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Astérisque, tome 179-180 (1989), p. 163-184
<http://www.numdam.org/item?id=AST_1989__179-180__163_0>

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Introduction

Many results about the topology of complex hypersurface singularities have a Hodge-theoretic counterpart. The monodromy theorem for isolated singularities combined with the Hodge filtration on the vanishing cohomology have led to the notion of the *spectrum* ([A],[St4], [V1]). The spectrum is a powerful invariant, giving necessary conditions for adjacency of singularities. In this paper, we define the spectrum for arbitrary (i.e. not necessarily isolated) hypersurface singularities and investigate some of its properties. In particular we conjecture a Thom-Sebastiani type theorem about the spectrum. This formula has recently been proven by M. Saito using the description of the mixed Hodge structure on the cohomology of the Milnor fibre via his theory of mixed Hodge modules [Sa1]. Moreover, we investigate the behaviour of the spectrum under certain deformations. We consider a hypersurface $\{f = 0\}$ in $\mathbb{C}^{n+1}$ whose singular locus is of dimension one and compare this with a hypersurface $\{f + \varepsilon\ell^k = 0\}$ where ε is sufficiently small and ℓ is a linear form which is not tangent to any component of the critical locus of f. We conjecture a formula for the spectrum of $f + \varepsilon\ell^k$ which generalizes a formula of Yomdin [Y] for the Milnor number. We are able to prove this formula in certain cases, which are listed in §2. M. Saito has recently given a proof in the general case [Sa 2]. The corresponding formula for the characteristic polynomial of the monodromy has been proven by D. Siersma [Si 2].

As an application, we give an example, found together with J. Stevens, of two isolated plane curve singularities which have different topological types but equal spectra. This gives a negative answer to a question mentioned by W. Neumann [N1], namely whether the real monodromy and Seifert form determine the (embedded) topology of an isolated complex hypersurface singularity. We also give an example in dimension two, which shows that even the topological type of the hypersurface singularity itself is not determined by these data. A detailed discussion of this will appear elsewhere.

It should be remarked that the spectrum of the affine cone over a projective hypersurface with only isolated singular points is independent of the position of these points. On the other hand, the Betti numbers of the Milnor fibre do depend on this position in general (a phenomenon, usually indicated by the word 'defect'). A. Dimca communicated to me, that he has a method to compute exact Betti numbers for projective hypersurfaces with arbitrary isolated singularities.

The author is indebted to Theo de Jong, Duco van Straten, Lê Dũng Tráng and Steve Zucker for stimulating discussions. He also thanks Dirk Siersma, M. Saito and Theo de Jong for pointing out some errors in an earlier draft of this paper.

§1. Spectra of hypersurface singularities

A *spectrum* is a set of rational numbers, counted with certain multiplicities. These multiplicities may be negative. Let $\mathcal{S} = \mathbb{Z}^{(\mathbb{Q})}$, the free abelian group on generators (α), $\alpha \in \mathbb{Q}$. A typical element of $\mathcal{S}$ will be denoted as $\sum n_\alpha(\alpha)$. We will consider spectra as elements of $\mathcal{S}$.

Let $\mathcal{C}$ denote the category whose objects are $\mathbb{C}[t]$-modules of finite length equipped with t-stable decreasing filtrations on which t acts as an automorphism of finite order and whose morphisms are $\mathbb{C}[t]$-linear maps which are compatible with the given filtrations. A typical object of $\mathcal{C}$ will be denoted as (H,F,γ) where F is the filtration and γ the automorphism given by the action of t. In the main application, H is the cohomology group of the Milnor fibre of an isolated hypersurface singularity, F is its Hodge filtration and γ corresponds to the action of the semisimple part of the monodromy (the monodromy itself is not compatible with F). A sequence

$$0 \longrightarrow H' \xrightarrow{\alpha} H \xrightarrow{\beta} H'' \longrightarrow 0$$

in $\mathcal{C}$ will be called *exact* if the underlying sequence of vector spaces is exact and if α and β are strictly compatible with the filtrations, i.e. $\alpha(H') \cap F^pH = \alpha(F^pH')$ and $F^pH'' = \beta(F^pH)$ for all p. With this concept of exact sequence, $\mathcal{C}$ becomes an *exact category* (see [Q]).

The group $\mathcal{S}$ can be considered as the Grothendieck group of $\mathcal{C}$ in the following way. Fix an integer n and let (H,F,γ) be an object of $\mathcal{C}$. Observe that γ acts on the subquotients $Gr_F^p(H) = F^p/F^{p+1}$. One defines $Sp_n(H,F,\gamma)$ as follows. Define rational numbers $\alpha_1,\dots,\alpha_{s(p)}$, where $s(p) = \dim Gr_F^p(H)$, by

$$n-p-1 < \alpha_j \le n-p \ ;$$

$$\det(tI - \gamma;\ Gr_F^pH) = \prod_{j=1}^{s_p} (t - e^{-2i\pi\alpha_j}) \ .$$

Then

$$\mathrm{Sp}_n(H,F,\gamma) := \sum_p \sum_{j=1}^{s_p} (\alpha_j) .$$

For every integer n the map Sp_n induces an isomorphism between $K_0(\mathscr{C})$ and $\mathscr{S}$. Changing n into n+j or shifting the filtration index by -j corresponds to a shift $(\alpha) \longrightarrow (\alpha+j)$ in $\mathscr{S}$.

Let $f\colon (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ be a non-zero holomorphic function germ. Its Milnor fibre X(f) is defined by

$$X(f) = \{z \in \mathbb{C}^{n+1} \mid |z| < \eta \text{ and } f(z) = t\}$$

for $0 < |t| \ll \eta \ll 1$. The cohomology groups $H^*(X(f))$ carry a canonical mixed Hodge structure (see [St2] for the case that f has an isolated critical point at 0 and [Na]§14 for the general case). The semisimple part T_s of the monodromy acts as an automorphism of these mixed Hodge structures. In particular, it preserves the Hodge filtration F.

We define the *spectrum* of f by

$$\mathrm{Sp}(f) := \sum_{k=0}^{n} (-1)^{n-k}\, \mathrm{Sp}_n(\tilde{H}^k(X(f)),F,T_s)$$

In the case of isolated singularities, this reduces to the existing definition, because then $\tilde{H}^k(X(f)) = 0$ for $k \neq n$, as X(f) has the homotopy type of a wedge of n-spheres.

Examples. For quasi-homogeneous isolated hypersurface singularities the spectrum can be calculated in the following way. Choose a basis $\{z^\alpha\}_{\alpha\in A}$ of monomials for the Artinian ring $Q_f = \mathbb{C}\{z_0,\dots,z_n\}/(\partial_0 f,\dots,\partial_n f)$. For $\alpha \in A$ put $w(\alpha) = \sum_{i=0}^{n} (\alpha_i+1)w_i - 1$ where $w_0,\dots,w_n \in \mathbb{Q}$ are the weights , normalized in such a way that f has degree one. Then $\mathrm{Sp}(f) = \sum_{\alpha\in A} (w(\alpha))$ (see [St3] for a proof). In particular we obtain for the simple singularities:

type	normal form	spectrum
A_k	$z_0^{k+1} + z_1^2 + \dots + z_n^2$	$\sum_{i=1}^{k} (\frac{i}{k+1} + \frac{n}{2} - 1)$
D_k	$z_0^{k-1} + z_0z_1^2 + z_2^2 + \dots + z_n^2$	$\sum_{i=1}^{k-1} (\frac{n}{2} + \frac{i}{k-1} - 1) + (\frac{n-1}{2})$
E_6	$z_0^4 + z_1^3 + z_2^2 + \dots + z_n^2$	$\sum_{j\in\{1,4,5,7,8,11\}} (\frac{6n+j}{12} - 1)$
E_7	$z_0^3z_1 + z_1^3 + z_2^2 + \dots + z_n^2$	$\sum_{j\in\{1,5,7,9,11,13,17\}} (\frac{9n+j}{18} - 1)$
E_8	$z_0^5 + z_1^3 + z_2^2 + \dots + z_n^2$	$\sum_{j\in\{1,7,11,13,17,19,23,29\}} (\frac{15n+j}{30})$

For more spectra of isolated singularities from Arnol'd's lists see [G].

