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REPRESENTATIONS OF HECKE ALGEBRAS

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CONTENTS

Introduction.

The Hecke algebra $\mathbf{H}(G,B)$ of a finite Chevalley group G with respect to a Borel subgroup B was first investigated by Iwahori [18], and applied by him and others to decompose the permutation representation of G on the cosets of B. The algebra $\mathbf{H}(G,B)$ is a specialization of the generic Hecke algebra $\mathbf{H}$ of the Weyl group W of G, over the commutative ring $\mathbf{Q}[u]$, where u is an indeterminate. The representations of $\mathbf{H}(G,B)$, and their connections with the representations of $\mathbf{H}$ and W play a crucial role in the solution of the decomposition problem. The representation theory of $\mathbf{H}$ has also turned out to be useful for the study of the zeta functions of the Deligne-Lusztig varieties of reductive algebraic groups over finite fields and other geometric problems associated with them.

In their paper [22], Kazhdan and Lusztig introduced a new basis of $\mathbf{H}$, whose construction involves the Kazhdan-Lusztig polynomials. Using intersection homology theory and the theory of perverse sheaves, some deep positivity properties of the coefficients of the Kazhdan-Lusztig polynomials and the structure constants of $\mathbf{H}$ for the Kazhdan-Lusztig basis, have been established (see Kazhdan-Lusztig [23] and Springer [32]).

The main purpose of these notes is to examine the consequences of these positivity properties for the structure and representation theory of $\mathbf{H}$, following Lusztig [28], [29], [30]. These include properties of cells and certain distinguished involutions in W called the Duflo involutions, left cell modules of $\mathbf{H}$, Lusztig's isomorphism theorem [24], and the fact that $\mathbf{Q}(u^{1/2})$ is a splitting field for $\mathbf{H}$. The leading terms of the irreducible character values of $\mathbf{H}$ are essential for Lusztig's work [26] on the decomposition of the virtual characters $\{R_T(\theta)\}$. Using asymptotic methods based on the positivity results, Lusztig introduced an algebra J, which is a kind of asymptotic form of $\mathbf{H}$, and whose irreducible character values are precisely the leading terms of the character values of $\mathbf{H}$. These results are all proved in Chapters II and III, following to a great extent a reworking of Lusztig's results in some informal lecture notes by T.A. Springer, who kindly communicated them to me to use in preparing these lectures. An introductory Chapter I contains a survey, not always with proofs, of some of the earlier work on Hecke algebras and their representations, referring the reader to surveys such as [4], [5], or [8] for a fuller discussion.

No attempt has been made to give an account of the historical development of the ideas in Chapter II. There are important connections between these ideas and the classification of primitive ideals in enveloping algebras, especially through the work of Joseph, Barbasch and Vogan, and the solution of the Kazhdan-Lusztig conjectures by Beilinson-Bernstein and Brylinski-Kashiwara. References and fuller discussion of these matters are given in [22] and [26], and in Joseph's article in this volume.

These notes are based on a course given by the author, during the Special Period on Unipotent Orbits, Representations of Finite, Reductive, and p-adic Groups, and Representations of Hecke Algebras, at Paris and Marseille, in June and July, 1987. The author's contribution was supported in part, by Université Paris VII and the NSF.

CHAPTER I. Generic Hecke algebras and their specializations.

1. Applications of the specialized algebras $\mathbf{H}_{(q^{1/2})}$ to the representation theory of reductive groups over finite fields.

Throughout these notes, (W,S) denotes a finite Coxeter system. We let $R = \mathbf{Q}[t,t^{-1}]$ denote the commutative ring of Laurent polynomials with rational coefficients, with t an indeterminate. The generic Hecke algebra $\mathbf{H}$ associated with (W,S) is the R-algebra with a free R-basis $\{e_w\}_{w\in W}$ indexed by the elements of W, and multiplication defined by

$$(1.1) \qquad e_s e_w = \begin{cases} e_{sw} & \text{if } l(sw) > l(w) \\ ue_{sw} + (u-1)e_w & \text{if } l(sw) < l(w) \end{cases},$$

for all $s \in S$, $w \in W$, where $u = t^2$ and $l(w)$ is the length function on W with respect to the set of generators S. (For a proof that these relations define an R-algebra, see [3, Ch. 4, Ex. 23].)

We first state some consequences of the definition.

(1.2) The structure constants of $\mathbf{H}$ with respect to the basis $\{e_w\}_{w\in W}$ belong to $\mathbf{Q}[u]$:

$$e_w e_{w'} = \sum c_{ww'w''} e_{w''} \qquad \text{for } w, w', w'' \in W,$$

where $c_{w,w',w''} = c_{w,w',w''}(u)$ is a polynomial in u with integer coefficients. The polynomials $c_{w,w',w''}$ are known explicitly (see [19], and [6] for a geometric interpretation.) (We have introduced R instead of $\mathbf{Q}[u]$ as the ring of coefficients for $\mathbf{H}$ because the structure constants for the Kazhdan-Lusztig basis (see §4) are in R

but not in $\mathbf{Q}[u]$, and it will turn out, as a consequence of Lusztig's isomorphism theorem (see §8) that $\mathbf{Q}(t)$, the quotient field of R, is a splitting field for $\mathbf{H}$.)

Let $S = \{s_1, \ldots, s_n\}$, and, for $i \neq j$, let n_{ij} be the order of $s_i s_j$ in W. Then W has the presentation

$$W = \langle s_1, \ldots, s_n : s_i^2 = (s_i s_j)^{n_{ij}} = 1, \quad \text{for} \quad 1 \leq i,\ j \leq n \rangle.$$

The R-algebra $\mathbf{H}$ has an analogous presentation, as follows.

(1.3) $\mathbf{H}$ has a presentation as R-algebra with identity $1 = e_1$, generators $\{e_{s_i}\}_{s_i \in S}$, and defining relations

$$e_{s_i}^2 = ue_1 + (u-1)e_{s_i}, \quad \text{for} \quad s_i \in S,$$

and for $1 \leq i,\ j \leq n$,

$$\begin{cases} (e_{s_i} e_{s_j}) = (e_{s_j} e_{s_i})^{k_{ij}} & \text{if } n_{ij} = 2k_{ij}, \text{ and} \\ (e_{s_i} e_{s_j})^{k_{ij}} e_{s_i} = (e_{s_j} e_{s_i})^{k_{ij}} e_{s_j} & \text{if } n_{ij} = 2k_{ij} + 1. \end{cases}$$

Let $f: R \longrightarrow F$ be a homomorphism from R into a field F. Then, using the map f, we may view F as an (F,R)-bimodule, and obtain an F-algebra $F \otimes_R \mathbf{H}$, called a <u>specialized algebra</u> of $\mathbf{H}$, and denoted by $\mathbf{H}_{(a)}$, where $a = f(t)$. A basis of the specialized algebra is given by $\{1 \otimes e_w\}_{w \in W}$, and the structure constants by:

$$(1 \otimes e_w)(1 \otimes e_{w'}) = \Sigma_{w''} f(c_{w,w',w''})(1 \otimes e_{w''}), w, w', w'' \in W.$$

It is the representation theory of the specialized algebras, and its connection with the representation theory of $\mathbf{H}$, which has been important for applications of Hecke algebras. We shall illustrate this point in the rest of the chapter with some examples. A first observation is that

(1.4) $$\mathbf{H}_{(1)} \cong FW,$$

so we expect the representation theory of $\mathbf{H}$ to be somehow related to the representation theory of the Coxeter group W.

An important family of specializations, related to the situation investigated by Iwahori [18], can be described as follows. Let G denote a connected reductive algebraic group, defined and split over the finite field $\mathbf{F}_q$, whose rational structure is defined by a Frobenius endomorphism $F:G \longrightarrow G$ such that the group of fixed points $G^F = \{g \in G : F(g) = g\}$ is finite. We assume that the given Coxeter group W occurs as the Weyl group of G, with respect to an F-stable maximal torus T contained in an F-stable Borel subgroup B, so that $W \cong N_G(T)/T$. The finite groups G^F defined in this way will be called <u>finite groups of Lie type</u>. Each finite group of Lie type G^F has a BN-pair (or Tits system) defined by the subgroups B^F and N^F, with Weyl group W, where $N = N_G(T)$.

Let $\mathbf{H}(G^F,B^F)$ denote the subalgebra of the complex group algebra $\mathbf{C}G^F$ consisting of the functions $f:G^F \longrightarrow \mathbf{C}$ which are constant on the double cosets $B^F\backslash G^F/B^F$. By the Bruhat decomposition in G^F with respect to the Borel subgroup B^F, there is a bijection from $W \longrightarrow B^F\backslash G^F/B^F$, given by $w \longrightarrow B^F\dot{w}B^F$, where $\dot{w} \in N^F$ is a coset representative corresponding to $w \in W$. Then the algebra $\mathbf{H}(G^F,B^F)$ has a <u>standard basis</u> consisting of the normalized characteristic functions $\{a_w\}_{w\in W}$, where

$$a_w = |B^F|^{-1} \sum_{x \in B^F\dot{w}B^F} x, \quad w \in W. \tag{1.5}$$

Letting $(1_{B^F})^{G^F}$ denote the $\mathbf{C}G^F$-module afforded by the induced permutation representation of G^F on the cosets of B^F, we have:

(1.6) PROPOSITION. <u>There exists an isomorphism of</u> $\mathbf{C}$-<u>algebras</u>

$$\mathbf{H}(G^F,B^F) \cong \mathrm{End}_{\mathbf{C}G^F}((1_{B^F})^{G^F}),$$

<u>and an isomorphism</u>

$$\mathbf{H}(G^F,B^F) \cong \mathbf{H}_{(q^{1/2})},$$

<u>given by</u> $a_w \longrightarrow 1 \otimes e_w$, <u>where</u> $\mathbf{H}_{(q^{1/2})}$ <u>is the specialized algebra associated with the homomorphism</u> $f:R \longrightarrow \mathbf{C}$ <u>such that</u> $f(t) = q^{1/2}$.

The first isomorphism holds for all finite groups with BN-pairs (see [8, Chapter 8]). The second isomorphism also follows from the theory of finite groups with BN-pairs [8] and the fact that since G^F is of split type over $\mathbf{F}_q$, we have $\operatorname{ind} a_{s_i} = q$ for each $s_i \in S$. ([4]).

Specializations of generic Hecke algebras of Coxeter groups also occur in more general cases of the problem of decomposing induced modules of finite groups of Lie type. Let G, B, T, $F:G \to G$, etc. be as in the previous discussion, and assume that the center $Z(G)$ of G is connected. Let P be an arbitrary F-stable parabolic subgroup of G, with the Levi decomposition $P = MU$, with $U = R_u(P)$ (the unipotent radical of P), and M an F-stable Levi subgroup.

Let L denote a simple cuspidal $\mathbf{C}M^F$-module. Then the Harish-Chandra induction functor assigns to L the $\mathbf{C}G^F$-module $\operatorname{ind}_{P^F}^{G^F} \tilde{L}$ where $\tilde{L}$ is the $\mathbf{C}P^F$-module pulled back from L, with U^F in its kernel. The decomposition of the induced modules $\operatorname{ind}_{P^F}^{G^F} \tilde{L}$ is a basic problem in the representation theory of G^F. We have:

(1.7) THEOREM. (Howlett-Lehrer [17], Lusztig [26]. *Assume that* G *has a connected center. There exists an isomorphism of* $\mathbf{C}$-*algebras*

$$\operatorname{End}_{\mathbf{C}G^F}(\operatorname{ind}_{P^F}^{G^F} \tilde{L}) \cong \tilde{\mathbf{H}}_{(q^{1/2})},$$

where $\tilde{\mathbf{H}}$ *is the generic Hecke algebra of a certain finite Coxeter group* $\tilde{W}$ (*which depends on the triple* (P^F, M^F, L) *and is related to the stabilizer of* L *in the normalizer of* M *in* N). *The structure of the* R-*algebra* $\tilde{\mathbf{H}}$ *is defined by*

$$e_{\tilde{s}_i} e_{\tilde{w}} = \begin{cases} e_{\tilde{s}_i \tilde{w}} & \text{if } l(\tilde{s}_i \tilde{w}) > l(\tilde{w}),\ \tilde{w} \in \tilde{W}, \\ u^{c_i} e_{\tilde{s}_i \tilde{w}} + (u^{c_i} - 1) e_{\tilde{w}} & \text{if } l(\tilde{s}_i \tilde{w}) < l(\tilde{w}) \end{cases}$$

where $u = t^2$ *and* $\{\tilde{s}_i\}$ *is a set of distinguished generators of the Coxeter group* $\tilde{W}$. *The* $\{c_i\}$ *are positive integers, also depending on the triple* (P^F, M^F, L).

