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A GEOMETRIC CLASSIFICATION OF POSITIVELY CURVED SYMMETRIC SPACES AND THE ISOPARAMETRIC CONSTRUCTION OF THE CAYLEY PLANE.

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INTRODUCTION.

In a series of lectures in Rome (June 1986) I explained that surprisingly many examples to the areas of minimal submanifolds, positively curved spaces and Einstein manifolds arise rather closely together : in the backyard of isoparametric hypersurfaces in spheres.

Isoparametric hypersurfaces.

Cartan [Ca] started the study of hypersurfaces with constant principal curvatures. He called them isoparametric. In euclidean and hyperbolic space there are only distance tubes around totally geodesic subspaces. In spheres Cartan found other ones, all (four) with $g = 3$ different principal curvatures and two with $g = 4$. All homogeneous examples were classified [HL] and their geometric data determined [TT] : there are also infinitely many with $g = 4$ different principal curvatures and two with $g = 6$. The homogeneous ones occur as orbits of the isotropy groups of symmetric spaces of rank 2. — The general theory was developed in [Mü] and refined in [Ab] . The first nonhomogeneous examples were given in [OT] and later many more in [FKM] ; they all have $g = 4$ different principal curvatures. Isoparametric hypersurfaces come as families of parallel surfaces, as distance-tubes around two "opposite" focal submanifolds of higher codimension. These families can be described as levels of homogeneous polynomials of degree g , intersected with the unit sphere.

The manuscript was written while the author enjoyed the hospitality of the IHES and the criticism of N.H. Kuiper. Comments by D. Ferus of the first draft lead to several improvements. For the development of the ideas on symmetric spaces W. Ziller's presence at the Max-Planck Institute in Bonn was extremely valuable.

Minimal submanifolds.

The focal submanifolds of isoparametric families are always minimal [No] . One has for example nontrivial minimal embeddings of some products $S^m \times S^n$ and also of canonical sphere bundles over spheres connected with Clifford representations. The largest volume hypersurface in an isoparametric family is also minimal ; in fact, noncongruent examples occur which agree in all curvature data and the volume [OT,FKM]; some of these are even diffeomorphic [Wg]. The cone of a minimal hypersurface from the center of the sphere is a minimal cone in $\mathbb{R}^n$. Most of the minimal cones which come from isoparametric hypersurfaces can be embedded in a foliation of minimal hypersurfaces of $\mathbb{R}^n$ which are regular except for that one conical singularity; these cones are therefore absolutely volume minimizing [BDG,La,FK]. If one intersects this foliation with a sphere around the singularity one gets (up to a scaling) the original isoparametric family back, and this picture is used to construct the foliation. A spherical version of this idea [Hs1] led to the discovery of many (and rather different) minimal hypersurface embeddings of spheres (and even more minimal immersions [HT]) [Hs2,FK,To] . The cones over many of these are known to be at least stable [HS]. Minimal immersions of exotic spheres have been found with this technique [HHS] and also a very rich collection of imbedded constant mean curvature hyperspheres in spheres [St].

Positively curved spaces.

Apart from the positively curved symmetric spaces (the compact ones of rank 1) one knows two examples by Berger [Bg] , three by Wallach [Wl] and an infinite family of quotients of SU(3) by Aloff and Wallach [AW]; these are all the homogeneous ones [Be,Wl]. Nonhomogeneous manifolds with positively curved metrics were found by Eschenburg, one in dimension 6 and a twoparameter family of quotients of SU(3) , quotients by actions of 1-dimensional isometry groups, not subgroups. For all this, see [Es] and the bibliography there. Connections with isoparametric hypersurfaces are the following. Berger's second example is the 13-dimensional focal submanifold of the isoparametric family given as SU(5)-orbits in $\Lambda^2\mathbb{C}^5$ (or as levels of $\|\omega\wedge\omega\|^2$). The $g = 3$ isoparametric examples are the distance tubes around the Veronese embeddings of the projective planes over the division algebras $K \approx \mathbb{R}^m$ ($m = 1,2,4,8$); a rather simple change of the hypersurface metric gives positively curved metrics. If $m = 1$, one has Berger's metrics on $SO(3)$ and $m > 1$ gives Wallach's three examples. The other (positively curved) examples are quotients of $SP(2)$ (Berger) or $SU(3)$. $SP(2)$ is the 10-dim focal submanifold M_+ of the homogeneous $g = 4$ family in S^{15} and $SU(3)$ is the 8-dim focal submanifold M_+ of the homogeneous $g = 4$ family in S^{11}. It is not yet known which of the isometry groups, divided out by Aloff-Wallach and Eschenburg, is compatible with the isoparametric embeddings. Furthermore, Eschenburg finds the positively curved metrics with a rather small amount of computation as follows : Consider $H \subset K \subset G$ such that G/K is symmetric of rank 1 and the normal homogeneous metric on K/H has positive curvature; write $\mathfrak{g} = k + m, [m,m] \subset k$ and $k = h + p$; if for every $0 \neq X \in m$, $0 \neq Y \in p$ holds $[X,Y] \neq 0$, then there exists a positively curved metric on G/H. The main idea is to improve first the metric on G from the biinvariant one to the submersion metric of $\pi: G \times K \to G, (g,k) \to g.k^{-1}$. This simplifies the treatment of all homogeneous examples (except the $SP(2)$-quotient) very much. The improved metric is also crucial for the nonhomogeneous examples : The quotient metric is positively curved if no 2-dim flat torus (of G) is perpendicular to the orbits of the isometric action. - This renews the desire to understand the classification of positively curved symmetric spaces as geometrically as possible. The classification leads (via totally geodesic cut loci) rather quickly to the case where the positively curved symmetric space is like a projective plane. But then, we can read the division algebra directly from the geometry. This brings us back to the isoparametric hypersurfaces : The definition of the Cayley plane in terms of the Cayley numbers is somewhat lengthy while the isoparametric polynomial which gives the Cayley plane as focal submanifold of an isoparametric family can be written down directly.

Einstein manifolds.

Again, a simple change of the hypersurface metric of the $g = 3$ examples ($m = 1$ is trivial) produces two different Einstein metrics on each of them. This follows the strategy that in a k-parameter family of Riemannian metrics where Ricci has at most k different eigenvalues, it is worth to look for an Einstein metric. Similar attempts with the $g = 4$ examples are considerably more complicated; so far they have not been successful.

The following notes contain only those parts of my lectures which have not (in the way I said things) appeared in print. That is, I refer to the literature for minimal submanifolds. I give the geometric classification of positively curved symmetric spaces and the isoparametric construction of the Cayley plane. Curvature computations are expanded to give the positively curved metrics and the Einstein metrics on the tubes around the Veronese embeddings of the projective planes. A slightly simplified version of the treatment of division algebras in [GWZ] is included since details are used. I did explain in Rome why quotients by isometry groups are curvature increasing. Although the details were different from the usual O'Neil submersion treatment, they are not included here. I did not get to saying anything about Einstein manifolds in spite of the promise in the first lecture.

