{p}_{\mathbb{C}}$. These are also invariant under θ .

Thus, we can say that the roots of class (c) and half of those of class (b) are associated to $\underline{p}$.

For (4.8.8), we choose

$$\tilde{\mu} = \dot{\sum}^{+} + \dot{\sum}^{-} \tag{4.10.2}$$

where $\dot{\sum}^{+}$ is the sum of the positive roots associated to $\underline{p}_Q$ whose restrictions do not occur in σ, and $\dot{\sum}^{-}$ is the sum of those positive roots whose negatives occur.

(4.11) Recall that the goal is to verify that at least one of the assertions in (4.6.6) holds. So, suppose that in (4.6.3) $n_k = 0$ for all $k < \nu$ and $n_\nu > 0$,[5] and moreover that an eigenvalue of Δ_ρ in

$$\Lambda^p \underline{p}_Q^* \otimes H^q(\underline{u}_Q, E)$$

is zero. It must be shown that $p + q < j - \nu$, i.e.

$$p + q \leq j - \dim \underline{a}. \tag{4.11.1}$$

Let s denote the number of summands in $\dot{\sum}^{-}$ (from (4.10.2)). Then (4.11.1) is attained by verifying the inequalities

$$\text{(4.11.2)} \qquad \text{i)}\ \ p \leq \tfrac{1}{2}(\dim X_Q + \dim \underline{h}_0) + s,$$

$$\text{ii)}\ \ q \leq \tfrac{1}{2}(\dim \underline{u}_Q - \dim(\underline{h}_0 \oplus \underline{a})) - s;$$

since $\dim_{\mathbb{R}}(X_Q \times A \times N)$ is the real codimension of the boundary component in question (compare (2.4.3)), adding the two inequalities in (4.11.2) produces (4.11.1).

(4.12) In order to verify (4.11.2), one is forced to get one's hands dirty in the root structure.

The central quantity in the weight considerations is

$$\mu = \rho + \delta_Q = w(\lambda + \delta_G) - \hat{\delta}, \tag{4.12.1}$$

[5] The case where $n_\nu = 0$ will be ruled out later, in (4.16).

where δ_Q is the half-sum for $\underline{u}_Q$, and $\hat{\delta}$ is the sum of δ and the half-sum for the group M^B (associated to the boundary component), together with its restriction to $\underline{h}_k$:

$$\mu_k = w(\lambda+\delta_G)|_{\underline{h}_k} - \hat{\delta}. \tag{4.12.2}$$

Let $\Psi(\underline{p}_Q)$, $\Psi(P)$, $\Psi(Q)$, $\Psi(Q/P)$ denote the sets of positive roots in $\underline{q}'$ associated to $\underline{p}_Q$, occurring in $\underline{n} = \underline{u}_P$, occurring in $\underline{u}_Q$, and occurring in $\underline{u}_Q/\underline{u}_P$, resp. Note that $\Psi(Q)$ contains

$$\Phi_w = \{\text{positive root } \alpha : (\mu+\hat{\delta}, \alpha) < 0\} \tag{4.12.3}$$
$$= \{\alpha > 0 : w^{-1}(\alpha) < 0\}.$$

Break up each of the sets $\Psi(\ )$ into three pieces, according to the sign of

$$(\mu_k+\hat{\delta},\alpha), \tag{4.12.4}$$

and label the pieces $\Psi(\ )_+$, $\Psi(\ )_-$, and $\Psi(\ )_0$ correspondingly.

(4.13) We next recall an elementary fact about root systems:

(4.13.1) PROPOSITION. Let α and β be non-proportional roots in a root system. Then:

i) if $(\alpha,\beta) > 0$, $\alpha-\beta$ is a root,

ii) if $(\alpha,\beta) < 0$, $\alpha+\beta$ is a root,

iii) if $(\alpha,\beta) = 0$, then either both $\alpha\pm\beta$ are roots, or neither is a root.

(4.13.2) Definition: One says that α and β are strongly orthogonal if the latter alternative in (4.13.1, iii) holds.