Note that (1.6) is, in a sense, the extreme case of (1.7), involving the full Weyl group, corresponding to the triple

$(B^F, T^F, 1_{T^F})$, since the trivial representation of T^F is a cuspidal irreducible representation.

By (1.6) and (1.7), the basic theory of Hecke algebras ([7], §11D) shows that the decomposition of these induced modules and the calculation of the degrees and other character values of their simple components, can be obtained from a knowledge of the representation theory of the specialized algebras $\mathbf{H}_{(q^{1/2})}$ or $\widetilde{\mathbf{H}}_{(q^{1/2})}$, respectively.

2. Applications to the geometry of reductive groups over finite fields.

We shall sketch a second application of the representation theory of the specialized algebras $\mathbf{H}_{(q^{1/2})}$, this time to the zeta functions of the Deligne-Lusztig varieties. Let G, B, T, W, $F: G \to G$ be as in (1.6). The quotient G/B is a smooth projective variety on which G acts by translation, called the flag variety of G. Since B is its own normalizer, the points of G/B can be identified with the set X consisting of all Borel subgroups of G, using the bijection $G/B \to X$ given by $gB \mapsto {}^gB = gBg^{-1}$. We shall carry over the variety structure from G/B to X, and identify X with the flag variety, on which G acts by conjugation.

The diagonal action of G on the cartesian product $G/B \times G/B$ defines the set of G-orbits $G\backslash(G/B \times G/B)$, which are in bijective correspondence with the double cosets $B\backslash G/B$. Combining the Bruhat decomposition in G relative to B with the identification $G/B \leftrightarrow X$, it follows that there exist bijections

$$W \leftrightarrow B\backslash G/B \leftrightarrow G\backslash(G\backslash B \times G/B) \leftrightarrow G\backslash(X \times X),$$

given by

$$w \to B\dot{w}B \to \text{G-orbit of } (B, \dot{w}B) \text{ in } G/B \times G/B \to \text{G-orbit of } (B, {}^{\dot{w}}B) \text{ in } X \times X.$$

(2.1) DEFINITION. (Deligne-Lusztig [9].) Let $w \in W$, and let $\mathcal{O}(w)$ denote the G-orbit in $X \times X$ corresponding to $w \in W$. A pair (B', B'') of Borel subgroups are said to be in relative position w whenever $(B', B'') \in \mathcal{O}(w)$; in this case we write

$$B' \xrightarrow[w]{} B''.$$

We have $B' \xrightarrow[w]{} B''$ if and only if

$$(B', B'') = {}^{g}(B, {}^{\dot{w}}B),$$

for some $g \in G$.

The Frobenius endomorphism $F: G \longrightarrow G$ acts on G/B, and on the variety X. Letting X^F denote the set of fixed points in X relative to F, we have:

(2.2) PROPOSITION. (i) X^F _is a finite set, on which the finite group_ G^F _acts transitively by conjugation. The resulting_ $\mathbf{C}G^F$-_module is isomorphic to the induced module_ $(1_{B^F})^{G^F}$.

(ii) _Let_ V _denote the_ $\mathbf{C}G^F$-_module with a_ $\mathbf{C}$-_basis identified with the elements of_ X^F, _with the transitive_ G^F-_action as in part_ (i). _The commuting algebra_ $\mathrm{End}_{\mathbf{C}G^F}(V)$ _has a basis consisting of the endomorphisms_ $\{T_w\}_{w \in W}$, _where_

$$T_w(B') = \sum_{B' \xrightarrow[w]{} B''} B'',$$

for $B', B'' \in X^F$.

(iii) _Letting_ a_w _denote the standard basis element of the Hecke algebra_ $\mathbf{H}(G^F, B^F)$ _corresponding to_ w (see (1.5)), _the map_ $T_w \mapsto a_w$ _defines an isomorphism of_ $\mathbf{C}$-_algebras_ $\mathbf{H}(G^F, B^F) \cong \mathrm{End}_{\mathbf{C}G^F} V$.

Proof. (i) We apply Lang's Theorem, which asserts that if $F: H \longrightarrow H$ is an endomorphism of a connected algebraic group H such that the fixed-point subgroup H^F is finite, then the morphism $h \longrightarrow h^{-1}F(h)$ from $H \longrightarrow H$ is surjective. Since all the Borel subgroups in G are conjugates of B, it suffices to prove that if $F(gBg^{-1}) = gBg^{-1}$, then gBg^{-1} is conjugate to B by an element of G^F. The condition implies that $g^{-1}F(g) \in N_G(B) = B$. Applying Lang's Theorem to the connected group B, there exists an element $b \in B$ such that $b^{-1}F(b) = g^{-1}F(g)$. Then $gb^{-1} \in G^F$, and $gBg^{-1} = (gb^{-1})B(gb^{-1})^{-1}$ as required.

(ii) It is easily checked that the endomorphisms $\{T_w\}_{w \in W}$ are linearly independent, and belong to the G^F-endomorphism algebra of V. Since $V \cong (1_{B^F})^{G^F}$, the dimension of the endomorphism algebra is $|B^F \backslash G^F / B^F| = |W|$, and (ii) follows.

(iii) Using the definition of the standard basis elements $\{a_w\}_{w\in W}$, a computation of their convolution product shows that the structure constant $c_{w,w',w''}(q)$ is given by

$$|(B^F\dot{w}B^F\cap\dot{w}''B^F(\dot{w}')^{-1}B^F|/|B^F|$$

This is equal to the number of Borel subgroups $B'\in X^F$ such that

$$B'\xrightarrow[w']{}B\xrightarrow[w]{}{}^{w''}B',$$

which is also the structure constant for the basis $\{T_w\}_{w\in W}$ of the endomorphism algebra.

The varieties to which we shall apply the representation theory of Hecke algebras are defined as follows.

(2.3) DEFINITION. Let $w\in W$. The Deligne-Lusztig variety X_w is the subvariety of the flag variety X defined by

$$X_w = \{B'\in X:B'\xrightarrow[w]{}F(B')\},$$

where F is the Frobenius endomorphism (see [9]).

For each w, the variety X_w is a smooth, locally closed subvariety of X on which the finite group G^F acts by conjugation. It is easily checked that X_w is isomorphic to the subvariety of G/B given by

$$\{gB\in G/B:g^{-1}F(g)\in BwB\}.$$

The virtual representations of G^F defined by $H_c^*(X_w) = \sum(-1)^iH_c^i(X_w)$, where $H_c^i(X_w)$ denotes l-adic cohomology with compact supports provide the starting point for the Deligne-Lusztig approach to the representation theory of the groups G^F. In case $w = 1$, the higher cohomology groups on X_1 vanish, and the G^F-module $H_c^*(X_1)$ is isomorphic to the G^F-module $(1_{B^F})^{G^F}$ discussed previously.

The zeta function $Z(X_w,z)$ of the variety X_w is the formal series in the indeterminate z defined by

$$\frac{d}{dt}\log Z(X_w, z) = \sum_{m=1}^{\infty} |X_w^{F^m}| z^{m-1}$$

where for each $m \geq 1$, $X_w^{F^m}$ is the fixed point subset under the action of F^m, and coincides with the set of $\mathbf{F}_{q^m}$-rational points on the varieties X_w.

The starting point of the investigation of these zeta functions is an identity proved below, which relates the number of fixed points $|X_w^{F^m}|$ to the trace of an element of the commuting algebra of the G^{F^m}-module afforded by $X_w^{F^m}$. For explicit computations and applications to the decomposition of the virtual representations $H_c^*(X_w)$, see [2], [12], [13], [14], [15], [26].

We first require some facts about Shintani descent. The F-<u>conjugacy classes</u> in the finite group G^{F^m}, for $m \geq 1$, are the equivalence classes for the equivalence relation

$$x \sim_F y \quad \text{if} \quad x = gyF(g)^{-1}, \quad \text{for} \quad x,\ y \in G^{F^m}.$$

Shintani [31] proved in the case of GL_n, and Kawanaka and Digne-Michel proved in general ([20], [21], [12], [13], [15]) that there exists a bijection from the set of F-conjugacy classes of G^{F^m} to the set of G^F-conjugacy classes in G^F. This bijection is defined as follows. The F-conjugacy class containing $g \in G^{F^m}$ corresponds to the conjugacy class in G^F containing $\hat{g}$, where if g is represented by Lang's Theorem as $h^{-1}F(h)$ for some $h \in G$, the element $\hat{g}$ is given by $hF^m(h^{-1})$.

We now prove the following identity, due to Asai [2], Digne-Michel [15], and Lusztig [26], independently.

(2.4) PROPOSITION. <u>Let</u> $w \in W$, <u>and let</u> m <u>be a fixed positive integer</u>. <u>Let</u> $g \in G^{F^m}$, <u>and let</u> $\hat{g} \in G^F$ <u>correspond to</u> g <u>by the Shintani map</u>,

$$g = h^{-1}F(h) \longrightarrow \hat{g} = hF^m(h^{-1})$$

Then

$$|X_w^{gF^m}| = \text{Trace } (T_{w^{-1}}^{(m)} gF,\ V^{(m)})$$

where $V^{(m)}$ *is the* $\mathbf{C}G^{F^m}$*-module afforded by the* G^{F^m}*-action on* $X_w^{F^m}$, *and* $\{T_w^{(m)}\}_{w\in W}$ *is the* $\mathbf{C}$*-basis of the* G^{F^m}*-endomorphism algebra of* $V^{(m)}$ *defined in* (2.3).

Proof. For $g\in G^{F^m}$, $B'\in X^{F^m}$, we shall sometimes write $g\,B'$ for ${}^gB'$. Then

$$T_{w^{-1}}^{(m)} gF(B') = \sum_{\substack{B''\in X^{F^m} \\ gFB' \xrightarrow[w^{-1}]{} B''}} B''$$

It follows that

$$\begin{aligned} \text{Trace } (T_{w^{-1}}^{(m)} gF,\ V^{(m)}) &= \text{card}\{B'\in X^{F^m} : gFB' \xrightarrow[w^{-1}]{} B'\} \\ &= \text{card}\{B'\in X : F^m B' = B' \text{ and } gFB' \xrightarrow[w^{-1}]{} B'\}. \end{aligned}$$

Put $g = h^{-1}F(h)$ for $h\in G$, and $\hat{g} = hF^m(h^{-1})$. Then, setting $h^{-1}B'' = B'$, the formula for the trace becomes

$$\begin{aligned} &\text{card}\{h^{-1}B''\in X : F^m(h^{-1}B'') = h^{-1}B'' \text{ and } h^{-1}F(h)Fh^{-1}B'' \xrightarrow[w^{-1}]{} h^{-1}B''\} \\ = &\text{card}\{h^{-1}B''\in X : F^m(h^{-1}B'') = h^{-1}B'' \text{ and } F(B'') \xrightarrow[w^{-1}]{} B''\} \end{aligned}$$

The condition $F^m(h^{-1}B'') = h^{-1}B''$ is equivalent to $\hat{g}F^m B'' = B''$, since $\hat{g} = hF^m(h^{-1})$. Thus the trace is equal to

$$\text{card}\{B''\in X : {}^{\hat{g}}F^m B'' = B'' \text{ and } B'' \xrightarrow[w^{-1}]{} F(B'')\} = |X_w^{\hat{g}F^m}|,$$

as required.

3. Connections between representations of $\mathbf{H}$ and specialized algebras. Generic degrees.

The representations of the specialized Hecke algebras $\mathbf{H}_{(a)}$, whose applications to the geometry and representation theory of reductive groups over finite fields was sketched in §§1 and 2, are closely related to the representations of the generic algebra $\mathbf{H}$ in a suitable splitting field. These connections have all been discussed thoroughly elsewhere ([5], [8]), and are reviewed here only to the extent they are needed later.