Let $f\colon \mathbb{C}^3 \longrightarrow \mathbb{C}$ be given by $f(x,y,z) = xy$ (type A_∞ in the notation of [Si1]). Then $X(f)$ is homeomorphic to the affine variety $xy = 1$ in $\mathbb{C}^3$, i.e. to $\mathbb{C}^* \times \mathbb{C}$, and the monodromy is the identity. We obtain that $Sp(f) = -(1)$.

Let $f\colon \mathbb{C}^3 \longrightarrow \mathbb{C}$ be given by $f(x,y,z) = xyz$ (type $T_{\infty,\infty,\infty}$). The Milnor fibre of f is diffeomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ and the mixed Hodge structure on $H^i(X(f))$ is purely of type (i,i) for $i = 0$, 1 and 2. The monodromy operator is the identity. Hence we obtain

$$Sp(f) = (0) - 2(1).$$

Let $f\colon \mathbb{C}^n \longrightarrow \mathbb{C}$ be a germ and define $g\colon \mathbb{C}^{n+1} \longrightarrow \mathbb{C}$ by $g(z_0,\dots,z_n) = f(z_1,\dots,z_n)$. If $Sp(f) = \sum_\alpha n_\alpha(\alpha)$, then $Sp(g) = -\sum_\alpha n_\alpha(\alpha+1)$.

We recall a few properties of the spectrum for isolated hypersurface singularities. It is convenient to introduce the following notions. For any subset B of $\mathbb{Q}$ or $\mathbb{R}$ we obtain a group homomorphism

$$\deg_B\colon \mathcal{S} \longrightarrow \mathbb{Z}$$

given by

$$\deg_B(\textstyle\sum_\alpha n_\alpha(\alpha)) = \sum_{\alpha\in B} n_\alpha .$$

We define $\mu(f) = \deg_{\mathbb{Q}} Sp(f)$; for isolated singularities this is the Milnor number.

The *semicontinuity property* of the spectrum ([V1],[St4]) can be formulated as follows. Let $\{f_t\}_{t\in\Delta}$ be a family of functions parametrized by a disc such that f_0 has an isolated critical point at 0 with $f_0(0) = 0$. Suppose that there are continuous maps $x_i\colon (0,1] \longrightarrow \mathbb{C}^{n+1}$, $i = 1,\dots,r$, such that the $x_i(t)$ are critical points of f_t with the same critical value and that $\lim_{t\to 0} x_i(t) = 0$. We can compare the spectra of the germ of f_0 at 0 and of the f_t at the $x_i(t)$. The result is:

Theorem. *For any half-open interval* B *of length one in* $\mathbb{R}$

$$\deg_B Sp(f_0,0) \geq \sum_{i=1}^r \deg_B Sp(f_t,x_i(t)) .$$

It would be interesting to have such a semicontinuity result for certain deformations of non-isolated singularities too. As an important example, consider a surface singularity in $\mathbb{C}^3$ which is weakly normal [vS], i.e. which has generically only ordinary double curves. Consider a deformation which admits a simultaneous normalisation. By [dJ-vS] this is equivalent to the condition that the singular locus varies in a flat way. Under these conditions the spectrum should behave semicontinuously.

We define the *convolution operation* $*$ on $\mathcal{S}$ as the bilinear mapping $*\colon \mathcal{S} \times \mathcal{S} \longrightarrow \mathcal{S}$ given on generators by

$$(\alpha) * (\beta) = (\alpha+\beta+1)$$

Each pair of germs $f\colon (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$, $g\colon (\mathbb{C}^{m+1},0) \longrightarrow (\mathbb{C},0)$ defines a germ $f\oplus g\colon (\mathbb{C}^{n+1} \times \mathbb{C}^{m+1},(0,0)) \longrightarrow (\mathbb{C},0)$ by $(f\oplus g)(z,w) = f(z) + g(w)$. If f and g have an isolated singularity, the same is true for $f\oplus g$, and, by the Thom-Sebastiani theorem, $\mu(f\oplus g) = \mu(f)\mu(g)$ where μ denotes the Milnor number.

Theorem. *Let* f, g *be as above. Then*

$$\mathrm{Sp}(f\oplus g) = \mathrm{Sp}(f)*\mathrm{Sp}(g)$$

See [V1], Thm 7.3 and also [SS] for the case of isolated singularities. The general case is due to M. Saito (private communication). We may even include the case that f and/or g are zero: just define the spectrum of the zero function in n variables to be $(-1)^n(n)$.

In the isolated singularity case, the spectrum is invariant under the reflection of $\mathcal{S}$ defined by $(\alpha) \longrightarrow (n-1-\alpha)$. The examples of A_∞ and $T_{\infty,\infty,\infty}$ above show that this need not be true in general.

For isolated hypersurface singularities $(V,0)$ the *geometric genus* $p_g(V,0)$ is related to the spectrum by $p_g = \deg_{(-1,0]}\mathrm{Sp}(f)$ where f is a defining function. Van Straten [vS] has generalized this notion to the case of weakly normal surface singularities and verified a similar formula.

§2. Functions with a one-dimensional critical locus

Let $f\colon (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ be a holomorphic function germ. The critical locus Σ of f is the set of common zeros of the partial derivatives of f or, more precisely, the germ at 0 of this set. By Sard's theorem, $\Sigma \subseteq f^{-1}(0)$. We consider germs f for which Σ is of dimension 1.

Let $\Sigma_1,\dots,\Sigma_r$ be the irreducible components of Σ. For each i we choose a point $P_i \neq 0$ on Σ_i and a slice U_i through P_i transverse to Σ_i. Let $g_i = f_{|U_i}\colon (U_i,P_i) \longrightarrow (\mathbb{C},0)$. Then g_i is an isolated hypersurface singularity. Its analytic type will in general depend on the choices which have been made. However, two different choices give rise to germs which are μ-homotopic, i.e. which are connected by a family with constant Milnor number. Therefore the μ-class of g_i is an invariant of f; it is called the *transverse type of* f *along* Σ_i (cf. [Y] or [Lê 1] (1.3.1) and (1.3.2)). Because the spectrum of an isolated singularity depends only on its μ-class [V3], the spectrum of g_i is well-defined.

Recall that on Σ one has a *sheaf of vanishing cycles* Φ_f (cf. [D]). This is a constructible sheaf complex (in fact a perverse sheaf) whose cohomology sheaves at a point of Σ give the reduced cohomology of the Milnor fibre of the germ of f at the given point. Hence $\mathcal{H}^i(\Phi_f)_P = 0$ for $P \in \Sigma \setminus \{0\}$ and $i \neq n-1$ and $\mathcal{H}^{n-1}(\Phi_f)$ is a local system on each $\Sigma_i \setminus \{0\}$ whose fibre at P_i is $\tilde{H}^{n-1}(X(g_i))$. Remark that on $\tilde{H}^{n-1}(X(g_i))$ we have two monodromy transformations: the monodromy T_i of the germ g_i (which we call the *horizontal monodromy*) and the monodromy τ_i (the *vertical monodromy*) of the local system $\mathcal{H}^{n-1}(\Phi_f)_{(i)}$ which is the restriction to the punctured disk $\Sigma_i \setminus \{0\}$ of $\mathcal{H}^{n-1}(\Phi_f)$. These two monodromies commute with each other, because T_i is locally constant on $\mathcal{H}^{n-1}(\Phi_f)_{(i)}$.

Let ℓ be a sufficiently general linear form on $\mathbb{C}^{n+1}$. Then for all k sufficiently large and ε with $0 < |\varepsilon| \ll 1$ the germ $f_k = f+\varepsilon\ell^k$ has an isolated singularity at 0. Yomdin [Y] has proved the following formula for its Milnor number.

(2.1) **Theorem.** *For all k sufficiently large*

$$\mu(f_k) = \mu(f) + ke_0(\Sigma)$$

Here $e_0(\Sigma)$ denotes the multiplicity of Σ at 0.

The main subject of this article concerns the relation of the spectra of f and f_k. We formulate a conjecture which we then verify in certain cases. We keep the preceding notations and put $m_i = e_0(\Sigma_i)$, $\mu_i = \mu(g_i)$, $Sp(g_i) = \sum_{j=1}^{\mu_i} (\lambda_{ij})$. Moreover we write

$$\beta_m = \sum_{i=0}^{m-1} (-i/m) \in \mathcal{S} \text{ for } m \in \mathbb{N}.$$

(2.2) **Conjecture**. *For all* i *there exist non-negative rational numbers* α_{ij}, $j=1,\ldots,\mu_i$, *depending only on the vertical monodromy* τ_i, *such that for all* k *sufficiently large*

$$Sp(f_k) = Sp(f) + \sum_{i,j} (\lambda_{ij} - \alpha_{ij}/km_i)*\beta_{m_i k}.$$

In case $\tau_i = \mathrm{Id}$ *we have* $\alpha_{ij} = 0$ *for all* j.