1. POSITIVELY CURVED SYMMETRIC SPACES.

Definition. A Riemannian manifold M is called a symmetric space, if for each $p \in M$ there is an isometry σ_p (called symmetry at p) which satisfies

$$\sigma_p(p) = p \ , \ \partial\sigma_p\big|_p = -\mathrm{id} : T_pM \to T_pM.$$

Part of the following results (with different proofs) are due to Chavel [Ch], the arguments are closer to [GWZ]. See also [He].

1.1 $$DR = 0 \ ,$$

since $\partial\sigma_p = -\mathrm{id}$ leaves DR invariant, i.e.

$$-(D_xR)(U,V)W = (D_{-x}R)(-U,-V)-W.$$

1.2 M is homogeneous,
namely, let $p,q \in M$ be arbitrary and m the midpoint of a geodesic segment from p to q then

$$\sigma_m(p) = q \ .$$

1.3. Each parametrized geodesic C defines a family of translations $T_t = \sigma_{C(\frac{t}{2})} \circ \sigma_{C(0)}$. One has that the derivative ∂T_t equals Levi-Civita parallel translation $P(0,t)$ along C from $C(0)$ to $C(t)$.

Proof. $DR = 0$ implies that the Jacobi equation can be solved (up to Levi-Civita translation) by a power series in $R(\ ,\dot{C})\dot{C}$. In particular, for a Jacobi field with $J(0) = 0$ holds $J(-t) = -P(t,-t).J(t)$. Now let $X \in T_{C(0)}M$ be arbitrary and J be the Jacobi field along C with $J(\frac{t}{2}) = 0$ and $J(0) = -X$, then $\partial T_t \cdot X = J(t) = P(0,t) \cdot X$.

1.4. The translations along a geodesic C form a group since, because of (1.3), the differentials of the isometries $T_t \circ T_\tau$ and $T_{t+\tau}$ agree. The corresponding Killing vectorfield $X = \frac{\partial}{\partial t} T_t$ is called a transvection along C . Its existence shows that all geodesic loops are closed geodesics : $C(t_1) = C(0)$ implies
$C(0) = X(C(0)) = X(C(t_1)) = C(t_1)$.

1.5. (1) The differential DX of a transvection is 0 along C .

(2) At each $p \in M$ and for all $X,Y \in T_pM$ there is a Killing field Z with $Z(p) = 0$, $DZ|_p = R(X,Y)$.

Proof. (1) $D_uX = \frac{D}{\partial s} (\frac{\partial}{\partial t} T_t(\exp su)) = \frac{D}{\partial t} (\partial T_t \cdot u) = 0$ by (1.3).

(2) If X,Y are Killing fields with $DX|_p = DY|_p = 0$ then the Killing field $Z = [X,Y]$ satisfies $Z(p) = (D_XY - D_YX)|_p = 0$

$$D_uZ|_p = D^2_{u,x}Y - D^2_{u,y}X = -R(Y,u)X + R(X,u)Y = R(X,Y)u$$

by the differential equation for Killing fields and the first Bianchi identity. (2) follows since there are enough transvections.

1.6. If for one unit vector $v \in T_pM$ the Jacobi operator $J \to R(J,v)v$ has <u>positive</u> eigenvalues, then the isotropy group at p is transitive on the unit sphere. By homogeneity (1.2) this follows for all p.

Proof. The Jacobi operator is invertible, i.e. to every $y \in v^\perp$ we find y_v such that $R(y_v,v)v = y$. The isotropy Killing field Z from 1.5.2 with $DZ|_p = R(y_v,v)$ moves v in direction y . The orbit of the isotropy group at p therefore covers a neighborhood of v , hence is the whole unit sphere.

1.7. Because of (1.6) each cut locus is spherical. We normalize the metric to

$$\text{injectivity radius} = \text{diameter} = \frac{\pi}{2}\ .$$

1.8. If one and therefore all cutpoints of p are <u>not conjugate</u> then M is isometric to $\mathbb{R}P^n$ (curvature 1).

<u>Proof</u>. Each cut point q has at least two geodesic segments to p. Their angle at q is π since otherwise there were closer cut points. Then, by (1.4), also the angle at p is π. The symmetry σ_p therefore fixes all cut points : The cut locus is a totally geodesic submanifold and locally a distance sphere ! Let λ_i^2 be the eigenvalues of the Jacobi operator ; then $\kappa_i = \lambda_i \cdot \mathrm{ctg}(\lambda_i \cdot r)$ are the principal curvatures of any distance sphere of radius r. From $\kappa_i(\frac{\pi}{2}) = 0$ (totally geodesic) and no conjugate points before $\frac{\pi}{2}$ follows $\lambda_i = 1$. The space is of constant curvature 1 with
injectivity radius = diameter = $\frac{\pi}{2}$ i.e. it is isometric to $\mathbb{R}P^n$.

1.9. (1) If one cutpoint and therefore all <u>are conjugate</u> (at distance $\frac{\pi}{2}$) then the largest eigenvalue of the Jacobi operator is 4. Define for all unit vectors X the m-dimensional subspaces
$U_X^m := \mathrm{Span}\{X, \text{Eigenspace to eigenvalue 4 of } R(\ ,X)X\}$.

(2) For these we have $y \in U_X \Rightarrow U_Y = U_X$.

(3) All geodesics are closed and the cut locus is again a totally geodesic submanifold.

<u>Proof</u>. (1) follows from $DR = 0$ - Every 2-dim subspace of U_X which contains X has by definition curvature 4 ; $\exp_p(\frac{\pi}{2}\cdot\ )$ maps therefore every unit circle in U_X from X to $-X$ to a curve of length zero, i.e. the whole unit sphere in U_X is mapped to one conjugate point q. This implies first $U_Y \supset U_X$ and (2) follows. It also implies that the geodesic rays in the directions $\pm X$ meet at q - as a closed geodesic by (1.4). As before (in (1.8.1) we have that the symmetry σ_p fixes all cut points of p, i.e. the cut locus is totally geodesic.

1.10. Either $U_X = T_pM$ and M is a sphere of curvature 4, or else $R(\ ,X)X$ has precisely one more eigenvalue $\lambda_i^2 = 1$, i.e. M is $\frac{1}{4}$-pinched. In this latter case (1.9.2) says that the unit sphere in T_pM is foliated by the unit spheres of the subspaces U_X ("Hopf-foliation").

Proof. $U_X^{\perp}$ is spanned by the eigenspaces of $R(\ ,X)X$ to eigenvalues $\lambda_i^2 < 4$. The distance spheres have then the principal curvatures $\lambda_i \cdot \operatorname{ctg} \lambda_i r$, $2 \operatorname{ctg} 2r$; those for $\lambda_i < 2$ converge to principal curvatures of the totally geodesic cut locus - for the normals which are given by the tangents of the geodesics under consideration ($\exp_p s \cdot Y$, $Y \in U_X$). Again, $\operatorname{ctg}(\lambda_i \cdot \frac{\pi}{2}) = 0$ with $\lambda_i < 2$ implies $\lambda_i = 1$.

