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UNIPOTENT AUTOMORPHIC REPRESENTATIONS: CONJECTURES

James Arthur

Foreword.

In these notes, we shall attempt to make sense of the notions of semisimple and unipotent representations in the context of automorphic forms. Our goal is to formulate some conjectures, both local and global, which were originally motivated by the trace formula. Some of these conjectures were stated less generally in lectures [2] at the University of Maryland. The present paper is an update of these lectures. We have tried to incorporate subsequent mathematical developments into a more comprehensive discussion of the conjectures. Even so, we have been forced for several reasons to work at a level of generality at which there is yet little evidence. The reader may prefer to regard the conjectures as hypotheses, to be modified if necessary in the face of further developments.

We had originally intended to describe in detail how the conjectures are related to the spectral side of the trace formula. However, we decided instead to discuss the examples of Adams and Johnson (§5), and the applications of the conjectures to intertwining operators (§7) and to the cohomology of Shimura varieties (§8). We shall leave the global motivation for another paper [5].

I would like to thank Robert Kottwitz and Diana Shelstad for a number of very helpful conversations, particularly on the topic of endoscopy. Any remaining inaccuracies are due entirely to me.

Notational Conventions: Suppose that H is a locally compact group. We shall write $\Pi(H)$ for the set of equivalence classes of irreducible (continuous) representations of H, and $\Pi_{\mathrm{unit}}(H)$ for the subset of representations in $\Pi(H)$ which are unitarizable. The symbol $Z(H)$ will denote the center of H, and $\pi_0(H)$ will stand for the group of connected components of H.

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§1. Introduction.

Suppose that G is a connected reductive algebraic group over a field F. We shall always assume that F has characteristic 0. For sections 1 and 2 we shall also take F to be a number field. The adèles $\mathbf{A}_F$ of F form a locally compact ring, in which F is embedded diagonally as a subring. We can take the group $G(\mathbf{A}_F)$ of adèlic points of G, which contains $G(F)$ as a discrete subgroup. The basic analytic object is the regular representation

$$(R(y)\phi)(x) = \phi(xy), \qquad \phi\in L^2(G(F)\backslash G(\mathbf{A}_F)),\ x,y\in G(\mathbf{A}_F)\ .$$

It is a unitary representation of $G(\mathbf{A}_F)$ on the Hilbert space of square integrable functions on $G(F)\backslash G(\mathbf{A}_F)$ (relative to the right-invariant measure). A basic goal of the modern theory of automorphic forms is to deduce information about the decomposition of R into irreducible representations.

Let $\Pi(G)$ be the set of irreducible representations $\pi\in\Pi_{\mathrm{unit}}(G(\mathbf{A}_F))$ which occur in the decomposition of R. In general, there will be a part of R which decomposes discretely and a part which decomposes continuously, so the definition is somewhat informal. Nevertheless, the theory of Eisenstein series [28] reduces the study of the decomposition of R to that of the discrete spectrum. Set

$$G(\mathbf{A}_F)^1 = \{x\in G(\mathbf{A}_F):\ |\chi(x)| = 1,\ \chi\in X(G)_F\}\ ,$$

where $|\cdot|$ is the absolute value on $\mathbf{A}_F$, and $X(G)_F$ is the group of F-rational characters on G. For example, if $G = GL(n)$, $G(\mathbf{A}_F)^1$ is the group of matrices in $GL(n,\mathbf{A}_F)$ whose determinant has absolute value 1. In general, $G(\mathbf{A}_F)^1$ is a subgroup of $G(\mathbf{A}_F)$ which contains $G(F)$ as a discrete subgroup of finite co-volume. If π is any representation in $\Pi_{\mathrm{unit}}(G(\mathbf{A}_F))$, let $m_0(\pi)$ be the multiplicity with which the restriction of π to $G(\mathbf{A}_F)^1$ occurs as a direct summand in $L^2(G(F)\backslash G(\mathbf{A}_F)^1)$. The nonnegative integers $m_0(\pi)$, and their analogues for smaller groups, essentially determine the decomposition of R. More precisely, let $\Pi_0(G)$ be the set of representations $\pi \in \Pi_{unit}(G(\mathbf{A}_F))$ with $m_0(\pi) \neq 0$. The theory of Eisenstein series gives a decomposition of $\Pi(G)$ into induced representations

$$\mathcal{I}_P(\pi_1)\ , \qquad \pi_1 \in \Pi_0(M_P)\ ,$$

where $P = M_P N_P$ ranges over parabolic subgroups of G.

For each valuation v of F, let F_v be the completion of F at v. We can write $G(\mathbf{A}_F)$ as a restricted direct product of the local groups $G(F_v)$, and a given representation in $\Pi(G(\mathbf{A}_F))$ has a unique decomposition [11]

$$\pi = \bigotimes_v \pi_v\ , \qquad \pi_v \in \Pi(G(F_v))\ .$$

Moreover, almost all the representations π_v are unramified. This means that for each valuation v outside a finite set S, π_v is an irreducible quotient of the representation induced from an unramified quasi-character on a Borel subgroup. Any such π_v is determined by a unique semisimple conjugacy class $\sigma(\pi_v) = \sigma_v(\pi)$ in the L-group ${}^LG$ of G [8]. In other words, π defines a family

$$\sigma(\pi) = \{\sigma_v(\pi) : v \notin S\}$$

of semisimple conjugacy classes in the complex group ${}^LG$. Let us write $\Sigma(G)$ for the set of families $\sigma = \{\sigma_v : v \notin S\}$ of semisimple conjugacy classes in ${}^LG$ such that $\sigma = \sigma(\pi)$ for some representation π in $\Pi(G)$. (Strictly speaking, the elements in $\Sigma(G)$ are equivalence classes, two families σ and σ' being equivalent if $\sigma_v = \sigma'_v$ for almost all v.) The representations $\pi \in \Pi(G)$ are believed to contain arithmetic information of a fundamental nature. This will show up in the data needed to describe the different conjugacy classes in a family $\sigma(\pi)$.

If $G = GL(n)$ and π is cuspidal, the family $\sigma(\pi)$ uniquely determines π. This is the theorem of strong multiplicity one. In general, however, the map $\pi \to \sigma(\pi)$ from $\Pi(G)$ onto $\Sigma(G)$ is not injective. One could consider the problem of decomposing R in two stages, namely, to describe the set $\Sigma(G)$, and to determine the fibres of the map $\pi \to \sigma(\pi)$. This is a utopian view of what can actually be accomplished in practice, but it is a useful way to motivate some of the constructions in the subject. For example, the theory of endoscopy, due to Langlands and Shelstad, is aimed especially at the second aspect of the problem. One goal of the theory is to partition the representations in $\Pi(G(\mathbf{A}_F))$ into certain classes, L-packets, according to the arithmetic properties of the local representations $\Pi(G(F_v))$. The representations in the intersection of an L-packet with $\Pi(G)$ should then all lie in the same fibre.

The theory of endoscopy works best for tempered representations. Recall that the subset $\Pi_{temp}(G(F_v)) \subset \Pi_{unit}(G(F_v))$ of tempered representations consists of the irreducible constituents in the spectral decomposition of $L^2(G(F_v))$. (We refer the reader to [13, §25] and [14, §14] for the formal definition of a tempered representation.) Let $\Pi_{temp}(G(\mathbf{A}_F))$ be the subset of representations in $\Pi_{unit}(G(\mathbf{A}_F))$ of the form

$$\pi = \bigotimes_v \pi_v , \qquad \pi_v \in \Pi_{temp}(G(F_v)) .$$

The theory of endoscopy suggests conjectural formulas for the multiplicities $m_0(\pi)$, when π belongs to $\Pi_{temp}(G(\mathbf{A}_F))$. (See the examples in [24] and [38].) This amounts to a conjectural description of the tempered representations in $\Pi(G)$. However, the formulas break down for nontempered representations. The purpose of these notes is to describe a conjectural extension of the theory which would account for all the representations in $\Pi(G)$.

Much of this conference has been based on the dual nature of conjugacy classes and characters. In this spirit, we should think of the tempered representations in $\Pi(G)$ as semisimple automorphic representations. Our goal is to decide what constitutes a unipotent automorphic representation. More generally we would like to know how to build arbitrary representations in $\Pi(G)$ from semisimple and unipotent automorphic representations.

Stated slightly differently, our aims could be described as follows: Given a representation π in the complement of $\Pi_{temp}(G(\mathbf{A}_F))$ in $\Pi_{unit}(G(\mathbf{A}_F))$, describe $m_0(\pi)$ in terms of the multiplicities

$$m_0(\pi_1) , \qquad \pi_1 \in \Pi_{temp}(G_1(\mathbf{A}_F)) ,$$

for groups G_1 of dimension smaller than G. This is of course a global problem. Its local analogue is essentially that of the unitary dual: Classify the representations π_v in the complement of $\Pi_{temp}(G(F_v))$ in $\Pi_{unit}(G(F_v))$. The parameters we shall define (§4, §6, §8) seem to owe their existence to the global problem. For example, they suggest an immediate definition for a unipotent automorphic representation, while on the other hand, the definition of a unipotent representation for a local group is more subtle. (See [7].) However, the existence of nontempered automorphic forms does mean that the local and global problems are related. In particular, the global parameters should lead to many interesting nontempered representations of the local groups $G(F_v)$.

§2. The case of GL(n).

As motivation for what follows, we shall discuss the example of $GL(n)$. Here the situation is rather simple. We shall state the conjectural description of the discrete spectrum for $GL(n)$ in the form of two hypotheses.

We should first recall the space of cusp forms. Let $L^2_{cusp}(G(F)\backslash G(\mathbf{A}_F)^1)$ be the space of functions $\phi \in L^2(G(F)\backslash G(\mathbf{A}_F)^1)$ such that

$$\int_{N_P(F)\backslash N_P(\mathbf{A}_F)} \phi(nx)dn = 0$$

for almost all points $x \in G(\mathbf{A}_F)$, and for every proper parabolic subgroup $P = M_P N_P$ of G. It is known that this space is contained in the discrete spectrum. That is, the regular representation of $G(\mathbf{A}_F)^1$ on $L^2_{cusp}(G(F)\backslash G(\mathbf{A}_F)^1)$ decomposes into a direct sum of irreducible representations, with finite multiplicities. If π is any representation in $\Pi_{unit}(G(\mathbf{A}_F))$, let $m_{cusp}(\pi)$ be the multiplicity in $L^2_{cusp}(G(F)\backslash G(\mathbf{A}_F)^1)$ of the restriction of π to $G(\mathbf{A}_F)^1$. Then

$$m_{cusp}(\pi) \leq m_0(\pi) .$$

If $\Pi_{cusp}(G)$ denotes the set of π with $m_{cusp}(\pi) \neq 0$, we have

$$\Pi_{cusp}(G) \subset \Pi_0(G) \subset \Pi_{unit}(G(\mathbf{A}_F)) .$$

These definitions of course hold for any G. If $G = GL(n)$, the multiplicity one theorem asserts that $m_{cusp}(\pi)$ equals 0 or 1.

Hypothesis 2.1: Any unitary cuspidal automorphic representation of $GL(n)$ is tempered. That is, $\Pi_{cusp}(GL(n))$ is contained in $\Pi_{temp}(GL(n,\mathbf{A}_F))$. □

This is the generalized Ramanujan conjecture, whose statement we have taken from [29, §2]. For $GL(n)$, the global problem becomes that of describing $m_0(\pi)$ in terms of the cuspidal multiplicities $m_{cusp}(\pi_1)$.

Suppose that v is a valuation of F. The unitary dual of $GL(n,F_v)$ has been classified by Vogan [49] if v is Archimedean, and by Tadic [45] if v is discrete. However, one does not need the complete classification to describe the expected local constituents of representations in $\Pi_0(GL(n))$. Suppose that d is a divisor of n, and that $P_d = M_dN_d$ is the block upper triangular parabolic subgroup of $GL(n)$ attached to the partition

$$(\underbrace{d, d, \dots, d}_{m}) , \qquad n = dm ,$$

of n. Suppose that π_v is a representation in $\Pi_{temp}(GL_d(F_v))$. Then the representation

$$(\pi'_v \otimes \delta_d)(g) = \bigotimes_{i=1}^{m} \pi_v(g_i) \, |\det g_i|^{\frac{1}{2}(m-2i+1)} ,$$

defined for any element

$$g = \prod_{i=1}^{m} g_i \in \prod_{i=1}^{m} GL(d,F_v) = M_d(F_v) ,$$

belongs to $\Pi(M_d(F_v))$. Let $\mathcal{I}_{P_d}(\pi'_v \otimes \delta_d)$ be the corresponding induced representation of $GL(n,F_v)$. The Langlands quotient $\mathcal{J}_{P_d}(\pi'_v \otimes \delta_d)$ belongs to $\Pi(GL(n,F_v))$, and is the unique irreducible quotient of $\mathcal{I}_{P_d}(\pi'_v \otimes \delta_d)$.

Theorem: (Speh [42], Tadic [45]). The representation $\mathcal{J}_{P_d}(\pi'_v \otimes \delta_d)$ is unitary. □

Thus, if

$$\pi = \bigotimes_v \pi_v , \qquad \pi_v \in \Pi_{unit}(GL(d,F_v)) ,$$

is a representation in $\Pi_{unit}(GL(d,\mathbf{A}_F))$, we can form the unitary representation $\bigotimes_v \mathcal{J}_{P_d}(\pi'_v \otimes \delta_d)$ of $GL(n,\mathbf{A}_F)$. The following conjectural description of the discrete spectrum of $GL(n)$ is widely believed, but has not yet been established, even modulo Hypothesis 2.1. (For more information, see [16].)

Hypothesis 2.2: The set $\Pi_0(GL(n))$ is the disjoint union, over all divisors d of n and all representations $\pi \in \Pi_{cusp}(GL(d))$, of the representations

$$\bigotimes_v \mathcal{J}_{P_d}(\pi'_v \otimes \delta_d) \,. \tag{2.1}$$ □

The representations in $\Pi_{cusp}(GL(n))$ should be the semisimple elements in $\Pi_0(GL(n))$. Some of these are parametrized by certain irreducible complex representations

$$Gal(\overline{F}/F) \rightarrow GL(n,\mathbb{C})$$

of the Galois group of F. In fact, any such representation of the Galois group is thought to be attached to an automorphic representation. This is part of Langlands' functoriality principle. From this point of view, it makes sense to parametrize more general representations in $\Pi_0(GL(n))$ by equivalence classes of irreducible complex representations

$$\psi: Gal(\overline{F}/F) \times SL(2,\mathbb{C}) \rightarrow GL(n,\mathbb{C}) \,. \tag{2.2}$$

Indeed, any such ψ is a tensor product $\psi_{ss} \otimes \psi_{unip}$, where

$$\psi_{ss}: Gal(\overline{F}/F) \rightarrow GL(d,\mathbb{C})$$

and

$$\psi_{unip}: SL(2,\mathbb{C}) \rightarrow GL(m,\mathbb{C})$$

are irreducible representations, with $n = dm$. In particular, $\psi_{unip}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is the principal unipotent element in $GL(m,\mathbb{C})$, the one whose Jordan normal form has one block. If ψ_{ss} parametrizes the cuspidal automorphic representation $\pi \in \Pi_{cusp}(GL(d))$, ψ itself will parametrize the representation (2.1). The analogy with the Jordan decomposition for conjugacy classes is clear. In particular, a unipotent automorphic representation in $\Pi_0(GL(n))$ will be one for which ψ_{ss} is trivial. That is, ψ_{unip} corresponds to the principal unipotent conjugacy class in $GL(n,\mathbb{C})$. The associated representation (2.1) is just the trivial one dimensional representation of $GL(n,\mathbf{A}_F)$.

A similar parametrization could be used for the larger set $\Pi(GL(n))$. One would simply not insist that the n-dimensional representations (2.2) be irreducible. The unipotent automorphic representations in $\Pi(GL(n))$ are then the representations of $GL(n,\mathbf{A}_F)$ induced from trivial one dimensional representations of parabolic subgroups $P(\mathbf{A}_F)$ of $GL(n,\mathbf{A}_F)$. It will not be possible to parametrize all the representations in $\Pi(GL(n))$ (or $\Pi_0(GL(n))$) in this way. To do so would require replacing $Gal(\overline{F}/F)$ by some larger group. However, the point is irrelevant to the present purpose, which is to illustrate how one can describe nontempered automorphic representations in terms of tempered ones.

A general implication of the functoriality principle is the existence of a map from n-dimensional representations of the Weil group W_F of F to automorphic representations of GL(n). (The reader is referred to [46] for generalities on the Weil group, and to [8] for the functoriality principle.) How does this relate to our parameter ψ? The absolute value on the idèle class group of F provides a canonical map $w \to |w|$ of W_F to the positive real numbers. Moreover, any representation of $Gal(\overline{F}/F)$ lifts to a representation of W_F. For ψ as above, the map

$$\phi_\psi(w) = \psi\left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right), \qquad w \in W_F,$$

becomes an n-dimensional representation of the Weil group. Moreover, (2.1) is precisely the automorphic representation attached to ϕ_ψ by the functoriality principle. Keep in mind that the general automorphic representation of GL(n) does not belong to $\Pi_0(GL(n))$, or even to $\Pi(GL(n))$. The parameters (2.2) provide a convenient means to characterize those representations of W_F which are tied to these sets.

The group GL(n) is special, in that the decomposition of the discrete spectrum into cuspidal and residual components matches its decomposition into tempered and nontempered representations. (Of course, we are relying here on both Hypotheses 2.1 and 2.2.) This will not be true in general. For general G, the noncuspidal representations in the discrete spectrum are quite sparse. I do not know a good way to characterize them. On the other hand, after the examples of Kurokawa [23] and Howe and Piatetskii-Shapiro [15] for Sp(4), it was clear that there would be many nontempered cusp forms. For general G, the decomposition of the discrete spectrum into tempered and nontempered representations seems to be quite nice. It is this second decomposition, suitably interpreted, which runs parallel to that of GL(n).

§3. Endoscopy.

Before we can consider nontempered representations for general G, we must review some of the ideas connected with endoscopy. These ideas are part of a theory of Langlands and Shelstad, which was originally motivated by the trace formula and its conjectured relation to algebraic geometry [24], [31]. The theory is now developing a close connection with the harmonic analysis on local groups [40], [41], [33].

There are three notions to consider: stable distributions, endoscopic groups, and transfer of functions. We shall discuss them in turn.

Suppose first that F is a local field. Recall that a distribution on G(F) is ***invariant*** if it remains unchanged under conjugation by G(F). Typical examples are the invariant orbital integrals

$$f_G(\gamma) = \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)dx , \qquad f\in C_c^\infty(G(F)) ,$$

in which γ is a strongly regular element in $G(F)$. It can be shown that any invariant distribution on $G(F)$ lies in the closed linear span of the orbital integrals; that is, it annihilates functions f such that $f_G(\gamma)$ vanishes for all γ. (This property is most difficult to establish for Archimedean fields, and the proof has not been published. We have mentioned it only for motivation, however, and we will not need to use it in what follows.) For any γ, let γ_G be the associated stable conjugacy class. It is the intersection of $G(F)$ with the conjugacy class of γ in $G(\overline{F})$, and is a finite union of conjugacy classes $\{\gamma_i\}$ in $G(F)$. The ***stable orbital integral*** of f at γ_G is the sum

$$f^G(\gamma_G) = \sum_i f_G(\gamma_i) .$$

A ***stable distribution*** on $G(F)$ is any invariant distribution which lies in the closed linear span of the stable orbital integrals. That is, it annihilates any function f such that $f^G(\gamma_G)$ vanishes for every γ_G. The theory of endoscopy describes invariant distributions on G in terms of stable distributions on certain groups H of dimension less than or equal to G. It is enough to analyze invariant orbital integrals in terms of stable orbital integrals.

The groups H are the endoscopic groups for which the theory is named. They are defined if F is either local or global. As in [33, §1], we shall denote the L-group by

$${}^LG = \hat{G} \rtimes W_F ,$$

where $\hat{G}$ is the complex "dual group", and W_F is the Weil group of F. The Weil group acts on $\hat{G}$ through the Galois group $\Gamma = \mathrm{Gal}(\overline{F}/F)$. We shall also fix an inner twist

$$\eta: G \to G^* ,$$

where G^* is quasi-split over F. Then there is a canonical identification ${}^LG \xrightarrow{\sim} {}^LG^*$ between the L-groups of G and G^*. (See [33, (1.2)].)

An endoscopic group is part of an ***endoscopic datum*** $(H,\mathcal{H},s,\xi)$ for G, the definition of which we take from [33, (1.2)]. Then H is a quasi split group over F, $\mathcal{H}$ is a split extension

$$1 \to \hat{H} \to \mathcal{H} \to W_F \to 1 ,$$

s is a semisimple element in $\hat{G}$, and ξ is an L-embedding of $\mathcal{H}$ into ${}^LG$. It is required that $\xi(\hat{H})$ be the connected centralizer of s in $\hat{G}$, and that

$$s\xi(h)s^{-1} = a(w(h))\xi(h) , \qquad h\in \mathcal{H} ,$$

where $w(h)$ is the image of h in W_F, and $a(\cdot)$ is a 1-cocycle of W_F in $Z(\hat{G})$ which is trivial if F is local, and is locally trivial if F is global. It is also required that the actions of W_F on $\hat{H}$ defined by $\mathcal{H}$ and ${}^LH$ be the same modulo inner automorphism. Two endoscopic data $(H, \mathcal{H}, s, \xi)$ and $(H', \mathcal{H}', s', \xi')$ are said to be ***equivalent*** if there exist dual isomorphisms $\alpha: H \to H'$ and $\beta: \mathcal{H}' \to \mathcal{H}$, together with an element $g \in \hat{G}$, such that

$$g\xi(\beta(h'))g^{-1} = \xi'(h'), \qquad h' \in \mathcal{H}',$$

and

$$gsg^{-1} = z\zeta's',$$

where z belongs to $Z(\hat{G})$ and ζ' lies in the centralizer of $\xi'(\mathcal{H}')$ in $\hat{G}$. Finally, an endoscopic datum is said to be ***elliptic*** if $\xi(\mathcal{H})$ is not contained in any proper parabolic subgroup of ${}^LG$.