We verify this conjecture in the cases n=1 (§4), n=2 and the transverse type of f along each component of Σ is simple (§5), n arbitrary and f homogeneous with transverse singularities of Pham-Brieskorn type (§6). To make the latter result more useful we compute the spectrum of a homogeneous germ f with one-dimensional singular locus in §6. It is related to the spectrum of an isolated homogeneous singularity by the same formula as in the conjecture, when we substitute $k = d$. Moreover, in the homogeneous case, the correction coefficient α associated to a transverse spectrum number λ is given by $\alpha = d\lambda - [d\lambda]$.

Using Lê's 'carrousel' method, Siersma has been able to prove a formula for the zeta function of the monodromy of f_k which is compatible with our conjecture. His proof also makes it possible to specify what the numbers α_{ij} in the formula should be.

Choose a basis $\{e_{ij}\}_{j=1}^{\mu_i}$ for $\tilde{H}^{n-1}(X(g_i))$ on which both horizontal and vertical monodromy are given by an upper triangular matrix, with diagonal elements ξ_{ij} and η_{ij} respectively, such that $\exp(-2\pi i\lambda_{ij}) = \xi_{ij}$. Then α_{ij} should be given by

$$0 \le \alpha_{ij} < 1 \text{ and } \exp(-2\pi i\alpha_{ij}) = \eta_{ij}.$$

M. Saito has given a general proof of our conjecture, which however does not give the formula for the spectrum of a homogeneous germ f with one-dimensional singular locus [Sa 2].

We give some examples to illustrate the power of these theorems.

1. Let $f\colon (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ be of type A_∞. Then $Sp(f) = -(n/2)$ and f_k has type A_{k-1}. We obtain $Sp(A_{k-1}) = -(n/2) + (n/2 - 1)*\beta_k$ so $\alpha = 0$ works in this case. Indeed, the vertical monodromy is the identity here.

2. Take $f(x,y) = x^2y$. Then $Sp(f) = (0)$ and the vertical monodromy is $-Id$. We take $\alpha = 1/2$ to get exactly the spectrum of f_k which is of type D_{k+1}.

3. Take $f(x,y,z) = xyz$. We have seen in §1 that $Sp(f) = -2(1) + (0)$ and $\tau_i = Id$ for $i = 1,2,3$. The germ f_k has type $T_{k,k,k}$ with spectrum $Sp(f) + 3(0)*\beta_k$. More generally, $Sp(T_{p,q,r}) = Sp(f) + (0)*\beta_p + (0)*\beta_q + (0)*\beta_r$.

4. Let $f \in \mathbb{C}[z_0,\dots,z_n]$ be homogeneous of degree d such that Σ is of dimension one and the tranverse type along each component of Σ is A_1. Then the vertical monodromy is multiplicatión by $(-1)^{nd}$. The correction coefficient α is equal to 0 if nd is even and to 1/2 if nd is odd.

5. Varchenko [V1] has derived the following upper bound for the number of

double points on a complex projective hypersurface of dimension n-1 with only isolated singularities in terms of the degree d. The estimate is as follows. Consider a homogeneous polynomial $f_{n,d}$ of degree d in n+1 variables with an isolated singularity at 0. Then $Sp(f_{n,d}) = \gamma_d^{*(n+1)}$ (multiple join product) where $\gamma_d = \beta_d-(0)$.
For nd even let $I = (n/2 - 2 + 1/d, n/2 - 1 + 1/d)$ and for nd odd let $I = (n/2 - 2 + 1/2d, n/2 - 1 + 1/2d)$. Then the number of ordinary double points on a hypersurface of degree d in $\mathbb{P}^n(\mathbb{C})$ with no other singularities is not bigger than $\deg_I(\gamma_d^{*n})$.
This also follows from our Theorem (6.1) and (6.3). Let f define such a hypersurface with δ ordinary double points. Then f defines also a singularity with one-dimensional singular locus, consisting of δ lines through the origin and transverse type A_1. Let ℓ be a general linear form. Then $f + \varepsilon\ell^{d+1}$ is an isolated singularity for $\varepsilon > 0$ small enough, hence it has a spectrum which is effective (all its coefficients are nonnegative), because the Milnor fibre of an n-dimensional isolated hypersurface singularity is (n-1)-connected.

Suppose that nd is even. By our Theorem (6.1) and (6.3)

$$Sp(f + \varepsilon\ell^{d+1}) = \gamma_d^{*(n+1)} - \delta[\beta_d - \beta_{d+1}]*(n/2 - 1)$$

hence the coefficient of $(n/2 - 1 + 1/d)$ in $\gamma_d^{*(n+1)}$ must be at least δ. This coefficient is exactly equal to $\deg_I(\gamma_d^{*n})$. If nd is odd,

$$Sp(f + \varepsilon\ell^{d+1}) = \gamma_d^{*(n+1)} - \delta\beta_d*(n/2 - 1 - 1/2d) + \delta\beta_{d+1}*(n/2 - 1 - 1/2(d+1))$$

and we see that the coefficient of $(n/2 - 1 + 1/2d)$ in $\gamma_d^{*(n+1)}$ has to be at least δ. (This argument is due to Theo de Jong.)

§3. Some toric geometry

In this section we gather some results from toric geometry which will be used in the next sections. Our basic references are [Da 1] and [Da 2].

Let Δ be an (n+1)-simplex in $\mathbb{R}^{n+1}$ with vertices $v_0, \dots, v_{n+1}$ in $\mathbb{Z}^{n+1}$. The toric variety $\mathbb{P}_\Delta$ is the union of the affine open subsets $U_i = \mathrm{Spec}(A_i)$, $i = 0, \dots, n+1$, with

$$A_i = \mathbb{C}[M_i] \text{ and } M_i = \mathbb{Z}^{n+1} \cap \textstyle\sum_{j\neq i} \mathbb{R}_+(v_j - v_i).$$

The integral points of Δ correspond to a basis of the space of sections $L(\Delta)$ for a line bundle $\mathcal{L}$ on $\mathbb{P}_\Delta$. Each non-zero element g of $L(\Delta)$ defines a hypersurface $Z_{\Delta,g}$ in $\mathbb{P}_\Delta$. The variety $\mathbb{P}_\Delta$ has only quotient singularities. If $g \in L(\Delta)$ is sufficiently general, $Z_{\Delta,g}$ intersects the strata $\mathbb{P}_\tau$ of $\mathbb{P}_\Delta$ for τ a face of Δ transversally, and $Z_{\Delta,g}$ will have only quotient singularities too.

In this situation it is called a *quasi-smooth* hypersurface.

Assume from now on that $Z_{\Delta,g} = Z$ is quasi-smooth and that all monomials occurring in g with non-zero coefficient lie in $\{v_0\} \cup \Delta_0$ where Δ_0 is the face of Δ opposite to v_0. Then one can define an automorphism γ of Z as follows. Without loss of generality we may assume that $v_0 = 0$. Let ℓ denote the linear form on $\mathbb{R}^{n+1}$ which takes the value 1 on Δ_0. It takes rational values on the lattice of integral points. For a finite subset A of $\mathbb{Z}^{n+1}$ we define $\sigma(A) \in \mathcal{S}$ by

$$\sigma(A) = \textstyle\sum_{P\in A} (\ell(P)-1) .$$

Write $\mathbf{e}(\lambda) = \exp 2\pi i\lambda$. By construction, $Z \cap U_i = \operatorname{Spec} A_i/(g_i)$ where $g_i = z^{-v_i}.g$. The map $z^{\beta} \longrightarrow \mathbf{e}(\beta)z^{\beta}$ defines an automorphism of A_i which leaves g_i invariant and is the restriction of a global automorphism γ of Z.

By Lefschetz' theorems $H^i(\mathbb{P}_{\Delta},\mathbb{C}) \cong H^i(Z,\mathbb{C})$ for $i \neq n$, $2n+2$. We let $P^n(Z,\mathbb{C}) = \operatorname{Coker} [H^n(\mathbb{P}_{\Delta},\mathbb{C}) \longrightarrow H^n(Z,\mathbb{C})]$; this is the most interesting cohomology group of Z. It carries a pure Hodge structure of weight n. Danilov [Da 2] has calculated its Hodge numbers h_0^{pq} . We will derive a formula for the spectrum of $(P^n(Z,\mathbb{C}),F,\gamma^*)$ which is a little bit more explicit than Danilov's formulas (which apply to a more general situation). Our formula is similar to the formula of [St 3].