We keep the notation from §1. Let $K = \mathbf{Q}(t)$ be the quotient field of $R = \mathbf{Q}[t,t^{-1}]$, let K^* be an algebraic closure of K, and R^* the integral closure of R in K^*. We let $\mathbf{H}^{K^*}$ denote the K^*-algebra $K^*\otimes\mathbf{H}$ obtained by extension of coefficients from R to K^*, and let $\{e_w^*\}_{w\in W}$ denote the K^*-basis $\{1\otimes e_w\}_{w\in W}$ of $\mathbf{H}^{K^*}$.

(3.1) PROPOSITION. (i) $\mathbf{H}^{K^*}$ _is a split semisimple_ K^*-_algebra_.

(ii) _We have_ $\mu(e_w^*)\in\mathbf{R}^*$ _for each irreducible_ K^*-_character_ μ _of_ $\mathbf{H}^{K^*}$.

(iii) _Let_ $f:R\to F$ _be a homomorphism from_ R _to a field_ F, _such that the specialized algebra_ $\mathbf{H}_{(a)}$, _for_ $a = f(t)$, _is semisimple_. _Let_ F^* _denote an algebraic closure of_ F, _and let_ $f^*:R^*\to F^*$ _be an extension of the homomorphism_ f. _For each irreducible character_ μ _of_ $\mathbf{H}^{K^*}$, _define an_ F^*-_linear map_ $\mu_{(a)}:(\mathbf{H}_{(a)})^{F^*}\to F^*$ by

$$\mu_{(a)}(1\otimes e_w) = f^*(\mu(e_w^*)), \quad w\in W.$$

Then $\mu_{(a)}$ _is an irreducible character of_ $(\mathbf{H}_{(a)})^{F^*}$, _and the map_ $\mu\to\mu_{(a)}$ _defines a bijection of irreducible characters_ (_depending on the choice of the extension_ f^* _of_ f.)

For a proof, see [8], §68. As an application, we have:

(3.2) COROLLARY. _Let_ G^F _be a finite group of Lie type_, _as in_ (1.6). _Then there exists an isomorphism of_ $\mathbf{C}$-_algebras_

$$\mathbf{C}W \cong \mathbf{H}(G^F,B^F).$$

This result asserts that $\mathbf{H}_{(1)} \cong \mathbf{H}_{(q^{1/2})}$, and follows from part (iii) of (3.1) since both specialized algebras $\mathbf{H}_{(1)} \cong \mathbf{C}W$ and $\mathbf{H}_{(q^{1/2})} \cong \mathbf{H}(G^F, B^F)$ are semisimple, and consequently have the same sets of numerical invariants, by (3.1).

(3.3) COROLLARY. <u>Keep the notation of</u> (1.6). <u>Then there exist bijections</u>

$$E \longrightarrow \mu_E \longrightarrow \mu_{E,q^{1/2}} \rightarrow \zeta_{E,q^{1/2}}$$

<u>from the set of irreducible representations</u> $\{E\}$ <u>of</u> $\mathbf{C}W$, $\mathbf{H}^{K^*}$, $\mathbf{H}(G^F, B^F)$ <u>respectively, and the set of irreducible</u> $\mathbf{C}$-<u>representations</u> $\zeta_{E,q^{1/2}}$ <u>of</u> G^F <u>which occur with positive multiplicity in the permutation representation</u> $(1_{B^F})^{G^F}$.

The existence of the bijections $E \rightarrow \mu_E \rightarrow \mu_{E,q^{1/2}}$ follows from part (iii) of (3.1). The bijection from the set of irreducible representations $\{\mu_{E,q^{1/2}}\}$ of $\mathbf{H}(G^F, B^F)$ and the irreducible components $\zeta_{E,q^{1/2}}$ of $(1_{B^F})^{G^F}$ is a standard result about Hecke algebras of permutation representations ([7], §11D).

In Chapter II, we shall require the fact that the degrees of the representations $\{\zeta_{E,q^{1/2}}\}$ can be expressed as polynomials in q. This can be explained as follows. Let E be a simple $\mathbf{C}W$-module, and set

$$d_E = \frac{(\deg E)P(u)}{\sum_{w \in W} u^{-l(w)} \mu_E(e^*_w)\mu_E(e^*_{w^{-1}})}, \tag{3.4}$$

where $u = t^2$, and $P(u) = \sum_{w \in W} u^{l(w)}$ is the <u>Poincare polynomial</u> of the Coxeter group W.

(3.5) THEOREM. <u>For each simple</u> $\mathbf{C}W$-<u>module</u> E, <u>there exists a polynomial</u> $D_E \in \mathbf{Q}[u]$ <u>with the following properties</u>:

(i) $D_E(u) = d_E$, <u>where</u> d_E <u>is given by</u> (3.4).

(ii) $D_E(1) = \deg E$.

(iii) $D_E(q) = \deg \zeta_{E,q^{1/2}}$.

For a proof, see [8], §68. The polynomials $\{D_E\}$ in $\mathbf{Q}[u]$ are called the generic degrees (or formal degrees) associated with the Coxeter system (W,S). It can be proved that for each simple $\mathbf{C}W$-module E, D_E devides $u^{l(w_0)}P(u)$ in $\mathbf{Q}[u]$, where w_0 is the element of W of maximal length. The polynomials $\{D_E\}$ have been computed for each type of indecomposable Coxeter system (see [4] for tables for the indecomposable Weyl groups, and [1] for the sporadic Coxeter groups H_3 and H_4). We remark that in the case of H_3 and H_4, their significance remains a mystery, as these groups do not occur as Weyl groups of reductive groups over finite fields.

CHAPTER II. Cells.

In this chapter, we examine some basic equivalence relations on a finite Coxeter group W which are defined using the Kazhdan-Lusztig polynomials. The equivalence classes, called left cells, define an important family of modules for $\mathbf{H}$ and W, the left cell modules. These concepts were introduced by Kazhdan and Lusztig [22], and are important in Lusztig's work [26] on the decomposition of the virtual characters $\{R_T(\theta)\}$ associated with a reductive group over a finite field. Here we present a new approach to these results (Lusztig [28], [29], [30]), which is based on some deep positivity results concerning the coefficients of the Kazhdan-Lusztig polynomials and the structure constants of the generic Hecke algebra $\mathbf{H}$ with respect to a suitable basis.

4. The Kazhdan-Lusztig basis $\{b_x\}_{x\in W}$. The polynomials $\{R_{x,y}\}$ and $\{P_{x,y}\}$.

As in §1, (W,S) denotes a finite Coxeter system, $R = \mathbf{Q}[t,t^{-1}]$ and $\mathbf{H}$ the generic R-algebra associated with (W,S), with the basis $\{e_x\}_{x\in W}$. For x, $y\in W$ we let $x \le y$ denote the Bruhat order (see [10]). In this section we introduce an $\mathbf{R}$-basis of $\mathbf{H}$, called the Kazhdan-Lusztig basis $\{b_x\}_{x\in W}$, whose transition matrix from the basis $\{e_x\}$ involves the Kazhdan-Lusztig polynomials $\{P_{x,y}(u)\in\mathbf{Z}[u]\}$ for $x \le y$ in W, where $u = t^2$.

(4.1) LEMMA. (i) *The involution* $i:R\longrightarrow R$ *defined by* $i(u^{1/2}) = u^{-1/2}$ *extends to an* $\mathbf{R}$-*semilinear ring automorphism* $i:\mathbf{H}\longrightarrow\mathbf{H}$ *of order* 2, *given by*

$$i(\sum_{x\in W} r_x e_x) = \sum_{x\in W} i(r_x) e_{x^{-1}}^{-1}, \quad (\text{where} \quad r_x \in R).$$

(ii) <u>For each pair</u> $x, y \in W$, <u>there exists a polynomial</u> $R_{x,y} \in \mathbf{Z}[u]$, <u>with</u> $R_{x,y} = 0$ <u>unless</u> $x \le y$, $R_{y,y} = 1$, <u>and</u> $\deg R_{x,y} = l(y) - l(x)$, <u>with the property that, for each</u> $y \in W$, <u>we have</u>

$$e_{y^{-1}}^{-1} = u^{-l(y)} \sum_{x\le y} R_{x,y}(u) e_x.$$

(iii) <u>The polynomials</u> $\{R_{x,y}\}$ <u>are characterized by the conditions</u>

$$R_{x,sy} = \begin{cases} R_{sx,y} & \text{if} \quad sx < x, \quad sy > y \\ uR_{sx,y} - (u-1)R_{x,y} & \text{if} \quad sx > x, \quad sy > y \\ 0 & \text{if} \quad x \not\le sy \end{cases}$$

<u>Proof</u>. The map $i: \mathbf{H} \longrightarrow \mathbf{H}$ clearly has order 2 and preserves the defining relations (1.3) of $\mathbf{H}$, from which (i) follows.

For the proof of (ii), use induction on $l(y)$. Since

$$e_s^{-1} = u^{-1}e_s - (1 - u^{-1}), \quad \text{if} \quad s \in S,$$

the result follows if $l(y) = 1$. Assume $l(y) > 1$ and choose $s \in S$ such that $sy > y$. Then $y^{-1}s > y^{-1}$ and we have, by the induction hypothesis,

$$\begin{aligned} e_{(sy)^{-1}}^{-1} &= e_{y^{-1}s}^{-1} = (e_{y^{-1}}e_s)^{-1} = e_s^{-1}e_{y^{-1}}^{-1} \\ &= (u^{-1}e_s - (1-u^{-1}))u^{-l(y)} \sum_{x\le y} R_{x,y}e_x \\ &= u^{-l(y)-1}[\sum_{x\le y} R_{x,y}e_s e_x - (u-1)\sum_{x\le y} R_{x,y}e_x] \\ &= u^{-l(y)-1}[\sum_{\substack{x\le y\\ sx>x}} R_{x,y}e_{sx} + \sum_{\substack{x\le y\\ sx<x}} R_{x,y}(ue_{sx} + (u-1)e_x) - (u-1)\sum_{x\le y} R_{x,y}e_x] \\ &= u^{-l(y)-1}[\sum_{\substack{x\le y\\ sx<x}} R_{sx,y}e_x + \sum_{\substack{x\le y\\ sx>x}} (uR_{sx,y} - (u-1)R_{x,y})e_x] \end{aligned}$$

Statements (ii) and (iii) follow directly from this calculation.

Since the involution i is R-semilinear, it is possible to have a basis of $\mathbf{H}$ consisting of elements fixed by i. Using this idea, we have:

(4.2) THEOREM. For each element $w \in W$, there exists a unique nonzero element $b_w \in \mathbf{H}$ such that $i(b_w) = b_w$ and

$$b_w = u^{-l(w)/2} \sum_{x \leq w} P_{x,w}(u) e_x$$

where the polynomials $P_{x,w} \in \mathbf{Z}[u]$, and satisfy the conditions $P_{w,w} = 1$ and $\deg P_{x,w}(u) \leq \frac{1}{2}(l(w) - l(x) - 1)$ for all $x < w$ in W. The elements $\{b_w\}_{w \in W}$ form an R-basis of $\mathbf{H}$.

Proof. We first prove uniqueness of the elements $\{b_w\}$. Let $\{b'_w\}$ be another set of elements fixed by the involution i and given by

$$b'_w = u^{-l(w)/2} \sum_{x \leq w} Q_{x,w}(u) e_x$$

with polynomials $Q_{x,w} \in \mathbf{Z}[u]$ and $\deg Q_{x,w} \leq \frac{1}{2}(l(w) - l(x) - 1)$. Then, for each $w \in W$,

$$\begin{aligned} b'_w = i(b'_w) &= u^{l(w)/2} \sum_{y \leq w} Q_{y,w}(u^{-1}) e_{y^{-1}}^{-1} \\ &= u^{l(w)/2} \sum_{y \leq w} Q_{y,w}(u^{-1}) u^{-l(y)} \sum_{x \leq y} R_{x,y}(u) e_x, \end{aligned}$$

by Lemma 4.1 Comparing the coefficients of e_x in b'_w and $i(b'_w)$ we obtain

$$u^{-l(w)/2} Q_{x,w}(u) = u^{l(w)/2 - l(x)} Q_{x,w}(u^{-1}) + \sum_{x < y \leq w} u^{l(w)/2 - l(y)} R_{x,y}(u) Q_{y,w}(u^{-1}).$$

If we assume that the polynomials $Q_{y,w}$ with $x < y \leq w$ are known, then this relation determines $Q_{x,w}$ uniquely. This follows since there is no power of u appearing with nonzero coefficient in both $u^{-l(w)/2}Q_{x,w}(u)$ and $u^{-l(w)/2-l(x)}Q_{x,w}(u^{-1})$ because of the restriction on $\deg Q_{x,w}$.