There is a simple class of examples one can keep in mind. Suppose that G is a split group of adjoint type. Then $\hat{G}$ is semisimple and simply connected. A theorem of Steinberg asserts that the centralizer of a semisimple element s in $\hat{G}$ is connected. It follows that for any endoscopic datum attached to s, the group H is also split. It is completely determined by s. The elliptic endoscopic data can thus be obtained in the familiar way from the extended Dynkin diagram. They are attached to vertices whose coefficient in the highest root is greater than one. For example, if $G = SO(2n+1)$, $\hat{G} = Sp(2n, \mathbb{C})$, and the diagram is

Deleting vertices with coefficient 2, we obtain

$$\hat{H} = Sp(2r, \mathbb{C}) \times Sp(2n-2r, \mathbb{C}), \qquad 0 < r < n,$$

so that the proper elliptic endoscopic groups are of the form

$$H = SO(2r+1) \times SO(2n-2r+1).$$

The group $\mathcal{H}$ need not be isomorphic to the L-group ${}^LH$. The minor complications that this causes are easily dealt with however [33, (4.4)], so we shall assume that for a given endoscopic datum, we have also been given an isomorphism of ${}^LH$ with $\mathcal{H}$. We shall also assume for the rest of this section that F is local. Langlands and Shelstad have defined a function $\Delta(\gamma_H, \gamma)$, where γ_H is a stable conjugacy class in $H(F)$ that is G-regular, and γ is a regular conjugacy class in $G(F)$ [33]. This function vanishes unless γ belongs to a certain stable conjugacy class γ_G in $G(F)$ (possibly empty), which is associated to γ_H. For any $f \in C_c^\infty(G(F))$, the finite sum

$$f^H(\gamma_H) = \sum_{\gamma} \Delta(\gamma_H, \gamma) f_G(\gamma)$$

then gives a function f^H on the set of classes $\{\gamma_H\}$.

For a given H, the transfer factor $\Delta(\gamma_H, \gamma)$ is canonically defined only up to a scalar multiple. The same is therefore true of the function f^H. However, if H equals G^*, $\Delta(\gamma_H, \gamma)$ is just a constant, so it can be normalized. Following the convention of [41], we shall set $\Delta(\gamma_{G^*}, \gamma)$

equal to the sign

$$e(G) = e(G,F)$$

defined by Kottwitz in [20]. For example, if $F = \mathbb{R}$,

$$e(G,F) = (-1)^{q(G)-q(G^*)} ,$$

where $q(G)$ equals one half the dimension of the symmetric space attached to G.

The functions $\Delta(\gamma_H,\gamma)$ are the transfer factors for orbital integrals. Langlands and Shelstad anticipate that there is a function $g \in C_c^\infty(H(F))$ such that

$$f^H(\gamma_H) = g^H(\gamma_H) .$$

If f is archimedean, the map f^H is the same as the one defined by Shelstad in [41]. In this case the function g is known to exist. For p-adic F, Langlands and Shelstad have shown how to reduce the existence of g to a local question in an invariant neighbourhood of 1 in $H(F)$. In any case, g will not be uniquely determined. However, if S is a stable distribution on $H(F)$, $S(g)$ will depend only on f^H.

The regular orbital integrals are a natural family of invariant distributions on $G(F)$. A second family is provided by the tempered characters. For each tempered representation $\pi \in \Pi_{temp}(G(F))$,

$$f_G(\pi) = \operatorname{tr} \pi(f) , \qquad f \in C_c^\infty(G(F)) ,$$

is obviously an invariant distribution. The tempered representations are also expected to provide a second natural family of stable distributions. This is known if F is archimedean. In fact, Shelstad [41] has shown that there is a theory of transfer of tempered characters which is parallel to that of orbital integrals. Let us recall her results.

Assume that $F = \mathbb{R}$. Recall [8] that

$$\Phi(G) = \Phi(G,\mathbb{R})$$

denotes the set of admissible maps

$$\phi: W_{\mathbb{R}} \to {}^LG ,$$

determined up to $\hat{G}$ conjugacy in ${}^LG$, while $\Phi_{temp}(G)$ denotes the subset of maps $\phi \in \Phi(G)$ whose image projects onto a bounded subset of $\hat{G}$. Associated to any $\phi \in \Phi(G)$ there is a finite packet Π_ϕ of irreducible representations. These representations are tempered if and only if ϕ belongs to $\Phi_{temp}(G)$. If ϕ does belong to $\Phi_{temp}(G)$, it turns out that the distribution

$$f^G(\phi) = \sum_{\pi \in \Pi_\phi} f_G(\pi) , \qquad f \in C_c^\infty(G(\mathbb{R})) ,$$

is stable.

Suppose that H is as above, and that ϕ_H is an element in $\Phi_{temp}(H)$. If f belongs to $C_c^\infty(G(\mathbb{R}))$, f^H is the image of a function in $C_c^\infty(H(\mathbb{R}))$ whose value on any stable distribution on $H(\mathbb{R})$ is uniquely determined. Therefore, $f^H(\phi_H)$ is well defined. Shelstad studies $f^H(\phi_H)$ as a function of f. She obtains a formula

$$f^H(\phi_H) = \sum_{\pi} \Delta(\phi_H,\pi) f_G(\pi) ,$$

for a certain complex valued function $\Delta(\phi_H,\cdot)$ on $\Pi_{temp}(G(\mathbb{R}))$. If $\phi \in \Phi_{temp}(G)$ is defined by the composition

$$W_{\mathbb{R}} \xrightarrow{\phi_H} {}^LH \rightarrow {}^LG ,$$

then $\Delta(\phi_H,\cdot)$ is supported on the finite subset Π_ϕ of $\Pi_{temp}(G(\mathbb{R}))$. The functions $\Delta(\phi_H,\pi)$ are dual analogues of the transfer factors for orbital integrals. They are closely related to the representation theory of a certain finite group.

Suppose that ϕ is an element in $\Phi_{temp}(G)$. Let S_ϕ denote the centralizer in $\hat{G}$ of the image $\phi(W_{\mathbb{R}})$. Set

$$\mathcal{S}_\phi = S_\phi / S_\phi^0 = \pi_0(S_\phi) ,$$

the finite group of connected components of S_ϕ. Now, suppose that s is a semisimple element in S_ϕ. Take $\hat{H}$ to be the connected centralizer of s in $\hat{G}$, and set

$$\mathcal{H} = \hat{H} \phi (W_{\mathbb{R}}) .$$

Then $\mathcal{H}$ is a split extension of $W_{\mathbb{R}}$ by $\hat{H}$. The action of $W_{\mathbb{R}}$ on $\hat{H}$ can be modified by inner automorphisms to yield an L-action. We can therefore identify $\hat{H}$ with the dual of a well defined quasi-split group $H = H_s$ over $\mathbb{R}$. Since $\mathcal{H}$ comes with an embedding into ${}^LG$, the element s determines an endoscopic datum. We shall assume for simplicity that $\mathcal{H}$ is isomorphic to ${}^LH$. Then for any such isomorphism there is a unique parameter $\phi_H \in \Phi_{temp}(H)$ such that the diagram

$$\begin{array}{ccc} & W_{\mathbb{R}} & \\ {}^{\phi_H}\swarrow & & \searrow^{\phi} \\ {}^LH \cong & \mathcal{H} \hookrightarrow & {}^LG \end{array}$$

is commutative. The distribution

$$f^H(\phi_H) , \qquad f \in C_c^\infty(G(\mathbb{R})) ,$$

is independent of the isomorphism. We therefore have a function

$$\delta(s,\pi) = \Delta(\phi_H,\pi)$$

on $S_\phi \times \Pi_\phi$, with the property that

$$f^H(\phi_H) = \sum_{\pi \in \Pi_\phi} \delta(s,\pi) f_G(\pi) , \qquad f \in C_c^\infty(G(\mathbb{R})) ,$$

for $H = H_s$.

The transfer factors are uniquely determined up to a constant multiple. It follows that for any fixed $\pi_1 \in \Pi_\phi$, the function

$$<\bar{s},\pi|\pi_1> \; = \; \delta(s,\pi)\delta(s,\pi_1)^{-1}\,, \qquad (s,\pi)\in S_\phi \times \Pi_\phi\,,$$

is canonically defined. One of the results of [41] asserts that, as the notation suggests, the function depends only on the image $\bar{s}$ of s in $\boldsymbol{S}_\phi$. Moreover,

$$<\bar{s},\pi|\pi_1>\,, \qquad \bar{s}\in \boldsymbol{S}_\phi\,,$$

is an irreducible character on $\boldsymbol{S}_\phi$. In fact, Shelstad shows that for any fixed π_1, the map $\pi \rightarrow <\cdot,\pi|\pi_1>$ is an an injection from Π_ϕ into the set $\Pi(\boldsymbol{S}_\phi)$ of (irreducible) characters on $\boldsymbol{S}_\phi$. This gives an elegant way to index the representations in the packet Π_ϕ.

We should recall that the group $\boldsymbol{S}_\phi$ is abelian. The quotient

$$S_\phi/S_\phi^0\, Z(\hat{G})^\Gamma \; = \; \boldsymbol{S}_\phi/\pi_0(Z(\hat{G})^\Gamma)$$

is in fact a product of several copies of $\mathbf{Z}/2\mathbf{Z}$. (Here, $Z(\hat{G})^\Gamma$ denotes the group of $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant elements in the center $Z(\hat{G})$.) Shelstad actually takes $\boldsymbol{S}_\phi$ to be this quotient, since the characters $<\cdot\,,\pi|\pi_1>$ are all trivial on the center. However, in more general situations one encounters nonabelian finite groups. A corresponding irreducible representation could have a central character which is essential, in the sense that it remains nontrivial under twisting by any one dimensional character. That one must allow for this possibility was pointed out to me by Vogan, and more recently, Kottwitz.

§4. Conjectures for real groups.

Endoscopy works beautifully for characters of real groups which are tempered. However, the theory breaks down for nontempered characters. For example, there seems to be no stable distribution naturally associated with a general irreducible character. What goes wrong?

We continue to take $F = \mathbb{R}$. Suppose that ϕ is an arbitrary parameter in $\Phi(G)$. Then the representations in Π_ϕ are Langlands quotients. More precisely, there is a parabolic subgroup $P = MN$ of G, a tempered parameter $\phi_M \in \Phi_{temp}(M)$, and a character

$$\chi_M\colon M(\mathbb{R}) \;\rightarrow\; \mathbb{R}^*\,,$$

which is positive on the chamber defined by P, such that

$$\Pi_\phi \; = \; \{\boldsymbol{J}_P(\pi_M \otimes \chi_M)\colon \pi_M \in \Pi_{\phi_M}\}\,.$$

Here $\boldsymbol{J}_P(\pi_M\otimes\chi_M)$ is the unique irreducible quotient of the induced representation $\boldsymbol{I}_P(\pi_M\otimes\chi_M)$. (Such induced representations are often called ***standard*** representations.) Now ϕ is just a twist of ϕ_M by the parameter of the character χ_M. It follows easily from the positivity of χ_M that the centralizer S_ϕ lies in $\hat{M}$, and in fact equals S_{ϕ_M}, the centralizer of $\phi_M(W_\mathbb{R})$ in $\hat{M}$. We set

$$\delta(s,\pi) = \delta(s,\pi_M) \qquad s \in S_\phi = S_{\phi_M},$$

for any representation

$$\pi = J_P(\pi_M \otimes \chi_M), \qquad \pi_M \in \Pi_{\phi_M},$$

in Π_ϕ. Thus, the functions $\delta(s,\pi)$ can be defined for a nontempered parameter. We can also define the character

$$<\bar{s},\pi \mid \pi_1> = \delta(s,\pi)\delta(s,\pi_1)^{-1}, \qquad s \in S_\phi,$$

on $\boldsymbol{S}_\phi$, for any pair of representations π and π_1 in the nontempered packet Π_ϕ.

However, the distribution

$$\sum_{\pi \in \Pi_\phi} f_G(\pi), \qquad f \in C_c^\infty(G(\mathbb{R})),$$

is generally not stable. Moreover, even if we could find a point $s \in S_\phi$ such that the corresponding distribution on $H_s(\mathbb{R}) = H(\mathbb{R})$ was stable, the distribution $f^H(\phi_H)$ would not in general equal

$$\sum_{\pi \in \Pi_\phi} \delta(s,\pi) f_G(\pi).$$

The problem is that Π_ϕ contains Langlands quotients, the character theory of which requires the generalized Kazhdan-Lusztig algorithm, and is very complicated. On the other hand, the character theory of the standard representations

$$\tilde{\Pi}_\phi = \{\mathcal{I}_P(\pi_M \otimes \chi_M) : \pi_M \in \Pi_{\phi_M}\}$$

is similar to that of the representations in Π_{ϕ_M}. In particular, the two assertions above would hold if we replaced the packet Π_ϕ by $\tilde{\Pi}_\phi$.

To deal with nontempered representations, it is necessary to introduce new parameters. We define

$$\Psi(G) = \Psi(G,\mathbb{R})$$

to be the set of $\hat{G}$-conjugacy classes of maps

$$\psi: W_{\mathbb{R}} \times SL(2,\mathbb{C}) \to {}^LG$$

such that the restriction

$$W_{\mathbb{R}} \xrightarrow{\psi} {}^LG \xrightarrow{\sim} {}^LG^*$$

lies in $\Phi_{temp}(G^*)$. Notice that we do not impose the usual condition that ψ be relevant. (See [8, 8.2(ii)].) As a consequence, ψ will sometime parametrize an empty set of representations. We have adopted this level of generality with the global role of the parameters in mind, rather than their possible application to the classification of local representations. For each $\psi \in \Psi(G)$, we define a parameter $\phi_\psi \in \Phi(G^*)$ by setting

$$\phi_\psi(w), \qquad w \in W_{\mathbb{R}},$$

equal to the image of

$$\psi(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix})$$

in ${}^LG^*$. As we remarked in [2, p. 10], the Dynkin classification of unipotent elements in $\hat{G}$ implies that $\phi \to \phi_\psi$ is an injection from $\Psi(G)$ into $\Phi(G^*)$. In particular, in the case $G = G^*$, we have embeddings

$$\Phi_{temp}(G^*) \subset \Psi(G^*) \subset \Phi(G^*) .$$

Suppose that ψ is an arbitrary parameter in $\Psi(G)$. Set S_ψ equal to the centralizer in $\hat{G}$ of the image $\psi(W_{\mathbb{R}} \times SL(2,\mathbb{C}))$, and write $s \to \overline{s}$ for the projection from S_ψ onto the finite group

$$\mathcal{S}_\psi = S_\psi/S_\psi^0 = \pi_0(S_\psi) ,$$

of components. We have identified ${}^LG$ with ${}^LG^*$, so we also have the subgroup S_{ϕ_ψ} of $\hat{G}$. It obviously contains S_ψ. The reader can check that the corresponding map

$$\mathcal{S}_\psi \to \mathcal{S}_{\phi_\psi}$$

of component groups is actually surjective. Consequently, there is a dual map

$$\Pi(\mathcal{S}_{\phi_\psi}) \to \Pi(\mathcal{S}_\psi)$$

of irreducible representations which is injective. Notice that there is a canonical central element

$$s_\psi = \psi(1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix})$$

in S_ψ. Since it can be deformed to the identity through the connected subgroup

$$\{\psi(1, \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}) : z \in \mathbb{C}^*\}$$

of S_{ϕ_ψ}, the image of s_ψ in $\mathcal{S}_{\phi_\psi}$ is the identity.

For each element s in S_ψ we can define the endoscopic group $H = H_s$ as in the tempered case. Again we shall assume for simplicity that there is an isomorphism of ${}^LH$ with $\mathcal{H}$, and therefore by composition, a parameter $\psi_H \in \Psi(H)$. The local conjecture boils down to the assertion that the theory for tempered parameters can be generalized to the parameters in $\Psi(G)$. We shall discuss this informally for real groups, leaving a formal statement of the conjecture for §6, where we shall consider a more general setting.

First and foremost, we postulate for every quasi-split group G_1 and every parameter $\psi_1 \in \Psi(G_1)$, the existence of a stable distribution

$$f_1 \to f_1^{G_1}(\psi_1) , \qquad f_1 \in C_c^\infty(G(\mathbb{R})) ,$$

which is a finite linear combination of irreducible characters on $G_1(\mathbb{R})$. Now suppose that $\psi \in \Psi(G)$. If

$$H = H_s , \qquad s \in S_\psi ,$$

we can form the distribution

$$f \to f^H(\psi_H) , \qquad f \in C_c^\infty(G(\mathbb{R})) ,$$

as in the tempered case from the stable distribution on $H(\mathbb{R})$ attached to ψ_H. It will be a finite linear combination of irreducible characters on $G(\mathbb{R})$, which we can write in the form

$$f^H(\psi_H) = \sum_\pi \delta(s_\psi s, \pi) f_G(\pi) , \tag{4.1}$$

for uniquely determined complex numbers $\delta(s_\psi s, \pi)$. Let Π_ψ denote the set of $\pi \in \Pi(G(\mathbb{R}))$ such that $\delta(s,\pi) \neq 0$ for some $s \in S_\psi$. Then Π_ψ will be a finite "packet" of representations in $\Pi(G(\mathbb{R}))$. Remember that f^H is well defined if $H = G^*$, and is otherwise determined up to a scalar multiple. Therefore, the numbers

$$\{\delta(s,\pi): \pi \in \Pi_\psi\}$$

are uniquely determined if $s = s_\psi$, and are given up to scalar multiples for general s.

Our second postulate is that $\delta(\cdot,\cdot)$ is closely related to the character theory of $\mathcal{S}_\psi$. More precisely, we conjecture the existence of a nonvanishing complex valued function ρ on S_ψ with $\rho(s_\psi) = \pm 1$, and with the following further property. For each $\pi \in \Pi_\psi$, the function

$$\langle \bar{s}, \pi \mid \rho \rangle = \delta(s,\pi)\rho(s)^{-1} , \qquad s \in S_\psi , \tag{4.2}$$

depends only on the image $\bar{s}$ of s in $\mathcal{S}_\psi$, and is the character of a nonzero finite dimensional representation of $\mathcal{S}_\psi$. We do not ask that the character be irreducible. However, we shall assume that its constituents have the same central character under s_ψ. That is,

$$\langle \bar{s}_\psi \bar{s}, \pi \mid \rho \rangle = e_\psi(\bar{s}_\psi, \pi \mid \rho) \langle \bar{s}, \pi \mid \rho \rangle ,$$

where $e_\psi(\cdot, \pi \mid \rho)$ is a sign character on $\{1, \bar{s}_\psi\}$. Thus,

$$\delta(s_\psi, \pi) = e_\psi(\pi) d_\psi(\pi) ,$$

where

$$e_\psi(\pi) = e_\psi(\bar{s}_\psi, \pi \mid \rho)\rho(s_\psi) = \pm 1 ,$$

while the number

$$d_\psi(\pi) = |\delta(s_\psi, \pi)|$$

equals the degree of the character $\langle \cdot, \pi \mid \rho \rangle$. Suppose that there is a representation $\pi_1 \in \Pi_\psi$ with $d_\psi(\pi_1) = 1$. Then the function

$$(4.3) \qquad \langle \bar{s},\pi|\pi_1\rangle = \delta(s,\pi)\delta(s,\pi_1)^{-1}$$

can be written as

$$\langle \bar{s},\pi|\rho\rangle\langle \bar{s},\pi_1|\rho\rangle^{-1},$$

and is obviously a finite dimensional character on $\boldsymbol{S}_\psi$. Therefore, the function $\delta(s,\pi_1)$ satisfies the conditions of ρ.

We shall add a third postulate to the special case that $G = G^*$. In this situation, we are provided with a second packet Π_{ϕ_ψ} of representations in $\Pi(G(\mathbb{R}))$. We conjecture that Π_{ϕ_ψ} is a subset of Π_ψ consisting of representations π_1 with $\delta(s_\psi,\pi_1) = 1$. In particular, we can form the character (4.3) for any such π_1. We conjecture further that (4.3) is actually an irreducible character on $\boldsymbol{S}_\psi$ and that the corresponding diagram

$$\begin{array}{ccc} \Pi_\psi & \longrightarrow & \Pi(\boldsymbol{S}_\psi) \\ \uparrow & & \uparrow \\ \Pi_{\phi_\psi} & \longrightarrow & \Pi(\boldsymbol{S}_{\phi_\psi}) \end{array}$$

is commutative.

Taken together, the three postulates provide a mild generalization of the conjecture stated on page 11 of [2]. In the earlier version, we were too optimistic to think that the characters $\langle\cdot,\pi|\rho\rangle$ would be distinct. This has been shown to fail in the examples of Adams and Johnson (see §5). There also seems to be no reason to suppose that the characters $\langle\cdot,\pi|\rho\rangle$ are irreducible, but we have retained this assertion in the case that G is quasi-split.

Our conjecture is far from being the whole story. For example, it ought to include a prescription for characterizing the Langlands parameters $\phi\in\Phi(G)$ attached to the individual representations in Π_ψ. As it is stated here, the conjecture does not even determine the objects $f^{G_1}(\psi_1)$, Π_ψ and $\delta(\cdot,\cdot)$ uniquely. For we cannot use the inversion argument of [2], which was based on the incorrect assumption that the map $\pi \to \langle\cdot,\pi|\rho\rangle$ would always be injective. The formula (4.1) at least determines Π_ψ and $\delta(\cdot,\cdot)$ from the stable distributions. In particular, everything can be defined for general G in terms of data for quasi-split groups. However, something more is clearly needed. We could try to make the conjecture rigid by adding some plausible hypotheses, but it is perhaps better at this point to leave the matter open.

The most difficult case of the conjecture will be when the parameter ψ is unipotent; that is, when the projection of $\psi(W_{\mathbb{R}})$ onto $\hat{G}$ equals $\{1\}$. For a start, the definition of a unipotent representation (as opposed to a unipotent parameter) is not at all obvious. Unipotent representations have been studied extensively by Barbasch and Vogan. When G is a complex group, they define [7] packets for many unipotent parameters, and they establish character formulas which obey (4.2). Their results imply that the conjecture is valid for complex groups, at least for the parameters they study explicitly. We refer the reader to [50] and [51] for progress in the study of unipotent representations for real groups, and how these fit into the general theory of the unitary dual.

The representations in Π_ψ will all have the same infinitesimal character. The character formulas required to prove the conjecture are easiest to handle when the infinitesimal character is regular. This is the case in the example of representations with cohomology, which has been studied by Adams and Johnson. We shall discuss their results in §5.

The motivation for the conjecture comes from automorphic forms. The representations in the packets Π_ψ should be the Archimedean constituents of unitary automorphic forms. It is therefore reasonable to conjecture that the representations in Π_ψ are all unitary.

§5. An example: representations with cohomology.

As an example, we shall look at the results [1] of Adams and Johnson. They have studied a family of parameters $\{\psi\}$ in $\Psi(G)$. The corresponding representations $\{\Pi_\psi\}$ are the unitary representations of $G(\mathbb{R})$ with cohomology, classified first by Vogan and Zuckerman [52], and later shown to be unitary by Vogan [48].