(3.1) **Proposition**. *Let Δ, g, ℓ be as above, with $v_0 = 0$. Let*

$$D = \mathbb{Z}^{n+1} \cap \{ \textstyle\sum_{i=1}^{n+1} t_i v_i \mid 0 < t_i < 1 \text{ and } \sum_{i=1}^{n+1} t_i \notin \mathbb{Z} \}.$$

Then

$$\operatorname{Sp}_n(P^n(Z,\mathbb{C}),F,\gamma^*) = \sigma(D).$$

Proof. First observe that $\operatorname{Gr}_F^p P^n(Z,\mathbb{C}) = H^{p,n-p}_{\neq 1}(Z)$ in the notation of [Da 2], (4.10). (The subscript $\neq 1$ refers to the subspace on which γ^* acts with eigenvalues $\neq 1$). Let $e \in \mathbb{N}$ be defined by $\ell(\mathbb{Z}^{n+1}) = e^{-1}\mathbb{Z}$. Danilov computes the element $h^{p,n-p}$ in the group ring $\mathbb{Z}[e^{-1}\mathbb{Z}/\mathbb{Z}]$ corresponding to the representation of the group μ_e of e^{th} roots of unity on $H^{p,n-p}_{\neq 1}(Z)$. The answer can be formulated in the following way. For $A \subseteq \mathbb{Z}^{n+1}$ finite let

$$\lambda(A) = \textstyle\sum_{p\in A}(\ell(p) \bmod \mathbb{Z}) \in \mathbb{Z}[e^{-1}\mathbb{Z}/\mathbb{Z}].$$

For a t-simplex $\tau \subseteq \mathbb{R}^{n+1}$ with integral vertices $v_0,\dots,v_n$ and $m \in \mathbb{N}$ define

$$D_m(\tau) = \{ \textstyle\sum_{i=0}^{t} s_i v_i \mid 0 < s_i < 1,\ \sum_{i=0}^{t} s_i = m \}$$
$$\lambda_m(\tau) = \lambda(\operatorname{int} m\tau \cap \mathbb{Z}^{n+1})$$
$$\delta_m(\tau) = \lambda(D_m(\tau) \cap \mathbb{Z}^{n+1}).$$

Then

$$h^{p,n-p} = \sum_{\tau\le\Delta}\sum_{i\ge1} (-1)^{n-p+i-1}\begin{bmatrix}\dim\tau+1\\ p+i+1\end{bmatrix}\lambda_i(\tau) = \delta_{n+1-p}(\Delta).$$

The last equality is a nice exercise. The proof is completed by looking at the definition of the spectrum. □

§4. Proof of the conjecture for curves

Let $f\colon (\mathbb{C}^2,0) \longrightarrow (\mathbb{C},0)$ be a curve singularity. We decompose f into irreducible factors

$$f = f_1^{p_1}\cdots f_r^{p_r} f_{r+1}\cdots f_{r+s}$$

with $p_i > 1$ for $i=1,\dots,r$. Let Σ_i be the zero set of f_i. Then the critical locus of f is $\Sigma_1 \cup \dots \cup \Sigma_r \cup \{0\}$. The transverse type of f along Σ_i is A_{p_i-1}. The transverse Milnor fibre consists of p_i points which are permuted by the vertical monodromy τ_i. If $g_i = f.f_i^{-p_i}$ and $\nu_i = \mathrm{ord}_0(g_{i|\Sigma_i})$, then τ_i is the ν_i^{th} power of a cyclic permutation of these points. Hence τ_i depends only on ν_i mod p_i.

Let $\pi'\colon Z' \longrightarrow \mathbb{C}^2$ be a good embedded resolution of f, i.e. a sequence of blowing-ups in points such that $(f\pi')^{-1}(0)$ is a divisor with normal crossings on Z'. Write $\pi'^{-1}(0) = E = \bigcup_{\alpha\in V}E_\alpha$ with E_α irreducible. The E_α are isomorphic to $\mathbb{P}^1$. Let X_i be the strict transform of Σ_i under π', and write

$$\mathrm{div}(f\pi') = \textstyle\sum_i p_iX_i + \sum_{\alpha\in V} e_\alpha E_\alpha.$$

For each $i \in \{1,\dots,r+s\}$ there is a unique $\alpha(i) \in V$ such that $X_i \cap E_{\alpha(i)} = \{P_i\} \ne \emptyset$.

(4.1) **Lemma**: $\nu_i \equiv e_{\alpha(i)} \bmod p_i$.

Proof. We have $\nu_i = \mathrm{ord}_0(g_{i|\Sigma_i}) = \mathrm{ord}_{P_i}(g_i\circ\pi'_{|X_i})$ as $X_i \longrightarrow \Sigma_i$ is the normalization. Write

$$\mathrm{div}(f_i\circ\pi') = \textstyle\sum_{\alpha\in V} \beta_\alpha E_\alpha + X_i.$$

Then

$$\mathrm{div}(g_i\circ\pi') = \textstyle\sum_{\alpha\in V} (e_\alpha - p_i\beta_\alpha)E_\alpha + \sum_{j\ne i} p_jX_j.$$

For $i \ne j$ the components X_i and X_j do not intersect, hence $\nu_i = e_{\alpha(i)} - p_i\beta_{\alpha(i)} \equiv e_{\alpha(i)} \bmod p_i$. □

Let ℓ be a linear form on $\mathbb{C}^2$ such that the line $\ell = 0$ is not tangent to any branch of f. We write

$$\mathrm{div}(\ell\circ\pi') = L + \textstyle\sum_{\alpha\in V} m_\alpha E_\alpha.$$

Let $k \in \mathbb{N}$ be such that $km_\alpha > e_\alpha$ for all $\alpha \in V$. Define $f_k = f + \varepsilon\ell^k$ for $0 < |\varepsilon| \ll 1$ (so that all necessary transversality properties will hold). Then in suitable holomorphic coordinates (u,v) on Z' around P_i we have

$$\ell\circ\pi'(u,v) = u^{m_{\alpha(i)}}, \quad f\circ\pi'(u,v) = u^{e_{\alpha(i)}}v^{p_i}, \quad X_i : v = 0 .$$

(4.2) **Lemma**: *Let* $m_i = e_0(\Sigma_i)$. *Then* $m_i = m_{\alpha(i)}$.
Proof. As ℓ is transverse to Σ_i, $m_i = \mathrm{ord}_0(\ell_{|\Sigma_i}) = \mathrm{ord}_{P_i}(\ell\circ\pi'_{|X_i}) = m_{\alpha(i)}$.

We are going to construct a modification Z of Z' which gives a good partial resolution of f_k (for k fixed) in the sense that Z is admitted to have some cyclic quotient singularities. We use toric methods. The construction is analogous to the one in [Da 2, §3] to which we refer for proofs.

The local situation near P_i is of the following type. Let $f, \ell \in \mathbb{C}[u,v]$ be given by $f(u,v) = u^e v^p$, $\ell(u,v) = u^m$. For fixed $k \in \mathbb{N}$ we let $f_k(u,v) = f(u,v) + \varepsilon\ell(u,v)^k = u^e(v^p + \varepsilon u^\lambda)$ with $\lambda = km - e$. We suppose that $\lambda > 0$.

Let $\Delta \subset \mathbb{R}^2$ be the Newton diagram of f_k, i.e. the convex hull of $((e,p) + \mathbb{R}^2_+) \cup ((km,0) + \mathbb{R}^2_+)$. We denote its 1-dimensional compact face by Γ.

Let $M_1 = \mathbb{Z}^2 \cap \mathbb{R}_+.(\Delta - (km,0))$, $M_2 = \mathbb{Z}^2 \cap \mathbb{R}_+.(\Delta - (e,p))$. Then the toric variety $\mathbb{P}_\Delta$ is the union $U_1 \cup U_2$ with $U_i = \mathrm{Spec}(\mathbb{C}[M_i])$. The inclusions $\mathbb{Z}^2_+ \subset M_i$ define a proper morphism

$$\rho\colon \mathbb{P}_\Delta \longrightarrow \mathrm{Spec}\ \mathbb{C}[u,v]$$

such that $\rho^{-1}(0) = \mathbb{P}_\Gamma \cong \mathbb{P}^1$.