We next prove existence. We have $b_s = u^{-1/2}(e_s + 1)$. Assume b_x is defined for every x with $l(x) < l(w)$. Let $\mu(x,y)$ denote the coefficient of $u^{1/2(l(y)-l(x)-1)}$ in $P_{x,y}(u)$ for $x \le y < w$. Choose $s \in S$ such that $l(sw) < l(w)$, and define

$$b_w = b_s b_{sw} - \sum_{\substack{x<sw \\ sx<x}} \mu(x,sw) b_x. \tag{4.3}$$

It is easily checked, using the induction hypothesis, that b_w has the required properties.

Since the equations defining the basis elements $\{b_w\}$ can be solved for the basis elements $\{e_x\}_{x\in W}$ as R-linear combinations of the elements $\{b_w\}$, it follows that the elements $\{b_w\}_{w\in W}$ form an R-basis of **H**, completing the proof.

We shall call the basis $\{b_x\}_{x\in W}$ the Kazhdan-Lusztig basis of **H**. (In [22], two bases of **H** were defined, $\{c_x\}_{x\in W}$ and $\{c'_x\}_{x\in W}$; the basis $\{b_x\}$ corresponds to the second basis $\{c'_x\}$.)

Both sets of polynomials $\{R_{x,y}\}$ and $\{P_{x,y}\}$ are useful in various applications. The following result implies a simple connection between the polynomials $\{R_{x,y}\}$ and the structure constants for the standard basis $\{e_x\}$.

(4.4) PROPOSITION. *For all* $w \in W$, *we have*

$$e_w e_{w_0} = \sum_{x\le w} (-1)^{l(x)+l(w)} u^{l(x)} R_{x,w} e_{xw_0},$$

where w_0 *is the element of* W *of maximal length*.

The proof is an easy exercise, using induction on $l(w)$ and Lemma 4.1.

It follows from (4.4) that for all $x \le w$,

$$R_{x,w} = (-1)^{l(x)+l(w)} u^{-l(x)} c_{w,w_0,xw_0},$$

and hence are given explicitly by Kawanaka's formulas for the $c_{x,y,z}$ (see (1.2), [19], and [11] for a geometric interpretation.)

5. Cells in W. Left cell modules for **H** and W.

We first calculate some structure constants in $R = \mathbf{Q}[t,t^{-1}]$, (where $t^2 = u$) for the Kazhdan-Lusztig basis elements $\{b_x\}_{x\in W}$ of **H**.

(5.1) PROPOSITION. *Let* $w\in W$ *and* $s\in S$. *Then*

$$b_s b_w = \begin{cases} b_{sw} + \sum\limits_{\substack{x<w \\ sx<x}} \mu(x,w) b_x & \text{if } sw > w \\ (t + t^{-1}) b_w & \text{if } sw < w \end{cases},$$

where $\mu(x,w)$ *is the coefficient of* $t^{l(w)-l(x)-1}$ *in* $P_{x,w}$, *for* $x < w$.

The result follows from (4.3) using induction on $l(w)$, and the fact that

$$b_s^2 = (t + t^{-1}) b_s$$

since

$$b_s = t^{-1}(e_s + 1)$$

We next define three preorders $\leq_L$, $\leq_R$ and $\leq_{LR}$, on W. We shall use the notations, for $w\in W$:

$$\mathcal{L}(w) = \{s\in S : sw < w\} \quad \text{and} \quad \mathcal{R}(w) = \{s\in S : ws < w\}.$$

It will also be convenient to define

$$\begin{cases} \mu(x,y) = \mu(y,x) & \text{if either } x \leq y \text{ or } y \leq x \\ \mu(x,y) = 0 & \text{otherwise} \end{cases},$$

where $\mu(x,y)$ is the coefficient of $u^{1/2(l(y)-l(x)-1)}$ in $P_{x,y}(u)$. Note, in particular, that if $\mu(x,y) \neq 0$, then $l(y) - l(x) - 1$ is even.

(5.2) Definition. The preorder $\leq_L$ on W is defined by the elementary relations

$$x \leq_L w \quad \text{if} \quad \text{(a)} \quad x < w \text{ or } w < x \text{ and } \mu(x,w) \neq 0;$$
$$\text{(b)} \quad \mathcal{L}(x) \not\subset \mathcal{L}(w).$$

In other words, $x \leq_L y$ if and only if there exists a sequence $x_0,\ldots,x_k$ in W with $x_0 = x$, $x_k = y$, and for each i, an elementary relation $x_i \leq_L x_{i+1}$ holds. The preorder $x \leq_R w$ is defined by:

$$x \leq_R w \quad \text{provided that} \quad x^{-1} \leq_L w^{-1},$$

and $x \leq_{LR} w$ is defined by the combination of the preorders $x \leq_L y$ and $x \leq_R y$ (in the obvious way). The equivalence classes in W defined by the preorders $\leq_L$, $\leq_R$ and $\leq_{LR}$ are called left cells, right cells, and two sided cells, respectively. The corresponding equivalence relations are denoted by $x \sim_L y$, etc.

Some basic properties of the preorders $\leq_L$ and $\leq_R$ are summarized in:

(5.3) LEMMA. (i) *Let* $s \in S$, $y < w$, $sw \leq w$, $sy > y$ *and* $\mu(w,y) \neq 0$. *Then* $y = sw$ *and* $\mu(w,y) = 1$.

(ii) *An elementary relation* $x \leq_L w$ *holds if and only if* $x \neq w$ *and* b_x *appears with a nonzero coefficient in* $b_s b_w$, *for some* $s \in S$.

(iii) *If* $x \leq_L y$, *then* $\mathcal{R}(y) \subseteq \mathcal{R}(x)$. *If* $x \sim_L y$ *then* $\mathcal{R}(x) = \mathcal{R}(y)$.

Proof. (i) By comparing the coefficients of e_y, $sy > y$ in (5.1), we obtain

$$P_{y,w} = P_{sy,w}, \; y < w, \; sw < w, \; sy > y.$$

If $sy \neq w$, it follows that (taking degrees in u)

$$\deg P_{y,w} = \deg P_{sy,w} \leq \tfrac{1}{2}(l(w) - l(y) - 2) < \tfrac{1}{2}(l(w) - l(y) - 1)$$

contrary to the assumption that $\mu(y,w) \neq 0$. If $sy = w$, then $P_{y,w} = P_{sy,sy} = 1$, and we obtain $\mu(w,y) = 1$, completing the proof.

(ii) If an elementary relation $x \leq_L w$ holds, then $sx < x$ and $sw > w$ for some $s \in S$, and $\mu(x,w) \neq 0$. If $x < w$ then b_x appears with nonzero coefficient in $b_s b_w$ by (5.1). If, on the other hand, $x > w$, then $x = sw$ by part (i), so b_x appears with nonzero coefficient in $b_s b_w$ also in this case, by (5.1) The converse is easily proved using (5.1).

(iii) We first observe that if $s \in S$,

$$ys > y \Rightarrow \mathcal{L}(ys) \supset \mathcal{L}(y)$$

and

$$sx > x \Rightarrow \mathcal{R}(sx) \supset \mathcal{R}(x).$$

Now let $x \leq_L y$ be an elementary relation, with $x < y$. Then $\mathcal{L}(x) \not\subset \mathcal{L}(y)$, and $\mu(x,y) \neq 0$. We next obtain $x^{-1}y \notin S$ by the first implication above, and the assumption that $\mathcal{L}(x) \not\subset \mathcal{L}(y)$. Thus $\mathcal{R}(x) \supset \mathcal{R}(y)$, otherwise

$$x < y,\ xs > x,\ ys < y,\ \mu(x,y) \neq 0,$$

and by a version of part (i), we obtain $y = xs$, contrary to what has been shown.

On the other hand, if $x > y$, $\mathcal{L}(x) \not\subset \mathcal{L}(y)$, and $\mu(x,y) \neq 0$, then $sx < x$, $sy > y$ for some element $s \in S$, and we have $y = sx$ by part (i). Then $sy > y$ implies $\mathcal{R}(sy) \supset \mathcal{R}(y)$, and $\mathcal{R}(x) \supset \mathcal{R}(y)$ completing the proof.

From the preceding Lemma, we obtain at once:

(5.4) PROPOSITION. <u>Let</u> $x \in W$. <u>Then</u>:

(i) $\mathbf{H}b_x \subseteq \sum_{y \leq_L x} Rb_y$;

(ii) $b_x\mathbf{H} \subseteq \sum_{y \leq_R x} Rb_y$; and

(iii) $\mathbf{H}b_x\mathbf{H} \subseteq \sum_{y \leq_{LR} x} Rb_y$

(5.5) COROLLARY. <u>Let</u> Γ <u>be a left cell in</u> W, <u>and define</u>

$$I_\Gamma = \sum_{\substack{y \le_L x \\ x \in \Gamma}} Rb_y \quad \underline{\text{and}} \quad I'_\Gamma = \sum_{\substack{y \le_L x \\ y \notin \Gamma \\ x \in \Gamma}} Rb_y.$$

Then I_Γ and $I_{\Gamma'}$ are left ideals in **H**.

It follows from (5.5) that I_Γ/I'_Γ is a left **H**-module, with a free R-basis consisting of the elements $\{b_x + I'_\Gamma : x \in \Gamma\}$. We shall call the left **H**-modules obtained in this way left cell modules for **H**, and denote them by

$$M_\Gamma = I_\Gamma/I'_\Gamma,$$

for each left cell Γ in W.

The matrices of left multiplication by the generators $\{e_s\}_{s\in S}$ of **H** have entries in R, and satisfy the defining relations (1.3) of **H**. The matrices obtained by setting $t = 1$, for $s \in S$, clearly satisfy the Coxeter relations, and hence define the structure of a **Q**W-module on M_Γ, which we shall denote by $(M_\Gamma)_1$. The resulting **Q**W-modules $\{(M_\Gamma)_1\}$, for the various left cells of W, are called the left cell modules for W.

Lusztig has made a deep study of the left cell modules in his book [26], in connection with the problems of classifying the unipotent representations, and decomposing the virtual representations $\{R_T(\theta)\}$, for reductive groups over finite fields.

The rest of this section contains some remarks, without proofs, concerning the interpretation of left cell modules in terms of W-graphs, and some other connections with the representation theory of W.

We first introduce a second basis $\{c_x\}_{x\in W}$ of **H** which is related to the basis $\{b_x\}_{x\in W}$ by the relation $c_x = (-1)^{l(x)} j(b_x)$, where $j: \mathbf{H} \longrightarrow \mathbf{H}$ is the involution defined by

$$j(\Sigma a_w e_w) = \Sigma i(a_w)(-1)^{l(w)} t^{-2l(w)} e_w,$$

using the involution i of R defined previously. A simple calculation, using Lemma 5.3 and the fact that $l(w) - l(y) - 1$ is even if $\mu(y, w) \neq 0$, shows that we have

$$e_s c_w = \begin{cases} -c_w & \text{if } s \in \mathcal{L}(w) \\ t^2 c_w + t \displaystyle\sum_{\substack{y - w \\ s \in \mathcal{L}(y)}} \mu(y,w) c_y & \text{if } s \notin \mathcal{L}(w) \end{cases}$$

where $y — w$ means that $y < w$ or $w < y$ and $\mu(y,w) \neq 0$.

A W-graph (see [22]) is a combinatorial object which defines a representation of $\mathbf{H}$. It consists of a graph with a set of vertices X, and a subset of $X \times X$ consisting of edges. To each edge, with vertices $\{x,y\}$, there is assigned an integer $\mu(x,y) \neq 0$, and to each vertex x, there is associated a subset $I_x \subseteq S$. Let $E(X)$ denote the free R-module with a basis identified with the elements of X. For each $s \in S$, let τ_s be the R-endomorphism of $E(X)$ defined by

$$\tau_s x = \begin{cases} -x & \text{if } s \in I_x \\ t^2 x + t \displaystyle\sum_{\substack{y \in X \\ x - y \\ s \in I_y}} \mu(x,y) y & \text{if } s \notin I_x \end{cases}, \tag{5.6}$$

where $x — y$ means that $\{x,y\}$ is an edge in the graph. The preceding data defines a W-graph provided that the map $s \to \tau_s$ extends to an R-representation of $\mathbf{H}$ on $E(X)$.