As in §4, G is a connected reductive group defined over $F = \mathbb{R}$. We shall write $\mathfrak{g}$ for the (complex) Lie algebra of $G(\mathbb{C})$. For simplicity we shall also assume that $G(\mathbb{R})$ has a maximal torus $T(\mathbb{R})$ which is compact modulo $A_G(\mathbb{R})^0$, the split component of the center of $G(\mathbb{R})$. We can then fix a Cartan involution of the form

$$\theta: g \rightarrow t_0 g t_0^{-1}, \qquad g \in G(\mathbb{R}),$$

where t_0 is a point in T whose square is central in G. The group

$$K'_{\mathbb{R}} = \{g \in G(\mathbb{R}): \theta(g) = g\}$$

of fixed points contains $T(\mathbb{R})$, and $K'_{\mathbb{R}}/A_G(\mathbb{R})^0$ is a maximal compact subgroup of $G(\mathbb{R})/A_G(\mathbb{R})^0$. Let τ be a fixed irreducible finite dimensional representation of $G(\mathbb{R})$. We are interested in unitary representations $\pi \in \Pi(G(\mathbb{R}))$ whose Lie algebra cohomology

$$H^*(\mathfrak{g}, K'_{\mathbb{R}}; \pi \otimes \tau) = \bigoplus_k H^k(\mathfrak{g}, K'_{\mathbb{R}}; \pi \otimes \tau)$$

does not vanish.

What are the parameters $\psi \in \Psi(G)$ associated to representations with cohomology? To answer this question, we begin with the representation τ. Fix a Borel subgroup B of G which contains T, and let

$$\Lambda_\tau: T(\mathbb{R}) \rightarrow \mathbb{C}^*$$

be the highest weight of the contragredient $\tilde{\tau}$ of τ, relative to B. As a one-dimensional character of $T(\mathbb{R})$, Λ_τ corresponds to a map

$$\phi_\tau: W_{\mathbb{R}} \rightarrow {}^L T.$$

We shall also fix a Borel subgroup $\hat{B}$ of $\hat{G}$ and a maximal torus in $\hat{B}$, which we shall denote by $\hat{T}$ since the choice of B and $\hat{B}$ determines an identification of $\hat{T}$ with the dual torus of T. As in [40], we shall write σ for the nontrivial element in $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, σ_T for the action of σ on T and $\hat{T}$, and $(1\times\sigma)$ for a fixed element in $W_{\mathbb{R}}$ which projects onto σ and has square equal to (-1). The values of ϕ_τ on the subgroup $\mathbb{C}^*$ of $W_{\mathbb{R}}$ may be described by a formula

$$\check{\lambda}(\phi_\tau(z)) = z^{\langle\lambda_\tau,\check{\lambda}\rangle}\bar{z}^{\langle\sigma_T\lambda_\tau,\check{\lambda}\rangle}, \qquad z\in\mathbb{C}^*,\ \check{\lambda}\in X_*(T), \tag{5.1}$$

where λ_τ is an element in $X^*(T)\otimes\mathbb{C}$ such that $\lambda_\tau - \sigma_T\lambda_\tau$ lies in $X^*(T)$. We can always conjugate the image of ϕ_τ by an element in $\hat{T}$. Since σ_T maps positive roots to negative roots, we see easily that $\phi_\tau(1\times\sigma)$ may be assumed to lie in the subgroup $Z(\hat{G})\rtimes W_{\mathbb{R}}$ of ${}^LT$.

Suppose for a moment that the entire image of $W_{\mathbb{R}}$ under ϕ_τ lies in $Z(\hat{G})\rtimes W_{\mathbb{R}}$. This means that τ is a one-dimensional representation of $G(\mathbb{R})$. The L-action σ_G of σ on $\hat{G}$ has the same restriction to $Z(\hat{G})$ as σ_T, so $Z(\hat{G})\rtimes W_{\mathbb{R}}$ has a canonical embedding as a subgroup of both ${}^LG$ and ${}^LT$. In particular, ϕ_T can be regarded as a map of $W_{\mathbb{R}}$ into ${}^LG$. The centralizer of $Z(\hat{G})\rtimes W_{\mathbb{R}}$ in $\hat{G}$ contains a principal unipotent element. Therefore, there is a map

$$\psi_G\colon W_{\mathbb{R}}\times SL(2,\mathbb{C}) \to {}^LG$$

whose restriction to $W_{\mathbb{R}}$ equals ϕ_τ, and which maps $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ to a principal unipotent element in $\hat{G}$. For the packet Π_{ψ_G}, one takes a single representation, namely the one-dimensional character τ. It is the simplest of the representations with cohomology. We note, incidentally, that ψ_G can be chosen so that the image of the diagonal elements in $SL(2,\mathbb{C})$ are given by the formula

$$\check{\lambda}\left(\psi_G\begin{pmatrix}z & 0\\ 0 & z^{-1}\end{pmatrix}\right) = z^{\langle 2\delta_G,\check{\lambda}\rangle} = \prod_{\alpha} z^{\langle\alpha,\check{\lambda}\rangle}, \qquad z\in\mathbb{C}^*,\ \check{\lambda}\in X_*(T), \tag{5.2}$$

where δ_G equals one half the sum of the roots α of (B,T).

More generally, suppose that $L\supset T$ is the Levi component of a parabolic subgroup Q of G which is standard with respect to B. Then L is defined over $\mathbb{R}$. We can identify the dual group $\hat{L}$ with the corresponding Levi component in $\hat{G}$ which contains $\hat{T}$ and is standard with respect to $\hat{B}$. The L-action σ_L of σ on $\hat{L}$ can be determined directly by its restriction to $\hat{T}$. This is just the composition $\sigma_T\circ \mathrm{ad}\, n_L$, where n_L is a fixed element in the derived group of $\hat{L}$ which maps the positive roots of $(\hat{L},\hat{T})$ to negative roots. Now, suppose that ϕ_τ maps $W_{\mathbb{R}}$ into $Z(\hat{L})\rtimes W_{\mathbb{R}}$. The groups L with this property are in bijective correspondence with the subsets of

$$\{\check{\alpha}\in\check{\Delta} : \lambda_\tau(\check{\alpha}) = 0\}.$$

These are just the subsets of the simple co-roots $\dot{\Delta}^\vee$ which lie in the kernel of the highest weight Λ_τ. We can clearly define the one-dimensional parameter

$$\psi_L \colon W_{\mathbb{R}} \times SL(2,\mathbb{C}) \to {}^L L$$

as above. In a moment we shall see how to extend the injection $\hat{L} \subset \hat{G}$ to a canonical embedding $\xi_{G,L} \colon {}^L L \to {}^L G$ of L-groups. The composition

$$\psi = \xi_{G,L} \circ \psi_L \colon W_{\mathbb{R}} \times SL(2,\mathbb{C}) \to {}^L G \tag{5.3}$$

is then a parameter for unitary representations with cohomology. Conversely, any such parameter will be of this form.

To describe the embedding $\xi_{G,L}$, we first recall how $\hat{T} \subset \hat{L}$ can be extended to an embedding of L-groups. There is a homomorphism

$$\psi_{L,T} \colon W_{\mathbb{R}} \to {}^L L \ ,$$

which maps $\mathbb{C}^*$ into $\hat{T}$ in such a way that

$$\check{\lambda}(\xi_{L,T}(z)) = z^{<\delta_L,\check{\lambda}>} \bar{z}^{-<\delta_L,\check{\lambda}>} \ , \qquad z \in \mathbb{C}^*,\ \check{\lambda} \in X_*(T) \ ,$$

and such that

$$\xi_{L,T}(1 \times \sigma) = n_L \rtimes (1 \times \sigma) \ .$$

This follows from [40, Proposition 1.3.5], which is in turn based on [27, Lemma 3.2]. (See the remark in [40] following the proof of the proposition.) As in §1 of [40], the map

$$\xi_{L,T} \colon t \rtimes w \to t\xi_{L,T}(w) \ , \qquad t \in \hat{T},\ w \in W_{\mathbb{R}} \ ,$$

then gives an embedding of ${}^L T$ into ${}^L L$. Observe that we had no use for $\xi_{L,T}$ in the construction above. We simply extended the co-domain of ϕ_τ to ${}^L L$ through the natural injections of $Z(\hat{L}) \rtimes W_{\mathbb{R}}$ into ${}^L T$ and ${}^L L$. However, an identical argument to that of [40, Proposition 1.3.5] and [27, Lemma 3.2] gives the embedding $\xi_{G,L}$. One simply replaces δ_L by $\delta_Q = \delta_G - \delta_L$, and n_L by $n_Q = n_G n_L^{-1}$. Once we have defined $\xi_{G,L}$, we see immediately from (5.1) and the definition (5.3) that

$$\check{\lambda}(\psi(z)) = z^{<\delta_Q+\lambda_\tau,\check{\lambda}>} \bar{z}^{<-\delta_Q+\sigma_T\lambda_\tau,\check{\lambda}>} \ , \qquad \check{\lambda} \in X_*(T) \ , \tag{5.4}$$

for any $z \in \mathbb{C}^* \subset W_{\mathbb{R}}$.

For another perspective on what we have discussed so far, let L_* be any group over $\mathbb{R}$ whose L-group is the given group ${}^L L$. One can of course parametrize the one-dimensional representations of $L_*(\mathbb{R})$ by certain elements $\phi_* \in \Phi(L_*)$, according to the Langlands classification. For any such ϕ_*, the packet Π_{ϕ_*} contains a single one-dimensional representation. However, one can also parametrize the one-dimensional representations of $L_*(\mathbb{R})$ by different maps $\phi \colon W_{\mathbb{R}} \to {}^L L$. Indeed, the tensor product with a fixed one-dimensional representation defines a bijection on $\Pi(L_*(\mathbb{R}))$. The corresponding bijection on $\Phi(L_*)$ is given by the product of a parameter in $\Phi(L_*)$ with a fixed map $\phi\colon W_{\mathbb{R}} \to {}^L L$ whose image lies in $Z(\hat{L}) \rtimes W_{\mathbb{R}}$. For the given L_*, we thus have a bijection $\phi \to \phi_*$ between the two different kinds of one-dimensional parameters. In the case at hand, we already have a parameter ϕ_τ whose image lies in $Z(\hat{L}) \rtimes W_{\mathbb{R}}$. For any L_* there will be an associated parameter $\phi_{\tau,*} \in \Phi(L_*)$. For example, if L_* is anisotropic modulo the center, then $\phi_{\tau,*}$ equals the composition of ϕ_τ, regarded now as an element in $\Phi(T)$, with the embedding $\xi_{L,T}$. If L_* is a quasi-split group, $\phi_{\tau,*}$ equals ϕ_{ψ_L}, the parameter in $\Phi(L_*)$ obtained from ψ_L.

We shall now discuss the objects attached to the parameter (5.3). Consider first the centralizer S_ψ. If $\lambda^\vee$ belongs to $X_*(T)$ and $z \in \mathbb{C}^*$, we have

$$\begin{aligned}\lambda^\vee(\phi_\psi(z)) &= \lambda^\vee(\psi(z))\lambda^\vee\left(\psi\begin{pmatrix}(z\bar{z})^{1/2} & 0\\ 0 & (z\bar{z})^{-1/2}\end{pmatrix}\right)\\ &= z^{<\delta_Q+\lambda_\tau,\lambda^\vee>}\,\bar{z}^{<-\delta_Q+\sigma_T\lambda_\tau,\lambda^\vee>}\,(z\bar{z})^{<\delta_L,\lambda^\vee>},\end{aligned}$$

by (5.4) and (5.2). Suppose that $\lambda^\vee$ lies in the span of the co-roots of (G,T) and that z is purely imaginary. Then

$$\begin{aligned}\lambda^\vee(\phi_\psi(z)) &= z^{<\delta_Q+\lambda_\tau,\lambda^\vee>}\,z^{-<-\delta_Q-\lambda_\tau,\lambda^\vee>}\\ &= z^{2<\delta_Q+\lambda_\tau,\lambda^\vee>}.\end{aligned}$$

Since λ_τ is dominant with respect to $B \subset Q$, we can choose z so that the centralizer of $\phi_\psi(z)$ in $\hat{G}$ equals $\hat{L}$. If z is a positive real number,

$$\lambda^\vee(\phi_\psi(z)) = z^{2<\delta_L,\lambda^\vee>},$$

and the centralizer of $\phi_\psi(z)$ in $\hat{L}$ equals $\hat{T}$. It follows that

$$S_\psi \subset S_{\phi_\psi} \subset \hat{T}.$$

Now, any point in $\hat{T}$ which commutes with the principal unipotent element $\psi\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ of $\hat{L}$ must lie in the center $Z(\hat{L})$. Moreover, $\psi(W_{\mathbb{R}})$ acts by conjugation on $Z(\hat{L})$ through the action of the Galois group $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\hat{L}$. It follows that S_ψ is contained in $Z(\hat{L})^\Gamma$. On the other hand, the elements in $Z(\hat{L})^\Gamma$ obviously commute with those in the image of ψ. It follows that

$$S_\psi = Z(\hat{L})^\Gamma.$$

The Galois action on $\hat{L}$ is such that the connected component of 1 in $Z(\hat{L})^\Gamma$ is identical to that in $Z(\hat{G})^\Gamma$. Therefore, the parameter ψ is elliptic, in the sense that its image does not lie in any proper parabolic subgroup of ${}^L G$. We see also that

$$\boldsymbol{S}_\psi = \pi_0(Z(\hat{L})) = Z(\hat{L})^\Gamma/(Z(\hat{G})^\Gamma)^0.$$

The packet Π_ψ constructed by Adams and Johnson takes the following form. Let W(G,T) and W(L,T) be the Weyl groups of G and L. Let $W_{\mathbb{R}}(G,T)$ be the real Weyl group of G, or equivalently, the Weyl group of $K'_{\mathbb{R}}$. The representations in Π_ψ are parametrized by the double cosets

$$\Sigma = W(L,T)\backslash W(G,T)/W_{\mathbb{R}}(G,T) .$$

For any $w \in \Sigma$, the group

$$L_w = w^{-1}Lw$$

is also defined over $\mathbb{R}$, and is a Levi subgroup of the θ-stable parabolic $Q_w = w^{-1}Qw$. The map ad(w) from L_w to L is an inner twist [1, Lemma 2.5], and can be used to identify ${}^LL$ with the L-group of L_w. The representations in Π_ψ are the derived functor modules

$$\pi_w = A_{Q_w}(w^{-1}\lambda_\tau) = \mathcal{R}_{Q_w}^{i(w)}(w^{-1}\lambda_\tau) , \qquad w \in \Sigma ,$$

where

$$i(w) = \frac{1}{2}(K'_{\mathbb{R}} \cap L_w \backslash K'_{\mathbb{R}}) .$$

(See [47, p. 344].) They have also been characterized in terms of the Langlands parameters [49, Theorem 6.16]. One can in fact show that π_w is a certain representation in the ordinary L-packet Π_{ϕ_w}, where $\phi_w \in \Phi(G)$ is the composition $\xi_{G,L} \circ \phi_{\tau,w}$. Here, $\phi_{\tau,w} \in \Phi(L_w)$ is the one-dimensional parameter corresponding to ϕ_τ in the manner described above.

Before describing the pairing on $\mathcal{S}_\psi \times \Pi_\psi$, we need to recall that there is a bijective map from $W(G,T)/W_{\mathbb{R}}(G,T)$ onto the set of elements in $H^1(\mathbb{R},T)$ whose image in $H^1(\mathbb{R},G)$ is trivial. Composed with the Tate-Nakayama map, this yields an injection $w \to t(w)$ from $W(G,T)/W_{\mathbb{R}}(G,T)$ into the quotient

$$X_*(T_{sc})/X_*(T_{sc}) \cap \{\lambda - \sigma_T\lambda : \lambda \in X_*(T)\} .$$

Here $X_*(T_{sc})$ is the submodule of $X_*(T)$ generated by the co-roots of (G,T). The map t is the starting point for the theory of endoscopy ([30, p. 702], [39, §2]). It is uniquely determined by the cocycle condition

$$t(w_1w_2) = t(w_1)w_1(t(w_2)) , \qquad w_1,w_2 \in W(G,T)/W_{\mathbb{R}}(G,T) , \tag{5.5}$$

and the formula

$$t(w_\beta) = \begin{Bmatrix} \beta^\vee , & \beta \text{ is noncompact,} \\ 0 , & \beta \text{ is compact,} \end{Bmatrix} \tag{5.6}$$

for its value on the reflection about a simple root β of (G,T) ([39, Propositions 2.1 and 3.1]).

Now, the natural map from $H^1(\mathbb{R},T)$ to $H^1(\mathbb{R},L)$ is surjective ([22, Lemma 10.2]). If two elements in $W(G,T)/W_{\mathbb{R}}(G,T)$ differ by left translation by an element in W(L,T), they have the same image in $H^1(\mathbb{R},L)$. Moreover, Kottwitz has established a generalization of the Tate-

Nakayama isomorphism which provides a canonical map from $H^1(\mathbb{R},L)$ to $\pi_0(Z(\hat{L})^\Gamma)^*$, the unitary dual of the finite group of connected components of $Z(\hat{L})^\Gamma$ [22, Theorem 2.1]. The classes which are trivial in $H^1(\mathbb{R},G)$ map to characters on $\pi_0(Z(\hat{L})^\Gamma)$ which are actually trivial on the subgroup $\pi_0(Z(\hat{G})^\Gamma)$. Since $\mathcal{S}_\psi = \pi_0(Z(\hat{L})^\Gamma)$, we shall interpret $w \to t(w)$ as a map from Σ into the group of characters of the finite abelian group $\mathcal{S}_\psi$ which are trivial on the subgroup $\pi_0(Z(\hat{G})^\Gamma)$. We can take the representation $\pi_1 = A_Q(\lambda_\tau)$ as our base point. Then if $\pi = \pi_w$ is any representation in Π_ψ, define

$$\langle x, \pi | \pi_1 \rangle = \langle x, t(w) \rangle, \qquad x \in \mathcal{S}_\psi, \tag{5.7}$$

the character on $\mathcal{S}_\psi$ determined by the element $w \in \Sigma$. This is the coefficient which occurs in the character formula of Adams and Johnson.

We can now see why several representations $\pi \in \Pi_\psi$ might give the same character on $\mathcal{S}_\psi$. According to [22, Theorem 1.2], the set of classes in $H^1(\mathbb{R},L)$ which map to the identity character on $\pi_0(Z(\hat{L})^\Gamma)$ is just the image of $H^1(\mathbb{R},L_{sc})$ in $H^1(\mathbb{R},L)$. Here, L_{sc} is the simply connected cover of the derived group of L. The representations $\pi \in \Pi_\psi$ for which the character $\langle \cdot, \pi | \pi_1 \rangle$ is trivial are precisely the ones whose corresponding element $w \in \Sigma$ maps to the image of $H^1(\mathbb{R},L_{sc})$ in $H^1(\mathbb{R},L)$. There is a similar description of the other fibres of the map

$$\pi \to \langle \cdot, \pi | \pi_1 \rangle, \qquad \pi \in \Pi_\psi .$$

Adams and Johnson state their character identities in terms of a certain sign

$$(-1)^{\gamma(w)}, \qquad w \in \Sigma,$$

where

$$\gamma(w) = \frac{1}{2}\dim(L_w/L_w \cap K'_{\mathbb{R}}) = q(L_w).$$

They first show that the distribution

$$f \to f^G(\psi) = \sum_{w \in \Sigma} (-1)^{\gamma(w)} \cdot f_G(\pi_w), \qquad f \in C_c^\infty(G(\mathbb{R})), \tag{5.8}$$

is stable, even when G is not quasi-split [1, Theorem 2.13]. They then establish the character formula

$$f^H(\psi_H) = \varepsilon_s \sum_{w \in \Sigma} (-1)^{\gamma(w)} \langle \bar{s}, t(w) \rangle f_G(\pi_w), \tag{5.9}$$

for $H = H_s$, $s \in S_\psi$, as in (4.1) [1, Theorem 2.21]. Here, ε_s is a certain constant which came out of Shelstad's earlier definition of the transfer factors for real groups [41]. Since f^H is defined only up to a scalar when $H \neq G^*$, ε_s is significant for us only when $s = 1$, in which case it equals 1. To deal with the signs $(-1)^{\gamma(w)}$, we need a lemma.

Lemma 5.1. $$(-1)^{\gamma(w)} = (-1)^{q(L)} \langle \bar{s}_\psi, t(w) \rangle, \qquad w \in \Sigma.$$

Proof. The lemma is easily reduced to a special case of a construction [20] of Kottwitz. For the convenience of the reader, we shall give a direct proof.

Recall that $K'_{\mathbb{R}}$ is the centralizer in $G(\mathbb{R})$ of an element $t_0 \in T$ whose square is central in G. It follows that if β is any root of (G,T),

$$\beta(t_0) = \begin{cases} -1, & \text{if } \beta \text{ is noncompact,} \\ 1, & \text{if } \beta \text{ is compact.} \end{cases}$$

Since $\gamma(w)$ equals the number of positive noncompact roots of (L_w,T), we see that

$$(-1)^{\gamma(w)} = \prod_{\alpha} (w^{-1}\alpha)(t_0) ,$$

the product being extended over the roots α of $(L \cap B, T)$. On the other hand, we recall that $s_\psi = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. It follows from (5.2) that

$$\check{\lambda}(s_\psi) = \prod_{\alpha} (-1)^{\langle\alpha,\check{\lambda}\rangle} , \qquad \check{\lambda} \in X_*(T) , \tag{5.10}$$

with the product taken over the same set of roots.

Each side of the required formula makes sense for any element $w \in W(G,T)$, but each side depends only on the image of w in Σ. We shall prove the lemma by induction on $l(w)$, the length of w. If w is the identity, $\langle s_\psi, t(w)\rangle = 1$ and $\gamma(w) = q(L)$, so there is nothing to prove. Suppose then that $w = w_\beta w_1$, where w_β is the reflection about a simple root β of (G,T), and $l(w_1)$ is less than $l(w)$. If α is a root of $(L \cap B, T)$,

$$\begin{aligned} (w^{-1}\alpha)(t_0) &= (w_1^{-1}w_\beta^{-1}\alpha)(t_0) \\ &= [w_1^{-1}(\alpha - \langle\alpha,\beta\rangle\beta)](t_0) \\ &= (w_1^{-1}\alpha)(t_0) \cdot (\beta_1(t_0))^{\langle\alpha,\beta\rangle} , \end{aligned}$$

where $\beta_1 = w_1^{-1}\beta$. Therefore,

$$(-1)^{\gamma(w)} = (-1)^{\gamma(w_1)} e(w_1,\beta)$$

where

$$e(w_1,\beta) = \begin{cases} \prod_{\alpha} (-1)^{\langle\alpha,\beta\rangle} , & \text{if } \beta_1 \text{ is noncompact,} \\ 1 , & \text{if } \beta_1 \text{ is compact.} \end{cases}$$

On the other hand,

$$\begin{aligned} \langle \bar{s}_\psi, t(w)\rangle &= \langle \bar{s}_\psi, t(w_1 w_{\beta_1})\rangle \\ &= \langle \bar{s}_\psi, t(w_1)\rangle\langle \bar{s}_\psi, w_1(t(w_{\beta_1}))\rangle \end{aligned}$$

by (5.5), while

$$<\overline{s}_\psi, w_1(t(w_{\beta_1}))> \; = \; \begin{cases} \overset{\vee}{\beta}(s_\psi)\ , & \text{if } \beta_1 \text{ is noncompact,} \\ 1\ , & \text{if } \beta_1 \text{ is compact,} \end{cases}$$

by (5.6). Applying (5.10), we obtain

$$<\overline{s}_\psi, t(w)> \; = \; <\overline{s}_\psi, t(w_1)>e(w_1,\beta)\ .$$

The lemma then follows by induction. □

If we apply the lemma to (5.9), we obtain

$$\begin{aligned} f^H(\psi_H) &= \varepsilon_s(-1)^{q(L)} \sum_{w\in\Sigma} <\overline{s}_\psi\overline{s}, t(w)>f_G(\pi_w) \\ &= \varepsilon_s(-1)^{q(L)} \sum_{\pi\in\Pi_\psi} <\overline{s}_\psi\overline{s}, \pi\,|\,\pi_1>f_G(\pi)\ . \end{aligned}$$

Therefore, the required formula (4.1) holds with

$$\delta(s_\psi s,\pi) \; = \; \varepsilon_s(-1)^{q(L)}<\overline{s}_\psi\overline{s}, \pi\,|\,\pi_1>\ .$$

§6. Some generalizations.

