Astérisque

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A special decomposition of the nilpotent cone of a classical Lie algebra

A SPECIAL DECOMPOSITION OF THE NILPOTENT CONE OF A CLASSICAL LIE ALGEBRA

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1. Introduction

1.1.   The *special unipotent classes* of a simple group were introduced by Lusztig in [Lu1] (last remark of the paper), in relation with *special representations* of Weyl groups. At the same time Spaltenstein remarked in [Sp1] that there is a natural involution on a certain subset of the unipotent classes of a simple group (cf. [Sp2, chap. II.1]). In case of a classical group it is induced by the standard involution of the unipotent classes in GL_n (defined via the duality of the corresponding partitions) and has a purely combinatorial description [loc.cit]. It is not hard to see that this subset is exactly the set of special classes of Lusztig. Spaltenstein also notes that the sets

$$\widehat{C} := \overline{C} \setminus \bigcup_{\substack{C' \text{ special} \\ \overline{C'} \subset \overline{C}}} \overline{C'}$$

where C runs through the special classes, form a partition of the unipotent variety, i.e. any unipotent class is contained in a $\widehat{C}$ for a unique special class C. For completness we will prove this in section 4 (Proposition 4.2).

1.2.   Lusztig conjectures in [Lu2, Conjecture 3] that the varieties $\widehat{C}$ for special classes C are *rational homology manifolds*. This has been verified for "minimal" special classes in [loc.cit], and for E_6, E_7, E_8 in a remark in [BS]. The case F_4 can be handled using [Sh], and for G_2 it is quite easy.

   The purpose of this paper is to prove it for the classical groups.

Theorem.   *Let C be a special unipotent conjugacy class of a classical group. Define $\widehat{C}$ as above. Then $\widehat{C}$ consists of 2^d conjugacy classes, where d is the number of irreducible components of $\widehat{C} \setminus C$. There is a smooth variety Y with an action of the group $\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$, d copies, and an isomorphism*

$$Y/\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2 \xrightarrow{\sim} \widehat{C}$$

which identifies the stratification of C by conjugacy classes with the stratification of the quotient by isotropy groups. (These are the 2^d subproducts of $\mathbf{Z}_2{}^d$.) In particular $\widehat{C}$ is a rational homology manifold.

1.3. The partition of the unipotent variety into subsets $\widehat{C}$ can also be justified by the following result of Lusztig.

Proposition. *Let C_1, C_2 be two unipotent classes and let ρ_1, ρ_2 be the Springer representation of the Weyl group W corresponding to $(C_1, 1)$, $(C_2, 1)$, where 1 is the constant sheaf. Then C_1 and C_2 belong to the same $\widehat{C}$ if and only if ρ_1 and ρ_2 belong to the same two sided cell of W.*

1.4. The paper depends in an essential way from the results and methods of [KP2] of which it should be considered as a continuation. We try to help the reader to extract the necessary information from there by recalling as we proceed the main definitions.

2. Conjugacy Classes and Partitions

2.1. Let ε be $+1$ or -1. A finite dimensional vector space V with a non-degenerate form $(\ ,\ )$ such that $(u, v) = \varepsilon(v, u)$ for all $u, v \in V$ will be called a *quadratic space of type ε* (shortly an *orthogonal space* in case $\varepsilon = 1$, a *symplectic space* in case $\varepsilon = -1$). We denote by $G(V)$ the subgroup of $\mathrm{GL}(V)$ leaving the form invariant. So we have $G(V) = \mathrm{O}_n$ or $G(V) = \mathrm{Sp}_n$, $n = \dim V$, according to $\varepsilon = 1$ or $\varepsilon = -1$. Similarly $\mathbf{g}(V)$ denotes the Lie algebra of $G(V)$.

