Astérisque

FRAYDOUN REZAKHANLOU
S. JAMES TAYLOR

The packing measure of the graph of a stable process

Astérisque, tome 157-158 (1988), p. 341-362
<http://www.numdam.org/item?id=AST_1988__157-158__341_0>

© Société mathématique de France, 1988, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (http://smf4.emath.fr/Publications/Asterisque/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme Numérisation de documents anciens mathématiques
http://www.numdam.org/

Société Mathématique de France
Astérisque 157-158 (1988)

The packing measure of the graph of a stable process

by Fraydoun REZAKHANLOU and S. James TAYLOR[1]

Processes in $\mathbb{R}^d$ with stationary independent increments are now called Lévy processes. Many of these determine fractal sets in $\mathbb{R}^d$. During the period 1955-60, the speaker (James Taylor) corresponded with Paul Lévy about the measure properties of the zero set of a Brownian path in $\mathbb{R}$. Lévy had already studied the stochastic nature of this set in great detail and clearly felt that it would be appropriate to find the right measure for telling the local time. The problem we worked on in the 1950's did not get resolved then because the needed techniques of Hausdorff measure theory were not yet developed. The expected result for Hausdorff measure was proved in [17] and the corresponding results for packing measure were obtained in [13].

In general, the sample path $X_t = X(t)$, $0 \leq t < +\infty$, of a Lévy process determines a trajectory in $\mathbb{R}^d$, a graph $G_t = (t, X_t)$ in $\mathbb{R}^{d+1}$, and occupation time sets in $\mathbb{R}$. For a particular process these sets are often fractals. We call a set $E \subset \mathbb{R}^d$ a fractal if its Hausdorff dimension dim E is the same as the packing dimension Dim E as defined in [15], and the common value is greater than the topological dimension. For the stable process of index α in $\mathbb{R}^d$, it is easily proved that:

if $\alpha < d$, the range is a fractal of dimension α,

if $\alpha > d = 1$, the level sets are fractals of dimension $1 - \frac{1}{\alpha}$,

if $0 < \alpha \leq 2$, the graph is a fractal of dimension $\max[1, 2 - \frac{1}{\alpha}]$.

In [13] the packing dimension γ' was obtained for the range of a general Lévy process, and in [9] it is shown that

[1]The research on which this paper was based was carried out with the support of NSF under contract #DMS-83-17815.

$$\frac{1}{2}\beta \le \gamma' \le \min(d,\beta)$$

and that these inequalities are sharp. Here β is the upper index of the process which can be defined using the Lévy measure ν by

$$\beta = \inf[\alpha > 0: r^{\alpha}\nu\{y: |y| > r\} \to \infty \text{ as } r \to 0].$$

This implies that some trajectories of Lévy processes are not fractals in our sense. In fact, given constants a,b such that $0 \le a \le b \le 2$, there is a Lévy process in $\mathbb{R}^2$ with $\dim E = a$, $\operatorname{Dim} E = b$, where E is the range of the process.

In the lecture we sought to survey the developing study of fractals arising from a Lévy process since 1953 when the first paper [5] was published by Paul Lévy. This development was described in detail in [14] so we do not repeat it in this paper. Instead, we give precise packing measure results for the case of stable processes where there is much more detailed information available about the relevant distributions. The Hausdorff measure results for these sets are essentially complete, and the packing measure of the trajectory was found in [15] for transient Brownian motion, in [4] for planar Brownian motion, and in [13] for strictly stable processes $\alpha < d$. In this paper we will obtain the correct packing measure function $\varphi(s) = s/|\log s|$ for the range of an asymmetric Cauchy process in $\mathbb{R}^d$ $(d \ge 2)$. Together with [8] this shows that this random fractal is unusually regular as both Hausdorff and packing measures are finite and positive.

Our main object in this paper is to investigate the packing measure of the graph $G_t = (t,X_t)$ of each stable process X_t in $\mathbb{R}^d$. We settle all the strictly stable cases in Section 2 except for the symmetric Cauchy process in $\mathbb{R}$ and planar Brownian motion--these cases will require some different techniques. In Section 3, we show that $\varphi(s) = s/|\log s|$ is the right function to give finite positive packing measure to the graph of every asymmetric Cauchy process. The corresponding Hausdorff measure results were obtained in [8].

In Section 1, we collect the precise definitions and preliminary estimates, most of which come from previous work. We will adopt the convenient practice of using c and C with suffices to denote finite positive constants whose value may depend on the process being considered but not on the sample path ω.

1. Preliminaries

Our detailed results will all relate to the class of Lévy processes in which each increment has a stable distribution in $\mathbb{R}^d$ with characteristic function

$$E \exp i \langle X_{s+t}-X_s, u\rangle = \exp t\,\Psi(u)$$

where, for $\alpha = 2$, $\Psi(u) = -\frac{1}{2}|u|^2$, for $0 < \alpha < 2$,

$$\Psi(u) = -|u|^\alpha \int_{S_d} w_\alpha(u,\theta)\mu(d\theta),$$

$$w_\alpha(u,\theta) = [1-i\ \mathrm{sgn}(u,\theta)\tan \pi\alpha/2]\ |\langle u/|u|,\theta\rangle|^\alpha, \quad \alpha \neq 1,$$

$$w_1(u,\theta) = |\langle u/|u|,\theta\rangle| + (2i/\pi)\langle u/|u|,\theta\rangle \log|\langle u,\theta\rangle|,$$

and μ is a probability measure on the surface of the unit sphere S_d in $\mathbb{R}^d$. To ensure that the process is genuinely d-dimensional, we assume that μ is not supported by a proper subspace of $\mathbb{R}^d$. The elimination of the linear drift made will not affect the results in the present paper.