The theory of endoscopy was motivated by the trace formula. One would like an extended theory to provide for applications of the twisted trace formula as well. Anticipating future work of Kottwitz and Shelstad, let us describe the likely form of some of the twisted analogues of the objects in §3 and §4.

One can get away with minimal changes in the notation if one takes G to be a connected component of a (nonconnected) reductive group over F. We shall assume this from now on. We shall write G^+ for the reductive group generated by G, and G^0 for the identity component of G^+. We shall also assume that we have an inner twist

$$\eta\colon\ G \to G^*\ ,$$

where G^* is a component such that $(G^*)^0$ is quasi-split, and such that $G^*(F)$ contains an element which preserves some F-splitting of $(G^*)^0$ under conjugation. Then η extends to an isomorphism of G^+ onto $(G^*)^+$ such that for any $\sigma\in \mathrm{Gal}(\overline{F}/F)$, the map

$$\eta\cdot\sigma(\eta^{-1})\colon\ G^* \to G^*$$

is an inner automorphism by an element in $(G^*)^0$. One can attach an L-group

$${}^LG^+ \; = \; \hat{G}^+ \rtimes W_F$$

[5, §1], which is a finite extension of the usual L-group

$${}^LG^0 \; = \; \hat{G}^0 \rtimes W_F$$

of the connected component G^0. Corresponding to G, we then have the "L-coset"

$${}^LG = \hat{G} \rtimes W_F ,$$

a coset of ${}^LG^0$ in ${}^LG$. Observe that $\hat{G}$ is a coset of the complex connected group $\hat{G}^0$ in $\hat{G}^+$.

Endoscopic data $(H, \boldsymbol{H}, s, \xi)$ can be defined as before. The semisimple element s lies in $\hat{G}$, which is now just a coset. Again H is a connected quasi-split group, and $\xi(\hat{H})$ is the connected centralizer of s in $\hat{G}^0$. Equivalence of endoscopic data can also be defined as before, the element g lying in the connected component $\hat{G}^0$. Finally, the endoscopic datum will be called elliptic if the set

$$\xi(\boldsymbol{H})s$$

is not contained in any proper parabolic subset of ${}^LG$. (A parabolic subset of ${}^LG$ is any nonempty set which is the normalizer in ${}^LG$ of a parabolic subgroup of ${}^LG^0$.) As before, we shall make the simplifying assumption that there is an isomorphism of ${}^LH$ with $\boldsymbol{H}$.

Suppose that F is local. We shall assume that the transfer factors $\Delta(\gamma_H,\gamma)$ and the functions

$$f^H(\gamma_H) = \sum_{\gamma} \Delta(\gamma_H,\gamma) f_G(\gamma)$$

have been defined as in the connected case. Here γ stands for a strongly regular $G^0(F)$-orbit in $G(F)$, γ_H is a stable conjugacy class in $H(F)$ obtained from γ by a norm mapping, and

$$f_G(\gamma) = \int_{G_\gamma(F)\backslash G^0(F)} f(x^{-1}\gamma x)dx .$$

Again, we shall assume that f^H is actually the stable orbital integral of a function on $H(F)$.

One would like to be able to define parameter sets $\Phi_{temp}(G)$, $\Phi(G)$ and $\Psi(G)$. However, if F is nonarchimedean, we must replace the local Weil group W_F by something larger. We shall use the Langlands group

$$L_F = \begin{cases} W_F \times SU(2,\mathbb{R}) , & F \text{ nonarchimedean,} \\ W_F , & F \text{ archimedean,} \end{cases}$$

which is the variant of the Weil-Deligne group suggested on p. 647 of [21]. (See also [29, p. 209].) The group $SU(2,\mathbb{R})$ here is to account for the discrete series which are not supercuspidal, and should not be confused with the group used to define the Ψ-parameters. For the Ψ parameters, it is necessary to add another factor, namely $SL(2,\mathbb{C})$, to L_F. We are also dealing now with the possibility that $G \neq G^+$, and we would like the representations of $G^0(F)$ in the packets to have a chance of extending to $G^+(F)$. This is accomplished by asking that the image of a parameter centralize some element in the set $\hat{G}$.

We shall thus define

$$\Psi(G) = \Psi(G,F)$$

to be the set of $\hat{G}^0$-orbits of maps

$$\psi:\ L_F \times SL(2,\mathbb{C}) \rightarrow {}^L G^0$$

such that the projection of the image of L_F onto $\hat{G}^0$ is bounded, and such that the set

$$S_\psi = S_\psi(G) = \mathrm{Cent}(\psi(L_F \times SL(2,\mathbb{C})), \hat{G})$$

is nonempty. We also ask that the restriction of ψ to L_F have the usual reasonable behaviour; it should satisfy conditions similar to (1)-(4) on p. 57 of [32], although not the relevance condition (5). Observe that S_ψ is a coset of the subgroup

$$S_\psi(G^0) = \mathrm{Cent}(\psi(L_F \times SL(2,\mathbb{C})), \hat{G}^0)$$

in

$$S_\psi^+ = S_\psi(G^+) = \mathrm{Cent}(\psi(L_F \times SL(2,\mathbb{C})), \hat{G}^+) .$$

We shall write S_ψ^0 for the connected component of 1 in $S_\psi(G^0)$. Then

$$\boldsymbol{S}_\psi = \boldsymbol{S}_\psi(G) = S_\psi(G)/S_\psi^0$$

is a coset of the finite group

$$\boldsymbol{S}_\psi(G^0) = S_\psi(G^0)/S_\psi^0 = \pi_0(S_\psi(G^0))$$

in

$$\boldsymbol{S}_\psi^+ = \boldsymbol{S}_\psi(G^+) = S_\psi(G^+)/S_\psi^0 = \pi_0(S_\psi^+) .$$

One defines the sets $\Phi(G)$ and $\Phi_{temp}(G)$ of maps $\phi: L_F \rightarrow {}^L G^0$ in a similar fashion, but with a condition of relevance when G is not quasi-split. The image of ϕ is not allowed to lie in a parabolic subgroup of ${}^L G^0$ unless the corresponding parabolic subgroup of G^0 is defined over F. Suppose that $\psi \in \Psi(G)$. Then the restriction of ψ to L_F belongs to $\Phi_{temp}(G^*)$. Similarly, as in §4, we can define the objects $\phi_\psi \in \Phi(G^*)$ and $s_\psi \in S_\psi(G^0)$. There is a surjective map

$$\boldsymbol{S}_\psi \rightarrow \boldsymbol{S}_{\phi_\psi} ,$$

and a dual injective map

$$\Pi(\boldsymbol{S}_{\phi_\psi}) \rightarrow \Pi(\boldsymbol{S}_\psi) ,$$

in which $\Pi(\boldsymbol{S}_\psi)$ denotes the subset of representation in $\Pi(\boldsymbol{S}_\psi^+)$ whose restriction to $\boldsymbol{S}_\psi(G^0)$ remains irreducible.

For the component G, one is interested in the irreducible representations of $G^0(F)$ which extend to $G^+(F)$. Let $\Pi(G(F))$ denote the set of (equivalence classes of) irreducible representations of $G^+(F)$ whose restrictions to $G^0(F)$ are irreducible. The dual

$$\pi_0(G^+)^* = \mathrm{Hom}(G^+/G^0, \mathbb{C}^*)$$

of the component group acts freely on $\Pi(G(F))$ by

$$(\zeta\pi)(x) = \zeta(\overline{x})\pi(x), \qquad x\in G^+(F),\ \zeta\in\pi_0(G^+)^*,$$

where $\overline{x}$ denotes the image of x in $\pi_0(G^+)$. It is clear that there is a bijection between the set $\{\Pi(G(F))\}$ of orbits of $\pi_0(G^+)^*$ in $\Pi(G(F))$ and the representations in $\Pi(G^0(F))$ which are fixed under conjugation by $G(F)$. More generally, suppose that G' is an arbitrary connected component in G^+. Then $\pi_0((G')^+)$ is a subgroup of $\pi_0(G^+)$. If π is a representation in $\Pi(G(F))$, the restriction π' of π to $(G')^+(F)$ belongs to $\Pi(G'(F))$. The map $\pi\to\pi'$ is a bijection from the orbits of $(\pi_0(G^+)/\pi_0((G')^+))^*$ in $\Pi(G(F))$ to the set of representations in $\Pi(G'(F))$ which are fixed under conjugation by $G(F)$.

As in §4, we are going to postulate the existence of a finite subset Π_ψ of $\Pi(G(F))$ for every $\psi\in\Psi(G)$. This includes the question of defining the tempered packets

$$\{\Pi_\phi:\ \phi\in\Phi_{temp}(G)\},$$

which is itself far from being known. (See the hypothesis in [32, §IV.2].) It is conceivable that such a packet could be empty; perhaps none of the representations in the corresponding packet for G^0 extend to $G^+(F)$. We would at least like this problem not to occur in the quasi-split case. In particular, for each $\psi\in\Psi(G)$, we would always like to be able to choose a representation $\pi_1\in\Pi_{\phi_\psi}$ to serve as a base point. The theory of Whitaker models suggests that this is always possible.

Suppose that $(B^*,T^*,\{x_\alpha\})$ is an F-splitting for $(G^*)^0$. Here, x_α denotes the additive one parameter subgroup of G^* attached to a simple root α of (B^*,T^*). Any element in the unipotent radical $N_{B^*}(F)$ of $B^*(F)$ is therefore of the form

$$u = \Big(\prod_\alpha x_\alpha(t_\alpha)\Big)u', \qquad t_\alpha\in F,$$

where u' lies in the derived subgroup of $N_{B^*}(F)$. If ψ_F is a nontrivial additive character on F,

$$\chi(u) = \prod_\alpha \psi_F(t_\alpha)$$

is a nondegenerate character on $N_{B^*}(F)$. For any representation $\pi_1\in\Pi((G^*)^0(F))$, the space $V_\chi(\pi_1)$ of χ-Whitaker functionals

$$\{\Lambda : \Lambda(\pi_1(u)v) = \chi(u)\Lambda(v),\ u\in N_{B^*}(F)\},$$

is known to have dimension at most 1. Moreover, each tempered packet

$$\{\Pi_\phi : \phi\in\Pi_{temp}((G^*)^0)\}$$

is expected to contain precisely one representation π_1 such that $V_\chi(\pi_1)\neq\{0\}$. Assume that this is so. We claim that if ϕ actually belongs to $\Phi_{temp}(G^*)$, that is, if $S_\phi(G^*)\neq\varnothing$, then π_1 should extend to $(G^*)^+(F)$. Indeed, our assumption on G^* implies that there is an element $n_G\in G^*(F)$ which preserves the splitting. Consequently,

$$\chi(n_G u n_G^{-1}) = \chi(u), \qquad u \in N_{B^*}(F).$$

The condition $S_\phi(G^*) \neq \emptyset$ should translate to the dual property that n_G acts as a permutation on Π_ϕ. In particular, n_G must transform π_1 to some representation in the packet Π_ϕ, so by uniqueness, π_1 is fixed by n_G. This establishes the claim.

Now, suppose that ψ belongs to $\Psi(G)$. Regarding ϕ_ψ for a moment as an element in $\Phi((G^*)^0)$ (rather than $\Phi((G^*))$, we take $\pi_1 \in \Pi((G^*)^0(F))$ to be the representation in the packet Π_{ϕ_ψ} whose associated standard representation $\tilde{\pi}_1$ has a χ-Whitaker model. Then $\tilde{\pi}_1$ will extend to a representation of $(G^*)^+(F)$. From this, it is not hard to see that π_1 also extends to a representation π_χ of $(G^*)^+(F)$. Thus, the packet

$$\Pi_{\phi_\psi} = \Pi_{\phi_\psi}(G^*) \subset \Pi(G^*(F))$$

should be nonempty. For each nondegenerate character χ there should be a representation $\pi_\chi \in \Pi_{\phi_\psi}$, whose restriction π_χ^0 to $(G^*)^0(F)$ is uniquely determined.

We shall now state the general local conjecture. It is just an extrapolation of the limited information now available, and should be treated as such. Our purpose is simply to suggest that the general theory for tempered parameters, whatever its ultimate form, will have a natural extension to the nontempered parameters in $\Psi(G)$. As in the special case described in §4 , the conjecture postulates the existence of three objects. The first is attached to any parameter $\psi_1 \in \Psi(G_1)$ in which G_1 is a connected quasi-split group over F, while the second and third are attached to parameters $\psi \in \Psi(G)$ where G is an arbitrary component.

Conjecture 6.1. For each ψ_1 there is a stable distribution $f_1 \to f_1^{G_1}(\psi_1)$ on $C_c^\infty(G_1(F))$, while for each ψ there is a finite subset $\Pi_\psi = \Pi_\psi(G)$ of $\Pi(G(F))$ and a function δ on $S_\psi^+ \times \Pi_\psi$, such that the following properties hold.

(i)
$$\delta(s,\zeta\pi) = \zeta(G)^{-1}\delta(s,\pi), \qquad s\in S_\psi,\ \zeta \in \pi_0(G^+)^*.$$

(ii)
$$f^H(\psi_H) = \sum_{\pi \in \{\Pi_\psi\}} \delta(s_\psi s, \pi) f_G(\pi), \qquad f \in C_c^\infty(G(F)),$$

where $H = H_s$, for a given semisimple element $s \in S_\psi$.

(iii) There is a nonvanishing normalizing function ρ on S_ψ^+, with $\rho(s_\psi) = \pm 1$, such that for any $\pi \in \Pi_\psi$, the function

$$\langle \bar{s}, \pi \mid \rho\rangle = \delta(s,\pi)\rho(s)^{-1}, \qquad s \in S_\psi^+,$$

is a positive definite class function on S_ψ^+. Furthermore,

$$\langle \bar{s}_\psi \bar{s}, \pi \mid \rho\rangle = e_\psi(\bar{s}_\psi, \pi \mid \rho)\langle \bar{s}, \pi\mid\rho\rangle,$$

where $e_\psi(\cdot, \pi \mid \rho)$ is a sign character on $\{1, \bar{s}_\psi\}$.

(iv) In the special case that $G = G^*$, there is a commutative diagram

$$\begin{array}{ccc} \Pi_\psi & \longrightarrow & \Pi(\boldsymbol{S}_\psi) \\ \uparrow & & \uparrow \\ \cup & & \cup \\ \Pi_{\phi_\psi} & \longrightarrow & \Pi(\boldsymbol{S}_{\phi_\psi}) \end{array}$$

in which the vertical arrows stand for the maps

$$\pi \rightarrow \langle \cdot, \pi | \pi_\chi \rangle = \langle \cdot, \pi \rangle \langle \cdot, \pi_\chi \rangle^{-1} ,$$

and $\pi_\chi \in \Pi_{\phi_\chi} \subset \Pi_\psi$ is the representation described above. In particular, π_χ is such that

$$d_\psi(\pi_\chi) = |\delta(s_\psi, \pi_\chi)| = 1 .$$

(v) If G' is any connected component of G^+, write ψ' for the parameter ψ, regarded as an element in $\Psi(G')$. Then the restriction map $\pi \rightarrow \pi'$ sends Π_ψ onto the set of representations in $\Pi_{\psi'}$ which are fixed under conjugation by $G(F)$, and $\delta(\cdot,\pi')$ is the restriction of $\delta(\cdot,\pi)$ to $S^+_{\psi'}$. □

Remarks. 1. By the first condition,

$$\delta(s_\psi s, \zeta\pi) f_G(\zeta\pi) = \delta(s_\psi s, \pi) f_G(\pi) ,$$

for any $\zeta \in \pi_0(G^+)^*$ and $\pi \in \Pi_\psi$. Therefore, the sum in (ii) really can be taken over the orbits $\{\Pi_\psi\}$ of $\pi_0(G^+)^*$ in Π_ψ.

2. As in §4, the conjecture is not rigid. However, the conditions do determine everything uniquely once the stable distributions $f_1^{G_1}(\psi_1)$ have been defined. One would like to strengthen condition (iv) in a way that would characterize the distributions $f_1^{G_1}(\psi_1)$ uniquely, at least modulo their analogues for tempered parameters.

3. The third condition asserts that there are ***nonnegative*** real numbers

$$\langle \lambda, \pi | \rho \rangle, \qquad \lambda \in \Pi(\boldsymbol{S}^+_\psi),\ \pi \in \Pi_\psi,$$

such that

$$\langle \overline{s}, \pi | \rho \rangle = \sum_{\lambda \in \Pi(\boldsymbol{S}^+_\psi)} \langle \lambda, \pi | \rho \rangle \operatorname{tr}(\lambda(\overline{s})), \qquad \overline{s} \in \boldsymbol{S}^+_\psi .$$

The usual case should be that of (iv), in which

$$\langle \lambda, \pi | \rho \rangle = \begin{cases} 1, & \lambda = \lambda_{\pi|\rho} \\ 0, & \lambda \neq \lambda_{\pi|\rho}, \end{cases}$$

for some $\lambda_{\pi|\rho} \in \Pi(\boldsymbol{S}_\psi)$. However, the weaker assertion is already required by the examples in [24] for p-adic quaternion algebras.

4. Suppose that $H = H_s$ and $H_1 = H_{tst^{-1}}$, for a semisimple element $s \in S_\psi(G)$ and $t \in S_\psi(G^0)$. The transfer of functions will be such that $f^H(\psi_H)$ equals $f^{H_1}(\psi_{H_1})$. It follows from condition (ii) that

$$\delta(tst^{-1}, \pi) = \delta(s,\pi), \qquad \pi \in \Pi_\psi .$$

In other words, $\delta(\cdot,\pi)$ is a class function.

5. Condition (ii) should also imply that

$$\delta(sts^{-1}, \xi\pi^0) = \delta(t,\pi^0), \qquad t \in S_\psi(G^0),\ \pi^0 \in \Pi_\psi(G^0),$$

where $s \in S_\psi(G)$ and $\xi \in G(F)$, and where

$$(\xi\pi^0)(g) = \pi^0(\xi^{-1}g\xi), \qquad g \in G^0(F) .$$

This is compatible with condition (v).

Conjecture 6.2. For every parameter $\psi \in \Psi(G)$, the representations in Π_ψ are unitary. □

§7. Intertwining operators and R-groups.

Intertwining operators play an important role in the discussion. They occur naturally in the trace formula and provide part of the global motivation for the conjectures. We shall discuss this in the next paper [5]. Closely tied to the global considerations are a number of local questions. These questions are interesting even for tempered parameters, where they have been studied by Shahidi [36], [37] and Keys and Shahidi [18]. For the nontempered parameters $\psi \in \Psi(G)$, the implication of the conjectures is that much of the tempered theory carries over. It is therefore reasonable to propose a nontempered analogue of the R-group.

Recall that G is now a connected component of a reductive group over F. In this paragraph, F will be a local field (of characteristic 0). We shall say that a parameter $\psi \in \Psi(G)$ is ***elliptic*** if the image of ψ in ${}^LG^0$ is contained in no proper parabolic subgroup. This is equivalent to saying that S_ψ is finite modulo the center, or more precisely, that S_ψ^0 is contained in $Z(\hat{G}^0)^\Gamma$. We would like to deduce information about arbitrary parameters from information on elliptic parameters. In particular, we would like a method of constructing the packet Π_ψ and the function $\delta(x,\pi)$, for arbitrary ψ, from the corresponding objects for elliptic parameters.

Fix a parameter $\psi \in \Psi(G)$. There are several finite groups associated with the centralizer S_ψ. For simplicity, we shall describe them first in the case that $G = G^0$. Then S_ψ is a complex reductive group. Fix a maximal torus T_ψ in S_ψ^0, and let N_ψ be the normalizer of T_ψ in S_ψ. The quotient

$$\boldsymbol{N}_\psi = N_\psi/T_\psi = \pi_0(N_\psi)$$

is a finite group. Notice that there is a surjective map from $\boldsymbol{N}_\psi$ to the group $\boldsymbol{S}_\psi = S_\psi/S_\psi^0$ of components. The kernel is just the Weyl group W_ψ^0 of (S_ψ^0, T_ψ). Every element of $\boldsymbol{N}_\psi$ may be regarded as an automorphism of T_ψ, so we also have a surjective map of $\boldsymbol{N}_\psi$ onto the Weyl group W_ψ of (S_ψ, T_ψ). The kernel of this second map consists of the elements in $\boldsymbol{N}_\psi$ which centralize T_ψ. Since every such element belongs to a unique coset in $\boldsymbol{S}_\psi$, the kernel is canonically isomorphic to the subgroup $\boldsymbol{S}_\psi^1$ of cosets in $\boldsymbol{S}_\psi$ which act on S_ψ^0 by inner automorphisms. Notice that $\boldsymbol{S}_\psi^1$ is also a normal subgroup of $\boldsymbol{S}_\psi$. The quotient

$$R_\psi = \boldsymbol{S}_\psi / \boldsymbol{S}_\psi^1$$

is the ***R*-group** of ψ. It can be regarded as a finite group of outer automorphisms of S_ψ^0, and can also be identified with the quotient of W_ψ by W_ψ^0. We can summarize these remarks in a commutative diagram of finite groups

$$\begin{array}{ccccccccc} & & & & 1 & & 1 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & W_\psi^0 & = & W_\psi^0 & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \to & \boldsymbol{S}_\psi^1 & \to & \boldsymbol{N}_\psi & \to & W_\psi & \to & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \to & \boldsymbol{S}_\psi^1 & \to & \boldsymbol{S}_\psi & \to & R_\psi & \to & 1 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 1 & & 1 & & \end{array} \tag{7.1}$$

The dotted arrows stand for splittings of short exact sequences determined by a fixed Borel subgroup of S_ψ^0 containing T_ψ.