2.2. For any quadratic space V of type ε the conjugacy class C_D of a nilpotent element $D \in \mathbf{g}(V)$ is completely determined by its conjugacy class in $\mathrm{End}(V)$, hence by its associated partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$,

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_s \geq 0, \quad |\lambda| := \sum_{i=1}^{s} \lambda_i = \dim V,$$

given by the sizes of the blocks of the Jordan normal form of D in $\mathrm{End}(V)$ (cf.[KP2, 2.1]. If we denote by $\widehat{\lambda} = (\widehat{\lambda}_1, \widehat{\lambda}_2, \ldots, \widehat{\lambda}_s)$ the dual partition (i.e. $\widehat{\lambda}_i := \#\{j \mid \lambda_j \geq i\}$) we have for all j

$$\dim \ker D^j = \sum_{i=1}^{j} \widehat{\lambda}_i.$$

2.3. The partition λ associated to a nilpotent $D \in \mathbf{g}(V)$ satisfies the following condition (cf. [KP2, 2.2]):

(Y_ε) *If V is orthogonal (V is symplectic) every row of even length (of odd length) occurs an even number of times.*

Such partitions will be called ε-*partitions.* We denote by $C_{\varepsilon, \lambda}$ the associated nilpotent conjugacy class in $\mathbf{g}(V)$, where V is a quadratic space of type ε of dimension $|\lambda|$.

2.4. It is convenient to represent the partitions geometrically as Young-diagrams with rows consisting of $\lambda_1, \lambda_2, \ldots, \lambda_s$ boxes respectively. Then the dual partition $\widehat{\lambda}$ is defined by setting $\widehat{\lambda}_j$ equal to the length of the jth column of the diagram of λ.

Also the inclusion behavior $C_{\varepsilon,\sigma} \subset \overline{C}_{\varepsilon,\lambda}$ has a purely combinatorial description: A necessary and sufficient condition is that the diagram σ is obtained from λ by moving down a number of boxes (cf. [KP2, 2.5]). In this case we write $\sigma \leq \lambda$ and call σ an ε-*degeneration* of λ.

3. Special Conjugacy Classes

3.1. Let $C = C_{\varepsilon,\lambda}$ be a nilpotent conjugacy class. It is not difficult to determine in terms of the partition λ whether the class C is *special* in the sense of Lusztig [Lu1] (cf. Introduction). We use the following definition which is easily seen to be equivalent to the one given by Spaltenstein [Sp2] (see also [Ke, §6]).

Definition. Let λ be an ε-partition. Define the sequence $(s_1, s_2, s_3, \ldots)$ by

$$s_i = \begin{cases} \lambda_1 + \lambda_2 + \ldots + \lambda_i, & \text{if } |\lambda| \text{ is even;} \\ 1 + \lambda_1 + \lambda_2 + \ldots + \lambda_{i-1}, & \text{if } |\lambda| \text{ is odd.} \end{cases}$$

Then λ is called *special* if the numbers $s_2, s_4, s_6, \ldots$ are all even.

3.2. Remark. The following two assertions are easily deduced from the definitions.

(a) Let σ be an orthogonal partition with $|\sigma|$ even and let σ' be the symplectic partition obtained from σ by removing the first column. Then $|\sigma'|$ is even, and σ' is special if and only if σ is special.

(b) Let σ be a orthogonal partition with $|\sigma|$ odd and assume that the first row σ_1 is odd. Denote by σ° the orthogonal partition obtained from σ by removing the first row. Then $|\sigma^\circ|$ is even and σ° is special if and only if σ is special.

3.3. Lemma. *Let σ be a symplectic partition. Then σ is special if and only if $\widehat{\sigma}$ is symplectic and special.*

PROOF: For a symplectic partition σ it is clear that $\widehat{\sigma}$ is special, i.e. that all sums $\widehat{\sigma}_1 + \widehat{\sigma}_2 + \ldots + \widehat{\sigma}_{2i}$ are even, since since every odd row has to occur an even number of times. To prove that for a special partition σ the dual $\widehat{\sigma}$ is symplectic, it suffices to show that either $\widehat{\sigma}_1 = \widehat{\sigma}_2$ or that $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ both are even. In fact this implies that the partition σ'' obtained from σ by removing the first two columns, is again special (and symplectic), and by induction we may assume that its dual $\widehat{\sigma''}$ is symplectic. Since $\widehat{\sigma}$ is obtained from $\widehat{\sigma''}$ by adding two rows of either the same length or both of even length the partition $\widehat{\sigma}$ is symplectic too. It remains to show that for $\widehat{\sigma}_1 \neq \widehat{\sigma}_2$ both $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ are even. Now $l := \widehat{\sigma}_1 > \widehat{\sigma}_2$ and $s_l = |\sigma|$ is even. Hence $s_{l-1} = |\sigma| - 1$ is odd and therefore l has to be even. Since σ is symplectic $\widehat{\sigma}_1 - \widehat{\sigma}_2 = \#$rows of length one is even and the claim follows. ∎