1.11. Assume that M is not a sphere of curvature 4.

(1) Each cut locus of M is then - as nontrivial totally geodesic submanifold (1.9.3) - again a positively curved symmetric space M'.

(2) $Z \perp U_X$ implies $U_Z \perp U_X$,
in particular, if $Z \in T_q M'$ then $U_Z \subset T_q M'$.

(3) Each cut locus M' is because of (1),(2) either an m-sphere of curvature 4 or else its cut locus is a positively curved symmetric space M'', still with m-dim Hopf-subspaces U_X''. Under each further repetition the dimension decreases by m - until the cut locus is an m-sphere.

Proof of (1.11.2). For $Z \perp U_X$ and $Y \in U_X$ we have $U_X = U_Y$ (1.9.2) hence from (1.10) $K(Y \wedge Z) = 1$ and therefore (again 1.10) $Y \perp U_Z$, i.e. $U_X \perp U_Z$. - If $Z \in T_q M'$ then Z is perpendicular to the normal space of M', but the normal space of a cut locus is the tangent space of a sphere of curvature 4, i.e. some U_X (see 1.9.1 or 1.10).

1.12. Let c be a geodesic, $c(0) = q$ and let T_t be the group of translations of c (1.4). Let $v \perp U_{\dot c}$, $|v| = 1$. Then

(1) $p := \exp_q \frac{\pi}{2} v$ is a fixed point of T_t.

(2) As q is moved by T_t along c to the cut point $\tilde q = c(\frac{\pi}{2})$ the initial direction at p of the geodesic pq is rotated by 90° in the 2-plane which is obtained by parallel translation of $\dot c(0)$ from q to p along $\exp_q sv$.

Proof. The Killing field $X = \frac{\partial}{\partial t} T_t$ satisfies $DX|_q = 0$, $X(q) = \dot c(0)$.

Along the geodesic $\gamma(s) = \exp_q s.v$ it restricts to a Jacobi field J, which because of (1.11.2) is in the +1 eigenspace of $R(\ ,\gamma')\gamma'$. Up to parallel translation we therefore have $J(s) = \cos s\ J(0)$, i.e. $J(\frac{\pi}{2}) = X(p) = 0$ proves (1). (2) follows since $-J'(\frac{\pi}{2})$ equals parallel translation of $\dot{c}(0)$.

1.13. Projective properties. Even before the positively curved symmetric spaces are explicitly determined one can quite well deal with their projective structure.

Definition. Call the m-dim totally geodesic spheres of curvature 4 "projective lines" and call the cut loci of points "projective hyperplanes". If $\dim M = 2m$ call M "projective plane".

(1) Any two points in M can be joined by a geodesic, hence by the projective line it defines via $\exp U_{\dot{c}}$.

(2) Projective hyperplanes are indeed projective subspaces since a projective line (1) which joins two if its points is contained in the hyperplane by (1.11.2).

(3) A line L which is not contained in a hyperplane meets it at most in one point. - Because of (1) it is enough to show that two different lines meet at most in one point. If two different lines meet they meet transversally because of (1.9.2) hence in isolated points. But the radial geodesics which leave such a point p and meet again (before they return to p) belong to the same projective line.

(4) A projective line L and a hyperplane H = (cut locus of p) always intersect. - Either $L \subset H$; or else, if $p \in L$, then the antipodal point q of p on the m-sphere L is in $C(p)$; or finally the largest distance from L to H is positive and $< \frac{\pi}{2}$, hence this distance is realized by a geodesic c which is perpendicular to L and H. Because of (1.11.2) we then have $U_{\dot{c}}$ perpendicular to L and to H; since $\dim H + \dim U_{\dot{c}} = \dim M$ we can therefore use translation along c (1.4) to move L into H! Because of (1.12.1) these translations have a fixed point on L which then must also be in H. (The indirect argument with the shortest distance gives slightly less information). If $\dim M = 2m$ then (1) - (4) are the axioms of a projective plane.

Finally

1.14 The projective planes have dimension $2m \in \{2,4,8,16\}$ since they determine an m-dimensional division algebra as follows :

For some $p \in M$ let S^m be the cut locus of p and let $q,\tilde{q}$ be antipodal points on S^m . Define

$V_1 := \nu_q S^m$ (normal space of S^m at q) ,

$V_2 := T_q S^m$, $V_3 := \nu_{\tilde{q}} S^m$.

Between these we define a norm preserving bilinear map

$\Pi : V_1 \times V_2 = V_3$, $|\Pi(y_1,y_2)| = |y_1|\cdot|y_2|$

as follows :

$\Pi(y_1,y_2) :=$ Parallel translation of $|y_2|\cdot y_1$ along the meridian of S^m from q to $\tilde{q}$ which has initial direction y_2 .

Clearly $\Pi(.,y_2) : V_1 \times \{y_2\} \to V_3$ is (for unit vectors y_2) a linear isometry in the first argument. But (1.12.2) shows that the map $\Pi(y_1,.) : \{y_1\} \times V_2 \to V_3$ can be described first as parallel translation of V_2 in direction y_1 from q to p , giving the intermediate result $\tilde{V}_2 \subset T_p M$; $\tilde{V}_2$ is the tangent space of another curvature 4 m-sphere $\tilde{S}^m$ with antipodal points $p,\tilde{q}$; secondly, map $\tilde{V}_2$ to $T_{\tilde{q}}\tilde{S}^m$ by the antipodal map, the composition gives $\Pi(y_1,.)$ (1.12.2).

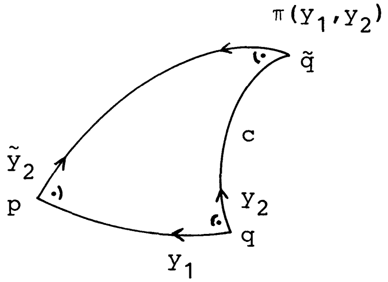

Illustration to (1.12.2) and (1.14). $U_{y_1} \perp U_{y_2}$.

The next section starts with the choice of bases which identify the V_j with $\mathbb{R}^m$ and make Π the product of a division algebra. These are then shown to be $\mathbb{R},\mathbb{C},\mathbb{H}$ or $\mathbb{C}a$. With that classification we have shown that the positively curved symmetric spaces are necessarily projective spaces over these division algebras. For $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ these are easily constructed together with their natural embeddings in (4.1). The non associativity of the Cayley numbers forces the Cayley plane to be nondesarguian (which is not reproved here); the Cayley plane can therefore not be projective subspace of a higher dimensional projective space since such an embedding trivially gives the desarguian property. This excludes the existence of higher dimensional projective spaces over the Cayley numbers and also shows that the existence proof for the Cayley plane cannot use the dimension independent procedure which worked for $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$. -The isoparametric construction of the projective planes which we also give in section 4 works for all division algebras in the same way.