Now suppose that G is an arbitrary component. The commutative diagram and the definitions above still make sense if interpreted in the obvious way. For example, $\boldsymbol{N}_\psi$ is now only a set of cosets in $\hat{G}$. However, $\boldsymbol{S}_\psi^1$ will consist of components in $\boldsymbol{S}_\psi^+ \cap \hat{G}^0$, and will remain a group. The groups W_ψ^0 and $\boldsymbol{S}_\psi^1$ operate freely on $\boldsymbol{N}_\psi$, and $\boldsymbol{S}_\psi$ and W_ψ become the sets of orbits. The ***R*-set** R_ψ is the set of orbits of $\boldsymbol{S}_\psi^1$ in $\boldsymbol{S}_\psi$ and, at the same time, the set of orbits of W_ψ^0 in W_ψ. If it is necessary to indicate the dependence on the component G, we can always write $\boldsymbol{N}_\psi(G) = \boldsymbol{N}_\psi$, $R_\psi(G) = R_\psi$, etc., as we did earlier for $\boldsymbol{S}_\psi$. Thus, $\boldsymbol{N}_\psi(G)$ is a coset of $\boldsymbol{N}_\psi(G^0)$ in a finite group $\boldsymbol{N}_\psi(G^+)$.

Consider the centralizer of T_ψ in ${}^LG^0$. Since it meets every coset of $\hat{G}^0$ in ${}^LG^0$, it is of the form

$${}^LM = \hat{M} \rtimes W_F .$$

This group is a Levi component of a parabolic subgroup ${}^LP$ of ${}^LG^0$. It is also the L-group of a Levi component M of a parabolic subgroup P of G^0 which is defined over F. There may be no element in G which normalizes P, so P may not be attached to a parabolic subset [3, §1] of G. At any rate, the image of ψ lies in ${}^LM$. Therefore, ψ can be regarded as an element in $\Psi(M)$, which is determined up to conjugation by the normalizer of $\hat{M}$ in $\hat{G}$. Obviously T_ψ equals the identity component of

$$S_\psi(M) = \mathrm{Cent}(\psi(L_F \times SL(2,\mathbb{C})), \hat{M}),$$

and the group

$$\boldsymbol{S}_\psi(M) = S_\psi(M)/T_\psi = \pi_0(S_\psi(M))$$

is just equal to $\boldsymbol{S}_\psi^1$. In particular, as an element in $\Psi(M)$, ψ is elliptic.

According to Conjecture 6.1, ψ determines a finite packet $\Pi_\psi(M) \subset \Pi(M(F))$. It is not hard to guess how we might obtain the packet $\Pi_\psi(G) \subset \Pi(G(F))$ from $\Pi_\psi(M)$. For each $\sigma \in \Pi_\psi(M)$, we shall let $\boldsymbol{I}_P(\sigma)$ denote the representation $G^+(F)$ obtained from σ by induction from $P(F)$. It acts on a Hilbert space $\boldsymbol{H}_P^+(\sigma)$. Observe that P is connected while G^+ is generally not connected; this simply enhances the reducibility of $\boldsymbol{I}_P(\sigma)$. Let $\Pi_\sigma(G)$ denote the set of representations in $\Pi(G(F))$ which occur as irreducible constituents of $\boldsymbol{I}_P(\sigma)$. Then $\Pi_\psi(G)$ should be the union over all $\sigma \in \Pi_\psi(M)$ of the sets $\Pi_\sigma(G)$.

It is more delicate to construct the function

$$\delta(x,\pi), \qquad x \in S_\psi(G),\ \pi \in \Pi_\psi(G).$$

The first ingredients will be the interwining operators. For any representation $\sigma \in \Pi(M(F))$, we can define the unnormalized intertwining operators

$$J_{P'|P}(\sigma_\lambda): \boldsymbol{H}_P^+(\sigma) \rightarrow \boldsymbol{H}_{P'}^+(\sigma), \qquad P' \in \boldsymbol{P}(M),\ \lambda \in \boldsymbol{a}_{M,\mathbb{C}}^*,$$

as, for example, in [4, §1]. Langlands has proposed normalizing these operators by a certain quotient of L-functions [28, Appendix 2]. This can be established for real groups [4, Theorem 2.1], and in certain cases for p-adic groups [35], [18]. In the present context, Langlands' normalizing factors are the functions

$$r_{P'|P}(\psi_\lambda) = L(0, \tilde{\rho}_{P'|P} \circ \phi_{\psi,\lambda})(\varepsilon(0, \tilde{\rho}_{P'|P} \circ \phi_{\psi,\lambda}, \psi_F) L(1, \tilde{\rho}_{P'|P} \circ \phi_{\psi,\lambda}))^{-1}, \tag{7.2}$$

where

$$\phi_{\psi,\lambda} : L_F \rightarrow {}^LM$$

is the twist of ϕ_ψ by the element λ in

$$\boldsymbol{a}_{M,\mathbb{C}}^* = X^*(M)_F \otimes \mathbb{C} \cong X_*(T_\psi) \otimes \mathbb{C},$$

and $\tilde{\rho}_{P'|P}$ is the contragredient of the adjoint representation of ${}^LM$ on

$${}^L\boldsymbol{n}_{P'}/{}^L\boldsymbol{n}_{P'} \cap {}^L\boldsymbol{n}_P,$$

a quotient of the Lie algebra of the unipotent radical of ${}^LP'$. (We refer the reader to [46] for the definition of the L and ε-factors. At the risk of some confusion, we have used ψ_F to denote a fixed nontrivial additive character of F.) We shall assume in what follows that the operators

$$R_{P'|P}(\sigma_\lambda,\psi_\lambda) = J_{P'|P}(\sigma_\lambda)r_{P'|P}(\psi_\lambda)^{-1}, \qquad \sigma\in\Pi_\psi(M),$$

have the properties one expects of normalized intertwining operators. (See for example the conditions in [4, Theorem 2.1]. Langlands' original suggestion applies here only to the case that σ belongs to $\Pi_{\phi_\psi}(M)$. However, Proposition 5.2 of [4] and the part of Lemma II.2.1 of [6] that deals with inner twisting suggest how one could deal with arbitrary representations σ in $\Pi_\psi(M)$.)

The choice of groups ${}^LP\in\boldsymbol{P}({}^LM)$ and $P\in\boldsymbol{P}(M)$ allows us to identify W_ψ with a subset of

$$W(G,A_M) = \{g\in G: gA_Mg^{-1} = A_M\}/M.$$

(As usual, A_M denotes the split component of the center of M.) Regarding a given $w\in W_\psi$ as an element in $W(G,A_M)$, we can form the component

$$M_w = Mw$$

of a nonconnected reductive group. Let M_w^* be the image of M_w under our inner twist η. We may assume that the group

$$M^* = \eta(M) = (M_w^*)^0$$

is quasi-split, and that the restriction of η to M_w is an inner twist.

We would like to know that $M_w^*(F)$ contains an element which preserves a splitting of M^*. Suppose that $(B^*,T^*,\{x_\alpha\})$ is an F-splitting of $(G^*)^0$. It is convenient to assume that T^* is contained in M^*, and that the ***opposite*** Borel subgroup $\overline{B}^*$ is contained in $P^* = \eta(P)$. The element $\eta(w)$ lies in the Weyl set $W(G^*,A_{M^*})$. It has a unique representative w_1 in the Weyl set of (G^*,T^*) which maps the simple roots of $(B^*\cap M^*,T^*)$ to simple roots. By the hypothesis on G, there is an element $n_G\in G^*(F)$ such that $\mathrm{ad}(n_G)$ preserves our splitting. Then the element

$$w_0 = \mathrm{ad}(n_G)^{-1}w_1$$

belongs to the Weyl group of $((G^*)^0,T^*)$, and maps the simple roots of $(B^*\cap M^*,T^*)$ to simple roots. Now the choice of a splitting also determines a canonical function

$$w^* \to n(w^*)$$

from the Weyl group of $((G^*)^0,T^*)$ into $(G^*)^0(F)$ ([43], [33, p. 228]). Define

$$n_w = n_Gn(w_0). \tag{7.3}$$

It is a consequence of [43, Proposition 11.2.11] that

$$n_wx_\alpha(1)n_w^{-1} = x_{w\alpha}(1),$$

for any simple root α of $(B^*\cap M^*,T^*)$. In other words, n_w preserves the splitting of M^*. We have shown that the component M_w satisfies the same conditions as G, so we shall assume that it also satisfies Conjecture 6.1.

The Weyl set $W(G,A_M)$ operates in the usual way,

$$(w\sigma)(m) = \sigma(w^{-1}mw) , \qquad w\in W(G,A_M),\ \sigma\in\Pi(M(F)),\ m\in M(F) ,$$

on $\Pi(M(F))$. The image of W_ψ will be identified with the subset of elements in $W(G,A_M)$ which map $\Pi_\psi(M)$ to itself. For any $\sigma\in\Pi_\psi(M)$, set

$$W_{\psi,\sigma} = \{w\in W_\psi \subset W(G,A_M): w\sigma = \sigma\} .$$

We then obtain an embedding

$$\begin{array}{ccccccccc} 1 & \rightarrow & W^0_{\psi,\sigma} & \rightarrow & W_{\psi,\sigma} & \rightarrow & R_{\psi,\sigma} & \rightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \rightarrow & W^0_\psi & \rightarrow & W_\psi & \rightarrow & R_\psi & \rightarrow & 1 \end{array}$$

of short exact sequences. If $G = G^0$ and ψ is tempered, $R_{\psi,\sigma}$ will be the usual R-group [19, §2-3], [41, §5], [17, §2]. In general, it should be closely tied to the reducibility of the induced representation $\mathcal{I}_P(\sigma)$.

Fix a representation $\sigma\in\Pi_\psi(M)$ and an element $w\in W_{\psi,\sigma}$. Then M_w is a component of a reductive group such that $M_w^0 = M$. Since $w\sigma$ is equivalent to σ, there is a representation $\sigma_w\in\Pi(M_w(F))$ whose restriction to $M^0(F)$ equals σ. The extension σ_w is of course not unique, for it can be replaced by $\zeta\sigma_w$, for any element $\zeta\in\pi_0(M_w^+)^*$. Nevertheless, we can define an isomorphism

$$A(\sigma_w): \mathcal{H}^+_{w^{-1}Pw}(\sigma) \rightarrow \mathcal{H}^+_P(\sigma)$$

by setting

$$(A(\sigma_w)\phi')(x) = \sigma_w(m)\phi'(m^{-1}x) , \qquad \phi'\in\mathcal{H}_{w^{-1}Pw}(\sigma),\ x\in G(F) ,$$

for any element $m\in M_w(F)$. This map is an intertwining operator from $\mathcal{I}_{w^{-1}Pw}(\sigma)$ to $\mathcal{I}_P(\sigma)$ which is independent of the representative m. In particular,

$$R_P(\sigma_w,\psi) = \lim_{\lambda\to 0}\left(A(\sigma_w)R_{w^{-1}Pw|P}(\sigma_\lambda,\psi_\lambda)\right) \tag{7.4}$$

is an operator on $\mathcal{H}^+_P(\sigma)$ which intertwines $\mathcal{I}_P(\sigma)$. Conjecture 6.1 implies that σ is unitary. Combined with [4, Theorem 2.1 (R_4) and Proposition 5.2], this would imply the unitarity of $R_P(\sigma_w,\psi)$ and the existence of the limit in (7.4). One would like a nice formula for

$$\mathrm{tr}\left(R_P(\sigma_w,\psi)\mathcal{I}_P(\sigma,f)\right) , \qquad f\in C_c^\infty(G(F)) . \tag{7.5}$$

However, it is clear that

$$R_P(\zeta\sigma_w,\psi) = \zeta(M_w)R_P(\sigma_w,\psi) , \qquad \zeta\in\pi_0(M_w^+)^* ,$$

so the trace will depend on the extension σ_w.

Since w belongs to W_ψ, there is a point in the coset

$$(M_w)^{\hat{}} = \hat{M}_w$$

which centralizes the image of ψ. In other words, ψ may also be regarded as a parameter in $\Psi(M_w)$. By Conjecture 6.1(5), the representation σ_w belongs to the packet $\Pi_\psi(M_w)$. Notice, however, that

$$\mathcal{S}_\psi(M_w) = \mathcal{S}_\psi(M)w = \mathcal{S}_\psi^1 w .$$

The conjecture thus associates to the component M_w and the representations σ_w, a character

$$<u, \sigma_w | \rho> = \delta(s, \sigma_w)\rho(s)^{-1} , \qquad u \in \mathcal{S}_\psi^1 w ,$$

where u is the image of a point $s \in S_\psi(M_w)$. Since

$$<u, \zeta\sigma_w | \rho> = \zeta(M_w)^{-1} <u, \sigma_w | \rho> , \qquad \zeta \in \pi_0(M_w^+)^* ,$$

the product of $<u, \sigma_w | \rho>$ with (7.5) will be independent of the extension σ_w of the representation σ. It is for this product that we should seek a formula. We shall describe a candidate.

The splitting $(B^*, T^*, \{x_\alpha\})$ described above provides elements $n_w \in M_w^*(F)$ and $n_G \in G^*(F)$. Combined with the additive character ψ_F, the splitting also determines a nondegenerate character χ on $N_{B^*}(F)$, as in §6. The elements n_w and n_G preserve χ, regarded as a nondegenerate character on $N_{B^*}(F) \cap M^*(F)$ and $N_{B^*}(F)$ respectively. Let σ_χ be a representation in $\Pi_{\phi_\psi}(M_w^*)$ whose associated standard representation $\tilde{\sigma}_\chi$ has a χ-Whitaker model. Then there is a nonzero complex number $c(\sigma_\chi, n_w)$ such that

$$\Lambda(\tilde{\sigma}_\chi(n_w)v) = c(\sigma_\chi, n_w)\Lambda(v) , \tag{7.6}$$

for any Λ in the one dimensional space $V_\chi(\tilde{\sigma}_\chi)$ of χ-Whitaker functionals, and any v in the underlying space of $\tilde{\sigma}_\chi$. Similarly, let π_χ be a representation in $\Pi_{\phi_\psi}(G^*)$ such that $\tilde{\pi}_\chi$ has a χ-Whitaker model. Then there is a nonzero complex number $c(\pi_\chi, n_G)$ such that

$$\Lambda(\tilde{\pi}_\chi(n_G)v) = c(\pi_\chi, n_G)\Lambda(v) , \tag{7.7}$$

for any Λ in $V_\chi(\tilde{\pi}_\chi)$ and any v in the underlying space of $\tilde{\pi}_\chi$.

The work of Shahidi suggests one final ingredient for our conjectural formula. If E is any finite extension of F, let $\lambda(E/F, \psi_F)$ be the complex number defined in [26] to describe the behaviour of the ε-factors under induction. Now, let $A_{B^*} \subset T^*$ be the split component of B^*, regarded as a parabolic subgroup of $(G^*)^0$ over F. Let $\Sigma^r(B^*; M^*)$ be the set of reduced roots of (B^*, A_{B^*}) whose restriction to A_{M^*} is nonzero. Any root β in this set gives rise to a Levi subgroup G_β of $(G^*)^0$ of semisimple rank one. Let $G_{\beta,sc}$ be the simply connected covering of the derived group of G_β. Then there are two possibilities. Either $G_{\beta,sc} = \mathrm{Res}_{F_\beta/F}(SL_2)$, or $G_{\beta,sc} = \mathrm{Res}_{F_\beta/F}(SU(2,1))$, for a finite extension F_β of F. In the first case, set

$$\lambda_\beta(\psi_F) = \lambda(F_\beta/F, \psi_F) .$$

In the second case, set

$$\lambda_\beta(\psi_F) = \lambda(E_\beta/F, \psi_F)^2\lambda(F_\beta/F, \psi_F)^{-1},$$

where E_β is the smallest extension of F_β over which $G_{\beta,sc}$ splits. For any element w in $W(G,A_M)$, set

$$\lambda_w(\psi_F) = \prod_{\{\beta: w_1\beta<0\}} \lambda_\beta(\psi_F), \tag{7.8}$$

where β ranges over the roots in $\Sigma^r(B^*; M^*)$, and w_1 is the representative of w described earlier.

The formula we seek is supposed to depend on an element u in $\mathcal{S}_\psi(M_w) = \mathcal{S}_\psi^1 w$. Recall that the coset $\mathcal{S}_\psi^1 w$ is a subset of $\mathcal{N}_\psi$ and that $\mathcal{N}_\psi$ in turn maps onto $\mathcal{S}_\psi$. Let $\bar{u}$ denote the image of u in $\mathcal{S}_\psi$. We want an expansion for the product of (7.5) and $\langle u, \sigma_w | \rho\rangle$ in terms of the characters $\langle \bar{u}, \pi | \bar{\rho}\rangle$, $\pi \in \Pi_\sigma(G)$. The expansion should be accompanied by a prescription for determining the normalizing function $\bar{\rho}$ for G from the normalizing function ρ for M_w.

We shall first assume that $G = G^*$ is quasi-split. Here we have the theory of Whitaker models, and we can take

$$\rho(s) = \delta(s,\sigma_\chi).$$

The normalizing function for G should then be

$$\bar{\rho}(s) = \delta(s,\pi_\chi).$$

Conjecture 7.1 (Special case). Suppose that $G = G^*$ is quasi-split. Then the expression

$$c(\sigma_\chi, n_w)^{-1}\langle u, \sigma_w | \sigma_\chi\rangle \operatorname{tr}(R_P(\sigma_w, \psi)\mathcal{I}_P(\sigma,f))$$

equals

$$\lambda_w(\psi_F)c(\pi_\chi, n_G)^{-1} \sum_{\pi\in\Pi_\sigma(G)} \langle \bar{u}, \pi | \pi_\chi\rangle f_G(\pi),$$

for any $u \in \mathcal{S}_\psi^1 w$ and any $f \in C_c^\infty(G(F))$. □

The conjectural formula agrees with the results of [36], [37] and [18]. Moreover, the two sides are balanced in their dependence on the various objects, σ_w, σ_χ, π_χ, n_G, ψ_F, the splitting, etc. which are not uniquely defined. Beyond these aesthetic considerations, however, there is a shortage of evidence even in the quasi-split case, and the formula should perhaps be regarded as simply a working hypothesis.

We return to the case that G is arbitrary. Here it is necessary to normalize the ratio of the transfer factors for G and M_w in a way that is compatible with the corresponding ratio for G^*. We shall sketch a variant of an argument of Kottwitz and Shelstad, which was in turn motivated by an idea of Vogan. The argument relies heavily on the techniques of [33], or rather their anticipated extension to nonconnected groups.

Let G^*_{sc} be the simply connected cover of the derived group of $(G^*)^0$, and let M^*_{sc} be the preimage of $\eta(M)$ in G^*_{sc}. We can assume that

$$\eta\sigma(\eta)^{-1} = \mathrm{ad}(u(\sigma)) , \qquad \sigma\in \mathrm{Gal}(\overline{F}/F) ,$$

where $u(\sigma)$ is an element in M^*_{sc}. Suppose that s is a semisimple element in $\hat{M}_w$. Let $(H,\mathcal{H},s,\xi)$ and $(H_w,\mathcal{H}_w,s,\xi_w)$ be compatible (twisted) endoscopic data for G and M_w. These can also serve as endoscopic data for G^* and M^*_w. Suppose that γ_H is a strongly G-regular stable conjugacy class in $H(F)$ which is the image of elements $\gamma\in G(F)$ and $\gamma^*\in G^*(F)$ [33, §(1.3)]. Let h be a point in $G^*_{sc}(\overline{F})$ such that $h\eta(\gamma)h^{-1} = \gamma^*$. Then the elements

$$v(\sigma) = hu(\sigma)\sigma(h)^{-1} , \qquad \sigma\in \mathrm{Gal}(\overline{F}/F) ,$$

belong to

$$T^* = \{t\in G^*_{sc}\colon t^{-1}\gamma^* t = \gamma^*\} ,$$

a group which is connected [44, Theorem 8.1], and hence a torus. Similarly, if γ_{H_w} is a strongly M_w-regular stable conjugacy class in $H_w(F)$ which is the image of elements $\gamma_w\in M_w(F)$ and $\gamma^*_w\in M^*_w(F)$, we can define points

$$v_w(\sigma) = h_w u(\sigma)\sigma(h_w)^{-1} , \qquad \sigma\in \mathrm{Gal}(\overline{F}/F) ,$$

in

$$T^*_w = \{t\in M^*_{sc}\colon t^{-1}\gamma^*_w t = \gamma^*_w\} .$$

The pair

$$(v^{-1}, v_w)\colon \sigma \to (v(\sigma)^{-1}, v_w(\sigma)) , \qquad \sigma\in \mathrm{Gal}(\overline{F}/F) ,$$

defines an element in $H^1(F,U)$, where U is the torus

$$T^*\times T^*_w/\{(z^{-1},z)\colon z\in Z(G^*_{sc})\} .$$

On the other hand, attached to s there is a character $s_U\in \pi_0(\hat{U}^\Gamma)^*$ on the component group of the dual torus. (See [33, p. 246] in the untwisted case.) The Tate-Nakayama pairing then gives a function

$$\lambda_H(\gamma,\gamma^*;\gamma_w,\gamma^*_w) = \langle s_U, (v^{-1},v_w)\rangle .$$

Suppose that the transfer factors $\Delta(\gamma_H,\gamma^*)$, $\Delta(\gamma_{H_w},\gamma^*_w)$ and $\Delta(\gamma_{H_w},\gamma_w)$ for (G^*,H), (M^*_w,H_w) and (M_w,H_w) have all be defined. Set

$$\Delta(\gamma_H,\gamma) = \lambda_H(\gamma,\gamma^*;\gamma_w,\gamma^*_w)\Delta(\gamma_{H_w},\gamma_w)\Delta(\gamma_{H_w},\gamma^*_w)^{-1}\Delta(\gamma_H,\gamma^*) . \tag{7.9}$$

The local hypothesis [33, Lemma 4.2A], or rather its extension to nonconnected groups, presumably implies that $\Delta(\gamma_H,\gamma)$ is the transfer factor for (G,H). Remember that the transfer factors are uniquely determined up to a scalar multiple. The point here is that (7.9) normalizes this scalar in terms of the other three transfer factors.

Now, suppose that $\psi\in\Psi(G)$ is as above. According to the Conjecture 6.1, there is a normalizing function $\rho(s)$ on $S_\psi(M_w)$ such that

$$\langle u, \sigma_w \mid \rho\rangle = \delta(s,\sigma_w)\rho(s)^{-1}, \qquad u\in \mathcal{S}^1_\psi w,$$

is a character in $\mathcal{S}^1_\psi w$. We can expect that

$$\rho_\chi(s) = \rho(s)\delta(s,\sigma_\chi)^{-1}\delta(s,\pi_\chi), \qquad s\in S_\psi(M_w), \tag{7.10}$$

is the restriction to $S_\psi(M_w)$ of a normalizing function on $S_\psi(G)$ for G. In particular, each function

$$\langle \bar{u}, \pi \mid \rho_\chi\rangle = \delta(s,\pi)\rho_\chi(s)^{-1}$$

should be the restriction of a character on $\mathcal{S}_\psi$.