3.4. Lemma. *Let σ be an orthogonal partition with $|\sigma|$ odd. Then σ is special if and only if $\widehat{\sigma}$ is orthogonal and special.*

PROOF: Clearly $\hat{\sigma}_1$ is odd. Now add a new row of odd length $> \sigma_1$ and remove the first column, which is now even, to obtain a symplectic partition ν. It follows from (a) and (b) that ν is special in case σ is special, and then $\hat{\nu}$ is special and symplectic by (c). Now $\hat{\sigma}$ is obtained from $\hat{\nu}$ by adding first an even column and then an odd row, hence $\hat{\sigma}$ is special and orthogonal. ∎

4. Small Degenerations

4.1. A degeneration $\sigma \leq \lambda$ is called *minimal* if $\sigma \leq \mu \leq \lambda$ implies $\mu = \sigma$ or $\mu = \lambda$. In [KP2] we have established an equivalence relation between degenerations. Under this relation the minimal degenerations fall into 8 classes a, b, c,..., h which are presented in [KP2, 3.4 table I]. In the following we will use degenerations of type a and g which are equivalent to the pairs $(1^{2n}) \leq (2, 1^{2n-2})$ of symplectic partitions.

In addition we have introduced in [KP2, 13.6] the concept of a degeneration $\sigma \leq \lambda$ being *decomposed into independent degeneration*. This means that we can decompose λ and σ into blocks consisting of consecutive rows

$$\lambda = \begin{matrix} \lambda^{(1)} \\ \lambda^{(2)} \\ \vdots \\ \lambda^{(s)} \end{matrix} \qquad \sigma = \begin{matrix} \sigma^{(1)} \\ \sigma^{(2)} \\ \vdots \\ \sigma^{(s)} \end{matrix}$$

such that all $\lambda^{(i)}$, $\sigma^{(i)}$ are ε-partitions, that $\lambda^{(i)}$ has the same number of rows as $\sigma^{(i)}$ except perhaps the last $\lambda^{(s)}$, and that $\sigma^{(i)}$ is a degeneration of $\lambda^{(i)}$.

Definition. An ε-degeneration $\sigma \leq \lambda$ is called *small* if σ is obtained from λ by independent minimal degenerations of type a or g.

Similarly we say that a degeneration $C' \subset \overline{C}$ of nilpotent conjugacy classes in $\mathbf{g}(V)$ is *small* if the degeneration of the corresponding ε-partitions is small.

4.2. Proposition (Spaltenstein). *Given an ε-partition σ there is a uniquely determined minimal special ε-partition λ such that $\sigma \leq \lambda$. In addition the ε-degeneration $\sigma \leq \lambda$ is small.*

PROOF: (1) We first consider the case where σ is symplectic. Assume that s_{2i} is even for $i = 1, \ldots, r - 1$ and that s_{2r} is odd. (See 3.1 for the definition of s_i.) Then σ_{2r-1} is even and σ_{2r} is odd, and there is an $r' > r$ such that $\sigma_{2r} = \sigma_{2r+1} = \ldots = \sigma_{2r'-1} > \sigma_{2r'}$. Now define

$$\sigma'_i = \begin{cases} \sigma_{2r} + 1, & \text{for } i = 2r; \\ \sigma_{2r'-1} - 1, & \text{for } i = 2r' - 1; \\ \sigma_i, & \text{in all other cases.} \end{cases}$$