The following assertion turns out to be crucial:

(4.13.3) PROPOSITION. $\underline{h}_0 \oplus \underline{a}$ has a basis S consisting of strongly orthogonal roots in $\Psi(P)$; in fact, these occur in $\underline{n}_2$.

The reason for this will be given in (4.15). On the other hand:

(4.13.4) PROPOSITION [36: 11.6]. No root of $\Psi(\underline{p}_Q)$ is strongly orthogonal to every element of S.

This allows the construction of an embedding

(4.13.5) $$T : \Psi(\underline{p}_Q) \cup \Psi(Q/P) \hookrightarrow \Psi(P),$$

defined by putting

(4.13.6) $$T(\alpha) = \gamma(\alpha) - \alpha,$$

for a certain $\gamma(\alpha) \in S$ that is strongly orthogonal to α. It is clear from the definition that $T(\alpha)$ gives the opposite sign in (4.12.4) from that for α.

(4.14) We can now proceed to verify (4.11.1). It is trivial that $S \subset \Psi(P)_0$. The following is not difficult to check:

(4.14.1) PROPOSITION. Assume that:

a) $\rho|_{\underline{h}_0} = 0$,

b) $(\rho|_{\underline{a}}, \alpha) \geq 0$ <u>if</u> $\alpha \in \Psi(P)$,

c) $(\rho, \alpha) = 0$ <u>if</u> $\alpha \in \Psi(\underline{p}_Q)$ <u>occurs in</u> $\sum^-$.

<u>Then</u>,

i) $\Phi_w \cap \Psi(P) \subset \Psi(P)_-$,

ii) $T(\Phi_w \cap \Psi(Q/P)_+) \cap (\Phi_w \cap \Psi(P)) = \phi$,

iii) <u>If</u> α <u>occurs in</u> $\sum^-$, <u>then</u> $T(\alpha) \in \Psi(P)_- - (\Phi_w \cap \Psi(P))$.

(4.14.2) COROLLARY. $|\Phi_w \cap \Psi(Q/P)_+| + s \leq |\Psi(P)_-| - |\Phi_w \cap \Psi(P)|$; <u>that is</u> $|\Phi_w \cap \Psi(P)| \leq \frac{1}{2}|\Psi(P)| - \frac{1}{2}|\Psi(P)_0| - s - |\Phi_w \cap \Psi(Q/P)_+|$.

The second estimate needed is even cruder. Certainly,

(4.14.3) $$|\Phi_w \cap \Psi(Q/P)| \leq |\Psi(Q/P)_-| + |\Psi(Q/P)_0| + |\Phi_w \cap \Psi(Q/P)_+|$$
$$= \frac{1}{2}|\Psi(Q/P)| + \frac{1}{2}|\Psi(Q/P)_0| + |\Phi_w \cap \Psi(Q/P)_+|.$$

Adding this to (4.14.2), we obtain

(4.14.4) $$|\Phi_w| \leq \frac{1}{2}|\Psi(P)| - \frac{1}{2}|\Psi(P)_0| - s + \frac{1}{2}|\Psi(Q/P)| + \frac{1}{2}|\Psi(Q/P)_0|$$

$$= \frac{1}{2}\dim \underline{u}_Q + \frac{1}{2}|\Psi(Q/P)_0| - \frac{1}{2}|\Psi(P)_0| - s.$$

In addition, $T(S \cup \Psi(Q/P)_0) \subset \Psi(P)_0$, so (4.14.4) becomes

(4.14.5) $$|\Phi_w| \leq \frac{1}{2}\dim \underline{u}_Q - \frac{1}{2}|S| - s,$$

which is precisely (4.11.2, ii).

Inequality (4.11.2, i) is actually very easy:

$$\begin{aligned} p &\leq |\Psi(\underline{p}_Q)| + s + \dim \underline{h}_0 \\ &= \frac{1}{2}(2|\Psi(\underline{p}_Q)| + \dim \underline{h}_0) + \frac{1}{2}\dim \underline{h}_0 + s \\ &= \frac{1}{2}\dim X_Q + \frac{1}{2}\dim \underline{h}_0 + s. \end{aligned}$$

As (4.14.1, b) is a consequence of the assumptions on the n_k's, we now have (4.11.1).