Conjecture 7.1 (General case). Suppose that the transfer factors and normalizing functions for G are given in terms of the corresponding objects for M_w by (7.9) and (7.10). Then the expression

$$c(\sigma_\chi,n_w)^{-1}\langle u,\sigma_w \mid \rho\rangle \mathrm{tr}(R_P(\sigma_w,\psi)\mathcal{I}_P(\sigma,f))$$

equals

$$\lambda_w(\psi_F)c(\pi_\chi,n_G)^{-1} \sum_{\pi\in\Pi_\sigma(G)} \langle \bar{u},\pi \mid \rho_\chi\rangle f_G(\pi)$$

for any $u\in \mathcal{S}^1_\psi w$ and any $f\in C_c^\infty(G(F))$. □

Remarks. 1. We have assumed that the parameter ψ is elliptic for M. This is clearly not necessary. One could make the same conjecture if M is any Levi subgroup of G^0 such that ψ factors through ${}^LM$.

2. If ψ is tempered, which is to say ψ is trivial on $SL(2,\mathbb{C})$, the sets $\Pi_\sigma(G)$, $\sigma\in\Pi_\psi(M)$, are disjoint. We have assumed implicitly in the conjecture that this holds for any ψ. However, there is no particular reason for this to be so. If it fails, it will mean that the character $\langle \bar{u},\pi \mid \rho_\chi\rangle$ is a sum of several characters, corresponding to the representations σ such that π is contained in $\Pi_\sigma(G)$. The conjectured formula would become an identity between the sum over σ of the first expression, and the second expression, but with $\Pi_\sigma(G)$ replaced by the full set $\Pi_\psi(G)$.

§8. Conjectures for automorphic forms.

The local conjectures we have stated were motivated by global considerations. The basic global question of course concerns the multiplicities of representations in spaces of automorphic forms. The global version of the conjectures will give a formula for the multiplicity of an irreducible representation of an adèle group in the discrete spectrum. For tempered representations, the global conjecture is implicit in the paper [24] of Labesse and Langlands. The formula we shall state could be regarded as a procedure for determining the multiplicity of an arbitrary representation in terms of the corresponding multiplicities for tempered representations.

From now on, F will be a number field. We continue to allow G to be an arbitrary connected component of a reductive group over F. Notice that the group $G(\mathbf{A}_F)^+$ generated by $G(\mathbf{A}_F)$ is usually a proper subgroup of $G^+(\mathbf{A}_F)$. We shall write $\Pi(G(\mathbf{A}_F))$ (resp. $\Pi_{unit}(G(\mathbf{A}_F))$) for the set of equivalence classes of representations (resp. unitary representations) of $G(\mathbf{A}_F)^+$ whose restriction to $G^0(\mathbf{A}_F)$ is irreducible. There is a canonical extension of the regular representation of $G^0(\mathbf{A}_F)$ to $G(\mathbf{A}_F)^+$ which is given by

$$(R(y)\phi)(x) = \phi(\xi^{-1}xy), \qquad \phi\in L^2(G^0(F)\backslash G^0(\mathbf{A}_F)),$$

for $x\in G^0(\mathbf{A}_F)$, $y\in G(\mathbf{A}_F)^+$, and for any point $\xi\in G^+(F)$ such that $\xi^{-1}y$ belongs to $G^0(\mathbf{A}_F)$. We are interested in how often a given representation $\pi\in\Pi_{unit}(G(\mathbf{A}_F))$ occurs in R.

In the paper [25], Langlands conjectured that there would be automorphic representations attached to maps $W_F \to {}^LG^0$ of the global Weil group into the L-group. Tempered automorphic representations would correspond to maps with bounded image in $\hat{G}^0$. However, it was clear that unlike the local situation, the set of representations obtained in this way would be rather small. In the later article [29], Langlands pointed out that if the tempered automorphic representations of $GL(n)$ had certain properties, they could be parametrized by the n-dimensional representations of a group which is larger than W_F. This could either take the form of a complex, reductive pro-algebraic group, as was suggested in [29], or a locally compact group L_F proposed in [21, §12]. We shall adopt the latter point of view.

We thus assume the existence of the hypothetical group L_F. It is to be an extension of W_F by a compact group. For each valuation v of F, there should be a homomorphism

$$L_{F_v} \to L_F,$$

where

$$L_{F_v} = \begin{cases} W_{F_v}, & v \text{ archimedean},\\ W_{F_v}\times SU(2,\mathbb{R}), & v \text{ nonarchimedean},\end{cases}$$

as in §6. According to Hypothesis 1.1, the cuspidal automorphic representations of $GL(n,\mathbf{A}_F)$ should all be tempered. These should be in natural bijection with the irreducible n-dimensional representations of L_F. More generally, the cuspidal tempered automorphic representations of

$G^0(\mathbf{A}_F)$ should occur in packets parametrized by elliptic maps of L_F to ${}^LG^0$. (See [21, §12].) Our goal is to try to enlarge this point of view so that it will account for the entire discrete spectrum.

As in the local situation, we must replace L_F by its product with $SL(2,\mathbb{C})$. We shall be interested in admissible maps

$$\psi: L_F \times SL(2,\mathbb{C}) \to {}^LG^0$$

such that the image of L_F in $\hat{G}^0$ is bounded. In this context, admissible shall mean that each of the elements

$$\psi(w), \qquad w \in L_F,$$

in ${}^LG^0$ is semisimple, and also that ψ is globally relevant. Its image is not allowed to lie in a parabolic subgroup of ${}^LG^0$ unless the corresponding parabolic subgroup of G^0 is defined over the global field F. Motivated by [21, §10], we define

$$D_\psi = D_\psi(G)$$

to be the set of s in $\hat{G}$ such that the point

$$s\psi(x)s^{-1}\psi(x)^{-1}$$

belongs to $Z(\hat{G}^0)$, for every $x \in L_F \times SL(2,\mathbb{C})$. This set could of course be empty if $G \neq G^0$. However, if s is an element in D_ψ, the cocycle

$$z_w = s\psi(w)s^{-1}\psi(w)^{-1}, \qquad w \in L_F,$$

defines an element in $H^1(L_F, Z(\hat{G}^0))$. Let $S_\psi = S_\psi(G)$ be the subset of elements $s \in D_\psi$ for which the corresponding class z_w is locally trivial, that is to say, z_w belongs to the kernel of the map

$$H^1(L_F, Z(\hat{G}^0)) \to \prod_v H^1(L_{F_v}, Z(\hat{G}^0)).$$

We can define the group $S_\psi^+ = S_\psi(G^+)$ in a similar fashion, and S_ψ becomes a coset of $S_\psi(G^0)$ in S_ψ^+. We can also define the coset

$$\mathcal{S}_\psi = \mathcal{S}_\psi(G) = S_\psi/S_\psi^0 \cdot Z(\hat{G}^0)$$

of $\mathcal{S}_\psi(G^0)$ in the finite group

$$\mathcal{S}_\psi^+ = \mathcal{S}_\psi(G^+) = S_\psi(G^+)/S_\psi^0 Z(\hat{G}^0).$$

(Notice that, unlike in the local case, we have divided out by the center $Z(\hat{G}^0)$.) We shall say that two maps

$$\psi_i: L_F \times SL(2,\mathbb{C}) \to {}^LG^0, \qquad i = 1,2,$$

are ***equivalent*** if there is an element $g \in \hat{G}^0$ such that

$$g^{-1}\psi_1(w,u)g = \psi_2(w,u)z_w, \qquad (w,u)\in L_F\times SL(2,\mathbb{C}),$$

where z_w is a 1-cocycle of L_F in $Z(\hat{G}^0)$ whose class in $H^1(L_F, Z(\hat{G}^0))$ is locally trivial.

Define

$$\Psi(G) = \Psi(G,F)$$

to be the set of equivalence classes of admissible maps

$$\psi: L_F\times SL(2,\mathbb{C}) \to {}^LG^0$$

such that the image of L_F in $\hat{G}^0$ is bounded, and such that the set S_ψ is nonempty. Since W_F is a quotient of L_F, we can copy other definitions from the local case. In particular, we can define the global parameter sets $\Phi(G)$ and $\Phi_{temp}(G)$, and the map $\psi \to \phi_\psi$ of $\Psi(G)$ into $\Phi(G^*)$. For each $\psi\in\Psi(G)$, we can also define the element $s_\psi\in \boldsymbol{S}_\psi(G^0)$ and the surjective map

$$\boldsymbol{S}_\psi \to \boldsymbol{S}_{\phi_\psi}.$$

Suppose that ψ is a parameter in $\Psi(G)$. Then for every valuation v we have the restricted map ψ_v in $\Psi(G,F_v)$. It follows from the definitions that there is an injection $s \to s_v$ from S_ψ^+ to $S_{\psi_v}^+Z(\hat{G}^0)$. Now we are assuming that Conjecture 6.1 holds. In particular, we have the finite local packets Π_{ψ_v}. We define the global packet $\Pi_\psi = \Pi_\psi(G)$ to be the set of representations in $\Pi(G(\mathbf{A}_F))$ obtained by restricting the representations

$$\{\pi = \bigotimes_v \pi_v : \pi_v\in\Pi_{\psi_v}\}$$

to $G(\mathbf{A}_F)^+$. For almost all v, the packets Π_{ψ_v} will contain unramified representations, and it is understood that these must be the local constituents of π for almost all v. Thus, Π_ψ is a set (usually infinite) of representations in $\Pi(G(\mathbf{A}_F))$, which according to Conjecture 6.2 are all unitary.

Our global conjecture will assert that any irreducible representation in $\Pi(G(\mathbf{A}_F))$ which occurs in $L^2(G^0(F)\backslash G^0(\mathbf{A}_F))$ must belong to one of the packets Π_ψ. It also provides a multiplicity formula, which requires some further description.

The local transfer factors, defined in [33] when $G = G^0$, are determined only up to a scalar multiple. However, the global transfer factors, which are products of the local ones, are canonically defined [33, §6]. More precisely, suppose that $\psi\in\Psi(G)$, and that $H = H_s$ is the endoscopic datum for G/F corresponding to a given point $s\in S_\psi$. Then the map

$$f \to f^H = \prod_v f_v^{H_v}, \qquad f = \prod_v f_v\in C_c^\infty(G(\mathbf{A}_F)),$$

is canonically defined. We shall assume this to be the case for any component G. Suppose that $\pi = \bigotimes_v \pi_v$ is any representation in Π_ψ. The functions $\delta(\cdot, \pi_v)$ on $S_{\psi_v}^+$ will be invariant under

$Z(\hat{G}^0)^{\Gamma_v}$, and since

$$S^+_{\psi_v}/ Z(\hat{G}^0)^{\Gamma_v} \cong S^+_{\psi_v}Z(\hat{G}^0)/ Z(\hat{G}^0) ,$$

$\delta(\cdot, \pi_v)$ can be identified with a $Z(\hat{G}^0)$-invariant function on $S^+_{\psi_v}Z(\hat{G}^0)$. We may therefore define

$$<\bar{s}, \pi> = \prod_v \delta(s_v, \pi_v), \qquad s \in S^+_\psi.$$

Almost all the terms in the product will be 1, and the product itself will be canonically defined. We shall also anticipate that the normalizing functions ρ_v on $S^+_{\psi_v}$, postulated in Conjecture 6.1 (iii), can be extended to $S^+_{\psi_v}Z(\hat{G}^0)$ in such a way that

$$\prod_v \rho_v(s_v) = 1, \qquad s \in S^+_\psi,$$

with almost all the terms in the product being equal to 1, and so that the function

$$<\bar{s}_v, \pi_v | \rho_v> = \delta(s_v,\pi_v)\rho_v(s_v)^{-1} , \qquad s_v \in S^+_{\psi_v}Z(\hat{G}^0)/S^0_{\psi_v} ,$$

remains positive definite. We obtain

(8.1) $$<\bar{s}, \pi> = \prod_v <\bar{s}_v, \pi_v | \rho_v>, \qquad s \in S^+_\psi.$$

The two formulas, together with Conjecture 6.1 (iii), imply that $<\bar{s}, \pi>$ does depend only on the image $\bar{s}$ of s in $\mathcal{S}^+_\psi$, and is a positive definite function on $\mathcal{S}^+_\psi$. It should in fact turn out to be the character of a nonzero finite dimensional representation of $\mathcal{S}^+_\psi$. On the other hand, if

$$f^H(\psi_H) = \prod_v f_v^{H_v}(\psi_{v,H_v}) , \qquad f = \prod_v f_v ,$$

for $H = H_s$, with $s \in S_\psi$, then

(8.2) $$f^H(\psi_H) = \sum_{\pi \in \{\Pi_\psi\}} <\bar{s}_\psi \bar{s}, \pi> f_G(\pi) ,$$

by Conjecture 6.1(ii). As before, $\{\Pi_\psi\}$ denotes the set of orbits of $\pi_0(G^+)^*$ in Π_ψ.

An intriguing aspect of the conjectured multiplicity formula is a connection with global root numbers. Let $\hat{\mathfrak{g}}$ denote the Lie algebra of $\hat{G}^0$. Then for any $\psi \in \Psi(G)$, we can define a finite dimensional representation

$$\tau_\psi : S_\psi(G^+) \times L_F \times SL(2,\mathbb{C}) \to GL(\hat{\mathfrak{g}})$$

by

$$\tau_\psi(s,w,u) = \mathrm{Ad}(s\psi(w,u)) , \qquad (s,w,u) \in S_\psi(G^+) \times L_F \times SL(2,\mathbb{C}) .$$

Decomposing into irreducible constituents, we write

(8.3) $$\tau_\psi = \bigoplus_k \tau_k = \bigoplus_k (\lambda_k \otimes \mu_k \otimes \nu_k) ,$$

where λ_k, μ_k and ν_k are irreducible representations of $S_\psi(G^+)$, L_F and $SL(2,\mathbb{C})$ respectively. Observe that τ_ψ preserves the Killing form on $\hat{\mathfrak{g}}$, so that τ_ψ is equivalent to its own contragredient. It follows that the contragredient $\tau_k \to \tilde{\tau}_k$ gives a permutation on the constituents of τ_ψ. The global L-function $L(s,\mu_k)$ will be defined as a product of local L-functions. We can expect the functional equation

$$L(s,\mu_k) = \varepsilon(s,\mu_k)L(1-s,\tilde{\mu}_k) ,$$

where $\varepsilon(s,\mu_k)$ is a finite product of local root numbers. Suppose that τ_k equals its contragredient $\tilde{\tau}_k$. Then $\mu_k = \tilde{\mu}_k$, and the functional equation implies that

$$\varepsilon(\tfrac{1}{2},\mu_k) = \pm 1 .$$

Under this condition, the image of μ_k must be contained in either the orthogonal group or the symplectic group. If μ_k is orthogonal, it is known [12] that $\varepsilon(\tfrac{1}{2},\mu_k) = 1$, provided that μ_k comes from a representation of the Galois group of F. This should hold for any orthogonal representation of L_F. On the other hand, if μ_k is symplectic, the sign of $\varepsilon(\tfrac{1}{2},\mu_k)$ is known to be quite subtle.

Given ψ, we shall say that a constituent τ_k of (8.3) is ***special*** if $\tau_k = \tilde{\tau}_k$, and if $\varepsilon(\tfrac{1}{2},\mu_k) = -1$. We define

(8.4) $$\varepsilon_\psi(s) = \prod_{\tau_k \text{ special}} \det\lambda_k(s) , \qquad s\in S_\psi(G^+) .$$

It is clear that ε_ψ is a one dimensional sign character of the group $\mathcal{S}_\psi^+$, which factors to a character of the quotient $\mathcal{S}_\psi^+$. Now, suppose that π is a representation in $\Pi_{\text{unit}}(G(\mathbf{A}_F))$. If π belongs to the packet Π_ψ, set

(8.5) $$m_\psi(\pi) = |\mathcal{S}_\psi^+|^{-1} \sum_{x\in \mathcal{S}_\psi^+} \varepsilon_\psi(x)\langle x,\pi\rangle .$$

Since $\langle\cdot,\pi\rangle$ is supposed to be the character of a finite dimensional representation of $\mathcal{S}_\psi^+$, this number is a nonnegative integer. It is just the multiplicity of the sign character ε_ψ in $\langle\cdot,\pi\rangle$. If π does not belong to Π_ψ, we shall simply set $m_\psi(\pi) = 0$.

In considering whether π occurs discretely in R, we are faced with the minor irritation of the split component of the center of G^+. However, the definitions of §1 are easily extended to the case that $G \neq G^0$. For example, we can write

$$G(\mathbf{A}_F)^1 = \{x\in G(\mathbf{A}_F): |\chi(x)| = 1,\ \chi\in X(G^+)_F\} .$$

Let $(G(\mathbf{A}_F)^1)^+$ be the group generated by $G(\mathbf{A}_F)^1$, and set

$$G^0(\mathbf{A}_F)^1 = G^0(\mathbf{A}_F) \cap (G(\mathbf{A}_F)^1)^+ .$$

Then for any $\pi\in\Pi_{\text{unit}}(G(\mathbf{A}_F))$, we shall write $m_0(\pi)$ for the multiplicity with which the restriction of π to $(G(\mathbf{A}_F)^1)^+$ occurs as a direct summand of $L^2(G^0(F)\backslash G^0(\mathbf{A}_F)^1)$. We can also define

$$\Pi_0(G) = \{\pi \in \Pi_{unit}(G(\mathbf{A}_F)) : m_0(\pi) \neq 0\} .$$

In addition, we shall write R_0 for the subrepresentation of R whose restriction to $(G(\mathbf{A}_F)^1)^+$ decomposes discretely. Finally, let $\Psi_0(G)$ be the subset of parameters $\psi \in \Psi(G)$ such that S_ψ^0 is contained in $Z(\hat{G}^0)$.

Conjecture 8.1. The formula

$$m_0(\pi) = \sum_{\psi \in \Psi_0(G)} m_\psi(\pi)$$

holds for any $\pi \in \Pi_{unit}(G(\mathbf{A}_F))$. □

Remarks. 1. The conjecture implies that any irreducible constituent of R_0 belongs to a packet Π_ψ, $\psi \in \Psi_0(G)$. Actually these packets should usually be disjoint, with the multiplicity formula reducing simply to

$$m_0(\pi) = m_\psi(\pi) , \qquad \pi \in \Pi_\psi .$$

2. Even though R has a continuous spectrum it should be possible to define the multiplicity $m(\pi)$ of any π in R. One would first need to define the Schwartz space on $G^0(F)\backslash G^0(\mathbf{A}_F)$. The group $G(\mathbf{A}_F)^+$ will act on this space, and also on the corresponding space of tempered distributions. One could then define $m(\pi)$ as the multiplicity of π in the space of tempered distributions on $G^0(F)\backslash G^0(\mathbf{A}_F)$. This incidentally would lead to a formal definition

$$\Pi(G) = \{\pi \in \Pi_{unit}(G(\mathbf{A}_F)) : m(\pi) \neq 0\}$$

for the set mentioned in §1. Conjecture 8.1 could then be generalized to a multiplicity formula

$$m(\pi) = \sum_{\psi \in \Psi(G)} m_\psi(\pi) , \qquad \pi \in \Pi_{unit}(G(\mathbf{A}_F)) . \tag{8.6}$$

Conjecture 8.1 agrees with the conjectural multiplicity formula for tempered parameters stated in [21, §12]. This was based on the original multiplicity formulas in [24] for SL(2) and related groups. However, at the moment there is not a great deal of direct evidence to support the conjecture. In [2] we discussed some examples for the group PSp(4), due to Piatetski - Shapiro and Waldspurger, that were compatible with the conjecture. The largest group for which there are complete results is now U(3). Rogawski's multiplicity formulas [34] for the discrete spectrum of this group are also compatible with the conjecture.

Suppose that G is the split group of type G_2. By examining the residues of Eisenstein series, Langlands discovered an interesting automorphic representation which occurs in the discrete spectrum [28, Appendix 3]. Our description of this example in [2] was incorrect. It is true that there are three equivalence classes of elliptic endoscopic groups

$$\hat{H}_i \subset \hat{G} , \qquad i = 1,2,3,$$

with

$$\hat{H}_1 = \hat{G}_1 ,$$
$$\hat{H}_2 \cong SL(2,\mathbb{C}) \times SL(2,\mathbb{C})/\{\pm 1\} ,$$

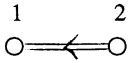
and

$$\hat{H}_3 \cong SL(3,\mathbb{C}) .$$

In each case, the principal unipotent element in $\hat{H}_i$ gives rise to a parameter

$$\psi_i : SL(2,\mathbb{C}) \rightarrow \hat{H}_i \rightarrow \hat{G}$$

in $\Psi(G)$ which is trivial on L_F. However, the principal unipotent element in $\hat{H}_2$ lies in a proper Levi subgroup of $\hat{G}$. The parameter ψ_2 factors through this subgroup, and consequently does not belong to $\Psi_0(G)$. It has nothing to do with the discrete spectrum of G. The parameters ψ_1 and ψ_3 do lie in $\Psi_0(G)$. The first one is attached to the principal unipotent, and gives the trivial one dimensional representation of $G(\mathbf{A}_F)$. The other one is attached to the unipotent class with diagram



.

The Langlands' representation should belong to the packet Π_{ψ_3}. It is in fact the unique element in $\Pi_{\phi_{\psi_3}}$.

The notions of semisimple and unipotent in the context of automorphic forms will by now be clear. Let π be a representation in $\Pi_{unit}(G(\mathbf{A}_F))$. We shall say that π is a ***semisimple automorphic representation*** if $m_\psi(\pi) \neq 0$ for some parameter $\psi \in \Psi(G)$ which is trivial on $SL(2,\mathbb{C})$. We shall say that π is a ***unipotent automorphic representation*** if $G = G^0$, and if there is a parameter $\psi \in \Psi(G)$, with $m_\psi(\pi) \neq 0$, such that the projection of $\psi(L_F)$ onto $\hat{G} = \hat{G}^0$ equals $\{1\}$. Let us also say that an automorphic representation is ***elliptic*** if it belongs to the set $\Pi_0(G)$ defined above. The trivial representation of $G(\mathbf{A}_F)$ is an elliptic unipotent automorphic representation. It seems that the only other elliptic unipotent representation which is known to exist is the Langlands' representation for G_2.

Recall that a representation $\pi \in \Pi(G(\mathbf{A}_F))$ gives a family $\sigma(\pi) = \{\sigma_v(\pi) : v \in S\}$ of semisimple conjugacy classes in ${}^LG^0$. The families associated to two representations in the same packet Π_ψ are equal at almost all v. We therefore obtain surjective maps

$$\Pi(G) \rightarrow \Psi(G) \rightarrow \Sigma(G) .$$

For many G, the second map will actually be a bijection. This is nice, because it would give an elementary interpretation of the parameters $\Psi(G)$. They would describe the generalization from $GL(n)$ to G of strong multiplicity one.

§9. L^2-cohomology of Shimura varieties.