It is easy to see that σ' is again symplectic and that $\sigma' \leq \sigma$ is a minimal degeneration of type a (if $r' = r + 1$) or g (if $r' > r + 1$) (see [KP2, 3.4 table I]). In particular it is a small degeneration. Furthermore the sequence $(s'_1, s'_2, s'_3, \ldots)$ for σ' (i.e. $s'_i := \sigma'_1 + \sigma'_2 + \ldots + \sigma'_i$) has the property that s'_{2i} is even for $i = 1, 2, \ldots, r' - 1$. To finish the proof in the symplectic case it suffices by induction to show that for any special symplectic partition $\lambda \geq \sigma$ we also have $\lambda \geq \sigma'$. Let $t_i := \lambda_1 + \lambda_2 + \ldots + \lambda_i$. By assumption $\lambda_i \geq \sigma_i$ for all i and λ_{2j} is even for all j. By definition we have

$$s'_i = \begin{cases} s_i, & \text{for } i \leq 2r - 1 \text{ and } i > 2r' - 1; \\ s_i + 1, & \text{for } 2r \leq i \leq 2r' - 1. \end{cases}$$

Hence we have to show that

$$(*) \qquad\qquad t_j > s_j \text{ for } 2r \leq j < 2r' - 1.$$

By assumption $s_{2r}, s_{2r+2}, \ldots, s_{2r'-2}$ are all odd and so $(*)$ is satisfied for the even j's. Now assume that $t_{2k+1} = s_{2k+1}$ for some k with $r \leq k < r'-1$. Then $\lambda_{2k+1} = t_{2k+1} - t_{2k} < s_{2k+1} - s_{2k}$, hence $\lambda_{2k+2} \leq \lambda_{2k+1} < \sigma_{2k+1} = \sigma_{2k+2}$, and so $t_{2k+2} = t_{2k+1} + \lambda_{2k+2} < s_{2k+1} + \sigma_{2k+2} = s_{2k+2}$ which is a contradiction.

(2) Now let σ be orthogonal and $|\sigma|$ even and denote by σ° the symplectic partition obtained from σ by adding a column of length $\hat{\sigma}_1$. We have seen in (1) that there is a unique minimal special symplectic partition $\lambda^\circ \geq \sigma^\circ$ and that the degeneration $\sigma^\circ \leq \lambda^\circ$ is small. Since the first two columns of σ° have the same length it follows that $\widehat{\lambda^\circ}_1 = \widehat{\sigma^\circ}_1 = \hat{\sigma}_1$. Removing the first column we obtain a special orthogonal partition λ (Remark (a)) such that σ is a small degeneration of λ, and clearly λ is also minimal under these conditions.

(3) Finally assume that σ is orthogonal with $|\sigma|$ odd. This time we add a first row of odd length $\geq \sigma_1 + 2$ to obtain an orthogonal partition σ° with $|\sigma^\circ|$ even, and proceed like in (2), using Remark (b). ∎

4.3. Proposition. *The map $\sigma \mapsto \hat{\sigma}$ induces a duality on the set $\mathbf{P}_{2n}^{symp}$ of special symplectic partitions of $2n$ and on the set $\mathbf{P}_{2n+1}^{orth}$ of special orthogonal partitions of $2n+1$. In addition there is a bijection*

$$\mathbf{P}_{2n+1}^{orth} \xrightarrow{\sim} \mathbf{P}_{2n}^{symp}$$

respecting the partial ordering and the duality.

PROOF: The first part is a reformulation of Lemma 3.3 and 3.4. Furthermore we define a map $\mathbf{P}_{2n+1}^{orth} \longrightarrow \mathbf{P}_{2n}^{symp}$ in the following way: Remove the last box from the special orthogonal partition σ to get a partition σ' of $2n$. Now there is a uniquely determined maximal symplectic partition $\lambda \leq \sigma'$ and one shows that this λ is special. We leave the details to the reader. ∎

5. Geometry of Small Degenerations

5.1. Let us start with a quadratic space V of type ε and a nilpotent element $D \in \mathbf{g}(V)$ with conjugacy class $C_D = C_{\varepsilon,\lambda}$. The image $U := D(V)$ is in a natural way a quadratic space of type $-\varepsilon$ and $D' := D|_U$ is a nilpotent element of $\mathbf{g}(U)$ with conjugacy class $C_{D'} = C_{-\varepsilon,\lambda'}$, where λ' is obtained from λ by removing the first column [KP2, 4.1]. Furthermore we get the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}(V,U) & \xrightarrow{\pi} & \mathbf{g}(U) \\ \downarrow{\scriptstyle\rho} & & \\ \mathbf{g}(V) & & \end{array}$$

The two maps are defined by $\pi(X) := XX^*$ and $\rho(X) := X^*X$, where $X^* : U \to V$ is the dual map of X. (We identify U^* with U and V^* with V using the non-degenerate form $(\ ,\ )$.) π and ρ are the quotient maps under $G(V)$ and $G(U)$ respectively [KP2, 1.2].