The transition density $p(t,x)$ of X_t is continuous and bounded for each $t > 0$. Whenever it satisfies the scaling property

$$p(t,x) = p(rt, r^{1/\alpha}x)r^{d/\alpha}$$

for all $r > 0$, we say that the process is strictly stable. In this case, $r^{-1/\alpha}X_{rt}$ is another version of X_t. When $\alpha \neq 1$, all our stable processes are strictly stable. When $\alpha = 1$, if

$$\xi = \int_{S_d} \theta\mu(d\theta) \neq 0,$$

we say the Cauchy process is strictly asymmetric. Standard Brownian motion in $\mathbb{R}^d$ is the case $\alpha = 2$. The asymmetric Cauchy process satisfies a modified scaling property:

$$rX_t \text{ and } X_{rt} - (\tfrac{2}{\pi}\, rt \log r)\xi$$

have the same distribution for $r > 0$, $t > 0$.

Whenever $d = 1$, $1 < \alpha \le 2$, it is well known that X_t has a jointly continuous local time $L(x,t)$ such that, for Borel E,

$$|\{s \in (0,t): X_s \in E\}| = \int_E L(x,t)dx$$

$$= \int_0^t 1_E(X_s)ds.$$

Recently, Barlow [2] has obtained the exact modulus of continuity in x for $L(x,s)$, $0 \le s \le t$. We do not see how to use this directly because we need the asymptotic behavior as $x \to 0$ and $t \downarrow 0$, so we state the fundamental estimate used by Barlow in his proof.

Lemma 1.1. *Let Y_t be the right continuous inverse of $L(0,t)$. Then for all $t > 0$, $\lambda > 0$,*

$$P\left\{\sup_{0\le s\le Y_t} L(x,s)-L(0,s) > \lambda\right\} \le \exp\left(-c\,\frac{\lambda^2}{t|x|^{\alpha-1}}\right).$$

This is Proposition 2.7 of Barlow [1], applied to the strictly stable process of index $\alpha > 1$, where Barlow's calculations show that if

$$h(x) = E\,L(0,H_x),\ H_x = \inf\{t \ge 0: X_t = x\},$$

then

$$c_1|x|^{\alpha-1} \le h(x) \le c_2|x|^{\alpha-1}.$$

For the strictly asymmetric Cauchy process in $\mathbb{R}^d$, we will need to estimate both tails of the distribution of X_t.

Lemma 1.2. *If $X = X_1$ is a random vector in $\mathbb{R}^k$ with a Cauchy distribution, there are constants c and C such that for $\lambda \ge 1$,*

$$c\lambda^{-1} \le P\{|X| \ge \lambda\} \le C\lambda^{-1}.$$

Lemma 1.3. *If X_t is a strictly asymmetric Cauchy process in $\mathbb{R}^d$, there are positive constants a_0, c such that*

(a) if $0 < a \leq a_0$ and $0 < t \leq a|\log a|^{-1}$, then

$$P\{|X_t| \geq a\} \leq cta^{-1}; \text{ \underline{and}}$$

(b) $$P\{\sup_{0 \leq s \leq t} |X_s| > a\} \leq cta^{-1};$$

(c) <u>if</u> $ca|\log a|^{-1} \leq t \leq t_0$, <u>then</u>

$$P\{|X_t| \leq a\} \leq \frac{ca}{t \log^2 t}.$$

Lemma 1.2 was proved as Lemma 1 of [8] and Lemmas 1.3(a),(c) as Lemma 2 of [8]. We now prove (b). Define the stopping time

$$\tau = \inf\{s > 0: |X_s| > a\}.$$

Let

$$E = \{\sup_{0 \leq s \leq t} |X_s| > a\} = \{\tau < t\}$$

$$F = \{|X_t| \geq \tfrac{1}{2} a\}.$$

Then

$$P(E) = P(E \cap F) + P(E \cap F^c) \leq P(F) + P(E)\, P\{|X_t - X_\tau| > \tfrac{1}{2} a\}$$

using the strong Markov property. By (a) we have $P\{|X_t - X_\tau| > \frac{1}{2} a\} < \frac{1}{2}$ for $0 < a \leq a_0$, so

$$P(E) \leq P(F) + \tfrac{1}{2} P(E),$$

and

$$P(E) \leq 2P(F) \leq 4cta^{-1}, \qquad \text{by (a)}.$$

We remark that this lemma implies that there are constants c_1, c_2 such that as $t \downarrow 0$, $P\{c_1 t|\log t| \leq |X_t| \leq c_2 t|\log t|\} \to 1$. In fact, the local behavior of X_t is very close to being deterministic.