Now let Γ be a left cell in W, and let M_Γ denote the left cell module for $\mathbf{H}$ associated with Γ. Then it is easily checked that M_Γ is the $\mathbf{H}$-module associated with the W-graph consisting of the set of vertices Γ, the set of edges $\{(x,y) \in \Gamma \times \Gamma : \mu(x,y) \neq 0\}$, the integers $\mu(x,y)$ defined as above for each edge, and the subsets of S defined by $I_x = \mathcal{L}(x)$, for $x \in \Gamma$. The basis of $E(\Gamma)$ satisfying the condition (5.6) is given by $\{c_x : x \in \Gamma\}$.

Gyoja has proved [16] that every simple $\mathbf{H}^{K^*}$-module is isomorphic to $E(X)^{K^*} = K^* \otimes_R E(X)$ for some W-graph X. In case W is of type A_n, the left cell modules themselves (or the W-graphs associated with them) provide a full set of simple $\mathbf{H}^{K^*}$-modules ([22]).

In general, the left cell modules $\{M_\Gamma^{K^*}\}$ are not simple modules. The $\mathbf{C}W$-composition factors of the associated left cell modules

$\{(M_\Gamma^{K^*})_1\}$, however, have been determined by Lusztig [27]. This information is derived from the following inductive description of the $\mathbf{C}W$-modules $\{(M_\Gamma^K)_1\}$. In what follows, W denotes a Weyl group (associated with the root system of a semisimple Lie algebra over $\mathbf{C}$.)

For each simple $\mathbf{C}W$-module E, let $D_E \in \mathbf{Q}[u]$ be the generic degree associated with it (see (3.5)), and let

(5.7) $$D_E = \alpha_E u^{a_E} + \cdots + \beta_E u^{A_E},$$

with α_E, β_E nonzero rational numbers, and $\alpha_E u^{a_E}$ and $a_E u^{A_E}$ the terms of lowest, and highest, degree, respectively.

(5.8) Definition. Let W_I, for $I \subseteq S$, be a parabolic subgroup of W. The operation of truncated induction $j_{W_I}^W$ assigns to each simple $\mathbf{C}W_I$-module E' the $\mathbf{C}W$-module $j_{W_I}^W(E')$, which is the direct sum of simple modules for W defined by

$$\sum_{\substack{E \in \mathrm{Irr} W \\ a_E = a_{E'}}} (E, \mathrm{ind}_{W_I}^W E')E$$

where $(E, \mathrm{ind}_{W_I}^W E')$ is the multiplicity of the simple W-module E in the induced module $\mathrm{ind}_{W_I}^W E'$, and a_E, $a_{E'}$ are the exponents of u in the terms of lowest degree in the generic degrees D_E and $D_{E'}$, respectively. The operation $j_{W_I}^W$ is extended to arbitrary $\mathbf{C}W_I$-modules by taking direct sums.

Using these operations, the constructible representations of W are defined recursively as follows. If $W = \{1\}$, only the trivial representation is constructible. If $W \neq \{1\}$, the set of constructible representations of W consists of all representations

$$j_{W_I}^W(E') \quad \text{and} \quad \mathrm{sgn} \otimes j_{W_I}^W(E'),$$

where sgn is the sign representation of W, and E' is a constructible representation of W_I, for some proper parabolic subgroup W_I of W.

We now state:

(5.9) THEOREM. (Lusztig) Let M be a $\mathbf{C}W$-module, for a Weyl group W. Then M affords a constructible representation of W if and only if $M \cong (M_\Gamma)_1$, for some left cell module M_Γ of $\mathbf{H}$.

The proof, which involves a case by case analysis, is given in [27].

6. Asymptotic methods. The a-function, left cells, and Duflo involutions.

Throughout this section, W denotes a Weyl group (associated with the root system of a semisimple Lie algebra over $\mathbf{C}$). We first recall some elementary facts. Let $\tau:\mathbf{H} \longrightarrow R$ be the R-linear map defined by

$$\tau(e_x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases},$$

where $\{e_x\}_{x\in W}$ is the R-basis satisfying (1.1) The resulting R-bilinear form (called the τ-form),

$$(a,b) = \tau(ab), \quad a,\ b \in \mathbf{H},$$

is associative, symmetric, and nondegenerate. The bases

$$\{e_x\}_{x\in W} \quad \text{and} \quad \{u^{-l(x)} e_{x^{-1}}\}_{x\in W}$$

are dual with respect to the τ-form, and can be used to obtain the primitive central idempotents and orthogonality relations for the irreducible characters of the split semisimple algebra $\mathbf{H}^{K^*}$ (see [8, §68]).

(6.1) LEMMA. *Let* $x, y \in W$ *and let* b_x, b_y *be the Kazhdan-Lusztig basis elements corresponding to them*. *Then*

$$\tau(b_x b_y) = \delta_{x,y^{-1}} + \sum_{i\geq 1} a_i t^{-i}, \quad \textit{with} \quad a_i \in \mathbf{Q},$$

where $\delta_{x,y^{-1}}$ *is the Kronecker* δ.

The proof is readily obtained, using induction on $l(x)$ and (5.1).

We now introduce the structure constants $\{h_{x,y,z}\}$ for the Kazhdan-Lusztig basis $\{b_x\}_{x\in W}$, which will be a main focus of attention in what follows. We set:

$$(6.2) \qquad b_x b_y = \sum_{z\in W} h_{x,y,z} b_z, \quad \text{for} \quad x, y \in W.$$

Then the structure constants $h_{x,y,z} \in \mathbf{Z}[t,t^{-1}] \subseteq R$, and are symmetric in t:

$$h_{x,y,z}(t) = h_{x,y,z}(t^{-1}),$$

by (5.1).

In order to proceed, we require the following nonelementary positivity properties of the polynomials $P_{x,y}$ and $h_{x,y,z}$, all viewed from now on as Laurent polynomials in t.

(6.3) THEOREM. (i) *The coefficients of* $P_{x,y}$ *are nonnegative integers, for all* x,y *in* W, *with* $x \leq y$.

(ii) *The coefficients of* $h_{x,y,z}$ *are nonnegative integers for all* $x,y,z \in W$.

Part (i) is due to Kazhdan and Lusztig [23]; both parts are proved in Springer's Bourbaki Seminar article [32], using intersection cohomology theory and the theory of perverse sheaves.

(6.4) DEFINITIONS. Let $z \in W$, and define

$$a(z) = \max_{x,y\in W} \deg h_{x,y,z}$$

where the degree is taken with respect to t. For $x,y,z\in W$, put

$$\gamma_{x,y,z} = \text{coefficient of } t^{a(z)} \text{ in } h_{x,y,z}.$$

(6.5) LEMMA. (i) $a(z)$ *is the least nonnegative integer with the property that*

$$t^{a(z)}h_{x,y,z}\in\mathbf{Q}[t] \quad \textit{for all} \quad x,y\in W.$$

(ii) $h_{y^{-1},x^{-1},z^{-1}} = h_{x,y,z}$ *for all* $x,y,z\in W$.

(iii) $a(z) = a(z^{-1})$.

Part (i) is clear from the definition, and the fact that $h_{x,y,z}$ is symmetric in t and t^{-1}. For the proof of part (ii), we make use of the R-algebra antiautomorphism of $\mathbf{H}$ which takes $e_x \to e_{x^{-1}}$, for $x\in W$. It is easily checked that $b_x \to b_{x^{-1}}$ under this map, and (ii) follows by applying it to the structure equations (6.2). Part (iii) follows from part (ii).

(6.6) COROLLARY. $\gamma_{x,y,z} = \gamma_{y^{-1},x^{-1},z^{-1}}$ *for all* $x,y,z\in W$.

Another basic property of the a-function is that

(6.7) $a(z) = 0$ if and only if $z = 1$.

This fact can be proved without making use of the positivity results (6.2)(see [28], Prop. 2.3). It also follows directly from the results to follow (see (6.8)).

(6.8) THEOREM. (i) *Let* $\delta(z)$ *be the degree of* $P_{1,z}$ *in* u. *Then we have*

$$a(z) \le l(z) - 2\delta(z) \quad \textit{for all} \quad z\in W.$$

(ii) *Let* $\mathcal{D} = \{z\in W : a(z) = l(z) - 2\delta(z)$. *Let* $d\in\mathcal{D}$, *and assume that* $\gamma_{x,y,d} \neq 0$ *for some* $x,y\in W$. *Then*

$$x = y^{-1}, \quad \gamma_{x,x^{-1},d} = 1 \quad \textit{and} \quad d^2 = 1.$$

Moreover, we have

$$P_{1,d} = u^{\delta(d)} + \text{terms of lower degree.}$$

(iii) For each $x \in W$, there is a unique element $d \in \mathcal{D}$, such that

$$\gamma_{x,x^{-1},d} = 1.$$

Proof. Let $z \in W$. Throughout the proof, the abbreviation deg means the degree taken with respect to t. By (5.1) and the definition of τ, we have

$$\tau(b_z) = t^{-l(z)} P_{1,z}.$$

Then, by (6.2),

$$\tau(b_x b_y) = \Sigma_z h_{x,y,z} t^{-l(z)} P_{1,z}.$$

Since $\deg \tau(b_x b_y) \leq 0$ by (6.1) and all the coefficients of $h_{x,y,z}$ and $P_{1,z}$ are nonnegative by (6.3), it follows that

$$\deg h_{x,y,z} t^{-l(z)} P_{1,z} \leq 0, \quad \text{for all } x,y \in W.$$

Consequently,

$$\deg h_{x,y,z} \leq l(z) - 2\delta(z)$$

and (i) follows.

Now let $d \in \mathcal{D}$, so $a(d) = l(d) - 2\delta(d)$, and assume $\gamma_{x,y,d} \neq 0$. Consider the equation

$$\tau(b_x b_y) = \Sigma_z h_{x,y,z} t^{-l(z)} P_{1,z}.$$

From our assumption, it follows that the degree of the right side is 0. By (6.1), the left side has degree 0 only if $xy = 1$, and in that case the coefficient of the term in degree zero is 1. It follows that there is a unique z contributing to the degree zero term on the right side (since the coefficients of $h_{x,y,z}$ are integers), and this element z must equal the given $d\in\mathcal{D}$, so $y = x^{-1}$, and

$$\gamma_{x,x^{-1},d} = 1.$$

It then follows that the leading coefficient in u in $P_{1,z}$ is also equal to 1. This proves all the statements in (ii) except the fact that $d^2 = 1$, whose proof we postpone until after (iii) is established.

For each $x\in W$, we have $\deg \tau(b_x b_{x^{-1}}) = 0$, and it follows that

$$\deg\ (\Sigma_z h_{x,x^{-1},z} t^{-l(z)} P_{1,z}) = 0.$$

Therefore, by the positivity results again,

$$\deg\ h_{x,x^{-1},z} t^{-l(z)} P_{1,z} = 0$$

for a unique element z; then $z\in\mathcal{D}$ and $\gamma_{x,x^{-1},z} = 1$, proving (iii).

Finally, let $d\in\mathcal{D}$, and choose x such that $\gamma_{x,x^{-1},d} \neq 0$. By Corollary (6.6), we have

$$\gamma_{x,x^{-1},d} = \gamma_{x,x^{-1},d^{-1}},$$

so the uniqueness statement in (iii) implies that $d = d^{-1}$. Thus $d^2 = 1$ for each $d\in\mathcal{D}$, and the theorem is proved.

The elements of $\mathcal{D}$ will be called the <u>Duflo involutions</u> in W; it will turn out that there is a unique one in each left cell, (see [29]).

(6.9) LEMMA. (i) <u>Let</u> $z, w\in W$, $s\in S$, <u>and assume that</u> $sw < w$, $sz > z$, $\mu(w,z) \neq 0$. <u>Let</u> $x,y\in W$ <u>be such that</u> $\gamma_{x,y,z} \neq 0$. <u>Then there exists</u> $v\in W$ <u>such that</u> $\deg h_{v,y,w} \geq a(z)$. <u>In particular</u> $a(z) \leq a(w)$.

(ii) <u>If</u> $x \leq_{LR} y$ <u>then</u> $a(y) \leq a(x)$. <u>If</u> $x \sim_{LR} y$ <u>then</u> $a(x) = a(y)$.