We shall conclude with some remarks on the relation of the parameters ψ to the cohomology of Shimura varieties. Suppose that $G = G^0$ and $F = \mathbb{Q}$. We shall write $\mathbf{A} = \mathbf{A}_{\mathbb{Q}}$. Let R be the real reductive group obtained from $GL(1)$ by restricting scalars from $\mathbb{C}$ to $\mathbb{R}$. Then $R(\mathbb{R}) \cong \mathbb{C}^*$ and $R(\mathbb{C}) \cong \mathbb{C}^* \times \mathbb{C}^*$. A Shimura variety is associated to a $G(\mathbb{R})$-orbit X of maps $h: R \to G$ which are defined over $\mathbb{R}$ and which satisfy some further conditions [29]. For example, any $h \in X$ provides a decomposition

$$\mathfrak{g} = \mathfrak{p}_h^+ \oplus \mathfrak{k}_h \oplus \mathfrak{p}_h^-$$

of the complex Lie algebra of $G(\mathbb{C})$, in which $\mathfrak{p}_h^+$ and $\mathfrak{k}_h$ and $\mathfrak{p}_h^-$ are the subspaces of $\mathfrak{g}$ which transform under

$$\mathrm{Ad}(h(z_1,z_2)), \qquad z_1,z_2 \in \mathbb{C}^*,$$

according to the characters $z_1^{-1}z_2$, 1 and $z_1z_2^{-1}$. Notice that $\mathfrak{k}_h$ is the complex Lie algebra of the stabilizer K_h of h in $G(\mathbb{R})$, and that X can be identified with $G(\mathbb{R})/K_h$.

The space X has a natural complex structure. The complex points on the Shimura variety are of the form

$$S_K(\mathbb{C}) = G(\mathbb{Q}) \backslash X\, G(\mathbf{A}_{\mathrm{fin}})/K,$$

where K is any open compact subgroup of the group $G(\mathbf{A}_{\mathrm{fin}})$ of finite adèlic points. We take K to be sufficiently small that S_K is nonsingular. Suppose that (τ, V_τ) is an irreducible finite dimensional representation of G which is defined over $\mathbb{Q}$. Then

$$\mathcal{F}_\tau(\mathbb{C}) = V_\tau(\mathbb{C}) \underset{G(\mathbb{Q})}{\times} (X\, G(\mathbf{A}_{\mathrm{fin}})/K)$$

is a locally constant sheaf on $S_K(\mathbb{C})$. One is interested in the L^2-cohomology

$$H^*_{(2)}(S_K(\mathbb{C}), \mathcal{F}_\tau(\mathbb{C})) = \bigoplus_k H^k_{(2)}(S_K(\mathbb{C}), \mathcal{F}_\tau(\mathbb{C}))$$

with coefficients in $\mathcal{F}_\tau(\mathbb{C})$.

For any $h \in X$, the L^2-cohomology has a decomposition in terms of the $(\mathfrak{g}, K_h)$-cohomology of the spectral decomposition of $L^2(G(\mathbb{Q})\backslash G(\mathbf{A}))$. Assume Conjecture 8.1. Then the number

$$\sum_{\psi \in \Psi_0(G)} m_\psi(\pi), \qquad \pi \in \Pi_{\mathrm{unit}}(G(\mathbf{A})),$$

which is given by (8.5), equals the multiplicity with which π occurs discretely in the space of functions on $G(\mathbb{Q})\backslash G(\mathbf{A})$ with the appropriate central character. The spectral decomposition is

$$\begin{aligned} (9.1) \qquad & H^*_{(2)}(S_K(\mathbb{C}), \mathcal{F}_\tau(\mathbb{C})) \\ & = \bigoplus_{\psi \in \Psi_0(G)} \bigoplus_{\pi \in \Pi_\psi} m_\psi(\pi) H^*(\mathfrak{g}, K_h; \pi_{\mathbb{R}} \otimes \tau) \otimes \pi_{\mathrm{fin}}^K, \end{aligned}$$

where $\pi_{\mathbb{R}}$ and π_{fin} stand for the components of π at $\mathbb{R}$ and the finite adèles, and π_{fin}^{K} is the finite dimensional space of K-invariant vectors for π_{fin}. When $G(\mathbb{Q})\backslash G(\mathbf{A})$ is compact modulo the center, this decomposition is given in [10, Chapter VII]. For general G, it is contained in the results of [9]. Observe that the Hecke algebra

$$\boldsymbol{H}_K = C_c(K\backslash G(\mathbf{A}_{\text{fin}})/K)$$

operates on the L^2-cohomology through the space π_{fin}^{K}.

It will be convenient to fix an element $h_1 \in \boldsymbol{X}$. First of all, fix (T,B) and $(\hat{T},\hat{B})$ as in §5. Then choose the element $h_1 \in \boldsymbol{X}$ so that its image lies in T and so that the parabolic subalgebra $\boldsymbol{k}_{h_1} + \boldsymbol{p}_{h_1}^{+}$ of $\boldsymbol{g}$ is standard relative to B. We shall write $\boldsymbol{k}_1 = \boldsymbol{k}_{h_1}$, $\boldsymbol{p}_1^{\pm} = \boldsymbol{p}_{h_1}^{\pm}$ and $K_1 = K_{h_1}$. We shall also adopt the notation of §5, with $K'_{\mathbb{R}}$ the normalizer of K_1 in $G(\mathbb{R})$. The restriction of h_1 to the first factor in $\boldsymbol{R}(\mathbb{C}) \cong \mathbb{C}^* \times \mathbb{C}^*$ defines a co-weight in $X_*(T)$. Let $\mu_1 \in X^*(\hat{T})$ be the corresponding dual weight. It is a fundamental, minuscule weight for $\hat{G}$ which is anti-dominant relative to $\hat{B}$. One checks that

$$\lambda(h_1(z_1,z_2)) = z_1^{<\lambda,\mu_1>} z_2^{<\sigma_T\lambda,\mu_1>}, \qquad \lambda \in X^*(T) . \tag{9.2}$$

Having fixed h_1, one defines a finite dimensional vector space

$$V_\psi = \bigoplus_{\pi_{\mathbb{R}} \in \Pi_{\psi_{\mathbb{R}}}} H^*(\boldsymbol{g}, K_1; \pi_{\mathbb{R}} \otimes \tau)$$

for each $\psi \in \Psi_0(G)$. This space, which depends only on the image $\psi_{\mathbb{R}}$ of ψ in $\Psi(G,\mathbb{R})$, is convenient for working with the decomposition (9.1). If the space is nonzero, $\psi_{\mathbb{R}}$ is one of the parameters discussed in §5, and the group $S_{\psi_{\mathbb{R}}}$ is abelian. We shall define a representation ρ_ψ of $S_{\psi_{\mathbb{R}}}$ on V_ψ. Let $Q = LN_Q \supset B$ be the standard parabolic subgroup associated as in §5 to $\psi_{\mathbb{R}}$, so that $\pi_1 = A_Q(\lambda_\tau)$ is the representation in $\Pi_{\psi_{\mathbb{R}}}$ which served as a base point in §5. Then for any representation $\pi_{\mathbb{R}} \in \Pi_{\psi_{\mathbb{R}}}$, we have a one dimensional character

$$\rho_{\pi_{\mathbb{R}}}(s) = <\bar{s}, \pi_{\mathbb{R}} | \pi_1> \mu_1(s) , \qquad s \in S_{\psi_{\mathbb{R}}} ,$$

on $S_{\psi_{\mathbb{R}}}$. The representation ρ_ψ of $S_{\psi_{\mathbb{R}}}$ on V_ψ is given by

$$\rho_\psi(s) = \bigoplus_{\pi_{\mathbb{R}} \in \Pi_{\psi_{\mathbb{R}}}} \rho_{\pi_{\mathbb{R}}}(s) , \qquad s \in S_{\psi_{\mathbb{R}}} .$$

Recall that if $\pi = \pi_{\mathbb{R}} \otimes \pi_{\text{fin}}$ is any representation in the packet Π_ψ, it is assumed that $<x,\pi>$ is a canonical finite dimensional character on $\boldsymbol{S}_\psi$. That is,

$$<x,\pi> = \text{tr}(r_\pi(x)) , \qquad x \in \boldsymbol{S}_\psi ,$$

where r_π is a representation of $\boldsymbol{S}_\psi$ on a finite dimensional complex vector space U_π. In this case, $\boldsymbol{S}_{\psi_{\mathbb{R}}}$ is abelian, so that U_π really depends only on π_{fin}. In fact, we also have the finite dimensional representation

$$r_{\pi_{\mathrm{fin}}}(s) = \rho_{\pi_{\mathbb{R}}}(s_{\mathbb{R}})^{-1} r_{\pi}(\overline{s}) , \qquad s \in S_{\psi} ,$$

of S_{ψ} on U_{π}. Here, $s_{\mathbb{R}}$ and $\overline{s}$ stand for the images of s in $S_{\psi_{\mathbb{R}}} Z(\hat{G}^0)$ and $\boldsymbol{S}_{\psi}$. Set

$$U_{\psi}^{K} = \bigoplus_{\pi_{\mathrm{fin}}} (\pi_{\mathrm{fin}}^{K} \otimes U_{\pi}) ,$$

where π_{fin} ranges over the finite components of representations in Π_{ψ}. This is a finite dimensional space, equipped with actions of both $\boldsymbol{H}_K$ and S_{ψ}. There is a tensor product action of the group S_{ψ} on the finite dimensional space $V_{\psi} \otimes U_{\psi}^{K}$ which obviously factors to a representation of the quotient group $\boldsymbol{S}_{\psi}$. Recall the formula (8.5) for the conjectured multiplicity. It allows us to rewrite the spectral decomposition of cohomology as

$$H_{(2)}^{*}(S_K(\mathbb{C}), \boldsymbol{F}_{\tau}(\mathbb{C})) = \bigoplus_{\psi \in \Psi_0(G)} (V_{\psi} \otimes U_{\psi}^{K})_{\varepsilon_{\psi}} , \tag{9.3}$$

where $(\ )_{\varepsilon_{\psi}}$ denotes the subspace of vectors which transform under $\boldsymbol{S}_{\psi}$ by the character ε_{ψ}.

The space V_{ψ} has some further structure. The Shimura variety is defined over a certain number field $E = E(G,\boldsymbol{X})$ which comes with an embedding into $\mathbb{C}$. Let E_v be the completion of E with respect to the associated Archimedean valuation. Then E_v equals $\mathbb{R}$ or $\mathbb{C}$, and we can form the Weil group $W_{E_v} = W_{\mathbb{C}/E_v}$. It turns out that ρ_{ψ} extends to a representation of

$$S_{\psi_{\mathbb{R}}} \times W_{E_v} \times SL(2,\mathbb{C})$$

on V_{ψ}.

The representation of $SL(2,\mathbb{C})$ comes from Lefschetz theory, and in particular, the cup product with the Kähler form. Recall [10] that $H^{*}(\mathfrak{g}, K_1; \pi_{\mathbb{R}} \otimes \tau)$ vanishes unless the Casimir operator acts by zero on $\pi_{\mathbb{R}} \otimes \tau$. In the latter case

$$\begin{aligned} H^{*}(\mathfrak{g}, K_1; \pi_{\mathbb{R}} \otimes \tau) &= \mathrm{Hom}_{K_1}(\Lambda^{*}(\mathfrak{g}/\mathfrak{k}_1), \pi_{\mathbb{R}} \otimes \tau) \\ &= \mathrm{Hom}_{K_1}(\Lambda^{*}(\mathfrak{g}/\mathfrak{k}_1) \otimes \tilde{\tau}, \pi_{\mathbb{R}}) \\ &= \mathrm{Hom}_{K_1}(\Lambda^{*}\mathfrak{p}_1^{+} \otimes \Lambda^{*}\mathfrak{p}_1^{-} \otimes \tilde{\tau}, \pi_{\mathbb{R}}) \\ &= \bigoplus_{p,q} \mathrm{Hom}_{K_1}(\Lambda^{p}\mathfrak{p}_1^{+} \otimes \Lambda^{q}\mathfrak{p}_1^{-} \otimes \tilde{\tau}, \pi_{\mathbb{R}}) , \end{aligned}$$

where Λ^{*} denotes the exterior algebra, and $\tilde{\tau}$ is the contragredient of τ. The last formula gives a decomposition of the $(\mathfrak{g}, K_1)$ cohomology, from which one gets the Hodge decomposition of the L^2-cohomology of $S_K(\mathbb{C})$. The Killing form

$$(X_1^{+}, X_1^{-}) \rightarrow \mathrm{tr}(\mathrm{ad}\, X_1^{+} \cdot \mathrm{ad}\, X_1^{-}) , \qquad X_1^{\pm} \in \mathfrak{p}_1^{\pm} ,$$

is a nondegenerate, K_1-invariant pairing on $\mathfrak{p}_1^{+} \times \mathfrak{p}_1^{-}$. It can be regarded as an element in $\mathrm{Hom}_{K_1}(\mathfrak{p}_1^{+} \otimes \mathfrak{p}_1^{-}, \mathbb{C})$. The wedge product with this element defines an endomorphism X of $H^{*}(\mathfrak{g}, K_1; \pi_{\mathbb{R}} \otimes \tau)$ which maps the (p,q) component into the $(p+1, q+1)$ component. It is implicit in the results of [52] that for any $i \leq n = \dim_{\mathbb{C}}(S_K)$, the map

$$X^{n-i} : H^i(\mathfrak{g}, K_1; \pi_{\mathbb{R}} \otimes \tau) \rightarrow H^{2n-i}(\mathfrak{g}, K_1; \pi_{\mathbb{R}} \otimes \tau)$$

is an isomorphism. The representation theory of SL(2) then allows us to define an endomorphism Y of $H^*(\mathfrak{g}, K_1; \pi_{\mathbb{R}} \otimes \tau)$, which maps the (p,q) component into the (p−1, q−1) component, such that $H = XY - YX$ acts on $H^k(\mathfrak{g}, K_1; \pi_{\mathbb{R}} \otimes \tau)$ by multiplication by $\frac{1}{2}(k-n)$. The endomorphisms X,Y and H span the Lie algebra of SL(2), which therefore acts on

$$V_\psi = \bigoplus_{\pi_{\mathbb{R}}} H^*(\mathfrak{g}, K_1; \pi_{\mathbb{R}} \otimes \tau) .$$

Exponentiating to the group, we obtain a representation of $SL(2,\mathbb{C})$ on V_ψ.

The representation of W_{E_v} is the one defined by Langlands [29, p. 239] from Hodge theory, but modified to have (essentially) bounded image. If $z \in \mathbb{C}^*$, let $\eta'(z)$ be the operator on

$$\Lambda^*(\mathfrak{g}/\mathfrak{k}_1) = \bigoplus_{p,q} (\Lambda^p \mathfrak{p}_1^+ \otimes \Lambda^q \mathfrak{p}_1^-)$$

which multiplies a vector in $\Lambda^p \mathfrak{p}_1^+ \otimes \Lambda^q \mathfrak{p}_1^-$ by

$$(z/\bar{z})^{-p/2}(z/\bar{z})^{+q/2} .$$

We have noted that any element in $H^*(\mathfrak{g}, \mathfrak{k}_1; \pi_{\mathbb{R}} \otimes \tau)$ can be represented by a K_1-equivariant linear map

$$\phi : \Lambda^*(\mathfrak{g}/\mathfrak{k}_1) \otimes \tilde{V}_\tau \rightarrow V_{\pi_{\mathbb{R}}} ,$$

$\tilde{V}_\tau$ and $V_{\pi_{\mathbb{R}}}$ being the spaces on which $\tilde{\tau}$ and $\pi_{\mathbb{R}}$ act. Define

$$(\rho_\psi(z)\phi)(U \otimes \tilde{v}) = \phi(\eta'(z)U \otimes \tilde{\tau}(h_1(z,\bar{z}))^{-1}\tilde{v}) ,$$

for $U \in \Lambda^*(\mathfrak{g}/\mathfrak{k}_1)$ and $\tilde{v} \in \tilde{V}_\tau$. Since the image of h_1 lies in the center of K_1, the linear map $\rho_\psi(z)\phi$ is also K_1-equivariant. Therefore, ρ_ψ gives a representation of $\mathbb{C}^*$ on $H^*(\mathfrak{g}, \mathfrak{k}_1; \pi_{\mathbb{R}} \otimes \tau)$ which commutes with the action of $SL(2,\mathbb{C})$. This takes care of the full Weil group W_{E_v} if E is not contained in $\mathbb{R}$. If E is contained in $\mathbb{R}$, choose an element $(1 \times \sigma)$ in W_{E_v} as in §5, and set

$$(\rho_\psi(1 \times \sigma)\phi)(U \otimes \tilde{v}) = \pi_{\mathbb{R}}(n_1)\phi(\mathrm{Ad}(n_1^{-1})U \otimes \tilde{\tau}(n_1^{-1})\tilde{v})$$

as in [29]. Here ϕ, U and $\tilde{v}$ are as above, and n_1 is an element in $G(\mathbb{R})$ such that

$$n_1 h_1(z,\bar{z}) n_1^{-1} = h_1(\bar{z},z) , \qquad z \in \mathbb{C}^* .$$

We thus obtain a representation of W_{E_v} on V_ψ which commutes with action of $SL(2,\mathbb{C})$. Both of these actions obviously commute with that of $S_{\psi_{\mathbb{R}}}$, so ρ_ψ does indeed extend to a representation of $S_{\psi_{\mathbb{R}}} \times W_{E_v} \times SL(2,\mathbb{C})$ on V_ψ.

There is another canonical representation of this group. Let (r^0, V_{r^0}) be the irreducible representation of $\hat{G}$ with extremal weight equal to the element $\mu_1 \in X^*(\hat{T})$ defined above. The Shimura field E is the fixed field of the group of elements in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, acting on $\hat{G}$, which fixes μ_1. There is a unique extension of the representation r^0 to the group ${}^LG_E = \hat{G} \rtimes W_E$ such that W_E acts trivially on the weight space of μ_1. Now, ${}^LG_E$ is a subgroup of finite index in ${}^LG$, and the restriction ψ_v of $\psi_{\mathbb{R}}$ to $W_{E_v} \subset W_{\mathbb{R}}$ takes values in ${}^LG_E$. The groups $\psi_{\mathbb{R}}(SL(2,\mathbb{C}))$ and $S_{\psi_{\mathbb{R}}}$ are contained in $\hat{G}$, so we obtain a representation

$$\sigma_\psi\colon (s,w,u) \to r^0(s\psi_{\mathbb{R}}(w,u)), \qquad (s,w,u) \in S_{\psi_{\mathbb{R}}} \times W_{E_v} \times SL(2,\mathbb{C}),$$

of $S_{\psi_{\mathbb{R}}} \times W_{E_v} \times SL(2,\mathbb{C})$ on V_{r^0}.

The lemma on p. 240 of [29] suggests that the representations ρ_ψ and σ_ψ are equivalent. This could be regarded as a reciprocity law for Shimura varieties at the Archimedean place. It is of course much easier than the expected reciprocity laws at the finite places, which involve étale cohomology. We shall verify it with W_{E_v} replaced by the subgroup $\mathbb{C}^*$ (of index at most 2).

Proposition 9.1. The representations ρ_ψ and σ_ψ of $S_\psi \times \mathbb{C}^* \times SL(2,\mathbb{C})$ are equivalent.

Proof. This will be a straightforward comparison of the definitions in §5 with the results of [52]. Vogan and Zuckerman work with connected groups, but it is easy to adapt their results to $G(\mathbb{R})$.

We fixed the point $h_1 \in \boldsymbol{X}$ so that the parabolic subalgebra $\boldsymbol{k}_{h_1} + \boldsymbol{p}^+_{h_1} = \boldsymbol{k}_1 + \boldsymbol{p}^+_1$ is standard relative to B. We also chose the parabolic subgroup $Q = LN_Q$ to be standard. Recall that there is a bijection $w \to \pi_w$ between the double cosets

$$\Sigma = W(L,T)\backslash W(G,T)/W_{\mathbb{R}}(G,T)$$

and the packet $\Pi_{\psi_{\mathbb{R}}}$. Now, the group $K_1 = K_{h_1}$ need not meet every connected component of $G(\mathbb{R})$, and its Weyl group $W(K_1,T)$ is only a subgroup of $W_{\mathbb{R}}(G,T)$. There is a bijection $w \to \pi'_w$ between the double cosets

$$\Sigma' = W(L,T)\backslash W(G,T)/W(K_1,T)$$

and the set of irreducible representations of $G(\mathbb{R})' = G(\mathbb{R})^0 K_1$ which are constituents of restrictions to $G(\mathbb{R})'$ of the elements in $\Pi_{\psi_{\mathbb{R}}}$. Then

$$\begin{aligned} V_\psi &= \bigoplus_{w\in\Sigma} H^*(\boldsymbol{g}, K_1; \pi_w \otimes \tau) \\ &= \bigoplus_{w\in\Sigma} \mathrm{Hom}_{K_1}(\Lambda^*(\boldsymbol{g}/\boldsymbol{k}_1) \otimes \check{\tau}, \pi_w) \\ &= \bigoplus_{w\in\Sigma'} \mathrm{Hom}_{K_1}(\Lambda^*(\boldsymbol{g}/\boldsymbol{k}_1) \otimes \check{\tau}, \pi'_w). \end{aligned}$$

We shall represent the double cosets Σ' by elements $w \in W(G,T)$ of smallest length. For any such w, set $K_1^w = w^{-1}K_1 w$, and represent the cosets $W(L,T)/W(L\cap K_1^w,T)$ by elements in $W(L,T)$ of minimal length. Then any element in $W(G,T)$ can be written uniquely as rwt, with $w \in \Sigma'$, $t \in W(K_1,T)$ and $r \in W(L,T)/W(L\cap K_1^w,T)$. Observe that

$$rw, \qquad r \in W(L,T)/W(L\cap K_1^w,T),\ w \in \Sigma',$$

is a set of representatives of $W(G,T)/W(K_1,T)$.