We now recall the definition and properties of fractal measures that we will use. A measure function $h(s)$ is a mapping $(0,1) \to (0,1)$ which is monotone increasing with $h(0+) = 0$, continuous, and satisfies the smoothness condition

$$\frac{h(2s)}{h(s)} \leq c \quad \text{for all } s.$$

The Hausdorff h-measure of subsets $E \subset \mathbb{R}^d$ has been studied by many authors. In a recent paper [15], packing h-measure was defined as follows. First, we define a pre-measure

$$h\text{-}P(E) = \lim_{\delta \downarrow 0} \sup\{ \sum_{i=1}^{\infty} h(2r_i) : B(x_i, r_i) \text{ disjoint}, x_i \in E, r_i < \delta \} \tag{1}$$

where $B(x_i, r_i)$ denotes the open ball centered at x_i, radius r_i.

$$h\text{-}p(E) = \inf\{\Sigma\, h\text{-}P(E_i) : E \subset \cup E_i\} \tag{2}$$

now defines a metric outer measure in $\mathbb{R}^d$, which we call h-packing measure. Using the functions $h(s) = s^\beta$, $\beta > 0$ gives the fractal index

$$\text{Dim } E = \inf\{\beta > 0 : s^\beta\text{-}p(E) = 0\},$$

which we call the packing dimension of E.

In order to calculate the packing measures, we will use two technical results.

<u>Lemma 1.4</u>. <u>For each measure function</u> h, <u>there is a constant</u> c <u>such that for all</u> $E \subset \mathbb{R}^d$ and Borel measures μ with $0 < \|\mu\| = \mu(\mathbb{R}^d) < \infty$

$$c\, \mu(E) \inf_{x \in E} \left\{ \limsup_{r \downarrow 0} \frac{\varphi(2r)}{\mu(B(x,r))} \right\} \leq \varphi\text{-}p(E)$$

$$\leq \|\mu\| \sup_{x \in E} \left\{ \limsup_{r \downarrow 0} \frac{\varphi(2r)}{\mu(B(x,r))} \right\}.$$

This is Theorem 5.4 of [15]. Instead of packing a set E by disjoint balls with center in E, we can use the class Γ of semi-dyadic cubes in $\mathbb{R}^d$. $C \in \Gamma$ if for some integer k, C is a cube of side 2^{-k} and each projection on an axis is a half open interval of the form $[\frac{1}{2} n\, 2^{-k}, (\frac{1}{2} n+1) 2^{-k})$ for some $n \in \mathbb{Z}$. Each $x \in \mathbb{R}^d$ belongs to 2^d cubes of Γ with side 2^{-k}; of these we denote by $v_k(x)$ the unique cube in Γ whose complement is at distance 2^{-k-2} from the dyadic cube of side 2^{-k-2} which contains x. Now put

$$\Gamma_E = \{v_k(x) : k \in \mathbb{N},\ x \in E\}$$

and use the cubes in Γ_E to replace the balls $B(x,r)$ of definition (1), with diameter $(d^{1/2}\ 2^{-k})$ replacing $2r = \operatorname{diam} B(x,r)$. This gives a new pre-measure $h\text{-}P^{**}$ which is comparable to $h\text{-}P$, and the final step (2) gives the dyadic packing h-measure $h\text{-}p^{**}(E)$ with the same class of sets of zero measure or finite measure.

Lemma 1.5. *For each measure function h, there are constants c_1, c_2 such that for all $E \subset \mathbb{R}^d$*

$$c_1 h\text{-}p(E) \le h\text{-}p^{**}(E) \le c_2 h\text{-}p(E).$$

This is proved in [15]. In order to compute the packing measure of the trajectory of a strictly stable process of index α, the following result was proved as Lemma 5 in [13].

Lemma 1.6. *Suppose $h(s) = s^\alpha \Psi(s)$ where $\Psi(s)$ is a measure function, and X_t is strictly stable in $\mathbb{R}^d$ with index $\alpha < d$, then*

$$\liminf_{r \downarrow 0} \frac{T_1(r)+T_2(r)}{h(2r)} = \begin{matrix} 0 \\ \\ +\infty \end{matrix} \quad \text{according as} \quad \int_{0+} \frac{[\Psi(s)]^2}{s}\, ds \quad \begin{matrix} = +\infty \\ \\ < +\infty \end{matrix}$$

where $T_i(r)$ denotes the total time in $B(0,r)$ by the process X_t^i and X_t^1, X_t^2 are independent copies of X_t.

2. The Strictly Stable Case

We will deal with all values of α except for the critical cases $\alpha = 1$ and the Brownian case $\alpha = 2 = d$. Whenever $d = 1$ and $1 < \alpha \le 2$, X_t hits points and the process is known to have a continuous local time. We will be interested in

$$T(a,a) = \int_0^a 1_{(-a,a)}(X_s)\,ds = \int_{-a}^a L(x,a)\,dx$$

where $L(x,a)$ is a continuous version of the occupation time at x. Intuitively, when a is small, $|L(x,a)-L(0,a)|$ should have smaller order than $L(0,a)$. For a fixed $a > 0$, Barlow [2] has obtained very precise results about the modulus of continuity of $L(x,a)$ in x. We do not see how to use these directly because we are interested in the asymptotics of $T(a,a)$ as $a \downarrow 0$. Our first objective is to show that these are the same as those of $2a\, L(0,a)$.