We embed $\mathrm{Hom}(V,U)$ in $\mathrm{Hom}(V,U) \times \mathrm{Hom}(U,V)$ by $X \mapsto (X, X^*)$. The $G(V) \times G(U)$-orbits in $\mathrm{Hom}(V,U)$ are completely determined by the corresponding $\mathrm{GL}(V) \times \mathrm{GL}(U)$-orbits in

$\operatorname{Hom}(V, U) \times \operatorname{Hom}(U, V)$ which are classified by their ab-diagrams (cf. [KP2, §6] and [KP1, §4]). Given the ab-diagram of X, e.g.

$$
\begin{array}{l}
bababab \\
babab \\
babab \\
bab \\
ba \\
ab
\end{array}
$$

then the Young-diagrams of $\pi(X)$ and $\rho(X)$ are obtained from it by removing all a's (resp. all b's), e.g.

$$
\begin{array}{lll}
bbbb & & aaa \\
bbb & & aa \\
bbb & \text{and} & aa \\
bb & & a \\
b & & a \\
b & & a
\end{array}
$$

representing the partitions $(4, 3, 3, 2, 1, 1)$ and $(3, 2, 2, 1, 1, 1)$. This fact allows to discuss in detail the decomposition into orbits of $\pi^{-1}(C)$ for a given conjugacy class C; it will be used in the proof of Lemma 5.3 below.

Now put $N_{\varepsilon,\lambda} := \pi^{-1}(\overline{C}_{-\varepsilon,\lambda'})$.

5.2. Lemma. *Let $C_{\varepsilon,\sigma} \subset \overline{C}_{\varepsilon,\lambda}$ with $\hat{\sigma}_1 = \hat{\lambda}_1$. Then*

(a) $O_\sigma := \rho^{-1}(C_{\varepsilon,\sigma})$ *is an orbit under $G(U) \times G(V)$ contained in $N_{\varepsilon,\lambda}$.*

(b) $\pi(O_\sigma) = C_{-\varepsilon,\sigma'}$.

(c) $G(U)$ *acts freely on O_σ.*

(d) π *is smooth on O_σ.*

PROOF: (a) and (b) follow from [KP2, Lemma 4.3]. Since $\hat{\sigma}_1 = \hat{\lambda}_1$ the elements of $C_{\varepsilon,\sigma}$ have the same rank as those of $C_{\varepsilon,\lambda}$, namely $\dim U$, hence $\rho^{-1}(C_{\varepsilon,\sigma}) \subset \{X \in \operatorname{Hom}(V, U) \mid X \text{ surjective}\}$. Now (c) and (d) follow from [KP2, Proposition 11.1]. ∎

5.3. Lemma. *Assume V symplectic, hence U orthogonal, and let $C_{-1,\sigma} \subset \overline{C_{-1,\lambda}}$ with $\hat{\lambda}_1 = \hat{\sigma}_1 - 1$ and $\hat{\lambda}_2 = \hat{\sigma}_2 + 1$. Then*

(a) $O_\sigma := \rho^{-1}(C_{-1,\sigma})$ *is an orbit under $G(U) \times G(V)$ contained in $N_{-1,\lambda}$.*

(b) $\pi(O_\sigma) = C_{1,\sigma''}$, *where σ'' is obtained from σ by removing the first row and then adding one box.*

(c) *The connected component $G(U)^0 = SO(U)$ acts freely on O_σ, and the stabilizer of $X \in O_\sigma$ is $\mathbf{Z}_2$.*