We need to know the order of $f\circ\rho$ and $\ell\circ\rho$ along the divisor $\mathbb{P}_\Gamma$. Let $\delta = \gcd(km-e,p)$ and write $km-e = \delta b$, $p = \delta a$, so $\gcd(a,b) = 1$. Choose $\alpha,\beta \in \mathbb{Z}$ with $\alpha a + \beta b = 1$. Put

$$\xi = u^\alpha v^\beta, \quad \eta = u^b v^{-a}.$$

Then $U_1 \cap U_2 = \mathrm{Spec}\ \mathbb{C}[\xi,\eta,\eta^{-1}]$. The ideal of $\mathbb{P}_\Gamma \cap U_1 \cap U_2$ is equal to (ξ). Moreover, on $U_1 \cap U_2$:

$$\rho^*(u) = \xi^\alpha\eta^\beta, \quad \rho^*(v) = \xi^b\eta^{-a}$$

so

$$\rho^*(f) = \xi^{akm}\eta^{a\beta-\alpha p}, \quad \rho^*(\ell) = \xi^{am}\eta^{\beta m}, \quad \rho^*(f_k) = \xi^{akm}(\eta^{a\beta-\alpha p} + \varepsilon\eta^{\beta km})$$

hence $\mathrm{ord}_{\mathbb{P}_\Gamma}(f\circ\rho) = \mathrm{ord}_{\mathbb{P}_\Gamma}(f_k\circ\rho) = akm$, $\mathrm{ord}_{\mathbb{P}_\Gamma}(\ell\circ\rho) = am$.

We conclude from this that the components of the special fibre of $f_k\circ\rho$ have multiplicity 1, akm or e. Choose a common multplie d of e and akm. Let

$$F_k(u,v,w) = w^d - f_k(u,v) .$$

Let $\tilde\Delta$ be the Newton diagram of F_k. Then Δ is a face of $\tilde\Delta$. The form F_k defines a hypersurface $\tilde U$ in $\mathbb{P}_{\tilde\Delta}$ which is transverse to the strata of $\mathbb{P}_{\tilde\Delta}$ corresponding to its faces. $\tilde{U}$ does not pass through the point of $\mathbb{P}_{\tilde{\Delta}}$ corresponding to the vertex (0,0,d) which is the only point where $\mathbb{P}_{\tilde{\Delta}}$ is not quasismooth. Hence $\tilde{U}$ is quasismooth. We obtain a finite morphism

$$\sigma\colon \tilde{U} \longrightarrow \mathbb{P}_{\Delta}$$

which exhibits $\tilde{U}$ as a cyclic covering of $\mathbb{P}_{\Delta}$ with covering group μ_d, where $\zeta \in \mu_d$ acts via $(u,v,w) \longrightarrow (u,v,\zeta w)$. We have a commutative diagram

$$\begin{tikzcd} \tilde{U} \arrow[r, "\sigma"] \arrow[d, "w"] & \mathbb{P}_{\Delta} \arrow[d, "f_k\circ\rho"] \\ \tilde{\mathbb{C}} \arrow[r] & \mathbb{C} \\ t \arrow[r] & t^d \end{tikzcd}$$

and $\tilde{U}$ is the normalization of $\mathbb{P}_{\Delta} \times_{\mathbb{C}} \tilde{\mathbb{C}}$. The special fibre of w on $\tilde{U}$ is a reduced divisor with V-normal crossings in the sense of [St 2].

Performing this construction in a small neighborhood U_i' of each point P_i we obtain spaces U_i and $\tilde{U}_i$. The integer d can be chosen in a uniform way. Put $U_0 = Z' \setminus \{P_1,\dots,P_r\}$ and let $\tilde{U}_0$ be the normalization of $U_0 \times_{\mathbb{C}} \tilde{\mathbb{C}}$. This glues to the $\tilde{U}_i$ to give a diagram

$$\begin{tikzcd} \tilde{Z} = \bigcup_{i=0}^{r} \tilde{U}_i \arrow[r, "\sigma"] \arrow[dd, "w"] & Z = \bigcup_{i=0}^{r} U_i \arrow[d, "\rho"] \\ & Z' \arrow[d, "f_k\circ\pi'"] \\ \tilde{\mathbb{C}} \arrow[r] & \mathbb{C} \end{tikzcd}$$

We define $\pi = \pi'\circ\rho$. Observe that ρ just replaces the points p_i by the curves $\mathbb{P}_{\Gamma_i}$. Hence for each $\alpha \in V$, the strict transform of E_α under ρ is isomorphic to E_α. By abuse of language we denote it by E_α again. So

$$\operatorname{div}(f_k\circ\pi) = \textstyle\sum_{\alpha\in V} e_\alpha E_\alpha + \sum_{i=1}^{r} a_i k m_{\alpha(i)} \mathbb{P}_{\Gamma_i} + \sum_{i=1}^{r+s} X_i^{(k)}$$

where $X_i^{(k)}$ is the strict transform of the i^{th} branch of f_k. It is a small deformation of X_i for $i > r$ and looks like a p_i-fold ramified covering of X_i for $i \le r$.

Recall from [St-Z, §3] that for each union E' of compact components of $\operatorname{div}(f_k\circ\pi)$ there exists a filtered sheaf complex $K_{E'}$ supported on E' such that

$$\mathbb{H}^*(E', K_{E'}) \cong H^*(X(f_k) \cap U_{E'}, \mathbb{C})$$

where $X(f_k)$ is the Milnor fibre of f_k and $U_{E'}$ is a tubular neighborhood of E' in Z. For divisors $E'' \subseteq E'$ we have relative complexes $K_{E',E''}$ with support on the closure of $E' \setminus E''$. In the case $E' = E'' \cup E_\alpha$, these relative groups are easy to compute. Let D_α, D', D" be the inverse images of E_α, E' and E" in $\tilde{Z}$. In D_α we have the finite subsets $\Sigma_1 = D_\alpha \cap D''$ and $\Sigma_2 = D_\alpha \cap$ (closure of div(w) $\setminus$ D'). Then

$$H^*(E_\alpha, K_{E',E''}) \cong H^*(D_\alpha \setminus \Sigma_2, \Sigma_1; \mathbb{C}).$$

This is even an isomorphism of mixed Hodge structures compatible with the monodromy actions. On the right hand side the monodromy acts via the covering transformation $w \longrightarrow \mathbf{e}(1/d)w$.

We now choose $E'' = \bigcup_{\alpha\in V} E_\alpha$, $E' = E'' \cup \bigcup_{i=1}^r \mathbb{P}_{\Gamma_i}$.

(4.3) **Proposition.** $Sp(f_k) - Sp(f) = Sp(\mathbb{H}^1(K_{E',E''}),F,T)$.

Proof. By [St 2], we have $\mathbb{H}^*(E',K_{E'}) \cong H^*(X(f_k),\mathbb{C})$ as mixed Hodge structures. The groups $\mathbb{H}^*(E'',K_{E''})$ carry a mixed Hodge structure which in general will depend on ε. If ε varies, we obtain a variation of mixed Hodge structure over a punctured disc which has a limit when $\varepsilon \longrightarrow 0$. This limit is isomorphic to $H^*(X(f),\mathbb{C})$, again by the construction of [St 2]. As for each i both finite subsets $\Sigma_{1,i}$ and $\Sigma_{2,i}$ of each component over $\mathbb{P}_{\Gamma_i}$ are non-empty, $\mathbb{H}^k(K_{E',E''}) = 0$ unless $k = 1$. □

To prove the conjecture, we just have to compute $Sp(H^1(C_i \setminus \Sigma_{2,i}, \Sigma_{1,i})$ where C_i lies above $\mathbb{P}_{\Gamma_i}$ in $\tilde{Z}$. The result in §3 deals with $H^1(C_i)$. It is an easy exercise to take $\Sigma_{1,i}$ and $\Sigma_{2,i}$ into account.