(4.15) So, how does one produce sufficiently many strongly orthogonal roots?

Consider first the case where $Q' = P$, i.e., where $Q = M^A$ (the improper parabolic subgroup). From the Hermitian structure of D and D^B, the symmetric space of $M^A \times A_P$ is realized as an open cone C in $\underline{n}_2$ (the same cone that appears in (3.4)). On the infinitesimal level, this comes about via the adjoint action

(4.15.1) $$\mathrm{ad} : \underline{p}^A \otimes \underline{n}_2 \to \underline{n}_2,$$

where $\underline{p}^A$ denotes $\underline{p} \cap \underline{m}^A \oplus \underline{a}_P$. In view of (3.4.1), it is not so surprising that for generic $Y \in \underline{n}_2$, the mapping

(4.15.2) $$B_Y : \underline{p}^A \to \underline{n}_2$$

$$B_Y(Z) = \mathrm{ad}\,(Z \otimes Y) = [Z,Y]$$

is an isomorphism. Indeed, one has a K_P-invariant element $Y \in C$ with this property. This identifies $\underline{p}^A$ and $\underline{n}_2$ as K_P-modules. (See [1: III, (4.2)].)

Put

(4.15.3) $$\zeta = (B_Y)^{-1} : \underline{n}_2 \to \underline{p}^A .$$

The bilinear pairing on $\underline{n}_2$

(4.15.4) $$W_1 \cdot W_2 = [\zeta(W_1), W_2]$$

imparts to $\underline{n}_2$ the structure of a Jordan algebra (see [1: II]).

Via ζ, one can match features of $\underline{n}_2$ with those of $\underline{p}^A$. Corresponding to a set $\{e_1, \ldots, e_\ell\}$ of orthogonal idempotents in $\underline{n}_2$, such that

(4.15.5) $$\sum_{1 \leq j \leq \ell} e_j = 1$$

(in the sense of the Jordan algebra), is a set of generators for an abelian subspace $\underline{b}$ of $\underline{p}^A$, and all such subspaces arise in this manner. If

(4.15.6) $$\{2\varepsilon_j : 1 \leq j \leq \ell\}$$

is the basis dual to $\{\zeta(e_j)\}$ of $\underline{b}$, then for $i \leq j$, the weight space of $(\varepsilon_i + \varepsilon_j)$ in $\underline{n}_2$ is mapped to the $\underline{p}^A$-component of the $(\varepsilon_i - \varepsilon_j)$ weight space in $\underline{m}^A$. (The process of grouping the idempotents into sub-sums in (4.15.5) determines a subspace of $\underline{b}$, to which the corresponding subsets of the ε_j's restrict to a common value.) For $i = j$, the weight space of $2\varepsilon_j$ in $\underline{n}_2$ produces elements in $\underline{p}^A$ that commute with $\underline{b}$.

If $\underline{b}$ is now the non-compact part $(\underline{h}_0 \oplus \underline{a})$ of $\underline{h}_Q$, it commutes, of course, with $\underline{h}_k$. Since ζ is K_P-, a fortiori K_Q-equivariant, $\underline{h}_k$ also commutes with $\zeta^{-1}(\underline{b})$. It follows that $e_j \in \underline{n}_2$ is a root vector with respect to $\underline{h}_Q$, with $2\varepsilon_j$ (extended by zero to $\underline{h}_k$) as the root. Since $2\varepsilon_i \pm 2\varepsilon_j$ is never a weight, it follows that (4.15.6) gives the desired set S in (4.13.3).

(4.16) Finally, we must rule out the possibility that $n_k = 0$ for all k (equivalently, $\mu|_{\underline{a}} = 0$) when a zero eigenvalue for Δ_ρ occurs. But then $\mu|_{\underline{h}_0} = \rho|_{\underline{h}_0} = 0$ as well, so $\mu + \hat{\delta}$ would vanish on $\underline{h}_0 \oplus \underline{a}$, a fortiori on the roots of S. This contradicts the regularity of $w(\lambda + \delta_G)$.

§5. THE EXTENDED CONJECTURE OF BOREL; ADDITIONAL IDEAS AND RESULTS

(5.1) We now drop (1.1, e): we no longer assume that the symmetric space $D = G/K$ is Hermitian.