It follows from [52, Proposition 6.19] that for each $w \in \Sigma'$, the space

$$\mathrm{Hom}_{K_1}(\Lambda^*(\boldsymbol{g}/\boldsymbol{k}_1)\otimes\tilde{\tau}, \pi'_w) = \bigoplus_{p,q} \mathrm{Hom}_{K_1}(\Lambda^p \boldsymbol{p}_1^+ \otimes \Lambda^q \boldsymbol{p}_1^- \otimes \tilde{\tau}, \pi'_w) \tag{9.4}$$

has a basis

$$\{\phi_{rw}: r \in W(L,T)/W(L\cap K_1^w,T)\}$$

parametrized by the cosets in $W(G,T)/W(K_1,T)$ which lie in the double coset of w. Moreover, if $\boldsymbol{n}_w$ is the complex Lie algebra of $w^{-1}N_Q w$, an element ϕ_{rw} lies in the summand on the right of (9.4) for which

$$p = \boldsymbol{l}(r) + \dim_{\mathbb{C}}(\boldsymbol{n}_w \cap \boldsymbol{p}_1^+)$$

and

$$q = \boldsymbol{l}(r) + \dim_{\mathbb{C}}(\boldsymbol{n}_w \cap \boldsymbol{p}_1^-) .$$

Finally, the K_1-type in τ associated to any element in (9.4) is generated by an extremal vector in V_τ with weight $w^{-1}\lambda_\tau$. Combining these facts with the formula (9.2), we see that

$$\begin{aligned}\rho_\psi(z)\phi_{rw} &= (z/\bar{z})^{-p/2}(z/\bar{z})^{q/2}(w^{-1}\lambda_\tau)(h_1(z,\bar{z}))\phi_{rw} \\ &= (z/\bar{z})^{-\frac{1}{2}(p-q)} z^{<w^{-1}\lambda_\tau,\mu_1>}\bar{z}^{<\sigma_T w^{-1}\lambda_\tau,\mu_1>}\phi_{rw}, \quad z \in \mathbb{C} .\end{aligned}$$

Consider the number

$$-\frac{1}{2}(p-q) = \frac{1}{2}(-\dim_{\mathbb{C}}(\boldsymbol{n}_w \cap \boldsymbol{p}_1^+) + \dim_{\mathbb{C}}(\boldsymbol{n}_w \cap \boldsymbol{p}_1^-)) .$$

Observe that if α is any root of (G,T), $<\alpha,\mu_1>$ equals -1, 0, or 1, according to whether the root vector of α lies in $\boldsymbol{p}_1^+$, $\boldsymbol{k}_1$ or $\boldsymbol{p}_1^-$. Therefore,

$$-\frac{1}{2}(p-q) = <w^{-1}\delta_Q,\mu_1> = <\delta_Q, w\mu_1> ,$$

since $2\delta_Q$ is just the sum of those roots whose root vectors lie in $\boldsymbol{n}_Q$. Notice also that

$$<\sigma_T w^{-1}\lambda_\tau,\mu_1> = <w^{-1}\sigma_T\lambda_\tau,\mu_1> = <\sigma_T\lambda_\tau, w\mu_1> .$$

It follows that

$$\rho_\psi(z)\phi_{rw} = z^{<\delta_Q+\lambda_\tau,w\mu_1>}\bar{z}^{<-\delta_Q+\sigma_T\lambda_\tau,w\mu_1>}\phi_{rw} . \tag{9.5}$$

On the other hand, r^0 is an irreducible representation whose extremal weight μ_1 is minuscule. It is well known that the weights of any such representation form one Weyl orbit. Since $W(K_1,T)$ is the stabilizer of μ_1 in $W(G,T)$, we can choose a basis of V_τ consisting of weight vectors

$$v_{rw}, \qquad w \in \Sigma',\ r \in W(L,T)/W(L\cap K_1^w,T) ,$$

such that

$$r^0(t)v_{rw} = (rw\mu_1)(t), \qquad t\in\hat{T}.$$

Suppose that $z\in\mathbb{C}^*$. Then

$$\begin{aligned}\sigma_\psi(z)v_{rw} &= r^0(\psi(z))v_{rw}\\ &= (rw\mu_1)(\psi(z))v_{rw}\\ &= z^{<\delta_Q+\lambda_\tau,rw\mu_1>}\bar{z}^{<-\delta_Q+\sigma_T\lambda_\tau,rw\mu_1>}v_{rw},\end{aligned}$$

by (5.4). The properties of σ_T, δ_Q and λ_τ allow us to remove r from the exponent. We obtain

$$\sigma_\psi(z)v_{rw} = z^{<\delta_Q+\lambda_\tau,w\mu_1>}\bar{z}^{<-\delta_Q+\sigma_T\lambda_\tau,w\mu_1>}v_{rw}. \tag{9.6}$$

We tentatively define an isomorphism of V_ψ with V_{r^0} by extending the bijection $\phi_{rw}\longleftrightarrow v_{rw}$ between basis vectors. Formulas (9.5) and (9.6) show that the isomorphism commutes with the action of $\mathbb{C}^*$.

The next step is to show that the isomorphism commutes with the action of $S_{\psi_\mathbb{R}}$. The representation π_1 mentioned above corresponds to $w = 1$. It follows from (5.7) that

$$\rho_\psi(s)\phi_{rw} = <\bar{s},\pi_w|\pi_1>\mu_1(s)\phi_{rw} = <\bar{s},t(w)>\mu_1(s)\phi_{rw},$$

for any basis vector ϕ_{rw} and any $s\in S_{\psi_\mathbb{R}}$. On the other hand,

$$\sigma_\psi(s)v_{rw} = r^0(s)v_{rw} = (rw\mu_1)(s)v_{rw} = (w\mu_1)(s)v_{rw},$$

since $S_{\psi_\mathbb{R}}$ is contained in the torus $\hat{T}$. It is therefore sufficient to show that

$$w\mu_1 - \mu_1 = t(w).$$

This follows easily by induction on the length of w, together with the properties (5.6) and (5.7) of t.

We must finally show that the isomorphism commutes with the action of $SL(2,\mathbb{C})$. First of all, note that there are decompositions

$$V_\psi = \bigoplus_{w\in\Sigma'} V_{\psi,w}$$

and

$$V_{r^0} = \bigoplus_{w\in\Sigma'} V_{r^0,w},$$

where

$$V_{\psi,w} = \{\sum_r c_r\phi_{rw}: c_r\in\mathbb{C}\} = \mathrm{Hom}_{K_1}(\Lambda^*(\mathfrak{g}/\mathfrak{k}_1)\otimes\tilde{\tau}, \pi'_w),$$

and

$$V_{r^0,w} = \{\sum_r c_rv_{rw}: c_r\in\mathbb{C}\}.$$

The group $S_\psi \times \mathbb{C}^*$ acts on each of the spaces $V_{\psi,w}$ and $V_{r^0,w}$ by the same scalars, while the spaces remain invariant under $SL(2,\mathbb{C})$. Since we are free to modify our isomorphism by any element in

$$\prod_{w\in\Sigma'} GL(V_{r^0,w}) ,$$

it is enought to show that for a fixed $w\in\Sigma'$, the representations of $SL(2,\mathbb{C})$ on $V_{\psi,w}$ and $V_{r^0,w}$ are equivalent. For this it is sufficient to show that $V_{\psi,w}$ and $V_{r^0,w}$ have the same set of weights under the action of the diagonal element H in the Lie algebra of $SL(2,\mathbb{C})$.

Recall first that

$$\begin{aligned}\rho_\psi(H)\phi_{rw} &= \frac{1}{2}(p+q-n)\phi_{rw}\\ &= \frac{1}{2}(\dim_{\mathbb{C}}(\boldsymbol{n}_w\cap\boldsymbol{p}_1^+) + \dim_{\mathbb{C}}(\boldsymbol{n}_w\cap\boldsymbol{p}_1^-) + 2\boldsymbol{l}(r)-n)\phi_{rw} .\end{aligned}$$

We can write

$$\begin{aligned} n &= \dim_{\mathbb{C}}(S_K) = \dim_{\mathbb{C}}(\boldsymbol{p}_1^+)\\ &= \dim_{\mathbb{C}}(\boldsymbol{n}_w\cap\boldsymbol{p}_1^+) + \dim_{\mathbb{C}}(\bar{\boldsymbol{n}}_w\cap\boldsymbol{p}_1^+) + \dim_{\mathbb{C}}(\boldsymbol{l}_w\cap\boldsymbol{p}_1^+) ,\end{aligned}$$

where $\boldsymbol{l}_w$ and $\bar{\boldsymbol{n}}_w$ are the complex Lie algebras of $w^{-1}Lw$ and $w^{-1}\bar{N}_Q w$, the unipotent radical opposite to $w^{-1}N_Q w$. Obviously

$$\dim_{\mathbb{C}}(\bar{\boldsymbol{n}}_w\cap\boldsymbol{p}_1^+) = \dim_{\mathbb{C}}(\boldsymbol{n}_w\cap\boldsymbol{p}_1^-) .$$

Since μ_1 is a minuscule weight, and w^{-1} maps positive roots of (L,T) to positive roots, we have

$$\dim_{\mathbb{C}}(\boldsymbol{l}_w\cap\boldsymbol{p}_1^+) = -2\langle w^{-1}\delta_L,\mu_1\rangle = -2\langle\delta_L,w\mu_1\rangle .$$

Thus

$$\rho_\psi(H)\phi_{rw} = (\boldsymbol{l}(r) + \langle\delta_L,w\mu_1\rangle)\phi_{rw} .$$

On the other hand, the map of $SL(2,\mathbb{R})$ into $\hat{L}$ which corresponds to the principal unipotent element sends H to the vector δ_L. Therefore

$$\begin{aligned}\sigma_\psi(H)v_{rw} = r^0(\psi(H))v_{rw} &= \langle\delta_L,rw\mu_1\rangle v_{rw}\\ &= \langle r^{-1}\delta_L,w\mu_1\rangle v_{rw} .\end{aligned}$$

Our task then is to show that $\langle r^{-1}\delta_L-\delta_L, w\mu_1\rangle$ equals $\boldsymbol{l}(r)$. It is well known that $\delta_L - r^{-1}\delta_L$ equals the sum of those positive roots of (L,T) which are mapped to negative roots by r. The number of these roots equals $\boldsymbol{l}(r)$. Now r is a representative of shortest length in $W(L,T)$ of a coset in $W(L,T)/W(L\cap K_1^w,T)$, so it maps positive roots of (K_1^w,T) to positive roots. Therefore, the positive roots in the sum above have their root spaces in $\mathrm{Ad}(w)(\boldsymbol{p}_1^+)$. The number of these roots equals

$$-\langle\delta_L - r^{-1}\delta_L, w\mu_1\rangle .$$

In other words,

$$l(r) = \langle r^{-1}\delta_L - \delta_L, w\mu_1 \rangle ,$$

as required.

We have just established that $V_{\psi,w}$ and $V_{r^0,w}$ have the same set of weights under H. This was the last step, so the isomorphism from V_ψ to V_{r^0} can be defined so that it intertwines the actions of S_ψ, $\mathbb{C}^*$, and SL(2,$\mathbb{C}$). □

Most of this section has dealt only with the local conjecture of §4 and the examples of §5. We shall conclude by posing a question motivated by the global conjecture. In each of the groups

$$H^{n-d}_{(2)}(S_K(\mathbb{C}), \mathcal{F}_\tau(\mathbb{C})) , \qquad 0 \le d \le n ,$$

one can take the primitive cohomology. For example, there is the subspace $\overline{H}(S_K,\tau)$ of the middle dimensional cohomology corresponding to parameters ψ which are trivial on SL(2,$\mathbb{C}$). This is a subspace of the primitive cohomology in $H^n_{(2)}(S_K(\mathbb{C}), \mathcal{F}_\tau(\mathbb{C}))$. In general, one would like to attach motives to the primitive cohomology in various dimensions. Is it possible to identify pieces of primitive cohomology with spaces $\overline{H}(S'_{K'}, \tau')$, attached to Shimura varieties of smaller dimensions?

I have not looked at the question closely, but it should have a reasonable algebraic answer. For any parameter $\psi \in \Psi(G)$, let $\mathcal{G}_\psi$ denote the centralizer of $\psi(SL(2,\mathbb{C}))$ in ${}^LG$. Then $\mathcal{G}_\psi$ is an extension of W_Q by $\hat{G}_\psi = \mathcal{G}_\psi \cap \hat{G}$, and ψ provides a map of the Langlands group L_Q into $\mathcal{G}_\psi$. Leaving aside the question of whether or not $\mathcal{G}_\psi$ is an L-group, let us just look at $\mathcal{G}_\psi$ and $\hat{G}_\psi$.

Assume that ψ contributes to the cohomology of S_K. Then we have the Levi subgroups $L \subset G$ and $\hat{L} \subset \hat{G}$. The image $\psi(SL(2,\mathbb{C}))$ is just the principal three dimensional subgroup of $\hat{L}$, associated with the principal unipotent class. In particular, the groups $\mathcal{G}_\psi$ and $\hat{G}_\psi$ depend only on $\hat{L}$. The restriction of ψ to L_Q could be very complicated, but we do know that the image $\psi(L_Q)$ is a subgroup of $\mathcal{G}_\psi$ whose centralizer in $\hat{G}_\psi$ is finite modulo $Z(\hat{G})$. We can try to obtain information about ψ, and its contribution to cohomology, by simply studying the group $\hat{G}_\psi$. In fact, Proposition 9.1 tells us that we can determine its contribution to the primitive cohomology from the finite dimensional representations

$$\sigma_\psi(g,u) = r^0(g\psi(u)) , \qquad g \in \hat{G}_\psi,\ u \in SL(2,\mathbb{C}) ,$$

of $\hat{G}_\psi \times SL(2,\mathbb{C})$ on V_{r^0}. The question above is essentially that of describing the decomposition

$$\sigma_\psi = \bigoplus_k (\gamma_k \otimes \delta_k) , \qquad \gamma_k \in \Pi(\hat{G}_\psi),\ \delta_k \in \Pi(SL(2,\mathbb{C})) ,$$

of σ_ψ into irreducible constituents. In particular, are the irreducible finite dimensional representations γ_k of $\hat{G}_\psi$ minuscule?

The maximal torus of $\hat{G}_\psi^0$ is just $A_{\hat{L}}$, the split component of the Levi subgroup $\hat{L}$ of $\hat{G}$. Moreover, the Weyl group of $\hat{G}_\psi$ with respect to $A_{\hat{L}}$ equals

$$W(A_{\hat{L}}) = \mathrm{Norm}_{\hat{G}}(A_{\hat{L}})/\hat{L} .$$

Finally, the weights of the restriction of σ_ψ to $\hat{G}_\psi$ are the restricted characters

$$\mu_1(w,L): a \rightarrow (w\mu_1)(a) , \qquad a \in A_{\hat{L}} ,$$

parametrized by the elements $w \in W(G,T)/W(K_1,T)$. Our constituents γ_k will all be minuscule if for every pair $\mu_1(w,L)$ and $\mu_1(w',L)$ of nonzero weights, $\mu_1(w',L)$ lies outside the convex hull of

$$\{r\mu_1(w,L): w \in W(A_{\hat{L}})\} .$$

To obtain a necessary and sufficient condition, we would have to replace $W(A_{\hat{L}})$ by the less accessible subgroup of elements induced by the identity component $\hat{G}_\psi^0$ of $\hat{G}_\psi$. At any rate, it would be interesting to test the question on some examples.

Bibliography

1. J. Adams and J. Johnson, *Endoscopic groups and packets of non-tempered representations,* Compositio Math. 64 (1987), 271-309.

2. J. Arthur, *On some problems suggested by the trace formula,* in *Lie Group Representations II,* Lecture Notes in Math., vol. 1041, Springer-Verlag, 1983, pp. 1-49.

3. J. Arthur, *The local behaviour of weighted orbital integrals,* Duke J. Math. 56 (1988), 223-293.

4. J. Arthur, *Intertwining operators and residues I. Weighted characters,* to appear in J. Funct. Anal.

5. J. Arthur, *Unipotent automorphic representations. Global motivation,* in preparation.

6. J. Arthur and L. Clozel, *Simple Algebras, Base Change and the Advanced Theory of the Trace Formula,* to appear in Annals of Math. Studies, Princeton Univ. Press.

7. D. Barbasch and D. Vogan, *Unipotent representations of complex semisimple Lie groups,* Ann. of Math. 121 (1985), 41-110.

8. A. Borel, *Automorphic L-functions,* in *Automorphic Forms, Representations and L-functions,* Proc. Sympos. Pure Math. 33, Part II, A.M.S., Providence, 1979, pp. 27-62.

9. A. Borel and W. Casselman, *L^2-cohomology of locally symmetric manifolds of finite volume,* Duke Math. J. 50 (1983), 625-647.

10. A. Borel and N. Wallach, *Continuous Cohomology, Discrete Subgroups and Representations of Reductive Groups,* Annals of Math. Studies, 94, Princeton Univ. Press, 1980.

11. D. Flath, *Decomposition of representations into tensor products,* in *Automorphic Forms, Representations and L-functions,* Proc. Sympos. Pure Math. 33, Part I, A.M.S., Providence, 1979, pp. 179-184.

12. A. Fröhlich and J. Queyrut, *On the functional equations of the Artin L-function for characters of real representations,* Invent. Math. 20 (1973), 125-138.

13. Harish-Chandra, *Discrete series for semisimple Lie groups II,* Acta. Math. 116 (1966), 1-111.

14. Harish-Chandra, *Harmonic analysis on p-adic groups,* in *Harmonic Analysis on Homogeneous Spaces,* Proc. Sympos. Pure Math. 26, A.M.S. Providence, 1973, pp. 167-192.

15. R. Howe and I. Piatetskii-Shapiro, *A counterexample to the "generalized Ramanujan conjecture" for (quasi-)split groups,* in *Automorphic Forms, Representations and L-functions,* Proc. Sympos. Pure Math. 33, Part I, A.M.S., Providence, 1979, pp. 315-322.

16. H. Jacquet, *On the residual spectrum of GL(n),* in *Lie Group Representations II,* Lecture Notes in Math., vol. 1041, Springer-Verlag, 1983, pp. 1-49.

17. D. Keys, *L-indistinguishability and R-groups for quasi-split groups: unitary groups in even dimension,* Ann. Scient. Ec. Norm. Sup., 4[e] série, t. 31 (1987), 31-64.

18. D. Keys and F. Shahidi, *Artin L-functions and normalization of intertwining operators,* to appear in Ann. Scient. Ec. Norm. Sup.

19. A. Knapp and G. Zuckerman, *Normalizing factors, tempered representations, and L-groups,* in *Autormorphic Forms, Representations and L-functions,* Proc. Sympos. Pure Math. 33, Part I, A.M.S., Providence, 1979, pp. 93-106.

20. R. Kottwitz, *Sign changes in harmonic analysis on reductive groups,* Trans. Amer. Math. Soc. 278 (1983), 289-297.

21. R. Kottwitz, *Stable trace formula: cuspidal tempered terms,* Duke Math. J. 51 (1984), 611-650.

22. R. Kottwitz, *Stable trace formula: elliptic singular terms,* Math. Ann. 275 (1986), 365-399.

23. N. Kurokawa, *Examples of eigenvalues of Hecke operators Siegel cusp forms of degree two,* Invent. Math. 49 (1978), 149-165.

24. J.-P. Labesse and R. Langlands, *L-indistinguishability for SL(2),* Canad. J. Math. 31 (1979), 726-785.

25. R. Langlands, *Problems in the theory of automorphic forms,* in *Lectures in Modern Analysis and Applications,* Lecture Notes in Math., vol 170, Springer-Verlag, 1970, pp. 18-86.

26. R. Langlands, *On Artin's L-function,* Rice University Studies 56 (1970), 23-28.

27. R. Langlands, *On the classification of irreducible representations of real reductive groups,* to appear in *Representation Theory and Harmonic Analysis on Semisimple Lie Groups,* Math. Surveys and Monographs, 30, A.M.S., Providence.

28. R. Langlands, *On the Functional Equations Satisfied by Eisenstein Series,* Lecture Notes in Math., vol. 544, Springer-Verlag, 1976.

29. R. Langlands, *Automorphic representations, Shimura varieties and motives. Ein Märchen,* in *Automorphic Forms, Representations and L-functions,* Proc. Sympos. Pure Math. 33, Part II, A.M.S., Providence, 1979, pp. 205-246.

30. R. Langlands, *Stable conjugacy: definitions and lemmas,* Canad. J. Math. 31 (1979), 700-725.

31. R. Langlands, *On the zeta functions of some simple Shimura varieties,* Canad. J. Math. 31 (1979), 1121-1216.

32. R. Langlands, *Les Débuts d'une Formule des Traces Stable,* Publ. Math. Univ. Paris VII, Vol. 13, Paris (1983).

33. R. Langlands and D. Shelstad, *On the definition of transfer factors,* Math. Ann. 278 (1987), 219-271.

34. J. Rogawski, *Automorphic Representations of Unitary Groups in Three Variables,* to appear in Annals of Math. Studies, Princeton Univ. Press.

35. F. Shahidi, *Local coefficients and normalization of intertwining opertors for GL(n),* Compositio Math. 48 (1983), 271-295.

36. F. Shahidi, *Some results on L-indistinguishability for SL(r),* Canad. J. Math. 35 (1983), 1075-1109.

37. F. Shahidi, *Local coefficients as Artin factors for real groups,* Duke Math. J. 52 (1985), 973-1007.

38. D. Shelstad, *Notes on L-indistinguishability (based on a lecture by R.P. Langlands),* in *Automorphic Forms, Representations and L-functions,* Proc. Sympos. Pure Math. 33, Part II, A.M.S., Providence, 1979, pp. 193-204.

39. D. Shelstad, *Orbital integrals and a family of groups attached to a real reductive group,* Ann. Scient. Ec. Norm. Sup., 4e série, t. 12 (1979), 1-31.

40. D. Shelstad, *Embeddings of L-groups,* Canad. J. Math., 33 (1981), 513-558.

41. D. Shelstad, *L-indistinguishability for real groups,* Math. Ann. 259 (1982), 385-430.

42. B. Speh, *Unitary representations of $SL(n,\mathbb{R})$ and the cohomology of congruence subgroups,* in *Non-commutative Harmonic Analysis and Lie Groups,* Lecture Notes in Math., vol. 880, Springer-Verlag, 1981, pp. 483-505.

43. T. Springer, *Linear Algebraic Groups,* Prog. Math. 9, 1981, Birkhäuser.

44. R. Steinberg, *Endomorphisms of Linear Algebraic Groups,* Mem. Amer. Math. Soc. 80, 1968.

45. M. Tadic, *Classification of unitary representations in irreducible representations of general linear group (non-archimedean case),* Ann. Scient. Ec. Norm. Sup., 4e série, t. 19 (1986), 335-382. 13 (1985), 39-42.

46. J. Tate, *Number theoretic background,* in *Automorphic Forms, Representations and L-functions,* Proc. Sympos. Pure Math. 33, Part II, A.M.S., Providence, 1979, pp. 3-26.

47. D. Vogan, *Representations of Real Reductive Lie Groups,* Prog. Math. 15, 1981, Birkhäuser.

48. D. Vogan, *Unitarizability of certain series of representations,* Ann. of Math. 120 (1984), 141-187.

49. D. Vogan, *The unitary dual of GL(n) over an archimedean field,* Invent. Math. 83 (1986), 449-505.

50. D. Vogan, *Representations of reductive Lie groups,* Proceedings of the International Congress of Mathematicians, Berkeley, 1986, Vol. 1, 245-266.

51. D. Vogan, *Unitary Representations of Reductive Lie Groups,* to appear in Annals of Math. Studies, Princeton Univ. Press.

52. D. Vogan and G. Zuckerman, *Unitary representations with non-zero cohomology,* Compositio Math. 53 (1984), 51-90.

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