Write m for $m_{\alpha(i)}$, e for $e_{\alpha(i)}$ and p for p_i. Put

$$A = \mathbb{Z}^2 \cap \{ t_1(e,p) + t_2(km,0) \mid 0 < t_1 \le 1,\ 0 < t_2 < 1 \}.$$

Then $\#A = km(p-1)$. In the notation of §3, with $\ell(t_1(e,p) + t_2(km,0)) = t_1 + t_2$ we get

$$Sp\ H^1(C_i \setminus \Sigma_{2,i}, \Sigma_{1,i}) = \sigma(A).$$

To connect this with the formula of the conjecture, observe that A consists of the points (h,j) where $j = 1,\dots,p-1$ and $h \in \mathbb{Z} \cap (je/p, km + je/p]$. The transverse spectrum numbers are $\lambda_j = -1 + j/p$. Put $\alpha_j = je/p - [je/p]$. Then α_j depends only on the vertical monodromy along X_i (use Lemma 1). Moreover, $(h,j) = t_1(e,p) + t_2(km,0)$ with $t_1 = j/p$, $t_2 = (n - \alpha_j)/km$ where $n = h - [je/p]$. Hence

$$\sigma(A) = \textstyle\sum_{j=1}^{p-1} (\lambda_j - \alpha_j/km) * \beta_{km}.$$

This finishes the proof of the conjecture for curves.

(4.4) **Remark.** There exists a slight generalization of the theorem. To formulate this, we recall the notions of a polar curve and the polar ratios of f. Let ℓ be a sufficiently general linear form. We obtain a map germ $\Phi = (\ell, f)\colon (\mathbb{C}^2,0) \longrightarrow (\mathbb{C}^2,0)$. The polar curve of f (with respect to ℓ) is the union Γ of those components of the critical locus of Φ which are not contained in $f^{-1}(0)$. If (z,w) are coordinates in the target, then for each component Γ_i

of Γ the curve $\Delta_i = \Phi(\Gamma_i)$ is tangent to the z-axis and has a Puiseux series

$$z = a_i w^{r_i} + \text{higher order terms}$$

with $r_i < 1$. The polar ratios of f are the various r_i. They can be determined in terms of a good resolution of f as follows. Let E be the exceptional divisor of such a resolution. Write $E = \bigcup_{\alpha\in V} E_\alpha$, $e_\alpha = \mathrm{ord}_{E_\alpha}(f)$ and $m_\alpha = \mathrm{ord}_{E_\alpha}(\ell)$. Call $\alpha \in V$ a *rupture point* if E_α meets at least three components of $\bigcup_{\beta\neq\alpha} E_\beta \cup L \cup \tilde{X}$ where L (resp. $\tilde{X}$) is the strict transform of $\ell^{-1}(0)$ (resp $\ell^{-1}(0)$). Then the set of polar ratios of f is exactly the set of all m_α/e_α for α a rupture point of V. See [Lê 2] and [St-Z].

(4.5) **Theorem.** *Let* $f\colon (\mathbb{C}^2,0) \longrightarrow \mathbb{C}$ *be a germ of a plane curve singularity. Consider a germ* ϕ *with the property that*

(i) $\mathrm{ord}_0(\phi) > r_j^{-1}$ *for each polar ratio* r_j *of f;*

(ii) for $i = 1,\dots,r$ *we can write* $\phi = \phi_i^{(1)} + \phi_i^{(2)}$ *with* $\phi_i^{(1)}$ *divisible by* $f_i^{p_i}$ *and* $\mathrm{ord}_0(\phi_i^{(2)}|_{\Sigma_i}) = \mathrm{ord}_0(\phi_i^{(2)}).e_0(\Sigma_i)$ *(i.e. the tangent cone to the curve defined by* $\phi_i^{(2)}$ *is transverse to* Σ_i*).*

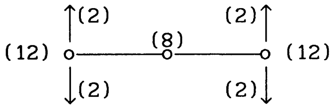
Then for $0 < |\varepsilon| \ll 1$*:*

$$\mathrm{Sp}(f + \varepsilon\phi) = \mathrm{Sp}(f) + \sum_{i=1}^{r}\sum_{j=1}^{p_i-1} (-j/p_i - \alpha_{ij}/k_i)*\beta_{k_i}$$

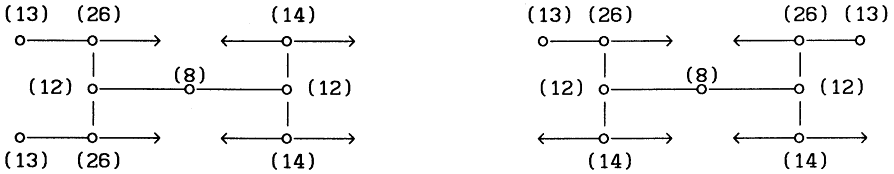
where $\alpha_{ij} = j\nu_i/p_i - [j\nu_i/p_i]$.

The proof is similar to the proof of the conjecture and will therefore be omitted. It should be remarked that the conditions of Theorem (4.5) are not as sharp as possible.

(4.6) **Example.** Consider the polynomial $f(x,y) = (x^4 - y^2)^2(x^2 - y^4)^2$. It has a resolution graph as follows (the numbers between brackets indicate the multiplicities >1, the arrows correspond to the non-compact components):



By a small perturbation one can deform the double components either to a smooth branch tangent to the exceptional divisor, or to a cusp which is transverse to the exceptional curve. We deform two of the double curves in the first way and the other two in the second way. This can be done in two essentially different ways: either the two cusp deformations take place on the same exceptional curve or not. This procedure leads to isolated plane curve singularities with the following resolution graphs:



These graphs are not isomorphic, so the topological types of these curve singularities are different. However, they have the same spectrum, as predicted by Theorem (4.5).

§5. The surface case

We will give an outline of the proof of the following

(5.1) **Theorem.** *Let* $f\colon (\mathbb{C}^3,0) \longrightarrow (\mathbb{C},0)$ *be a germ with 1-dimensional singular locus* Σ. *Suppose that the transverse type of* f *along each branch* Σ_i *of* Σ *is a simple plane curve singularity (in the sense of Arnol'd). Then Conjecture* (2.2) *holds for* f.

Proof. We will use the theory of embedded improvements due to Jan Stevens [Sv]. He shows that there exists a proper modification (sequence of blowing-ups) $\pi\colon Y \longrightarrow \mathbb{C}^3$ such that $\pi\colon Y \setminus \pi^{-1}(0) \longrightarrow \mathbb{C}^3 \setminus \{0\}$ is biholomorphic and such that the strict transform of $\{f = 0\}$ has only a very mild type of singularities. E.g. if the transverse type of f is A_1, only ordinary double curves and pinch points remain; the latter only occur in the case where the transverse monodromy is -1.

We need a slightly stronger result: a local normal form for $f\circ\pi$ and $\ell\circ\pi$ after a suitable improvement. As Stevens communicated to me, the methods of [Sv] lead to the following

Proposition. *Let* $f\colon (\mathbb{C}^3,0) \longrightarrow \mathbb{C}$ *be a square-free surface singularity with simple tranverse type. Then there exists an embedded improvement* $\pi\colon Y \longrightarrow \mathbb{C}^3$ *of* f *such that the singularities of* $f\circ\pi$ *are of the following types: one has normal crossings at the points of* $E = \pi^{-1}(0)$ *different from the* *intersection points with the strict transforms of the components of Σ, and near these intersection points only the following types can occur:*

transverse type	normal forms
A_k	$x^{\alpha}(z^2 - y^k)$, $x^{\alpha}(z^2 - xy^{2k})$
D_k	$x^{\alpha}(yz^2 - y^k)$, $x^{\alpha}(yz^2 - xy^{2k})$
	and for $k = 4$: $x^{\alpha}(z^3 - xy^3)$
E_6	$x^{\alpha}(z^3 - y^4)$
E_7	$x^{\alpha}(z^3 - zy^3)$
E_8	$x^{\alpha}(z^3 - y^5)$

Given a sufficiently general linear form ℓ, we may also assume that $\ell \circ \pi = x^{\beta}$.

Once these local forms have been obtained, the same toric methods as in §4 can be used to construct resolutions for f and f_k and to compare their spectra explicitly.