1.15. Remark. In his recent thesis [Ci] Quo-Shin Chi proves the following classification result : If the following two axioms for the curvature tensor of a Riemannian manifold M are satisfied, then M is covered by a rank 1 symmetric space :

(1) The Jacobi operator $R(\ ,v)v$ has for all unit vectors v precisely two eigenvalues b,c (also of constant multiplicity).

(2) The Hopf-spaces $U_v := \mathrm{Span}\{v, c\text{-eigenspace of } R(\ ,v)v\}$ satisfy $w \in U_v \Rightarrow U_w = U_v$.

Chi then constructs the occuring groups. Since the amount of work is considerable, I also want to recall that Tricerri and Vanhecke [TV] found an immediate argument to prove the following : If a Riemannian manifold has pointwise the curvature tensor of an irreducible symmetric space then it indeed is locally symmetric. Namely : First, M is Einstein and hence $D\,\mathrm{Ric} = 0$. Secondly, D^2R is symmetric since the skew symmetric part depends only on the curvature tensor and vanishes in symmetric spaces. Third, the first two results combine to give $\langle\langle \Sigma_{e_i} D^2_{e_i,e_i} R, R \rangle\rangle = 0$ and therefore $0 = \Delta \langle\langle R,R \rangle\rangle = 2 \langle\langle DR, DR \rangle\rangle$.

2. ALGEBRAS OVER $\mathbb{R}$ WITH NORM PRESERVING PRODUCT AND UNIT.

This section follows Gluck, Warner and Ziller [GWZ]. The formulas up to (2.4) will be used in section 3. The remaining part of the classification is included to make the discussion of positively curved symmetric spaces in section 1 selfcontained (where we needed the result not the details of this section).

By definition we have a scalar product (with its euclidean norm) and a product $\cdot$ on $A = \mathbb{R}^m$ such that

(2.1) $|x\cdot y| = |x| \cdot |y|$, $1\cdot x = x\cdot 1 = x$,

in particular, left and right multiplication by unit vectors are isometries, $\langle y\cdot x, y\cdot z\rangle = |y|^2 \cdot \langle x,z\rangle = \langle x\cdot y, z\cdot y\rangle$.

Remark. In geometry (e.g. sections 1.14 and 3.4) such normed algebras arise as bilinear norm preserving maps

$B : V_1 \times V_2 \to V_3$, $|B(x,y)| = |x|\cdot|y|$.

Choose an orthonormal basis $\{e_1,\dots,e_m\}$ for V_1 and pick a unit vector $f_1 \in V_2$. Then define an orthonormal basis $\{g_1,\dots,g_m\}$ for V_3 by $g_i := B(e_i,f_1)$, $i = 1,\dots,m$. Finally obtain an orthonormal basis $\{f_1,\dots,f_m\}$ for V_2 by requiring $B(e_1,f_i) = g_i$, $i = 1,\dots,m$. If one identifies V_j via these bases with $\mathbb{R}^m$, then B defines a product $\cdot$ as in (2.1) with the first basis vector as unit.

Now put

$\mathrm{Im}\, A := 1^\perp$ (imaginary part) ,

$x = x_{\mathbb{R}} + x' \in \mathbb{R}\cdot 1 \oplus \mathrm{Im}\, A$, $x_{\mathbb{R}} = \mathrm{Re}\, x$, $x' = \mathrm{Im}\, x$,

$\bar{x} := x_{\mathbb{R}} - x'$ (conjugate of x) .

Then starting from

$$(1 + |x'|^2)^2 = |(1 + x')\cdot(1 - x')|^2 = \langle 1 - x'^2, 1 - x'^2\rangle ,$$

one has immediately :

(2.2) $(x')^2 = -|x'|^2$, $|x|^2 = x\cdot\bar{x}$, $x^{-1} = |x|^{-2}\cdot\bar{x}$,

$\overline{x\cdot y} = \bar{y}\cdot\bar{x}$, $\mathrm{Re}(x\cdot\bar{y}) = \langle x,y\rangle$, $\mathrm{Re}(x\cdot y) = \mathrm{Re}(y\cdot x)$.

In particular we have inverses, hence the name division algebra. The most important formula is the following rest of associativity.

(2.3) $x\cdot(x^{-1}\cdot y) = y$.

Proof. Associativity for real factors holds by definition. Therefore it suffices to prove (2.3) if $x = x'$ is imaginary and $|x'| = 1$.

Left multiplication by x' is an isometry, even a skew isometry : $\langle x',1\rangle = 0 \Rightarrow \langle x'\cdot y,y\rangle = 0$. For any skew isometry L holds

$\langle y,w\rangle = \langle Ly,Lw\rangle = -\langle y,L(Lw)\rangle$ or $L(Lw) = -w$.

This is (2.3) since $(x')^{-1} = -x'$ because of (2.2).

Immediate consequences of (2.3) and (2.2) are

(2.4) $\mathrm{Re}((x\cdot y)\cdot z) = \langle x\cdot y,\bar{z}\rangle \underset{(2.3)}{=} \langle x,\bar{z}\cdot\bar{y}\rangle = \mathrm{Re}(x\cdot(y\cdot z)) = \mathrm{Re}((yz)x)$.

This would suffice for the classification. Usually the following tri-linear map is considered :

(2.5) $[x,y,z] := (x\cdot y)\cdot z - x\cdot(y\cdot z)$ is alternating.

Proof. The image is zero if one factor is real, hence

$[x,y,z] = [x',y',z']$. Then $[x',x',z] = 0$ and $[x,y',y'] = 0$

follow from $y' = \text{real}\cdot(y')^{-1}$ together with (2.3).

(2.6) Lemma for induction. Let $1 \in A \subset B$ be normed $\mathbb{R}$-algebras (2.1) and $\varepsilon \in A^{\perp}$, $|\varepsilon| = 1$. Then

$(a+b\varepsilon)\cdot(c+d\varepsilon) = (ac - \bar{d}b) + (da + b\bar{c})\varepsilon$.

Proof. $a(d\varepsilon) = (da)\varepsilon$, $(b\varepsilon)c = (b\bar{c})\varepsilon$ and $((b\varepsilon)\cdot(d\varepsilon))\varepsilon = -(\bar{d}b)$ are trivially true if one of the factors involved is real. Therefore it suffices to prove these relations for a,b,c,d imaginary. They follow, since (2.5) implies $x'(y'z) = -y'(x'z)$, with repeated use of (2.3).

Lemma (2.6) suggests to define, for any division algebra A with unit, a product on $A\times A$ by

(2.7) $(a,b)\cdot(c,d) := (ac - \bar{d}b, da + b\bar{c})$;

if this product is norm preserving we have made $A\times A$ into a division algebra, denoted $A \boxplus A$. In particular, if A is a nontrivial subalgebra of another division algebra B and if $\varepsilon \in A^{\perp}$, $|\varepsilon| = 1$, then $A \boxplus A$ is isomorphic to the subalgebra generated by A and ε $((1,0) \to 1, (0,1) \to \varepsilon)$. The product on $A \boxplus A$ turns out to be associative if and only if the product on A is commutative.