(5.1.1) Definition: D (or G) is said to be equal-rank if G admits a compact Cartan subgroup, i.e., if

$$\mathrm{rk}_{\mathbf{C}}\, K = \mathrm{rk}_{\mathbf{C}}\, G.$$

(5.1.2) Remarks: i) If D is Hermitian, then D is equal-rank (see [23: p. 383]).

ii) If D is equal-rank, then its dimension is even, $2m$, for the non-compact root spaces of $\underline{g}$ come in $\pm$ pairs, and give $\underline{g}/\underline{k}$. Thus, one may speak of m as the "complex dimension" of D.

(5.2) On the basis of the results in [10], Borel extended (1.6.1) to:

(5.2.1) CONJECTURE (1983). Let X be an arithmetic quotient of an equal-rank symmetric space, X^* a Satake compactification of X such that every rational boundary component is also equal-rank, and $\mathbf{E}$ a metrized local system as in (1.1, d). Then $\mathcal{L}^{\bullet}_{(2)}(X^*,\mathbf{E})$ is quasi-isomorphic to $\mathcal{IC}^{\bullet}(X^*,\mathbf{E})$.

(5.2.2) Remark: Under the hypothesis of (5.2.1), one knows at least that $H^{\bullet}_{(2)}(X,\mathbf{E})$ is finite-dimensional [10]; d has closed range.

We should say a little here about Satake compactifications, though we refer the reader to [47: (1.6), (A.2)] and [46]. A symmetric space D of rank r (i.e., $r = \mathrm{rk}_{\mathbf{R}} G$) admits 2^r-1 Satake compactifications, one for each non-empty subset Θ of a set of simple roots. The corresponding boundary components are described in terms of certain parabolic subgroups of G, depending on Θ (see [46: (2.10)]). Under two reasonable assumptions, one can define from this a compactification of arithmetic quotients of D, using only the rational boundary components ([46: §3]).

Consider the case where the $\mathbf{Q}$-rank of G equals the $\mathbf{R}$-rank r. To eliminate a few anomalous cases, assume that $r > 2$. The type-B real root system occurring in the Hermitian case (compare (2.4.2)) has one simple root β at one end of the Dynkin diagram, that is distinguished by its length. The Baily-Borel Satake compactification is the one determined by $\Theta = \{\beta\}$ (see [46: (3.11)]). Under the assumptions of this paragraph, Conjecture (5.2.1) adds to (1.6.1) only the analogous compactifications for $G = SO(p,q)$ with $p + q$ odd, and for $G = Sp(p,q)$, groups whose real root systems are also of type B. As far as I know, the conjecture has not been verified for these, even when $r = 3$.

(5.3) Before continuing, we observe that most of the considerations of (2.4)-(2.5) are quite general, and can be done for arbitrary semi-simple G; for $SO(p,q)$ or $Sp(p,q)$, it goes through verbatim.

In [15: (2.6)], Casselman showed, in effect, that the determination of the sign in the weight w_1^{α} (recall (2.7,i), (2.5.9)), which seems to depend on more delicate information than just the degree q (of $H^q(\underline{n},E)$), actually follows a very simple rule in cases of interest:

(5.3.1) PROPOSITION. Let G be a semi-simple algebraic group over $\mathbb{R}$, P a maximal $\mathbb{R}$-parabolic subgroup, E a finite-dimensional representation of G that is isomorphic to its complex conjugate contragredient. Write $P = M'A'N$ in a real Langlands decomposition, and use this to replace (2.4.1). Suppose that ρ is an irreducible constituent of $H^q(\underline{n},E)$ as a representation of M' that is also isomorphic to its conjugate contragredient. Then the coefficient in (2.5.9) is zero exactly when $q = \frac{1}{2}\dim \underline{n}$, and is positive exactly when $q > \frac{1}{2}\dim \underline{n}$.

The bearing of (5.3.1) on (5.2.1) comes via the following sequence of remarks. The condition that G be equal-rank is equivalent to the assertion that every representation E is isomorphic to its conjugate contragredient [10: §1]. On the other hand, M' is generally not equal-rank, so if ρ is generic, it would fail to satisfy this condition.