(5.2) **Remark**. In the case of transverse type A_1, we can do slightly better and derive a formula for $Sp(f + \varepsilon\phi)$ where ϕ is a germ such that $ord_0(\phi) > r_j^{-1}$ for each polar ratio r_j and which for each i can be written as $\phi_i^{(1)} + \phi_i^{(2)}$ with $\phi_i^{(1)} \in I(\Sigma_i)^2$ and $\phi_i^{(2)}$ has a tangent cone transverse to Σ_i. As before let α_i = 0 (resp 1/2) if τ_i = I (resp. -I). Then

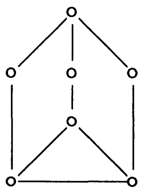
$$Sp(f + \varepsilon\phi) = Sp(f) + \textstyle\sum_{i=1}^{r} (-\alpha_i/\nu_i)^*\beta_{\nu_i} .$$

with $\nu_i = ord_0(\phi_{|\Sigma_i}) = ord_0(\phi_i^{(1)}).e_0(\Sigma_i)$. A similar formula should hold in general if one requires that $\phi_i^{(1)} \in I(\Sigma_i)^{m_i}$ where m_i is chosen in such a way that the transverse type of f along Σ_i does not change when one perturbes it by a germ of order m_i.

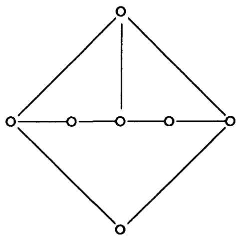
(5.3) **Example**. We will use the formula from (5.2) to show two isolated surface singularities which are not topologically equivalent but have the same spectrum. Let ℓ, $\ell_1,\dots,\ell_4$ be linear forms on $\mathbb{C}^3$ no three of which are linearly dependent. Put $f = \ell_1\ell_2\ell_3\ell_4$. The critical locus of f consists of the six lines L_{ij}: $\ell_i = \ell_j = 0$ for $i < j$ and the transverse type of f along these lines is A_1. Let $\phi_{ij} = \ell_r^2\ell_s^2$ where $\{i,j,r,s\} = \{1,2,3,4\}$. Then $\phi_{ij} \in I(L_{mn})^2$ if $\{i,j\} \neq \{m,n\}$ and the tangent cone to ϕ_{ij} is transverse to L_{ij}. Define

$$\phi_1 = \ell(\phi_{12} + \phi_{23} + \phi_{13}) + \ell^2(\phi_{14} + \phi_{24} + \phi_{34})$$
$$\phi_2 = \ell(\phi_{12} + \phi_{14} + \phi_{13}) + \ell^2(\phi_{23} + \phi_{24} + \phi_{34})$$

We put $f_i = f + \varepsilon\phi_i$, $i = 1,2$. It is clear from (5.2) that f_1 and f_2 have the same spectrum. However their resolution graphs are not even isomorphic. Hence the singularities of $f_1^{-1}(0)$ and $f_2^{-1}(0)$ at 0 are not homeomorphic by a result of W. Neumann [N2].



resolution graph of $f_1^{-1}(0)$



resolution graph of $f_2^{-1}(0)$

§6. The homogeneous case

In this section, we determine the spectrum of a homogeneous polynomial $f \in \mathbb{C}[z_0,\dots,z_n]$ with a 1-dimensional critical locus. Moreover, under the assumption that the projective hypersurface $\tilde{V}(f)$ defined by f has only singularities of Pham-Brieskorn type, we derive a formula for $Sp(f + \varepsilon\ell^k)$, $k > d$, which proves our conjecture in this case.

As in §4 we put $\gamma_d = \sum_{i=1}^{d-1} (-i/d) \in \mathscr{S}$. By the Thom-Sebastiani theorem for spectra of isolated singularities we see that the germ $\sum_{i=0}^{n} z_i^d$ has spectrum $\gamma_d^{*(n+1)}$. Because the spectrum stays constant under deformations with constant Milnor number, any homogeneous polynomial in $\mathbb{C}[z_0,\dots,z_n]$ with an isolated singularity at 0 has spectrum $\gamma_d^{*(n+1)}$.

(6.1) **Theorem**: *Let* $f \in \mathbb{C}[z_0,\dots,z_n]$ *be homogeneous of degree* d. *Suppose that* $\tilde{V}(f) \subset \mathbb{P}^n$ *has only isolated singularities, say* $P_1,\dots,P_r$. *Let* $g_i\colon (\mathbb{C}^n,0) \longrightarrow (\mathbb{C},0)$ *be a local equation for* $\tilde{V}(f)$ *near* P_i. *Write* $Sp(g_i) = \sum_{j=1}^{\mu_i} (\lambda_{ij})$. *Define* $\alpha_{ij} = d\lambda_{ij} - [d\lambda_{ij}]$. *Then*

$$Sp(f) = \gamma_d^{*(n+1)} - \textstyle\sum_{i,j} (\lambda_{ij} - \alpha_{ij}/d)*\beta_d.$$

Proof. We will use a one-parameter deformation f_t such that $f_0 = f$ and f_t has an isolated singularity at 0 for $t \neq 0$.

As f is homogeneous, the mixed Hodge structure on the cohomology of its Milnor fibre is isomorphic to the mixed Hodge structure on the cohomology of the affine hypersurface $V_0 \subset \mathbb{C}^{n+1}$ defined by the equation $f(z) = 1$. Let ℓ be a linear form on $\mathbb{C}^{n+1}$ such that the corresponding hyperplane in $\mathbb{P}^n$ does not pass through any of the P_i. By Sard's theorem, there exists $\varepsilon > 0$ such that for $t \in$

$\mathbb{C}$ with $0 < |t| < \varepsilon$ the varieties $Z_t = \tilde{V}(f + t\ell^d) \subset \mathbb{P}^n$ and $Y_t = \tilde{V}(f + t\ell^d - z_{n+1}^d) \subset \mathbb{P}^{n+1}$ are non-singular. Observe that Y_0 is the projective closure of V_0 and $Z_0 = Y_0 \setminus V_0$. We let $V_t = Y_t \setminus Z_t$. Moreover we define

$$\Delta_\varepsilon = \{\, t \in \mathbb{C} \mid |t| < \varepsilon \,\};\ Y = \bigcup_t (\{t\} \times Y_t) \subset \Delta_\varepsilon \times \mathbb{P}^{n+1};$$
$$Z = \bigcup_t (\{t\} \times Z_t) \subset \Delta_\varepsilon \times \mathbb{P}^n;\ V = Y \setminus Z.$$

We let π_Y, π_Z and π_V denote the corresponding projections to Δ_ε.

Let $\gamma\colon V_0 \longrightarrow V_0$ be defined by $z \longrightarrow \zeta z$ where $\zeta = \mathbf{e}(1/d)$. Then the monodromy operator T_f on $H^*(V_0)$ is given by $T_f(\omega) = (\gamma^*)^{-1}(\omega)$. This action extends to the whole of Y by

$$\gamma(z_0:\ldots:z_n:z_{n+1}) = (\zeta z_0:\ldots:\zeta z_n:z_{n+1})$$

and induces the identity on Z.

Though V_0 is smooth, Y_0 and Z_0 have isolated singularities at the points P_i. The spaces Y and Z are smooth and we will compute the vanishing cohomology of the families π_Y and π_Z to get hold of $H^*(V_0)$ with its γ-action. A complicating factor is the relation between the local monodromy operators $\tilde{T}_i$ and T_i of π_Y and π_Z at P_i and the action of γ.

The germs of π_Y and π_Z at P_i are equivalent to $g_i + z_{n+1}^d = \tilde{g}_i$ and g_i respectively. The Thom-Sebastiani theorem identifies $H^n(X(\tilde{g}_i))$ with $\tilde{H}^{n-1}(X(g_i))\otimes\Gamma_d$ where $\Gamma_d = \tilde{H}^0(X(z^d))$, and via this identification, $\tilde{T}_i = T_i \otimes T'$ with T' the monodromy of z^d. The main observation is that $\gamma^* = 1 \otimes T'$.

We define $W^p = Gr_F^{n-p}H^n(X(\tilde{g}_i)) \oplus Gr_F^{n-1-p}\tilde{H}^{n-1}(X(g_i))$ and let $W = W^0 \oplus \ldots \oplus W^n$. Its filtration F is given by $F^pW = W^p \oplus \ldots \oplus W^n$.

(6.2) **Lemma**. $Sp(f) = Sp(f_t) - Sp_n(W,F,\gamma^*)$.