If $\alpha > 1$, it is well known that the right continuous inverse of $L(0,a)$ is a subordinator of index $\beta = 1 - 1/\alpha$. This index will give us an adequate bound on $|L(x,a)-L(0,a)|$. We do not strive for exact answers but are satisfied with estimates sufficient for our main objective.

Lemma 2.1. *Suppose* $1 < \alpha \le 2$, $\beta = 1 - 1/\alpha$ *and* $L(x,s)$ *is the continuous local time of the stable process* X_t *of index* α *in* $\mathbb{R}$. *Then, if* $0 < \varepsilon < \frac{1}{4}(\alpha-2 + 1/\alpha)$,

$$\lim_{k\to\infty} \frac{L(y2^{-k},2^{-k})-L(0,2^{-k})}{2^{-k(\beta+\varepsilon)}} = 0 \quad \text{a.s.}$$

if $0 < \varepsilon < \frac{1}{4}(\alpha-2 + 1/\alpha)$ *and* $H(a) = \{\omega:\ \sup_x L(x,a) \le 1\}$, *there is an* $a_0 > 0$ *such that*

$$E\left(\frac{|T(a,a)-2aL(0,a)|}{a^{2-1/\alpha+\varepsilon}}\, 1_{H(a)}\right) \le c a^{\frac{1}{2}\varepsilon}$$

whenever $0 < a \le a_0$.

Proof. For fixed y in $[-1,1]$, let $x = ay$ and use Lemma 1.1 to estimate the large tail of the distribution of $|L(x,a)-L(0,a)|$. Let $Y_1(t)$ be the right continuous inverse of $L(0,s)$ and $Y_2(t)$ the inverse of $L(x,s)$. Now $L(x,s)$ starts to grow at the first hitting time of x, so defining

$$F(a) = \{Y_2(a^{\beta-\varepsilon}) \le a\} \cup \{Y_1(a^{\beta-\varepsilon}) \le a\},$$

and using Lemma 2.3 of [7] gives

$$P(F(a)) \le c_1 \exp(-c_2 a^{-c_3 \varepsilon}). \tag{3}$$

Now note that

$$E(|L(x,a)-L(0,a)| 1_{H(a)}) \le E(\sup_{0\le s\le a} |L(x,s)-L(0,s)| 1_{H(a)})$$

$$= E\left(\sup_{0\le s\le a} |L(x,s)-L(0,s)| 1_{H(a)} 1_{F(a)^c}\right)$$

$$+ E\left(\sup_{0\le s\le a} |L(x,s)-L(0,s)| 1_{H(a)} 1_{F(a)}\right)$$

$$\le E\left(\sup_{0\le s\le Y_1(a^{\beta-\varepsilon})} L(x,s)-L(0,s)\right)$$

$$+ E\left(\sup_{0\le s\le Y_2(a^{\beta-\varepsilon})} L(0,a)-L(x,s)\right)+P(F(a))$$

since, for $\omega \in F(a)$, $0 \le L(x,s) \le 1$ and $0 \le L(0,s) \le 1$ whenever $0 \le s \le a$

$$\le 2 \int_0^\infty \exp\left(-c \frac{u^2}{a^{\beta-\varepsilon}|x|^{\alpha-1}}\right) du+P(F(a))$$

using Lemma 1.1 and the identity

$$E(Z) = \int_0^\infty P(Z > u)du,$$

valid for any non-negative random varible. But

$$\int_0^\infty \exp(-c\lambda^2)d\lambda = c_4 c^{-1/2},$$

so

$$2\int_0^\infty \exp\left(-c \frac{u^2}{a^{\beta-\varepsilon}|x|^{\alpha-1}}\right) du = c_5\, a^{\frac{1}{2}(\beta-\varepsilon)} |x|^{\frac{1}{2}(\alpha-1)} \le c_5\, a^{\frac{1}{2}(\alpha - \frac{1}{2}) - \frac{1}{2}\varepsilon}.$$

When $a \le a_0$, (3) ensures that the term $P(F(a))$ is of smaller order so we obtain

$$E\left(\frac{|L(ay,a)-L(0,a)|}{a^{\beta+\varepsilon}}\,1_{H(a)}\right) \le c_6\, a^{(\alpha-\frac{1}{\alpha})\frac{1}{2}-1+1/\alpha-\frac{3}{2}\varepsilon}$$

$$= c_6\, a^{(\alpha+\frac{1}{\alpha}-2)\frac{1}{2}-\frac{3}{2}\varepsilon}.$$

Finally,

$$T(a,a)-2aL(0,a) = \int_{-a}^{a} (L(x,a)-L(0,a))dx,$$

so

$$E\left(\frac{|T(a,a)-2aL(0,a)|}{a^{2-1/\alpha+\varepsilon}}\,1_{H(a)}\right) \le \int_{-1}^{1} E\,\frac{|L(ay,a)-L(0,a)|}{a^{\beta+\varepsilon}}\,1_{H(a)}dy$$

$$\le 2\, c_6 a^{\frac{1}{2}\varepsilon}.$$

<u>Corollary 2.2</u>. <u>Under the conditions of Lemma 2.1</u>,

$$\lim_{k\to\infty} \frac{T(2^{-k},2^{-k})-2^{-k+1}L(0,2^{-k})}{2^{-k(1+\beta+\varepsilon)}} = 0 \quad \text{a.s.}$$