<u>Proof</u>. Since $\gamma_{x,y,z} \neq 0$, we have $h_{x,y,z} \neq 0$, so $z \leq_R x$ by (5.4), and hence $\mathcal{L}(x) \subseteq \mathcal{L}(z)$ by (5.3)(iii). It follows that $sx > x$. Upon writing out the associativity formula

$$(b_s b_x) b_y = b_s (b_x b_y),$$

we obtain for the coefficient of b_w on the left side (using (5.1)):

$$h_{sx,y,w} + \sum_{\substack{v<x \\ sv<v}} \mu(v,x) h_{v,y,w}$$

while the right side is

$$\sum_{sz'<z'} (t + t^{-1}) h_{x,y,z'} + \sum_{sz'>z'} h_{x,y,z'} (b_{sz'} + . \sum_{\substack{u<z' \\ su<u}} \mu(u,z') b_u).$$

If $\gamma_{x,y,z} \neq 0$ and $\mu(w,z) \neq 0$ as in the hypothesis, then the coefficient of w on the right side has degree $\geq a(z)$, since $w = sz$ if $w > z$ by (5.3i). It follows that $\deg h_{v,y,w} \geq a(z)$ for some v, which proves the first statement. Since the hypothesis of part (i) holds whenever an elementary relation $w \leq_L z$ occurs, part (ii) follows easily, using (6.5 iii).

(6.10) THEOREM. (i) <u>For all</u> x, y, z, <u>we have</u>

$$\gamma_{x,y,z} = \gamma_{y,z^{-1},x^{-1}} = \gamma_{z^{-1},x,y^{-1}}.$$

(ii) If $\gamma_{x,y,z} \neq 0$, <u>then</u> $x \sim_L y^{-1}$, $y \sim_L z$, $x \sim_R z$ <u>and</u> $a(x) = a(y) = a(z)$.

(iii) <u>If</u> $x \leq_L y$ <u>and</u> $a(x) = a(y)$ <u>then</u> $x \sim_L y$.

(iv) <u>If</u> $x \leq_L y$ <u>and</u> $x \sim_{LR} y$ <u>then</u> $x \sim_L y$.

<u>Proof</u>. Suppose $x, y, z \in W$, and $\gamma_{x,y,z} \neq 0$. Then, for some $d \in \mathcal{D}$, $\gamma_{z,z^{-1},d} = 1$, by (6.8 iii). Then $h_{z,z^{-1},d} \neq 0$, so $d \leq_R z$, by (5.4), and hence $a(z) \leq a(d)$, by Lemma 6.9.

Assume first that $a(z) = a(d) = a$, and $\gamma_{x,y,z} \neq 0$. The associativity formula for the basis $\{b_x\}$ implies that

$$\Sigma_u h_{x,y,u} h_{u,z^{-1},d} = \Sigma_v h_{y,z^{-1},v} h_{x,v,d}.$$

If $h_{u,z^{-1},d} \neq 0$ then $d \leq_R u$, so $a(u) \leq a$. Similarly $h_{x,v,d} \neq 0$ implies $d \leq_L v$ so $a(v) \leq a$. Then both sides are summed over elements u, v such that $a(u) \le a$ and $a(v) \le a$. We now calculate the leading terms on both sides. On the left, $h_{u,z^{-1},d}$ has leading coefficient 1 if $u = z$, and 0 if $u \ne z$. Since we are assuming $a(z) = a$, the left side has the leading term

$$\gamma_{x,y,z}\gamma_{z,z^{-1},d}t^{2a} = \gamma_{x,y,z}t^{2a}.$$

The coefficient of t^{2a} on the right side occurs when $v = x^{-1}$ and is

$$\gamma_{y,z^{-1},x^{-1}}\gamma_{x,x^{-1},d}t^{2a}$$

since $a(v) \le a$ for all v. Upon comparing these terms, we obtain $\gamma_{x,x^{-1},d} \ne 0$, so $\gamma_{x,x^{-1},d} = 1$, and

$$\gamma_{y,z^{-1},x^{-1}} = \gamma_{x,y,z},$$

proving the first statement in part (i).

Now assume that $a(z) < a(d)$, and define a sequence of elements of $\mathcal{D}$, $d_1 = d, d_2, d_3, \ldots$ with $\gamma_{d_i,d_i,d_{i+1}} \ne 0$ for each i, using (6.8iii). Then we have

$$a(d_1) \le a(d_2) \le \ldots.$$

If $a(d_1) = a(d_2)$, then

$$\gamma_{z,z^{-1},d} = \gamma_{z^{-1},d,z^{-1}} \ne 0$$

by what has been proved, hence $a(d) \le a(z^{-1}) = a(z)$ and we are in the first case. If $a(d_i) = a(d_{i+1})$, a similar argument shows that $a(d_i) = a(d_{i-1})$, so we must have $a(z) = a(d)$, and part (i) is proved, in case $\gamma_{x,y,z} \ne 0$. If any one of the elements in part (i) is $\ne 0$, then we have equality, by what has been proved, and this establishes part (i).

If $\gamma_{x,y,z} \ne 0$, then $z \le_R x$ and $z \le_L y$. From $\gamma_{y,z^{-1},x^{-1}} \ne 0$ we obtain $x^{-1} \le_L z^{-1}$ and hence $x \sim_R z$. Similar arguments prove the rest of part (ii).

For the proof of part (iii), first assume $x \le_L y$ is an elementary relation. Then there exists $s \in S$ such that $sx < x$,

$sy > y$, and $\mu(x,y) \neq 0$. Choose u,v such that $\gamma_{u,v,y} \neq 0$. Then, by Lemma 6.9, there exists $w \in W$ such that

$$\deg h_{w,v,x} \geq a(y).$$

Since $\deg h_{w,v,x} \leq a(x)$ and $a(x) = a(y)$, we obtain

$$\deg h_{w,v,x} = a(x)$$

and hence

$$\gamma_{w,v,x} \neq 0.$$

By part (ii), $\gamma_{u,v,y} \neq 0$ and $\gamma_{w,v,x} \neq 0$ imply $v \sim_L x$ and $v \sim_L y$, so that $x \sim_L y$. By applying this argument to a sequence of elementary relations, we obtain part (iii).

Now let $x \leq_L y$, and $x \sim_{LR} y$. Then $a(x) = a(y)$ by part (ii) of Lemma 6.9. Then $x \sim_L y$ by part (iii). This completes the proof of the Theorem.

Part (iv) of the Theorem is a powerful result, as we shall see in Chapter III. It was proved first by Lusztig [24], using the theory of enveloping algebras, and the fact that the Kazhdan-Lusztig conjectures hold.

(6.11) THEOREM. <u>Each left cell</u> Γ <u>contains a unique Duflo involution</u>.

<u>Proof</u>. Let $x, y \in \Gamma$. Choose $d, d' \in \mathcal{D}$ such that

$$\gamma_{x^{-1},x,d} = \gamma_{y^{-1},y,d'} = 1.$$

By the previous Theorem, we have $x \sim_L d$ and $y \sim_L d'$, so $d, d' \in \Gamma$. Moreover,

$$\gamma_{x,d,x} = \gamma_{y,d',y} = 1.$$

Now assume $x \leq_L y$ is an elementary relation. Then we can apply Lemma 6.9 to obtain $v \in W$ such that $\gamma_{v,d',x} \neq 0$. Then $\gamma_{x^{-1},v,d'} \neq 0$ by the Theorem, and it follows that $v = x$ and $d = d'$, completing the proof.

CHAPTER III. Representation theory of $\mathbf{H}$ in the field $\mathbf{Q}(t)$.

Throughout this chapter, $\mathbf{H}$ denotes the generic Hecke algebra over the commutative ring $R = \mathbf{Q}[t,t^{-1}]$ of a finite Weyl group (W,S) (a finite Coxeter group satisfying the crystallographic condition as in Chapter II). The main topic will be the irreducible representations and characters of the algebra $\mathbf{H}^{\mathbf{Q}(t)}$, where $\mathbf{Q}(t)$ is the quotient field of R. A particularly interesting feature is Lusztig's construction of a $\mathbf{Z}$-algebra J, which is a kind of asymptotic form of $\mathbf{H}$, and whose irreducible character values give the leading terms of the irreducible character values of $\mathbf{H}^{\mathbf{Q}(t)}$.

§7. An associativity formula.

We first recall some properties of the R-basis $\{b_x\}$ of $\mathbf{H}$ from §§4 and 5. We have, for $x,y\in W$,

$$b_x b_y = \sum_{z\in W} h_{x,y,z} b_z, \quad \text{with} \quad h_{x,y,z}\in\mathbf{Z}[t,t^{-1}],$$

where the polynomials $h_{x,y,z}$ are symmetric in t, and have coefficients ≥ 0. The a-function is defined by

$$a(z) = \max_{x,y\in W} \deg h_{x,y,z},$$

where the degree is taken with respect to t.

For each $x,y,z\in W$, we put

$$\gamma_{x,y,z} = \text{coefficient of } t^{a(z)} \text{ in } h_{x,y,z}.$$

We recall from §6 that

$$z \leq_{LR} z' \Rightarrow a(z') \leq a(z),$$

so the a-function is constant on two sided cells. We also require the results from Theorem 6.10 that:

(7.1) $\quad x \leq_L y$ and $a(x) = a(y) \Rightarrow x \sim_L y,$ and

(7.2) $\quad x \leq_L y$ and $x \sim_{LR} y \Rightarrow x \sim_L y.$

Now let $\mathbf{H}'$ be a copy of $\mathbf{H}$ over the ring $R' = \mathbf{Q}[t', t'^{-1}]$, where t' is a second indeterminate over $\mathbf{Q}$, and let $\{b'_x\}_{x\in W}$ denote the Kazhdan-Lusztig basis of $\mathbf{H}'$, with structure constants $h_{x,y,z}(t') \in \mathbf{Z}[t', (t')^{-1}]$. We now introduce a free module $B_{t,t'}$ over the ring $\mathbf{Q}[t, t^{-1}, t', (t')^{-1}]$ with a basis $\{\beta_x\}_{x\in W}$, such that $\mathbf{H}$ acts on $B_{t,t'}$ from the left and $\mathbf{H}'$ from the right. These actions are defined in such a way that

$$b_x\beta_y \text{ corresponds to } b_xb_y \text{ in } \mathbf{H}$$

and

$$\beta_y b'_z \text{ corresponds to } b'_y b'_z \text{ in } \mathbf{H}'.$$

These actions do not necessarily commute.

(7.3) ASSOCIATIVITY LEMMA. <u>For all</u> $x \in W$, <u>and</u> $s, s' \in S$, <u>the element</u>

$$(b_s\beta_x)b'_{s'} - b_s(\beta_x b'_{s'})$$

<u>is a linear combination of basis elements</u> β_y <u>for which</u> $a(y) > a(x)$.

<u>Proof</u>. <u>Case</u> 1. $sx < x$, $xs' < x$. Then

$$(b_s\beta_x)b'_{s'} = b_s(\beta_x b'_{s'}) = (t + t^{-1})(t' + t'^{-1})\beta_x,$$

by (5.1).

<u>Case</u> 2. $sx < x$, $xs' > x$. In this case,

$$(b_s\beta_x)b'_{s'} = (t + t^{-1})(\beta_{xs'} + \sum_{\substack{y<x \\ ys'<y}} \mu(y,x)\beta_y.$$

while

$$b_s(\beta_x b'_{s'}) = b_s(\beta_{xs'} + \sum_{\substack{y<x \\ ys'<y}} \mu(y,x)\beta_y).$$

The two expressions are equal, since the elements xs' and y with $\mu(y,x) \neq 0$ and $ys' < y$ are all $\leq_R x$, and hence $s \in \mathcal{L}(x) \subset \mathcal{L}(xs')$, $\mathcal{L}(y)$ for these elements, by Lemma 5.3iii.