(d) π *is smooth on O_σ.*

PROOF: Since $\dim U = \dim V - \hat{\lambda}_1$ there is a unique ab-diagram with $\dim U$ a's lying on top of λ, i.e. which by removing the a's gives λ. (It is obtained by inserting between all consecutive b's of λ an a.) Hence $O_\lambda := \rho^{-1}(C_{-1,\lambda})$ is the $G(U) \times G(V)$-orbit with this ab-diagram. By assumption the last parts of λ and σ have the form

$$
(\ldots, \underbrace{2, 1, \ldots, 1}_{2s-1}) \quad \text{and} \quad (\ldots, \underbrace{1, 1, \ldots, 1, 1}_{2s}),
$$

where $2s = \hat{\sigma}_1 - \hat{\sigma}_2$. In particular $\operatorname{rk} D = \dim U - 1$ for $D \in C_{-1,\sigma}$. As a consequence the ab-diagram of O_λ has the following last part:

$$
\text{last } 2s - 1 \text{ rows} \left\{ \begin{array}{l} bab \\ b \\ b \\ \vdots \\ b \end{array} \right.
$$

But then there is only one possibility for an ab-diagram on top of $C_{-1,\sigma}$: The first $\hat{\sigma}_2$ rows are completely determined by σ and λ and are the same as for O_λ, and the last part must have the following form (see [KP2, 6.3 table III]):

$$
\text{last } 2s + 1 \text{ rows} \left\{ \begin{array}{l} a \\ b \\ b \\ b \\ \vdots \\ b \end{array} \right.
$$

Hence $\rho^{-1}(C_{\varepsilon,\sigma})$ is an orbit and $\pi(\rho^{-1}(C_{\varepsilon,\sigma})) = C_{-\varepsilon,\sigma''}$, where σ'' is obtained from σ by removing the first row and then adding one box. This gives (a) and (b).

Now (c) follows from [KP2, Proposition 11.5] since

$$
\rho^{-1}(C_{\varepsilon,\sigma}) \subset L'' := \{X \in \operatorname{Hom}(V,U) \mid \operatorname{rk} \rho(X) \geq \dim U - 1\} \setminus L'
$$

where

$$
L' := \{X \in \operatorname{Hom}(V,U) \mid X \text{ surjective}\},
$$

and (d) follows from [KP2, Proposition 11.4], since

$$
L^0 := \{X \mid \operatorname{codim} X(V) \leq 1\} \supset L''
$$

because $\operatorname{rk} X^* X \leq \operatorname{rk} X$. ∎

5.4. We now recall the fundamental construction of [KP2] associated to a nilpotent element $D \in \mathfrak{g}(V)$ with conjugacy class $C_{\varepsilon,\lambda}$ [KP2, §5]. It is obtained by iterating the construction above. The spaces $V_i := D^i(V)$ are quadratic of type $(-1)^i \varepsilon$ and $D_i := D|_V \in \mathfrak{g}(V_i)$ is nilpotent with Young-diagram λ_i obtained from λ by removing the first i rows. By forming an iterated fiber product we get the following diagram:

$$
\begin{array}{ccccccc}
Z & \xrightarrow{\phi} & Z_1 & \longrightarrow & \cdots & \longrightarrow & \overline{C}_{D_t} = 0 \\
\downarrow & & \downarrow & & & & \\
\bullet & \longrightarrow & \bullet & \cdots & & & \\
\downarrow & & \vdots & & & & \\
\vdots & & \vdots & & & & \\
\bullet & \longrightarrow & \bullet & & & & \\
\downarrow & & \downarrow & & & & \\
N_D & \xrightarrow{\pi} & \overline{C}_{D_1} & & & & \\
\downarrow \rho & & & & & & \\
\overline{C}_D & & & & & &
\end{array}
$$

By construction all the squares in this diagram are fibre products. As a subdiagram we obtain