<u>Proof</u>. For each $\delta > 0$ we have

$$P\left\{\frac{|T(2^{-k},2^{-k})-2^{-k+1}L(0,2^{-k})|}{2^{-k(1+\beta+\varepsilon)}}\,I_{H(a)} > \delta\right\} \le 2\, c_6\delta^{-1}\, 2^{-\frac{1}{2}\varepsilon k}$$

whenever $2^{-k} < a$, and $\Sigma\, 2^{-\frac{1}{2}\varepsilon k}$ converges. The easy half of Borel Cantelli now gives

$$\lim_{k\to\infty} \frac{|T(2^{-k},2^{-k})-2^{-k+1}L(0,2^{-k})|}{2^{-k(1+\beta+\varepsilon)}}\,1_{H(a)} = 0 \quad \text{a.s.}$$

But the joint continuity of the local time ensures that $P(H(a)) \to 1$ as $a \downarrow 0$. Hence, for almost all ω we have $\omega \in H(a)$ for $a < a_1(\omega)$, so $1_{H(a)} = 1$ and the corollary follows.

<u>Corollary 2.3</u>. <u>Suppose</u> $h(s)$ <u>is monotone and such that</u> $s^{-\beta-\varepsilon}h(s) \to \infty$ <u>as</u> $s \downarrow 0$ <u>for some</u> ε <u>in</u> $(0,\frac{1}{4}(\alpha-2+1/\alpha))$, <u>then</u>

(a) $$\limsup_{a\downarrow 0} \frac{T(a,a)}{ah(a)} = 2 \limsup_{a\downarrow 0} \frac{L(0,a)}{h(a)} \quad \text{a.s.}$$

(b) $$\liminf_{a\downarrow 0} \frac{T(a,a)}{ah(a)} = 2 \liminf_{a\downarrow 0} \frac{L(0,a)}{h(a)} \quad \text{a.s.}$$

(c) $$\liminf_{a\downarrow 0} \frac{T_1(a,a)+T_2(a,a)}{ah(a)} = 2 \liminf_{a\downarrow 0} \frac{L_1(0,a)+L_2(0,a)}{h(a)} \quad \text{a.s.}$$

Proof. The results follow immediately from Lemma 2.2 whenever $a = 2^{-k}$, $k \to \infty$. If the limiting values are 0 or $+\infty$, the monotonicity of $T(a,a)$ or $L(0,a)$ will fill up the gaps. If the limit is a positive constant a.s., we can remove the gaps using the sequence $a = \rho^{-k}$, $\rho > 1$ and monotonicity. In (c), we are assuming two independent copies of the process.

Since the limiting behavior of $L(0,a)$ can be deduced from that of its inverse Y_t as $t \to 0$, and this process is a stable subordinator of index $\beta = 1 - 1/\alpha$, we can deduce from Corollary 2.3(a) that

$$\limsup_{a\downarrow 0} \frac{T(a,a)}{a^{2-1/\alpha}(\log|\log a|)^{1/\alpha}} = c \quad \text{a.s.}$$

and could even evaluate c whenever X_t is symmetric stable. This result is Theorem 5.1 of [7]. We will use Corollary 2.3(c) in the sequel but note that (a) could be used to prove

Theorem 2.4. *Suppose $1 < \alpha \le 2$, X_t is stable of index α in $\mathbb{R}$, and $T(a,a)$ denotes the total time spent in $[-a,a]$ up to $t = a$. If $\Psi(s)$ is monotone increasing, then*

$$\liminf_{a\downarrow 0} \frac{T(a,a)}{a^{2-1/\alpha}\Psi(a)} = \begin{matrix} 0 \\ \\ +\infty \end{matrix} \quad \text{according as} \int_{0+} \frac{\Psi(s)}{s}\, ds \begin{matrix} = \infty \\ \\ < \infty \end{matrix} .$$

We omit the proof which uses standard Borel-Cantelli arguments. Now we can state the main theorem of the section.

Theorem 2.5. *Suppose $1 < \alpha \le 2$, and $G_t = (t,X_t)$ is the graph of a strictly stable process X_t of index α in $\mathbb{R}$. Then if*

$h(s) = s^{2 - 1/\alpha}\Psi(s)$, <u>where</u> $\Psi(s)$ <u>is monotone increasing</u>,

$$h\text{-}p\ G[0,1] = \begin{matrix} 0 \\ \\ +\infty \end{matrix} \quad \underline{\text{according as}} \int_{0+} \frac{(\Psi(s))^2}{s}\,ds \begin{matrix} < \infty \\ \\ = +\infty \end{matrix} .$$

Remark. The case $\alpha = 2$ relates to the graph of the Weiner process. It is surprising that

$$s^{3/2}|\log s|^{-1/2}\text{-}p\ G[0,1] = \infty \quad \text{a.s.}$$

even though X_t has a uniform modulus of continuity $|X_{t+h}-X_t| \le ch^{1/2}|\log h|^{1/2}$. In [11] one of us considered the packing measure of the graphs of continuous functions and asked for conditions which ensure that a continuous function whose uniform modulus is $h\,\xi(h)$ will have a graph for which $h\,\xi^{-1}(h)$ is the correct packing measure function.