Case 3. $sx > x$, $xs' < x$. Set $t = t'$, $\mathbf{H}' = \mathbf{H}$. The resulting R-module $B_{t,t}$ affords the two sided regular representation of $\mathbf{H}$. The further specialization $t \longrightarrow 1$ yields a $\mathbf{Q}$-module $B_{1,1}$ with the structure of a $\mathbf{Q}W$, $\mathbf{Q}W$-bimodule. It follows that the specialized form of $(b_s\beta_x)b'_{s'} - b_s(\beta_x b'_{s'})$ is zero, and that we have for the original expression

$$(b_s\beta_x)b'_{s'} - b_s(\beta_x b'_{s'}) = (t + t^{-1} - 2) \sum_{\substack{ys'<y \\ sy<y \\ y\leq_L x}} \mu(y,x)\beta_y$$
$$- (t' + (t')^{-1} - 2) \sum_{\substack{ys'<y \\ sy<y \\ y\leq_R x}} \mu(y,x)\beta_y.$$

Consider a nonzero term in the first sum. Since $y \leq_L x$, we have $a(y) \geq a(x)$. If $a(y) = a(x)$, then $y \sim_L x$ by (7.1), and hence $\mathcal{R}(x) = \mathcal{R}(y)$ by 5.3iii), which is impossible. Therefore $a(y) > a(x)$ for all terms in the first expression. A similar argument applies to the second expression, starting from the condition $y \leq_R x$. This completes the proof.

8. Lusztig's isomorphism theorem.

Let B be the R-module with the basis $\{\beta_x\}_{x\in W}$, as in §7. Then B admits a left action by $\mathbf{H}$ and a right action by RW. The right action by W is defined for a generator s by setting $t' = 1$ in the matrix of the right multiplication by the standard basis element e'_s of $\mathbf{H}'$ with respect to the basis $\{\beta_x\}_{x\in W}$. Since B is a right **H**'-module, it follows that the right action by the generators $s\in S$ satisfy the defining relations of W, so that B becomes a right W-module, and hence a right RW-module. The left action by **H** and the right action by W do not commute, but we shall obtain commuting actions by **H** and RW on a suitable graded version of B.

For each two sided cell $\mathbf{c}$, put

$$B_{\mathbf{c}} = \sum_{\substack{y\leq_{LR}z \\ z\in\mathbf{c}}} R\beta_y \quad \text{and} \quad B'_{\mathbf{c}} = \sum_{\substack{y\leq_{LR}z \\ z\in\mathbf{c} \\ y\notin\mathbf{c}}} R\beta_y .$$

The submodules $\{B_{\mathbf{c}}\}$ define a kind of filtration of B using the preorder relation $y\leq_{LR} z$. The associated graded module $\operatorname{gr} B$ is defined by

$$\operatorname{gr} B = \oplus_{\mathbf{c}} B_{\mathbf{c}}/B'_{\mathbf{c}},$$

where the sum is taken over the two sided cells. Clearly $\operatorname{gr} B$ inherits a left **H**-action and a right RW-action. The left action by **H** also defines a left action by RW on B, by specialization, and it is clear that B, and hence $\operatorname{gr} B$, become (RW,RW)-bimodules.

(8.1) LEMMA. The graded R-module $\operatorname{gr} B$ is an $(\mathbf{H},RW)$-bimodule.

The proof is immediate from the associativity Lemma 7.3, since the a-function is constant on two sided cells by Lemma 6.9.

We can now state the main result of this section.

(8.2) THEOREM. (Lusztig [24]). (i) There exists a unique homomorphism $\eta.\ \mathbf{H}\rightarrow RW$ such that, for each $w\in W$ and $h\in\mathbf{H}$, we have

$$h\beta_w - \eta(h)\beta_w = \sum_{z\not\leq_{LR}w} r_z\beta_z$$

for some coefficients $r_z\in R$.

(ii) The extended map $1\otimes\eta : \mathbf{Q}(t)\otimes_R\mathbf{H}\rightarrow\mathbf{Q}(t)W$ is an isomorphism of $\mathbf{Q}(t)$-algebras.

Proof. Let $\mathrm{End}_W(\mathrm{gr}\, B)_W$ denote the algebra of R-endomorphisms of $\mathrm{gr}\, B$ which commute with the right action by W. By Lemma 8.1, there exists a homomorphism of R-algebras $\alpha:\mathbf{H} \longrightarrow \mathrm{End}_W(\mathrm{gr}\, B)_W$.

On the other hand, the commuting two sided action of RW on gr B defines a homomorphism of R-algebras $\beta:RW \longrightarrow \mathrm{End}_W(\mathrm{gr}\, B)_W$. We assert that β is an isomorphism. For this, it is sufficient to prove that gr B is isomorphic to RW as a two sided RW-module. This results from the fact that gr B, viewed as a $\mathbf{Q}$W-bimodule, affords the two sided regular representation of $\mathbf{Q}$W, because $\mathbf{Q}$W is a semisimple $\mathbf{Q}$-algebra

The first statement of Lusztig's theorem follows by taking $\eta = \beta^{-1} \circ \alpha$.

The second statement is proved by showing that if h belongs to the kernel of $1 \otimes \eta$, for $h \in \mathbf{H}^{\mathbf{Q}(t)}$, then the left multiplication by h is zero on gr B, and hence h acts as a nilpotent endomorphism of B. It follows that $1 \otimes \eta$ is injective, since $\mathbf{H}^{\mathbf{Q}(t)}$ is a semisimple algebra over $\mathbf{Q}(t)$, by the discussion in §3. The algebras $\mathbf{H}^{\mathbf{Q}(t)}$ and $\mathbf{Q}(t)W$ have the same dimension over $\mathbf{Q}(t)$, and it follows that $1 \otimes \eta$ is an isomorphism, completing the proof.

We remark that Lusztig's proof that gr B is an $(\mathbf{H}, RW)$-bimodule uses the theory of primitive ideals in enveloping algebras ([24], Lemma 4.1). The proof given here (using the Associativity Lemma) is based instead on the positivity theorem (6.3).

(8.3) Corollary. *The algebra* $\mathbf{H}^{\mathbf{Q}(t)}$ *is split semisimple, in other words,* $\mathbf{Q}(t)$ *is a splitting field for* $\mathbf{H}$.

This follows from part (ii) of the preceding theorem, and the fact that $\mathbf{Q}$ is a splitting field for each finite Weyl group W.

Following Lusztig [25], we shall describe an explicit connection between simple modules for $\mathbf{H}^{\mathbf{Q}(t)}$ and $\mathbf{Q}(t)W$, and their characters.

Let E be a simple $\mathbf{Q}(t)W$-module. There exists a unique two sided cell $\mathbf{c}$ such that E occurs as a composition factor of the direct summand

(8.4) $$M_{\mathbf{c}} = \mathbf{Q}(t) \otimes_R (B_{\mathbf{c}}/B'_{\mathbf{c}})$$

of $\mathbf{Q}(t)\otimes_R \mathrm{gr}\,B$, since $\mathbf{Q}(t)\otimes_R \mathrm{gr}\,B$ affords the two sided regular representation of $\mathbf{Q}(t)W$, by the proof of Theorem 8.2.

Since $M_{\mathbf{c}}$ is a two sided W-module, the inner tensor product $M_{\mathbf{c}}\otimes E$ is a W-module, and the set of W-fixed points

$$E(t) = \mathrm{inv}_W(M_{\mathbf{c}}\otimes E) \tag{8.5}$$

is a left $\mathbf{H}^{\mathbf{Q}(t)}$-module, by the proof of Theorem 8.2.

The following result follows readily from the preceding discussion. For a proof, and other facts connected with it, we refer the reader to [25].

(8.6) PROPOSITION. Let E be a simple $\mathbf{Q}(t)W$-module, and let $E(t)$ be the left $\mathbf{H}^{\mathbf{Q}(t)}$-module defined by (8.5). The following statements hold.

(i) $E(t)$ is an absolutely simple $\mathbf{H}^{\mathbf{Q}(t)}$-module, and $E \longrightarrow E(t)$ defines a bijection of isomorphism classes of simple modules, for the algebras $\mathbf{Q}(t)W$ and $\mathbf{H}^{\mathbf{Q}(t)}$, respectively.

(ii) For each $x\in W$, $\mathrm{Tr}(e_x, E(t))\in\mathbf{Z}[t]$, and we have

$$\mathrm{Tr}(e_x, E(t))_{t=1} = \mathrm{Tr}(x, E)$$

9. The algebra J.

We shall define a $\mathbf{Z}$-algebra J, with a basis whose structure constants are the leading terms $\gamma_{x,y,z}$ of the structure constants $\{b_{x,y,z}\}$ of $\mathbf{H}$ with respect to the Kazhdan-Lusztig basis. It will turn out that $\mathbf{Q}(t)\otimes_{\mathbf{Z}} J \cong \mathbf{H}^{\mathbf{Q}(t)}$, so that J can be viewed as an asymptotic form of $\mathbf{H}$. These results, all due to Lusztig, first appeared in [28]-[30].

For the definition of J, let $\{j_x\}_{x\in W}$ be a basis for a free $\mathbf{Z}$-module, and define a bilinear multiplication on J by setting

$$j_x j_y = \sum_{z\in W} \gamma_{xyz} j_z, \quad \text{for } x, y\in W,$$

where $\gamma_{x,y,z}$ is the coefficient of $t^{a(z)}$ in $h(x,y,z)$ (see (6.4)).

(9.1) THEOREM. *The* **Z**-*algebra* J *is associative, with identity element* $\sum_{d\in\mathcal{D}} j_d$, *where* $\mathcal{D}$ *is the set of Duflo involutions in* W (*see* (6.8)).

Proof. To check associativity, we have to prove that, for all $x,y,z,v\in W$,

$$\sum_{u\in W} \gamma_{x,y,u}\gamma_{u,z,v} = \sum_{u\in W} \gamma_{x,u,v}\gamma_{y,z,u} .$$

For the nonzero terms in this expression we have $a(x) = a(y) = a(z) = a(u) = a(v) = a$ by Theorem 6.10. From the associativity of **H**, we have

$$\Sigma_u\ h_{x,y,u}h_{u,z,v} = \Sigma_u\ h_{x,u,v}h_{y,z,u}.$$

For each nonzero term we obtain from (5.4),

$$v \le_R u \le_R x,$$

and hence $a(x) \le a(u) \le a(v)$ by Lemma 6.9. Therefore the sums above are taken over elements $u\in W$ for which $a(u) = a$. The desired associativity result follows by comparing the coefficients of t^{2a} on both sides of the equation.

In order to check that $\Sigma_{d\in\mathcal{D}}\ j_d$ is the identity element, we have to show, for example, that

$$j_x(\sum_{d\in\mathcal{D}} j_d) = j_x \quad \text{for} \quad x\in W.$$

This amounts to showing that

$$\sum_{d\in\mathcal{D}}\gamma_{x,d,y} = \begin{cases}1 & \text{if} \quad y = x\\ 0 & \text{if} \quad y \ne x\end{cases} .$$

By Theorem 6.10, we have

$$\gamma_{x,d,y} = \gamma_{y^{-1},x,d},$$

and this is 0 or 1 according as $y \neq x$ or $y = x$ by Theorem 6.8. A similar argument shows that $(\Sigma j_d) j_x = j_x$, completing the proof.

The connection between J and $\mathbf{H}$ is described in the following result.

(9.2) THEOREM. <u>The</u> R-<u>linear map</u> $\psi : \mathbf{H} \to R \otimes_{\mathbf{Z}} J$, <u>defined by</u>

$$\psi(b_x) = \sum_{\substack{z \in W \\ a(d)=a(z) \\ d \in \mathcal{D}}} h_{x,d,z} j_z, \quad x \in W$$

<u>is a homomorphism of</u> R-<u>algebras</u>, <u>and becomes an isomorphism when tensored with</u> $\mathbf{Q}(t)$.