Proof. As γ is the geometric monodromy of the germ f, the cohomological monodromy operator on $H^n(V_0)$ is γ^{*-1}. The following are exact sequences of mixed Hodge structures which are equivariant for the action of γ:

$$\ldots \longrightarrow H_c^k(V_t) \longrightarrow H^k(Y_t) \longrightarrow H^k(Z_t) \longrightarrow H_c^{k+1}(V_t) \longrightarrow \ldots$$
$$\ldots \longrightarrow H^k(Y_0) \longrightarrow H^k(Y_t) \longrightarrow \oplus\, \tilde{H}^k(X(\tilde{g}_i)) \longrightarrow H^{k+1}(Y_0) \longrightarrow \ldots$$
$$\ldots \longrightarrow H^k(Z_0) \longrightarrow H^k(Z_t) \longrightarrow \oplus\, \tilde{H}^k(X(g_i)) \longrightarrow H^{k+1}(Z_0) \longrightarrow \ldots$$

In the first sequence t can take all values in Δ_ε. In the second and third ones $H^k(Y_t)$ and $H^k(Z_t)$ carry the limit mixed Hodge structure associated to the degenerations π_Y and π_Z respectively. We get

$$Sp(f_t) - Sp(f) = Sp_n(H^n(V_t),F,\gamma^{*-1}) - Sp_n(H^n(V_0),F,\gamma^{*-1})$$

To compute this, we have to express the difference of $Gr_F^pH^n(V_t)$ and $Gr_F^pH^n(V_0)$ in the Grothendieck group of $\mathbb{C}[t]$-modules, where t acts as γ^{*-1}. For this we use the exact sequences above and the duality between $Gr_F^pH^n(V_t)$ and $Gr_F^{n-p}H_c^n(V_t)$. This duality gives rise to an isomorphism of $\mathbb{C}[t]$-modules, if we let t act as γ^* on $Gr_F^{n-p}H_c^n(V_t)$. This follows from the fact that γ^* preserves the cup product form and acts trivially on $H_c^{2n}(V_t)$. We obtain

$$\begin{aligned} Gr_F^pH^n(V_t) - Gr_F^pH^n(V_0) &= Gr_F^{n-p}H_c^n(V_t) - Gr_F^{n-p}H_c^n(V_0) \\ &= \textstyle\sum_k (-1)^k Gr_F^{n-p}H^{n+k}(Y_t) - \sum_k (-1)^k Gr_F^{n-p}H^{n+k}(Z_t) \\ &\quad - \textstyle\sum_k (-1)^k Gr_F^{n-p}H^{n+k}(Y_0) + \sum_k (-1)^k Gr_F^{n-p}H^{n+k}(Z_0) \\ &= \textstyle\sum_i Gr_F^{n-p}H^n(X(\tilde{g}_i)) + \sum_i Gr_F^{n-p-1}\tilde{H}^{n-1}(X(g_i)) \end{aligned}$$

in $K(\mathbb{C}[t])$, where t acts as γ^*. This proves our lemma.

For each i we choose a basis (ξ_{ij}), $j = 1,\dots,\mu_i$, for $\tilde{H}^{n-1}(X(g_i))$ in such a way that $T_i(\xi_{ij}) = \mathbf{e}(-\lambda_{ij})\xi_{ij}$ and that $F^p\tilde{H}^{n-1}(X(g_i))$ is spanned by the ξ_{ij} for which $n-p-2 < \lambda_{ij} \le n-p-1$. We also choose a basis $\vartheta_1,\dots,\vartheta_{d-1}$ for Γ_d such that $T'(\vartheta_k) = \mathbf{e}(-k/d)\vartheta_k$. Then a basis for W consists of the elements ξ_{ij} and $\xi_{ij}\otimes\vartheta_k$ for $i = 1,\dots,r$, $j = 1,\dots,\mu_i$ and $k = 1,\dots,d-1$. Observe that $\xi_{ij} \in W^p \Leftrightarrow \lambda_{ij} \in (p-1,p]$ and that $\gamma^*(\xi_{ij}) = \xi_{ij}$. By the Thom-Sebastiani result for the Hodge filtration (see [V2, Th. 7.3]) we find that $\xi_{ij}\otimes\vartheta_k \in W^p \Leftrightarrow \lambda_{ij} + k/d \in (p-1,p]$, and $\gamma^*(\xi_{ij}\otimes\vartheta_k) = \xi_{ij}\otimes T'(\vartheta_k) = \mathbf{e}(-k/d)\xi_{ij}\otimes\vartheta_k$.

For fixed i,j, consider the subspace W_{ij} of W spanned by ξ_{ij}, $\xi_{ij}\otimes\vartheta_1,\dots,\xi_{ij}\otimes\vartheta_{d-1}$. Let $\lambda'_{ij} = n-2-\lambda_{ij}$. Then $Sp(g_i) = \sum_j (\lambda'_{ij})$ because $Sp(g_i)$ is invariant under the reflection $(\alpha) \longrightarrow (n-2-\alpha)$ of $\mathcal{S}$ (see [V2, §1.7]. Choose p such that $\lambda_{ij} \in (p-2,p-1]$. Then $\lambda'_{ij} \in [n-p-1,n-p)$. Let $k_{ij} = \max\{k \in \mathbb{Z} \mid \lambda_{ij} + k/d \le p-1\}$. Then $\xi_{ij} \in W^{p-1}$ so $Sp(\mathbb{C}\xi_{ij},F,\gamma^*) = (n-p)$. For $k \le k_{ij}$, $\xi_{ij}\otimes\vartheta_k \in W^{p-1}$ and $Sp(\mathbb{C}\xi_{ij}\otimes\vartheta_k,F,\gamma^*) = (n-p+k/d)$. For $k > k_{ij}$, $\xi_{ij}\otimes\vartheta_k \in W^p$ and $Sp(\mathbb{C}\xi_{ij}\otimes\vartheta_k,F,\gamma^*) = (n-p-1+k/d)$. Adding these up we obtain

$$Sp(W_{ij},F,\gamma^*) = (n-p-1+k_{ij}/d)*\beta_d.$$

Put $\alpha'_{ij} = d\lambda'_{ij} - [d\lambda'_{ij}]$. Then one checks easily that

$$n-p-1+k_{ij}/d = \lambda'_{ij} - \alpha'_{ij}/d.$$

This finishes the proof of the theorem.

(6.3) We can now verify the conjecture in the case that f is homogeneous with one-dimensional singular locus such that each germ g_i (notations as above) is analytically equivalent to a Pham-Brieskorn polynomial, i.e. a polynomial of the form $\sum_{i=1}^n x_i^{a_i}$. The proof is similar to the case n = 1 so we just sketch the argument.

Let $f \in \mathbb{C}[z_0,\dots,z_n]$ be our polynomial. Let $\pi'\colon Z' \longrightarrow \mathbb{C}^{n+1}$ be the blowing up of the origin. Assume that the coordinates on $\mathbb{C}^{n+1}$ were chosen in such a way that $(1:0:\dots:0)$ is a singular point of $\tilde{V}(f)$. The strict transform

Σ' of the component of $\Sigma(f)$ corresponding to this singularity of $\tilde{V}(f)$ will intersect the exceptional divisor E_0 of π' in a point P. An affine coordinate neighborhood of P in Z' is Spec $\mathbb{C}[u_0,\dots,u_n]$ with $\pi'^*(z_0) = u_0$, $\pi'^*(z_j) = u_0u_j$ for $j = 1,\dots,n$. Then $\pi'^*(f) = u_0^d f(1,u_1,\dots,u_n)$. By hypothesis, there is an analytic coordinate transformation ϕ of $\mathbb{C}^n$ such that $f(1,u_1,\dots,u_n) = y_1^{a_1} + \dots + y_n^{a_n}$ for suitable $a_1,\dots,a_n$, $y_j = \phi^*(u_j)$. Thus, for each sufficiently general linear form ℓ on $\mathbb{C}^{n+1}$ we can find analytic coordinates $y_0,\dots,y_n$ centered at P such that $\pi'^*(f) = y_0^d(y_1^{a_1} + \dots + y_n^{a_n})$ and $\pi'^*(\ell) = y_0$. Now we can use the same toric methods as in §3 to blow up Z' further and verify the conjecture.

(6.4) The following argument shows that the α_{ij} in the formula depend only on λ_{ij} and the transverse monodromy along Σ_i. (Here the transverse type may be an arbitrary isolated singularity.) Near a point P as above, $\pi'^*(f)$ is of the form $y_0^d g(y_1,\dots,y_n)$. The transverse Milnor fibre is given by $g(y_1,\dots,y_n) = ty_0^{-d}$. The vertical monodromy τ which is induced by letting y_0 turn around 0 once in a counterclockwise direction, therefore is equal to T_g^{-d}, and its eigenvalues are the d^{th} powers of the eigenvalues of T_g. In particular, if τ is unipotent, each eigenvalue of T_g is a d^{th} root of unity and hence all α_{ij} are zero.

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