Proof. We use the standard trick of projecting Lebesgue measure from the time axis onto the trajectory of $G_t = (t,X_t)$. Define a random Borel measure in the plane by

$$\mu(E) = |\{t \in [0,1]: (t,X_t) \in E\}|.$$

In order to apply the density Lemma 1.4, we have to calculate

$$\liminf_{r\downarrow 0} \frac{\mu(B(x,r))}{r^{2 - 1/\alpha}\Psi(r)}$$

for $x = G_t$, $0 < t < 1$. Clearly,

$$T_x^1(\tfrac{1}{2}r,\tfrac{1}{2}r)+T_x^2(\tfrac{1}{2}r,\tfrac{1}{2}r) \le \mu(B(x,r)) \le T_x^1(r,r)+T_x^2(r,r)$$

where $x = (t,X_t)$ and

$$T_x^1(r,r) = |\{s \in (t,t+r): |X_s-X_t| \le r\}$$

$$T_x^2(r,r) = |\{s \in (t-r,t): |X_s-X_t| \le r\}.$$

Since X_s has stationary independent increments, once we fix $t \in (0,1)$, $T_x^1(r,r)+T_x^2(r,r)$ behaves as $r \downarrow 0$ like the sum of two independent copies of $T(r,r)$. By Corollary 3.3(c),

$$\liminf_{a\downarrow 0} \frac{T_1(a,a)+T_2(a,a)}{a^{2-1/\alpha}\Psi(a)} = \liminf_{a\downarrow 0} \frac{L_1(0,a)+L_2(0,a)}{a^{\beta}\Psi(a)}$$

where L_1, L_2 are local times of two independent stable processes of index α. Note that there is no loss of generality in assuming that for a suitable $\varepsilon > 0$, $s^{-\varepsilon}\Psi(s) \to +\infty$ as $s \downarrow 0$ and this allows us to satisfy the growth condition on $h(s)$ in Corollary 2.3.

$L_i(0,a)$ is also the time spent in $(0,a)$ by the subordinator Y_t^i which is strictly stable of index β. We can apply Lemma 1.6 to get

$$\liminf_{a\downarrow 0} \frac{L_1(0,a)+L_2(0,a)}{a^{\beta}\Psi(a)} = \begin{matrix} 0 \\ \\ +\infty \end{matrix} \quad \text{with} \int_{0+} \frac{\Psi(s)^2}{s}\,ds \begin{matrix} = +\infty \\ \\ \text{or} < \infty \end{matrix}.$$

Putting these results together gives, for $t \in (0,1)$

$$\liminf_{r\downarrow 0} \frac{h(2r)}{r^{2-1/\alpha}\Psi(r)} = +\infty. \tag{4}$$

If $F \subset [0,1]$ is the set of t satisfying (4), a Fubini argument tells us that $|F| = 1$, so that $\mu\, G(F) = 1$ a.s. By Lemma 1.4 we get

$$h\text{-}p\ G[0,1] \geq h\text{-}p\ G(F) = +\infty \text{ a.s.}$$

We have to work harder in the other direction. Assume that $\int s^{-1}\Psi(s)^2 ds$ converges, and denote by H the set of good points

$$H = \{t \in (0,1): \liminf \frac{\mu(B(X_t,r))}{h(2r)} = +\infty\}.$$

Then $|H| = 1$ a.s. and Lemma 1.4 tells us that $\varphi\text{-}p\ G(H) = 0$. We have to worry about the image of the bad points $[0,1]\backslash H = \bigcup_{n=1}^{\infty} H_n$, where $H_n = \{t \in (0,1): \liminf \frac{\mu(B(X_t,r))}{h(2r)} \leq n\}$. For $t \in H_n$, by monotonicity we have

$$\mu\ B(X_t, 2^{-k}) \le cn\ h(2^{-k}) \tag{5}$$

for infinitely many k. Consider a semi-dyadic square S of side 2^{-k} bad if G_S hits the inside square of side 2^{-k-2} but spends less than $c\ n\ h(2^{-k})$ in S. Any $t \in H_n$ will be in infinitely many such bad S. The probability that S is bad given that it is hit is at most

$$P\{T(2^{-k}, 2^{-k}) \le c\ n\ h(2^{-k})\}. \tag{6}$$

We now assume, without loss of generality, that $s^{-\frac{1}{2}\varepsilon}\Psi(s) \to \infty$ as $s \downarrow 0$ for some positive $\varepsilon < \frac{1}{4}(\alpha-2 + 1/\alpha)$.