$(**)$

$$
\begin{array}{ccc}
Z & \xrightarrow{\phi} & Z_1 \\
\downarrow{\theta'} & & \downarrow{\theta_1} \\
N_D & \xrightarrow{\pi} & \overline{C}_{D_1} \\
\downarrow{\rho} & & \\
\overline{C}_D & &
\end{array}
$$

where the composition $\theta := \rho \circ \theta' : Z \to \overline{C}_D$ is the quotient map under the group

$$
G := G(V_1) \times G(V_2) \times \ldots \times G(V_t).
$$

5.5. Proposition. *Let $C' \subset \overline{C}_D$ be a small degeneration. Then Z is smooth in $Z' := \theta^{-1}(C')$ and the connected component G^0 of the group G acts freely on Z'. Furthermore the stabilizer in G of any point of Z' is isomorphic to $\mathbf{Z}_2^d$, where d is the number of minimal degenerations needed to obtain the ε-diagram of C' from that of C_D.*

PROOF: Let $C' = C_{\varepsilon,\sigma}$ and $C_D = C_{\varepsilon,\lambda}$. Since $\sigma \leq \lambda$ is a small degeneration we have either $\hat{\sigma}_1 = \hat{\lambda}_1$ or V is symplectic (and V_1 orthogonal) and $\hat{\lambda}_1 = \hat{\sigma}_1 - 1 \geq \hat{\lambda}_2 = \hat{\sigma}_2 + 1$ (see Definition). It follows from Lemma 5.2 and 5.3 that π is smooth on $N' := \rho^{-1}(C')$ and that $G(V_1)^0$ acts freely on N'. The fibre product diagram $(**)$ implies that ϕ is smooth on Z' and that $G(V_1)^0$ acts freely on Z'. Furthermore $\pi(\rho^{-1}(C'))$ is a conjugacy class $C_1' \subset \overline{C}_{D_1}$ with ε-diagram σ' obtained from σ by cancelling the first row (and adding one box in case $\hat{\lambda}_1 = \hat{\sigma}_1 - 1$). It is clear from this that $C_1' \subset \overline{C}_{D_1}$ is again a small degeneration. From the following diagram

$$
\begin{array}{ccccccc}
Z' & \subset & Z & \xrightarrow{\phi} & Z_1 & \supset & Z_1' \\
\downarrow & & \downarrow{\theta'} & & \downarrow{\theta_1} & & \downarrow \\
M & \subset & N_D & \xrightarrow{\pi} & \overline{C}_{D_1} & \supset & C_1' \\
\downarrow & & \downarrow{\rho} & & & & \\
C' & \subset & \overline{C}_D & & & &
\end{array}
$$

we see that the outer square

$$
\begin{array}{ccc}
Z' & \xrightarrow{\phi} & Z_1' \\
\downarrow & & \downarrow \\
M & \xrightarrow{\pi} & C_1'
\end{array}
$$

is cartesian too, hence the claim follows by induction. $\blacksquare$

6. A Special Decomposition

6.1. Definition. For a special ε-partition λ we define

$$
S_\lambda := \bigcup_{\sigma \leq \lambda} C_{\varepsilon,\sigma},
$$

where the union is taken over all σ such that λ is the minimal special ε-partition $\geq \sigma$.

It follows from Proposition 2.1 that S_λ is a locally closed subvariety of $\mathbf{g}(V)$, and that

$$
S_\lambda = \bigcup_{\substack{\lambda \text{ special} \\ \lambda' < \lambda}} C_{\varepsilon,\lambda'}.
$$

6.2. Theorem. *Let* λ *be a special* ε*-degeneration and let* d *be the number of different minimal degenerations of* λ *of type a or g. Then* S_λ *consists of* 2^d *conjugacy classes, and there is a smooth variety* Y_λ *with an action of the group* $\mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$, *d copies, and an isomorphism*

$$S_\lambda \simeq Y_\lambda / \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$$

which identifies the stratification of S_λ *by conjugacy classes with the stratification by isotropy groups.*

PROOF: It follows from Proposition 4.2 that every conjugacy class $C' \subset S_\lambda$ is a small degeneration of C_λ. Hence $\theta^{-1}(S_\lambda)$ is a smooth open subvariety of Z with a free action of G° by Proposition 5.5. As a consequence $Y_\lambda := Z_\lambda/G^\circ$ is smooth too and $S_\lambda \simeq Z_\lambda/F$ with $F = G/G^\circ \simeq \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2$, d copies. The rest is clear. ∎

6.3. Corollary. *The variety* S_λ *is a rational homology manifold.*

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