When G is defined over $\mathbb{Q}$, it need not have the property that every maximal Q-parabolic subgroup remains maximal over $\mathbb{R}$. (In general, one would have the rational and real Langlands decompositions of P related by $M_P \supset M'$, $A_P \subset A'$.) However, if, for instance, G is Hermitian, and its Lie algebra is simple over $\mathbb{R}$, then it does have that property [3: (2.10)]; the non-simple case can be reduced to the preceding by the use of restriction of scalars functors [3: §3].

From [10: (2.2), (5.6)], the L^2-cohomology with coefficients associated to a representation that is not isomorphic to its conjugate contragredient vanishes completely. Thus, the condition on ρ in (5.3.1) is necessary for taking its contribution to

(5.3.2) $$H^{\bullet}_{(2)}(X^A, \mathbb{H}^{\bullet}(\underline{n},E))$$

seriously. Recall, however, that we have imposed the weights w_2^{α} in (2.7, ii).

(5.4) For simplicity, assume that we are in the generalized Baily-Borel setting in what follows. If one could ignore the weights w_2^α, then (5.3.1) would reduce (2.7) to

(5.4.1) PROPOSITION [8: (2.3)]. If $q < \frac{1}{2} \dim \underline{n}$, then

$$H^p_{(2)}(X^A, \mathbf{H}^q(\underline{n},E)) = 0 \quad \underline{\text{for}} \quad p \geq j - q.$$

Of course, we cannot just forget about w_2^α. The function f in (2.4.3) blows up at the boundary of D^A (see [47: (3.17)]). In view of the direct relation between w_2^α and w_1^α (2.5.9), one has an inclusion

$$(5.4.2) \qquad L^\bullet_{(2)}(X^A, \mathbf{H}^\bullet_\alpha(\underline{n},E)) \to L^\bullet_{(2)}(X^A, \mathbf{H}^\bullet_\alpha(\underline{n},E);\ w_2^\alpha)$$

whenever w_1^α is a negative exponential, and the reverse (dual) inclusion when w_1^α is positive exponential. By comparing the cohomology of the corresponding unweighted and weighted L^2 sheaves on a suitable compactification of X^A, it may be possible to convert (5.4.1) into a proof of the conjecture (cf. [47: (3.30)]).

(5.5) What is different about the non-Hermitian case (aside from the absence of the complex structure)? The considerations of (4.15) are no longer available to give the key Prop. (4.13.3). Perhaps the latter, or some close variant of it, remains valid.

A nice little formula, which was not explicitly needed in either §3 or §4, is that for Hermitian G, the coefficient of a simple root in the restriction of δ_G to a maximal Q-split torus gives the complex codimension of the corresponding singular stratum [47: p. 394, (A.1)]. This is proved recursively. The formula fails to hold already for SO(p,q) and Sp(p,q), for the induction breaks down, though only at the first step. Likewise, the equality (3.4.1) is false for these groups, with the left-hand side greater in the former case, smaller in the latter.

Furthermore, it seems that the structure of the restriction from the real to the rational root system (compare [3: (2.9), (3.2)]) needs to be understood before (5.2.1) can be proved in full generality.

(5.6) We close with a remark that might be applicable to simplify the Saper-Stern argument. Can one verify a single a priori estimate on X^A that proves the vanishing of the weighted L^2-cohomology in (2.7.2, ii)? (The answer "no" seems quite possible, by the way.)

To this end, we work out the general formula comparing a weighted Laplacian to the unweighted one, i.e., for a conformal change of metrization. Let $\mathbb{V}$ be a metrized local system on the orientable Riemannian m-manifold M. The star-operators

(5.6.1) $$\text{i)} \quad * : L_2^i(M,\mathbf{V}) \to L_2^{m-i}(M,\mathbf{V}^\vee)),$$
$$\text{ii)} \quad *_g : L_2^i(M,\mathbf{V};g) \to L_2^{m-i}(M,\mathbf{V}^\vee;g^{-1})$$

compare by

(5.6.2) $$*_g = g* \; .$$

Of course, d is intrinsic, but the adjoint depends on the metrization through (5.6.2), and