We now extend the idea we used in proving Lemma 2.1. Since $L(x,s)$ is continuous in (x,s) for $0 \le s \le 1$, and has compact support, it is uniformly continuous. We define the event

$$J(a) = \{\omega: |x| \le a,\ 0 \le t \le a \Rightarrow |L(y,s)-L(y+x,s+t)| \le 1\}.$$

Clearly, $J(a) \uparrow$ as $a \downarrow 0$ and $P(J(a)) \to 1$ as $a \downarrow 0$. Also, the local condition $H(a)$ will be satisfied at every $t \in [0,1]$ if $J(a)$ holds. For $2^{-k} < a$, our first object is to prove

$$P(\{T(2^{-k}, 2^{-k}) \le c_1\ h(2^{-k})\} \cap J(a)) \le c_2\Psi(2^{-k}). \tag{7}$$

The left hand side of (7) is bounded by

$$P\{2^{-k+1}L(0,2^{-k}) \le 2\ c_1\ h(2^{-k})\}$$

$$+ P\{|T(2^{-k}, 2^{-k})-2^{-k+1}L(0,2^{-k})|1_{J(a)} \ge c_1\ h(2^{-k})\}.$$

Using the results in [13] and the scaling property shows that the first term is

$$P\{L(0,1) \le c_1\Psi(2^{-k})\} \le c_3\Psi(2^{-k}),$$

and the second term is bounded by

$$P\left\{\frac{|T(2^{-k}, 2^{-k})-2^{-k+1}L(0,2^{-k})|}{2^{-k(1+\beta+\varepsilon)}}\ 1_{H(a)} \ge c_1\ \frac{\Psi(2^{-k})}{2^{-k\varepsilon}}\right\}$$

$$\le \frac{2^{-k\varepsilon}}{c_1\Psi(2^{-k})}\ E\left\{\frac{|T(2^{-k}, 2^{-k})-2^{-k+1}L(0,2^{-k})|}{2^{-k(1+\beta+\varepsilon)}}\ 1_{H(a)}\right\}$$

$$\leq c_4 \frac{2^{-\frac{3}{2}k\varepsilon}}{\psi(2^{-k})} < c_5\psi(2^{-k})$$

when k is large, because of the growth condition on $\psi(s)$. This completes the proof of (7).

Now denote by N_k the number of bad semi-dyadics entered by G_t. By Lemma 6.1 of [7], if $\omega \in J(a)$ and $2^{-k} < a$, the number of semi-dyadics entered by G_t is $O(2^{k(2 - 1/\alpha)})$ so

$$E(N_k) \leq c\ 2^{k(2-1/\alpha)}\ n\ \psi(2^{-k}).$$

If we use all possible bad semi-dyadics without worrying about disjointness we get, for $\omega \in J(a)$,

$$\begin{aligned} E\ h\text{-}p^{**}(G(H_n)) &\leq E\ h\text{-}P^{**}(G(H_n)) \\ &\leq c \sum_{k=k_0}^{\infty} E(N_k)h(2^{-k}) \\ &\leq c \sum_{k=k_0}^{\infty} \psi(2^{-k})^2 \end{aligned}$$

and this series converges. Letting $k_0 \to \infty$ gives

$$E\ h\text{-}p^{**}(G(H_n)) = 0$$

so that $h\text{-}p^{**}(G(\cup H_n)) \leq \sum_{n=1} h\text{-}p^{**}G(H_n) = 0$ a.s. By Lemma 1.5, $h\text{-}p(G(\cup H_n)) = 0$ and so $h\text{-}pG[0,1] = 0$ a.s. on $J(a)$. But a.s. for each ω there is an $a_1(\omega) > 0$ such that $\omega \in J(a)$ for $0 < a \leq a_1$, so we conclude finally that $h\text{-}p\ G[0,1] = 0$ a.s., and we have completed the proof of Theorem 2.5.

We now summarize the information regarding the other cases which can be solved by the same methods.

Theorem 2.6. *If $G_t = (t, X_t)$ is the graph of a strictly stable process X_t of index α in $\mathbb{R}^d$. Then*

(a) *If $d \geq 2$ and $1 < \alpha < 2$, $h(s) = s^{\alpha}\Psi(s)$*

$$h\text{-}pG[0,1] = \begin{matrix} 0 \\ \infty \end{matrix} \quad \text{according as} \quad \int_{0+} \frac{\Psi(s)^2}{s}\, ds \begin{matrix} < \infty \\ = +\infty \end{matrix}.$$

(b) *If $d \geq 3$ and $\alpha = 2$, $h(s) = s^2(\log|\log s|)^{-1}$, there is a finite c such that a.s. for every Borel set $E \subset [0,1]$, $h\text{-}pG(E) = c|E|$.*

(c) *If $d \geq 1$ and $\alpha < 1$, $h(s) = s$, then a.s. for every Borel set $E \subset [0,1]$, $h\text{-}p(G(E)) = |E|$.*

Proof. (a) This is the transient case. Since the projection of G_t on the state space is the range of X_t, we have $h\text{-}pG[0,1] \geq h\text{-}pX[0,1]$ and the case when the integral diverges follows from Theorem 2 of [13]. A simple rate of escape argument shows that a.s.

$$T(a,a) = T(a) \quad \text{for } 0 < a \leq a_0$$

so that the lim inf behavior of $T(a)$ and $T(a,a)$ will be the same. Whenever the integral converges the usual density argument shows that $h\text{-}pG(H) = 0$ where H is the set of good points. For bad points, since

$$P\{T(a) \neq T(a,a)\} \leq a^c,$$

the probability that $T_x(a,a)$ is small is of the same order as $T_x(a)$ small, so the argument on page 219 of [13] is valid without change to yield $h\text{-}pG[0,1] = 0$.