<u>Proof</u>. We first prove that $\psi(b_x b_y) = \psi(b_x)\psi(b_y)$, for $x, y \in W$. This comes down to proving that

$$(9.3) \qquad \sum_{\substack{u \in W \\ a(d)=a(z) \\ d \in \mathcal{D}}} h_{x,y,u} h_{u,d,z} = \sum_{\substack{d,e \in \mathcal{D} \\ a(d)=a(u) \\ a(e)=a(v) \\ u,v \in W}} h_{x,d,u} h_{y,e,v} \gamma_{u,v,z}.$$

In order to prove this identity, we begin with the fact that the R-module $\operatorname{gr} B$ defined in §8 is an $(\mathbf{H}, \mathbf{H}')$-bimodule, so that we have

$$\sum_{\substack{u \in W \\ a(u)=a(d)}} h_{x,d,u}(t) h_{u,v,z}(t') = \sum_{\substack{u \in W \\ a(u)=a(d)}} h_{x,u,z}(t) h_{d,v,u}(t'),$$

for a fixed $d \in \mathcal{D}$, and $x, v, z \in W$. The degree in t' of the polynomial on the right side is $\leq a(d)$, and since the structure constants have positive coefficients, the same is true for the left side. Multiplying both sides by $(t')^{-a(d)}$ and setting $t' = 0$, we obtain

$$\sum_{\substack{u\in W\\ a(u)=a(d)}} h_{x,d,u}\gamma_{u,v,z} = \sum_{\substack{u\in W\\ a(u)=a(d)}} h_{x,u,z}\gamma_{d,v,u} .$$

Since $\gamma_{d,v,u} = \gamma_{v,u^{-1},d}$ by (6.10), and is zero except when $u = v$ when it is 1, by (6.8), the whole expression becomes $h_{x,v,z}$, and (9.3) follows from the associativity formula in $\mathbf{H}$.

We prove next that $\psi(b_1)$ is the identity element in $J^{\mathbf{Q}(t)}$. We have

$$\psi(b_1) = \sum_{\substack{z\in W\\ a(d)=a(z)\\ d\in\mathcal{D}}} h_{1,d,z}j_z,$$

and the desired result follows from the fact that

$$h_{1,d,z} = \begin{cases} 1 & \text{if } z = d \\ 0 & \text{if } z \neq d \end{cases} .$$

Finally, we check that the extended map $\hat{\psi}:\mathbf{H}^{\mathbf{Q}(t)} \longrightarrow J^{\mathbf{Q}(t)}$ is an isomorphism. The algebra $J^{\mathbf{Q}(t)}$ has the basis $\{t^{-a(x)}(1\otimes j_x),\ x\in W\}$. We have

$$\hat{\psi}(1\otimes b_x) = \sum_{z\in W} \xi_{x,z}t^{-a(z)}(1\otimes j_z),$$

where the matrix $(\xi_{x,z})$ has entries in $\mathbf{Q}[t]$. Using the properties of the elements $\gamma_{x,d,z}$ from §6, it follows that $\xi_{x,z} - \delta_{x,z}\in t\mathbf{Q}[t]$ for all $x,z\in W$, and hence the matrix $(\xi_{x,z})$ is invertible over the ring of formal power series in t. Therefore $(\xi_{x,z})$ is invertible over $\mathbf{Q}(t)$, and we conclude that the map $\hat{\psi}$ is an isomorphism completing the proof of the Theorem.

(9.4) COROLLARY. <u>There exists an isomorphism of</u> $\mathbf{Q}$-<u>algebras</u> $J^{\mathbf{Q}} \cong \mathbf{Q}W$.

This is proved using the preceding theorem by setting $t = 1$. We have $\mathbf{H}_{(1)} \cong \mathbf{Q}W$. In order to prove that the specialized homomorphism $1\otimes\psi : \mathbf{H}_{(1)} \longrightarrow J^{\mathbf{Q}}$ is an isomorphism, it is sufficient to prove that the kernel of $1\otimes\psi$ is annihilated by all the irreducible characters of $\mathbf{H}_{(1)}$. This follows from the correspondence between the irreducible characters of $\mathbf{H}^{\mathbf{Q}(t)}$ and $J^{\mathbf{Q}(t)}$ given by the isomorphism $\hat{\psi}$, and the relation between these irreducible characters and those of the specialized algebras $\mathbf{H}_{(1)}$ and $J^{\mathbb{Q}}$, respectively (see (8.6), and a version of (3.1) for $J^{\mathbb{Q}(t)}$, proved in §68 of [8]).

The partition of W into left cells is related to the multiplication of the basis elements $\{j_x\}_{x\in W}$ by the next result.

(9.5) PROPOSITION. (i) Let Γ be a left cell containing the element $d\in\mathcal{D}$ (see (6.11)). Then we have

$$j_x j_d = \begin{cases} j_x & \text{if } x\in\Gamma \\ 0 & \text{if } x\notin\Gamma \end{cases}.$$

(ii) $j_x j_y \neq 0 \Leftrightarrow x \sim_L y^{-1}$.

(iii) $j_x j_z j_y \neq 0$ for some $z \Leftrightarrow x \sim_{LR} y$.

The proof follows easily from the connections between the cell relations and the elements $\gamma_{x,y,z}$ established in §6, and is left as an exercise for the reader.

(9.6) COROLLARY. (i) The elements $\{j_d\}_{d\in\mathcal{D}}$ are orthogonal idempotents in J, and give a decomposition of J as a direct sum of left ideals

$$J = \sum_{\Gamma=\text{left cell}} J_\Gamma,$$

where J_Γ is the left ideal generated by the idempotent j_d, for $d\in\mathcal{D}\cap\Gamma$, and has the $\mathbf{Z}$-basis $\{j_x : x\in\Gamma\}$.

(ii) Let Γ,Γ' be left cells in W. Then there is an isomorphism of $\mathbf{Z}$-modules.

$\mathrm{Hom}_J(J_\Gamma,J_{\Gamma'}) \cong j_d J j_{d'}$, where $d\in\mathcal{D}\cap\Gamma$, $d'\in\mathcal{D}\cap\Gamma'$, and $j_d J j_{d'}$ has a $\mathbf{Z}$-basis consisting of the elements $\{j_x : x\in\Gamma^{-1}\cap\Gamma'\}$.

We also obtain a decomposition of J as a direct sum of two sided ideals with bases indexed by the elements in the two sided cells. The submodules $\{j_d J j_{d'}\}$ behave like matrix units within these ideals.

Finally, we note that the $\mathbf{Z}$-linear map $\tilde{\tau}: J \rightarrow \mathbf{Z}$ defined by:

$$\tilde{\tau}(j_z) = \begin{cases} 1 & \text{if } z \in \mathcal{D} \\ 0 & \text{if } z \notin \mathcal{D} \end{cases}$$

defines a nondegenerate symmetric associative bilinear map from $J \times J \longrightarrow \mathbf{Z}$. The elements

$$\{j_x\}_{x \in W} \quad \text{and} \quad \{j_{x^{-1}}\}_{x \in W}$$

form a pair of dual bases with respect to the form. These remarks provide a basis for the usual connections between irreducible characters, orthogonality relations, and primitive central idempotents in the split semisimple $\mathbf{Q}(t)$-algebra $J^{\mathbf{Q}(t)}$, as in ([7], §9).

10. <u>Leading terms of irreducible character values</u>.

Let E be a simple $\mathbf{Q}(t)W$-module. Then E is associated to a unique two sided cell $\mathbf{c}$ of W, by the discussion in §8 (see (8.4)). By (8.5), E corresponds to an absolutely simple $\mathbf{H}^{\mathbf{Q}(t)}$-module $E(t)$, and to an absolutely simple $J^{\mathbf{Q}(t)}$-module $\tilde{E}$, by (9.4).

(10.1) THEOREM. <u>Let</u> $a(\mathbf{c})$ <u>denote the common value of the</u> a-<u>function on the elements of the two sided cell</u> $\mathbf{c}$ <u>corresponding to</u> E. <u>Let</u> e_x <u>be the standard basis element of</u> $\mathbf{H}$ <u>corresponding to</u> $x \in W$. <u>Then the leading term of the character value on</u> $E(t)$ <u>of</u> e_x <u>is given by</u>

(i) $\operatorname{Tr}(e_x, E(t)) = c_{x,E}t^{l(x)+a(\mathbf{c})}$ + <u>terms involving lower powers of</u> t, <u>where</u> $c_{x,E}$ <u>is an integer, for each</u> $x \in W$.

(ii) <u>We have</u> $c_{x,E} \neq 0$ <u>for some</u> $x \in \mathbf{c}$. <u>Moreover, for all</u> $x \in W$, $c_{x,E}$ <u>is itself a character value on the algebra</u> $J^{\mathbf{Q}(t)}$, <u>namely</u>

$$c_{x,E} = \operatorname{Tr}(j_x, \tilde{E}),$$

<u>where</u> j_x <u>is the basis element of</u> J <u>corresponding to</u> x.

<u>Proof</u>. By (8.6ii), we have $\operatorname{Tr}(e_x, E(t)) \in \mathbf{Z}[t]$. The characters of the simple modules $E(t)$ and $\tilde{E}$ are related by the isomorphism $\hat{\psi}: \mathbf{H}^{\mathbf{Q}(t)} \longrightarrow J^{\mathbf{Q}(t)}$ (see (9.2)). We obtain

$$(10.2)\qquad \mathrm{Tr}(b_x, E(t)) = \mathrm{Tr}(\hat{\psi}(b_x), \tilde{E}) = \sum_{\substack{z\in W\\ a(z)=a(d)\\ d\in\mathcal{D}}} h_{x,d,z}\mathrm{Tr}(j_z, \tilde{E}),$$

after identifying b_x with $1\otimes b_x$, and j_z with $1\otimes j_z$. Using the results in §9, it follows that $\mathrm{Tr}(j_z,\tilde{E}) = 0$ if $z\notin \mathbf{c}$. Furthermore $\mathrm{Tr}(j_z,\tilde{E})\in\mathbf{Z}$ for all z, by (9.4). The term of highest degree in t on the right side of (10.2) has degree $a(\mathbf{c})$, and the coefficient of $t^{a(\mathbf{c})}$ is

$$\sum_{\substack{z\in\mathbf{c}\\ a(d)=a(\mathbf{c})\\ d\in\mathcal{D}}} \gamma_{x,d,z}\mathrm{Tr}(j_z,\tilde{E}) = \mathrm{Tr}(j_x,\tilde{E})\in\mathbf{Z},$$

since $\gamma_{x,d,z} = \gamma_{z^{-1},x,d}$, and is 1 or 0 according as $z = x$ or $z \neq x$, by §6.

By the definition of the Kazhdan-Lusztig basis and the properties of the polynomials $P_{x,y}$, it follows that the transition matrix from the Kazhdan-Lusztig basis to the standard basis is triangular (using an ordering based on the Bruhat order), with diagonal elements $t^{-l(x)}$, for $x\in W$. Therefore, $\mathrm{Tr}(e_x,E(t)) = c_{x,E}t^{l(x)+a(\mathbf{c})}$ + terms involving lower powers of t, for some integer $c_{x,E}$. Combining these observations, we obtain a proof of the Theorem.

As an application, we show that the value of the a-function on the two sided cell $\mathbf{c}$ can be calculated from the generic degree D_E of the simple $\mathbf{Q}(t)W$-module E associated to $\mathbf{c}$.

(10.3) COROLLARY. Keep the preceding notation. Then

$$a(\mathbf{c}) = N - \deg D_E,$$

where N and $\deg D_E$ denote the degrees in $u = t^2$ of the Poincaré polynomial $P(u)$ of W, and the generic degree D_E of E, respectively (see §3).

Proof. The orthogonality relations for the characters of the modules $E(t)$ are

$$\sum_{x\in W} t^{-2l(x)}\mathrm{Tr}(e_x,E(t))\mathrm{Tr}(e_{x^{-1}},E'(t)) = \begin{cases} 0 & \text{if } E\not\cong E' \\ \dfrac{\dim E\cdot P(u)}{D_E(u)} & \text{if } E\cong E' \end{cases}$$

By the preceding Theorem, the left side is

$$t^{2a(\mathbf{c})} \sum_{x\in W} c_{x,E}c_{x^{-1},E} + \text{lower terms},$$

and the coefficient of $t^{2a(\mathbf{c})}$ is different from zero by the orthogonality relations for the irreducible characters of $J^{\mathbb{Q}(t)}$, by §9, using the fact that $c_{x,E} = \mathrm{Tr}(j_x,\tilde{E})$.

<u>Remark</u>. In ([28], (6.4)), Lusztig proved, using an a-function defined in terms of another basis of **H**, that the value of the a-function on the elements of the two sided cell **c** attached to E is a_E, the exponent of u in the lowest term of the generic degree. By the palindromic property of the generic degrees (see [8], (71.17)), the formula in Corollary 10.3 becomes

$$a(\mathbf{c}) = N - \deg D_E = a_{\mathrm{sgn}\otimes E},$$

thus reconciling (10.3) with Lusztig's formula, since $\mathrm{sgn}\otimes E$ is associated with the two sided cell $w_0\mathbf{c}$, where w_0 is the element of W of maximal length.

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