(2.8) Examples. $\mathbb{R} \boxplus \mathbb{R} = \mathbb{C}$, $\mathbb{C} \boxplus \mathbb{C} = \mathbb{H}$, $\mathbb{H} \boxplus \mathbb{H} = \mathbb{C}a$.

(2.9) Classification : The division algebras of (2.8) are the only ones.

Proof. If one computes threefold products with (2.7) then one finds : The product on $A \times A$ satisfies (2.5) (or (2.4)) if and only if the product on A is associative. Since the product for $\mathbb{C}a$ is not associative, the product on $\mathbb{C}a \times \mathbb{C}a$ cannot be norm preserving, i.e. $\mathbb{C}a \boxplus \mathbb{C}a$ does not exist.

3. WEYL IDENTITIES FOR ISOPARAMETRIC HYPERSURFACES.

Relations between the principal curvatures and the covariant derivative of the shape operator (second fundamental tensor) are derived by differentiating the Codazzi equations and combining with the Gauss equations. Such computations have a long history and can now be done in proper generality [Wa]. In our case of constant principal curvatures the computations are much simpler, the results are also more specific : the more generally valid formulas are sums over more terms. The usefulness of our shorter formulas is shown by giving quick proofs of two results of Cartan. Our main application is in sections 4 and 5.

Let k_i, k_j be two different principal curvatures of an isoparametric hypersurface in a space of constant curvature $\tilde{K}$; let E_i, E_j be the corresponding eigenspaces of the shape operator S and DS its covariant derivative ; let $X_i \in E_i$, $Y_j \in E_j$ and $\{e_\alpha\}$ an orthonormal eigenbasis of S . Then

$$(3.1)\qquad (\tilde{K} + k_i k_j)\cdot |X_i|^2 \cdot |Y_j|^2 = 2 \sum_{\substack{e_\alpha \\ e_\alpha \perp E_i, E_j}} \frac{\left\langle D_{X_i} S \cdot Y_j , e_\alpha \right\rangle^2}{(k_i - k_\alpha)\cdot(k_j - k_\alpha)} .$$

Clearly the sum is ≤ 0 if k_i, k_j are the smallest and the largest principal curvature and ≥ 0 if k_i, k_j are adjacent principal curvatures. It follows immediately if $\tilde{K} < 0$, and almost immediately if $\tilde{K} \leq 0$, that at most two different principal curvatures are possible (Cartan).

Let $\tilde{K} = 1$ and assume that there are $g = 3$ different principal curvatures. Then (3.1) specializes to

$$(3.2)\quad \frac{(1+k_ik_j)\cdot(k_i-k_k)\cdot(k_j-k_k)\cdot|X_i|^2}{2}\cdot\langle Y_j,W_j\rangle = \langle D_{X_i}S\cdot Y_j, D_{X_i}S\cdot W_j\rangle .$$

We shall see that (3.2) contains the classification of isoparametric hypersurfaces with 3 different principal curvatures. (3.2) is also essential in our construction of the Cayley plane and in the curvature computations.

<u>Proof of</u> (3.1). Let X_i, Y_j be eigenvectorfields of S. Differentiate $S\cdot Y_j = k_jY_j$ and use the Codazzi equation $D_XS\cdot Y = D_YS\cdot X$:

$$D_{X_i}\ S\cdot Y_j = (k_j-S)\cdot D_{X_i}\ Y_j \quad \perp \quad E_j$$

$$= D_{Y_j}\ S\cdot X_i = (k_i-S)\cdot D_{Y_j}\ X_i \quad \perp \quad E_i$$

Hence

$$(3.3)\quad D_{X_i}S\cdot Y_j \in (E_i \cup E_j)^{\perp} .$$

(3.4) <u>Remark</u>. In the case of 3 different principal curvatures (3.3) says $DS : E_i \times E_j \to E_k$. This map is - up to the constant factor on the left of (3.2) - norm preserving and therefore defines a division algebra (2.1). This leaves only the possibilities $\dim E_i \in \{1,2,4,8\}$ (Cartan).

We also get from (3.3) $\langle D_{X_i}S\cdot Y_i, Y_j\rangle = 0$, or

$$(3.5)\quad D_{X_i}S\cdot Y_i = (k_i - S)\cdot D_{X_i}Y_i = 0 ,$$

which says : The curvature distribution E_i is integrable and the curvature leafs are totally geodesic in the isoparametric hypersurface M .

We differentiate (3.5) and use Codazzi :

$$D^2_{Z_j,X_i}S\cdot Y_i + D_{Y_i}S\cdot D_{Z_j}X_i + D_{X_i}S\cdot D_{Z_j}Y_i = 0 .$$

The derivatives of the vectorfields are eliminated with $D_{Y_j}X_i = (k_i - S)^{-1}.D_{Y_j}S\cdot X_i$ (because of (3.3) we use the abbreviation $(k_i - S)^{-1} = ((k_i - S)|_{E_i^{\perp}})^{-1})$:

$$(3.6)\quad \left\langle D^2_{Z_j,X_i} S\cdot Y_i, W_k\right\rangle = -\left\langle D_{X_i} S\cdot W_k, (k_i - S)^{-1} D_{Z_j} S\cdot Y_i\right\rangle - \left\langle D_{Y_i} S\cdot W_k, (k_i - S)^{-1} D_{Z_j} S\cdot X_i\right\rangle .$$

From (3.6) we eliminate the second derivative of S by skewsymmetrization :

$$(3.7)\quad -D^2_{X_i,Y_j} S\cdot U + D^2_{Y_j,X_i} S\cdot U = S\cdot R(X_i,Y_j)U - R(X_i,Y_j)SU$$

and simplify with the Gauss equations ($\tilde{K}\in\{-1,0,+1\}$):

$$(3.8)\quad R(X,Y)Z = \tilde{K}\cdot(\langle Y,Z\rangle\cdot X - \langle X,Z\rangle\cdot Y) + \langle SY,Z\rangle SX - \langle SX,Z\rangle SY$$

to obtain (3.1):

$$-(k_j - k_i)\cdot(\tilde{K} + k_i k_j)\cdot |X_i|^2\cdot \langle Z_j, W_j\rangle \underset{(3.7),(3.8)}{=}$$

$$\left\langle D^2_{Z_j,X_i} S\cdot X_i, W_j\right\rangle - \left\langle D^2_{X_i,Z_j} S\cdot W_j, X_i\right\rangle \underset{(3.6)}{=}$$

$$-2\left\langle D_{X_i} S\cdot W_j, (k_i - S)^{-1} D_{Z_j} S\cdot X_i\right\rangle + 2\left\langle D_{X_i} S\cdot W_j, (k_j - S)^{-1} D_{X_i} S\cdot Z_j\right\rangle \underset{(S\cdot e_\alpha = k_\alpha e_\alpha)}{=}$$

$$-2(k_j - k_i)\cdot \sum_{\substack{e_\alpha \\ e\perp E_i, E_j}} \frac{\left\langle D_{X_i} S\cdot W_j, e_\alpha\right\rangle \left\langle e_\alpha, D_{X_i} S\cdot Z_j\right\rangle}{(k_i - k_\alpha)\cdot(k_j - k_\alpha)} .$$