(5.6.3) $$(d^*)_g \, \phi = d^*\phi - *^{-1}(d\lambda \wedge * \phi),$$

where $\lambda = \log g$. From here, one gets

(5.6.4) $$\Delta_g \phi = \Delta\phi - d*^{-1}(d\lambda \wedge *\phi) - *^{-1}(d\lambda \wedge *d\phi) \; .$$

A little manipulation shows that the conditions of (4.2.1) hold for the weighted L^2 complex, i.e., there is a constant $\kappa > 0$ with

(5.6.5) $$||\Delta_g \phi|| \geq \kappa ||\phi||$$

if one has

(5.6.6) $$||\Delta\phi|| \geq C||\phi||$$

for some $C > 2 \sup_M |d\lambda|^2$. In our case, $g = w_2^\alpha$, and the lower bound for C can be computed from the formula for f in (2.4.3); cf. [47: (3.17(2)), (3.18(1))].

References

[1] Ash, A., Mumford, D., Rapoport, M., Tai, Y.: Smooth Compactifications of Locally Symmetric Varieties. Mathematical Science Press, Brookline, MA, 1975.

[2] Baily, W.: Satake's compactification of V_n^*. Amer. J. Math. 80(1958), 348-364.

[3] Baily, W., Borel, A.: Compactifications of arithmetic quotients of bounded symmetric domains. Ann. of Math. 84(1966), 442-528.

[4] Beilinson, A., Bernstein, J., Deligne, P.: Faisceaux pervers. Astérisque 100(1982).

[5] Borel, A.: Stable real cohomology of arithmetic groups. Ann. Sci. ENS 7 (1974), 235-272.

[6] Borel, A.: Regularization theorems in Lie algebra cohomology. Applications. Duke Math. J. 50(1983), 605-624.

[7] Borel, A.: L^2-cohomology and intersection cohomology of certain arithmetic varieties. In: Emmy Noether in Bryn Mawr, Springer, 1983, 119-131.

[8] Borel, A.: A vanishing theorem in relative Lie algebra cohomology. In: Algebraic Groups: Utrecht, 1986, Lecture Notes in Math. 1271, Springer, 1987.

[9] Borel, A., et al.: Intersection Cohomology. Birkhäuser, Boston, 1984.

[10] Borel, A., Casselman, W.: L^2-cohomology of locally symmetric manifolds of finite volume. Duke Math. J. 50(1983), 625-647.

[11] Borel, A., Casselman, W.: Cohomologie d'intersection et L^2-cohomologie de variétés arithmétiques de rang rationnel 2. CR Acad. Sc. Paris 301(1985), 369-373.

[12] Borel, A., Garland, H.: Laplacian and discrete spectrum of an arithmetic group. Amer. J. Math. 105(1983), 309-335.

[13] Borel, A., Wallach, N.: Continuous Cohomology, Discrete Subgroups and Representations of Reductive Groups. Annals of Math. Studies 94, Princeton Univ. Press, 1980.

[14] Brylinski, J.-L., Labesse, J.-P.: Cohomologie d'intersection et fonctions L de certaines variétés de Shimura. Ann. Sci. ENS 17(1984), 361-412.

[15] Casselman, W.: L^2-cohomology for groups of real rank one. In: Representation Theory of Reductive Groups, Birkhäuser, 1983, 69-84.

[16] Casselman, W.: Introduction to the L^2-cohomology of arithmetic quotients of bounded symmetric domains. In: Complex Analytic Singularities, Adv. Stud. Pure Math. 8(1986), 69-93.

[17] Cattani, E., Kaplan, A., Schmid, W.: L^2 and intersection cohomologies for a polarizable variation of Hodge structure. Inventiones Math. 87 (1987), 217-252.

[18] Cheeger, J.: On the Hodge theory of Riemannian pseudomanifolds. AMS Proc. Symposia Pure Math. 36(1980), 91-146.

[19] Cheeger, J., Goresky, M., MacPherson, R.: L^2-cohomology and intersection homology of singular varieties. In: Seminar on Differential Geometry, Annals of Math. Studies 102(1982), 303-340.