4. EXPLICIT CONSTRUCTIONS.

4.1 The projective spaces over the fields $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$, are described easy enough together with their standard embeddings and symmetric space structure : As metric spaces one sees them most quickly by identifying the 1-dim. subspaces of $K^{n+1} \approx \mathbb{R}^{m(n+1)}$ with their intersection with the unit sphere ; as distance between these disjoint subspheres one takes their spherical distance in $S^{m(n+1)-1}$. The natural embedding goes into a sphere in the vector space of K-linear symmetric endomorphisms of K^{n+1} by sending the subspace $K\cdot v \subset K^{n+1}$ to the map L_v, defined as orthogonal projection of K^{n+1} to $K\cdot v$. In the complex case for example $L_v(x) := \langle v,x\rangle\cdot v + \langle iv,x\rangle\cdot iv$. The symmetric endomorphisms have the natural euclidean norm $\|S\|^2 = \operatorname{trace} S^2$ and one checks immediately that the map $K\cdot v \to L_v$ changes the metric only by a scale factor. Moreover the natural K-linear isometries U of K^{n+1}, which preserve the metric on the projective space, act also nicely on the image : $L_{U.v}(X) = U \circ L_v \circ U^{-1}(X)$. In particular the isometry U which is $+1$ on $K.v$ and -1 on $(K.v)^{\perp} K^{n+1}$ gives the symmetry at $K.v$ to make the projective spaces into symmetric spaces. Finally, since the metric is the distance between orbits of an isometric action on $S^{m(n+1)-1}$, namely multiplication by units in K, this natural metric has curvature ≥ 1.

4.2 The four projective planes are obtained from the corresponding division algebras $A = \mathbb{R}^m$ $(m = 1,2,4,8)$ in the following way : In terms of the multiplication $\cdot$ in A we write down in 4.3 a polynomial F on $\mathbb{R}^2 \times A \times A \times A = \mathbb{R}^{3m+2}$ and we check with property (2.4) of A that F is isoparametric This is the existence part to the classification remark (3.4). We then rewrite F in terms of the geometric data of the (now existing!) hypersurfaces, making use of section 3. In the rewritten form so many isometries of $\mathbb{R}^{3m+1}$ are apparent which leave F invariant that one has directly : The regular levels are homogeneous, the maximal and minimal (or focal) levels are symmetric (!) and they contain many totally geodesic m-spheres (4.6). To identify them completely one still needs that <u>all</u> curvatures are positive (5.6). I chose to do these computations in an expanded way which also gives the Einstein metrics and the positively curved metrics on the regular levels, in section 5.

4.3 The polynomial on $\mathbb{R}^2 \times A \times A \times A$:

$$\begin{aligned} F(n,N,Y_1,Y_2,Y_3) := & -n^3 + 3n N^2 \\ & - 3(n+\sqrt{3}\,N) \cdot \tfrac{1}{2} \langle Y_1, Y_1 \rangle_A \\ & + \quad 3n \quad \cdot \langle Y_2, Y_2 \rangle_A \\ & - 3(n-\sqrt{3}\,N) \cdot \tfrac{1}{2} \langle Y_3, Y_3 \rangle_A \\ & + 3 \cdot \sqrt{3} \cdot \mathrm{Re}((Y_1 \cdot Y_2) \cdot Y_3) \quad . \end{aligned}$$

With the help of (2.4) grad F is computed :

$$\frac{\partial F}{\partial n} = -3n^2 + 3N^2 - \tfrac{3}{2} \langle Y_1, Y_1 \rangle + 3 \langle Y_2, Y_2 \rangle - \tfrac{3}{2} \langle Y_3, Y_3 \rangle$$

$$\frac{\partial F}{\partial N} = 6nN - \frac{3\sqrt{3}}{2} (\langle Y_1, Y_1 \rangle - \langle Y_3, Y_3 \rangle)$$

$$\frac{\partial F}{\partial Y_1} = -3(n+\sqrt{3}N) \cdot y_1 + 3\sqrt{3}\, \overline{y_2 \cdot y_3}$$

$$\frac{\partial F}{\partial Y_2} = 6ny_2 + 3\sqrt{3}\, \overline{y_3 \cdot y_1}$$

$$\frac{\partial F}{\partial y_3} = -3(n-\sqrt{3}N)y_3 \qquad + \qquad 3\sqrt{3}\ \overline{y_1 \cdot y_2}$$

Again with (2.4) follows

$$\Delta F = 0 \quad \text{and} \ |\text{grad } F|^2 = 9 \cdot (N^2+n^2+|y_1|^2+|y_2|^2+|y_3|^2)^2 \ .$$

This implies that the spherical restriction f has $|\text{grad } f|$ and Δf constant on its levels. The levels are therefore a family of parallel hypersurfaces, all of constant mean curvature; therefore all principal curvatures are constant, F is an isoparametric polynomial. The unit circle in the n-N-plane is tangent to grad f, so that the Y_j-coordinates are tangential to the levels along the normal circle. It is therefore promising that the isometries $(y_i, y_j) \to (-y_i, -y_j), (n,N,y_k) \to (n,N,y_k)$ leave F invariant, but it does not really help since we cannot get such information outside the n-N-plane.

4.4 How is the polynomial F determined by the geometric data of the isoparametric hypersurfaces ?

(1) An isoparametric function with $g = 3$ different principal curvatures is a polynomial if one chooses the Cartan-Münzner normalization : If $c(\varphi)$ is any normal great circle then normalize such that : $f(c(\varphi)) = -\cos 3\varphi = -\cos^3\varphi + 3\cos\varphi \sin^2\varphi$; extend f by homogeneity of degree 3 to a function F on euclidean space, then F is a polynomial. In some cartesian coordinates (n,N) for the plane of the normal circle, the polynomial F is then given as

$$F(n,N,0,0,0) = -n^3 + 3 \cdot n \cdot N^2 \quad , \ |\text{grad } F|_c = 3 \ .$$

(2) Along a normal great circle the m-dim eigenspaces of the shape operator S are parallel. We now choose the Y_j variables in these eigenspaces. Then there can be no terms linear in the Y_j and quadratic in (n,N), since the gradient of such a term would not be in the n-N-plane. -The terms which are linear in (n,N) and quadratic in the Y_j will now be derived from the shape operators S (x denotes the position vector):

$f := F|_{S^n}$ $\qquad$ $\operatorname{grad} f = (\operatorname{grad} F)^{\tan} = \operatorname{grad} F - 3\cdot F\cdot x$

$|\operatorname{grad} F|^2{}_{|S^n} = 9 = |\operatorname{grad} f|^2 + 9\cdot f^2$, i.e. $|\operatorname{grad} f| = 3$ if $f = 0$.

$$\begin{aligned} -S\cdot y = \; &\partial_y \nu = \frac{1}{|\operatorname{grad} f|}\cdot \partial_y(\operatorname{grad} F - 3\cdot F\cdot x) \\ (y \perp x) \quad &= \frac{1}{|\operatorname{grad} f|}\cdot(\partial_y \operatorname{grad} F - 3\, F\cdot y)\Big|_{f=0} = \frac{1}{3}\,\partial_y \operatorname{grad} F \, . \end{aligned}$$

The eigenvalues of S at the level $f = 0$ are given by the distance to the focal levels $f = \pm 1$, hence they are

$\operatorname{ctg} 30° = \sqrt{3}$, $\operatorname{ctg} 90° = 0$, $\operatorname{ctg} 150° = -\operatorname{ctg} 30°$.

Two points on the normal great circle where $f = 0$ are $(n=0, N=1)$ and $(n=\frac{1}{2}\sqrt{3}, N=\frac{1}{2})$. We may number Y_1, Y_2, Y_3 in such a way that the eigenvalues of $\partial \operatorname{grad} F$ on the corresponding three copies of $\mathbb{R}^m$ are

$-3\sqrt{3}$, 0 , $3\cdot\sqrt{3}$ at $n = 0$, $N = 1$,

$-3\sqrt{3}$, $3\sqrt{3}$, 0 at $n = \frac{1}{2}\sqrt{3}$, $N = \frac{1}{2}$.

Note that grad f and the tangent vector of the normal great circle switch from parallel to antiparallel (and vice versa) as the great circle passes through a focal point, i.e. one switch between $(n,N) = (0,1)$ and $(\frac{1}{2}\sqrt{3},\frac{1}{2})$. Comparison with (4.3) shows that we have reconstructed all the terms in F of the form $\operatorname{lin}(n,N)\cdot\operatorname{quadratic}(Y_1,Y_2,Y_3)$; I repeat : this time along an arbitrary great circle and with a better geometric interpretation of the Y_j-variables as before in (4.3).

(3) To determine the terms which are cubic in the Y_j differentiate $-\langle S\cdot Y, Z\rangle = \frac{1}{3}\langle \partial_Y \operatorname{grad} F, Z\rangle$ once more in a direction X tangential to the $f = 0$ level. With D denoting covariant derivative in this hypersurface we may assume $D_X Y = 0$, $D_X Z = 0$, so that $\partial_X Y$, $\partial_X Z$ are in the n-N normal plane. Then

$$-\langle D_X S\cdot Y, Z\rangle = \frac{1}{3}\langle \partial^2_{X,Y} \operatorname{grad} F, Z\rangle \, .$$

The last term in (4.3) is therefore replaced by

$3\cdot\sqrt{3}\cdot\operatorname{Re}((Y_1\cdot Y_2)\cdot Y_3) \rightsquigarrow -3\cdot\langle D_{Y_1} S\cdot Y_2, Y_3\rangle$,

as expected from remark (3.4) and (3.2) which gives $|D_{Y_1} S\cdot Y_2|^2 = 3|Y_1|^2\cdot|Y_2|^2$, etc. as in (4.3).

4.5 The isometries of the levels of $f = F_{|S^n}$.

At any point of a regular level we can define an isometry U of S^n which clearly leaves f invariant by :

U restricted to two eigenspaces of S is $-\mathrm{id}$

U restricted to the orthogonal complement is $+\mathrm{id}$.

These are enough isometries so that they are transitive on each curvature leaf and therefore on each regular level. In fact, the even products of isometries which are $-\mathrm{id}$ on the tangent spaces along an equator of one curvature leaf (sphere) all fix the poles of that sphere and are transitive on the tangent directions of each pole. - On the focal submanifolds the isometries are not only transitive, but the isometry U above restricts to $-\mathrm{id}$ on the tangent space at the point where the normal great circle meets the focal point of the curvature leaf on which $U = \mathrm{id}$. The focal submanifolds (which are isometric to each other under the antipodal map) are therefore symmetric spaces, even extrinsically symmetric submanifolds.

4.6 As the regular levels converge to a focal level, distances converge ; in particular : of the three totally geodesic curvature foliations by m-spheres one has its leaves collapsed to points while the other two give totally geodesic m-spheres of equal size in the focal submanifold.

5. CURVATURE COMPUTATIONS.

The aim is to prove with the same family of metrics positivity of the focal submanifolds, the existence of two Einstein metrics on the regular levels and also the existence of positively curved metrics. The idea is to start with the minimal level ($f=0$) where the leaves of two curvature foliations are already spheres of the same size and change the relative size of the third foliation ; shrinking these third leaves to points gives a metric proportional to that of a focal submanifold.

Consider the family of metrics on the zero level :

5.1 $g_\rho(X,Y) = g((S^2+\rho)\cdot X,Y)$, $0 \leq \rho$;

the metric is changed by a factor $3+\rho$ on the leaves with principal curvature $\pm\sqrt{3}$ and by a factor ρ on the leaves with principal curvature 0 . We find :

For $\rho = 1$ (all leaves equal size) and for $\rho = 3\cdot(m-1)$ (if $m > 1$) g_ρ is Einstein (5.5) ; for $0 < \rho < 1$ g_ρ has positive curvature (5.7); the quotient metric g_0 has positive curvature (5.6).

5.2 Expressing the covariant derivative D^ρ of g_ρ as $D^\rho = D + \Gamma$,

$$g_\rho(\Gamma(X,Y,Z) = \frac{1}{2}(-(D_Z g_\rho)(X,Y) + (D_X g_\rho)(Y,Z) + (D_Y g_\rho)(Z,X))$$

we find $(D_Z g_\rho)(X,Y) = g(D_Z S\cdot SX + S\cdot D_Z S\cdot X, Y)$ and

5.2.1 $(\rho + S^2)\circ\Gamma(X,Y) = S\cdot D_X S\cdot Y$.

In particular : The curvature leaves stay totally geodesic for all ρ (this gives already 5.4.1).

We compute

$$(D_X\Gamma)(Y,Z) + \Gamma(X,\Gamma(Y,Z)) =$$

$$(S^2+\rho)^{-1}\cdot S\cdot D^2_{X,Y}S\cdot Z + (S^2+\rho)^{-1}\cdot D_X S\cdot\rho\cdot(S^2+\rho)^{-1}\cdot D_Z S\cdot Z$$

and with

$$D^2_{X,Y}S\cdot Z - D^2_{Y,X}S\cdot Z = R(X,Y)\cdot SZ - S\cdot R(X,Y)Z$$

have

$$(5.3)\quad (\rho+S^2)\cdot R^\rho(X,Y)Z = S\cdot R(X,Y)\cdot SZ + \rho^2\cdot R(X,Y)Z + \\ + D_X S\cdot\rho(S^2+\rho)^{-1}\cdot D_Y S\cdot Z - D_Y S\cdot\rho\cdot(S^2+\rho)^{-1}.D_X S\cdot Z .$$

(5.4) The eigendistributions of S are also eigendistributions of Ric^ρ .

Proof. This follows from (5.3) with (3.2) and (3.3) :

If X,Y,Z are from different eigenspaces of S then $R(X,Y)Z = 0$ and $D_X S\cdot D_Y S\cdot Z = 0$, hence $R^\rho(X,Y)Z = 0$.