[20] Gaffney, M.: Hilbert space methods in the theory of harmonic integrals. Transactions AMS 78(1955), 426-444.

[21] Goresky, M., and MacPherson, R.: Intersection homology, II. Inventiones Math. 72(1983), 77-129.

[22] Harder, G.: On the cohomology of discrete arithmetically defined subgroups. In: Proceedings of the International Colloquium on Discrete Subgroups of Lie Groups and Applications, Oxford Univ. Press, 1975, 129-160.

[23] Helgason, S.: Differential Geometry, Lie Groups, and Symmetric Spaces. Academic Press, New York-San Francisco-London, 1978.

[24] Hörmander, L.: L^2 estimates and existence theorems for the $\bar{\partial}$ operator. Acta Math. 113(1965), 89-152.

[25] Kashiwara, M., Kawai, T.: The Poincaré lemma for variations of polarized Hodge structure. Publ. RIMS 23(1987), 345-407.

[26] Kohn, J.: Harmonic integrals on strongly pseudo-convex manifolds, I. Ann. of Math. 78(1963), 112-148.

[27] Kostant, B.: Lie algebra cohomology and the generalized Borel-Weil theorem. Ann. of Math. 74(1961), 329-387.

[28] Looijenga, E.: L^2-cohomology of locally symmetric varieties. Compositio Math. 67(1988), 3-20.

[29] Matsushima, Y., and Murakami, S.: On vector bundle valued harmonic forms and automorphic forms on symmetric Riemannian manifolds. Ann. of Math. 78(1963), 365-416.

[30] Matsushima, Y., and Murakami, S.: On certain cohomology groups attached to Hermitian symmetric spaces. Osaka J. Math. 2(1965), 1-35.

[31] Raghunathan, M.: On the first cohomology of discrete subgroups of semi-simple Lie groups. Amer. J. Math. 87(1965), 103-139.

[32] Raghunathan, M.: Vanishing theorems for cohomology groups associated to discrete subgroups of semi-simple Lie groups. Osaka J. Math. 3 (1966), 243-256; 16(1979), 295-299.

[33] Saito, Morihiko: Modules de Hodge polarisables. Preprint RIMS-553, 1986.

[34] Saito, Morihiko: Mixed Hodge modules. Preprint RIMS-585, 1987; Proc. Japan Acad. 62(1986), Ser. A., 360-363.

[35] Saper, L., Stern, M.: L_2 cohomology of arithmetic varieties. Proc. Natl. Acad. Sci. USA 84(1987), 5516-5519.

[36] Saper, L., Stern, M.: L_2-cohomology of arithmetic varieties. Ann. of Math. (to appear).

[37] Satake, I.: On the compactification of the Siegel space. J. Indian Math. Soc. 20(1956), 259-281.

[38] Satake, I.: On compactifications of the quotient spaces for arithmetically defined discontinuous groups. Ann. of Math. 72(1960), 555-580.

[39] Séminaire H. Cartan, Fonctions Automorphes, ENS 1957/58, vol. 2.

[40] Vogan, D., Zuckerman, G.: Unitary representations with non-zero cohomology. Compositio Math. 53(1984), 51-90.

[41] Warner, G.: Harmonic Analysis on Semi-Simple Groups I. Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[42] Zucker, S.: Locally homogeneous variations of Hodge structure. L'Enseignement Math. 27(1981), 243-276.

[43] Zucker, S.: L_2 cohomology of warped products and arithmetic groups. Inventiones Math. 70(1982), 169-218.

[44] Zucker, S.: L_2-cohomology and intersection homology of locally symmetric varieties. AMS Proc. Symposia Pure Math. 40, Part 2(1983), 675-680.

[45] Zucker, S.: On a theorem of Chern. Math. Zeit. 182(1983), 87-94.

[46] Zucker, S.: Satake compactifications. Commentarii Math. Helv. 58(1983), 312-343.

[47] Zucker, S.: L_2-cohomology and intersection homology of locally symmetric varieties, II. Compositio Math. 59(1986), 339-398.

[48] Zucker, S.: The Hodge structures on the intersection homology of varieties with isolated singularities. Duke Math. J. 55(1987), 603-616.

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