If X,Y are from the same eigenspace of S then $R(X,Y)Y \sim X$, $D_X S\cdot Y = 0 = D_Y S\cdot Y$, hence $R^\rho(X,Y)Y \sim X$.

If X,Y are from different eigenspaces of S then $R(X,Y)Y \sim X$, $D_Y S\cdot Y = 0$, $D_Y S\cdot D_X S\cdot Y \sim X$ hence $R^\rho(X,Y)Y \sim X$.

More specifically we obtain from (5.3) with $k_1 = 0$, $k_{2,3} = \pm\sqrt{3}$ in (3.2)

5.4.1 $K(X_1,Y_1) = \frac{1}{\rho}$, $K(X_2,Y_2) = K(X_3,Y_3) = \frac{4}{3+\rho}$,

5.4.2 $K(X_1,Y_2) = K(X_1,Y_3) = \rho(3+\rho)^{-2}$, $K(X_2,X_3) = (3-2\rho)(3+\rho)^{-2}$.

The curvatures (5.4.1) agree with the size of the leaves. Because of (5.4) these sectional curvatures are enough to compute the eigenvalues of Ric^{ρ} :

(5.5) $\mathrm{Ric}\, X_1 = ((m-1)\rho^{-1} + 2m\rho(3+\rho)^{-2}) \cdot X_1$,

$\mathrm{Ric}\, X_{2,3} = (4(m-1)(3+\rho)^{-1} + m(3-\rho)\cdot(3+\rho)^{-2}) \cdot X_{2,3}$;

in particular : the eigenvalues are equal
if $\rho = 1$ or if $\rho = 3\cdot(m-1)$ (if $m > 1$).

This gives the Einstein metrics.

(5.6) In the limit $\rho \to 0$ we get for the quotient metric
(if $g_\rho(Y_2,Y_2) = 1 = g_\rho(Y_3,Y_3)$)

$R^{\rho}(X_2,Y_2)Y_2 = 4\cdot X_2$,

$R^{\rho}(X_3,Y_2)Y_2 = 1\cdot X_2$.

The eigenvalues of the Jacobi operator $R^{\rho}(\ ,Y_2)Y_2$ are therefore 4 and 1 . Since we know already that the focal submanifolds are symmetric spaces (4.5),(5.6) is because of (1.6) enough to prove that all Jacobi operators have these eigenvalues. We therefore have identified the focal submanifolds with the projective planes of section 1, in particular the Cayley plane whose existence was not covered by (4.1).

(5.7) The metrics g_ρ have for $0 < \rho < 1$ strictly positive curvature (the case $m = 1$ gives Berger metrics on $SO(3)$).

<u>Proof</u>. Let $X = X_1 + X_2 + X_3$, $Y = Y_1 + Y_2 + Y_3$, X_i, Y_j tangential to curvature distributions and $X \perp Y$. We can also assume $X_1 \perp Y_1$ and $\langle X_2,Y_2\rangle = -\langle X_3,Y_3\rangle$. The proof of (5.4) showed that $g_\rho(R^{\rho}(X,Y)Z,W)$ vanishes if at least three arguments are tangential to <u>different</u> distributions or if precisely three are tangent to the same and the fourth to a different distribution. Therefore

$$g_\rho(R^\rho(X,Y)Y,X) = \sum_i g_\rho(R^\rho(X_i,Y_i)Y_i,X_i) +$$

$$\sum_{i\neq j} g_\rho(R^\rho(X_i,Y_j)Y_j,X_i) + g_\rho(R^\rho(X_i,Y_j)Y_i,X_j) \quad (5.3)$$

$$\sum_i (k_i^2+\rho)(1+k_i^2)(|X_i|^2|Y_i|^2 - \langle X_i,Y_i\rangle^2)$$

$$+\sum_{i\neq j}(k_ik_j+\rho)(1+k_ik_j)\cdot(\tfrac{1}{2}(X_i^2Y_j^2+X_j^2Y_i^2) - \langle X_i,Y_i\rangle\langle X_j,Y_j\rangle)$$

$$-\sum_{i\neq j}\frac{\rho}{\rho+k_k^2}(\tfrac{1}{2}(|D_{X_i}SY_j|^2+|D_{X_j}SY_i|^2) - \langle D_{X_i}S\cdot X_j, D_{Y_i}S\cdot Y_j\rangle)$$

Again with $k_1 = 0$, $k_{2,3} = \pm\sqrt{3}$ and (3.2) we continue (the inequality comes from the last term $\langle D_{X_i}S\cdot X_j, D_{Y_i}S\cdot Y_j\rangle \geq -3|X_i||X_j||Y_i||Y_j|$) :

$$\geq \rho\cdot|X_1|^2\,|Y_1|^2 + 4(3+\rho)(|X_2|^2\,|Y_2|^2 - \langle X_2,Y_2\rangle^2 + |X_3|^2|Y_3|^2 - \langle X_3,Y_3\rangle^2)$$

$$+ \rho\cdot(X_1^2(Y_2^2+Y_3^2) + Y_1^2(X_2^2+X_3^2))$$

$$+ 2(3-\rho)(X_2^2Y_3^2 + X_3^2Y_2^2 - 2\langle X_2,Y_2\rangle\langle Y_3,Y_3\rangle)$$

$$- 3\cdot\{X_2^2Y_3^2 + X_3^2Y_2^2 + 2|X_2|\,|X_3|\,|Y_2|\,|Y_3| +$$

$$+ \frac{\rho}{3+\rho}(X_1^2(Y_2^2+Y_3^2) + Y_1^2(X_2^2+Y_3^2) + 2|X_1||Y_1|\cdot(|X_2||Y_2| + |X_3||Y_3|)\}$$

(Now observe $\langle X_2,Y_2\rangle = -\langle X_3,Y_3\rangle$,
hence $\langle X_2,Y_2\rangle^2 + \langle X_3,Y_3\rangle^2 \leq 2|X_2||Y_2||X_3||Y_3|$:)

$$\geq \frac{\rho^2}{3+\rho}(X_1^2Y_1^2 + X_1^2(Y_2^2+Y_3^2) + Y_1^2(X_2^2+X_3^2))$$

$$+ (12+3\rho)(X_2^2Y_2^2 + X_3^2Y_3^2) + (3-2\rho)(X_2^2Y_3^2 + X_3^2Y_2^2)$$

$$- (18+14\rho)\ |X_2||X_3||Y_2||X_3|$$

$$\geq \frac{\rho^2}{3+\rho}(X_1^2Y_1^2 + X_1^2(Y_2^2+Y_3^2) + Y_1^2(X_2^2+X_3^2)) + (3-3\rho)(X_2^2+X_3^2)(Y_2^2+Y_3^2)$$

$$\geq \min(3-3\rho, \frac{\rho^2}{3+\rho})\cdot|X|^2\cdot|Y|^2 \ . \text{ Q.E.D.}$$

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