(b) This can be obtained using the above modifications to the argument in [15].

(c) When $\alpha < 1$, $J = \sum_{t \in [0,1]} |X_t - X_{t-}|$ converges. Let $\overline{G}$ be the set in $\mathbb{R}^{d+1}$ obtained from the range of G_t by adding the countable set of line segments joining X_{t_i-} to X_{t_i} whenever X_t has a discontinuity at t_i. $\overline{G}$ is then a rectifiable arc in $\mathbb{R}^{d+1}$ of length $(1+J)$. We can parametrize this set using arc length s from $(0,0)$ in such a way that each jump of X_t at $t = t_i$ corresponds to an interval I_i of length $|X_{t_i} - X_{t_i-}| = J_i$ starting at $t_i + \sum_{t_j < t_i} J_j$. In [16] we considered the linear packing measure of subsets of rectifiable arcs in $\mathbb{R}^2$ and the result was extended in [10] to sets in $\mathbb{R}^{d+1}$. Using the length parametrization of an arc $f(s)$, we know that for any Borel set

$$F \subset [0,1+J],\ h\text{-}p\ f(F) = h\text{-}m\ f(F) = |F|.$$

In order to get subsets of G_t $(0 \le t \le 1)$, we first remove any points from $\cup I_i$, and then note that every $s \in F - \cup I_i$ corresponds to a t in $(0,1)$ and every $t \in (0,1)$ corresponds to a value $s \notin \cup I_i$. It follows that if E is Borel $\subset [0,1]$, we have a.s.

$$h\text{-}p\ X(E) = |E|.$$

Remark. The Hausdorff measure results corresponding to this theorem were obtained by Jain and Pruitt [3]. Our methods do not cover the strictly stable Cauchy case $\alpha = 1$ or planar Brownian motion $\alpha = 2 = d$. An analysis of the packing measure of the range of a symmetric Cauchy process in $\mathbb{R}$ is also missing from the literature. One expects the answer to be similar to that for planar Brownian motion [4].

3. The Strictly Asymmetric Cauchy Process

The Hausdorff measure properties of the graph of this process were obtained in [8]. We observed there that covering by equal balls did not increase the covering measure by more than a constant factor. The same phenomenon will be observed for packing. In fact, the same measure function $\varphi(s) = s|\log s|^{-1}$ is correct for both packing and covering so this random set has stronger regularity properties than any others previously studied. We have enough preliminary estimates to proceed to the main result.

Theorem 3.1. *Given a strictly asymmetric Cauchy process X_t in $\mathbb{R}^d$, there are finite positive constants c_1, c_2 such that*

(a) for $d \geq 1$, if $G_t = (t, X_t)$ is the graph of X_t a.s. for every Borel set $E \subset \mathbb{R}^+$, $\varphi\text{-}p(G(E)) = c_1|E|$,

(b) for $d \geq 2$, $\varphi\text{-}p(X(E)) = c_2|E|$.

Remark. Since the corresponding results were proved in [8] for Hausdorff φ-measure, we see that for every set E of positive Lebesgue measure

$$\varphi\text{-}p\ G(E)/\varphi\text{-}m\ G(E) = c.$$

It is interesting to ask, is $c = 1$? In [10] the authors show that, if $\varphi\text{-}p(E) = \varphi\text{-}m(E)$ for subsets of a set of finite positive s^α-measure, then α is an integer. Is this φ-measure sufficiently close to linear measure to allow the strong regularity implied by $c = 1$?

Proof. (a) As usual, we project Lebesgue measure on the sample path of G_t and invoke the density argument, Lemma 1.4. We already know that

$$\limsup_{a \downarrow 0} \frac{T(a,a)}{\varphi(2a)} = c_1,$$

so it is sufficient to show that

$$\liminf_{a \downarrow 0} \frac{T_1(a,a)+T_2(a,a)}{\varphi(2a)} = c_2.$$

We use the sequence $a_k = 2^{-k}$.

If $a_k \leq a < a_{k-1}$, then

$$\omega \in E_k = \left\{ \inf_{a_k \leq a \leq a_{k-1}} \frac{T_1(a,a)+T_2(a,a)}{\varphi(2a)} < c \right\}$$

$$\Rightarrow \frac{T_1(2^{-k},2^{-k})}{\varphi(2a_{k-1})} < c \quad \text{and} \quad \frac{T_1(2^{-k},2^{-k})}{\varphi(2a_{k-1})} < c.$$

By independence, when c is small enough

$$P(E_k) = 0(\frac{1}{k^2}) \tag{8}$$

and a simple Borel-Cantelli argument tells us that E_k happens finitely often with probability 1.

In order to prove (8), we note that

$$P\{T(2^{-k},2^{-k}) < c\,\varphi(2a_{k-1})\} \le P\{T(2^{-k},2^{-k}) < c_1 k^{-1} 2^{-k}\}$$

$$\le P\{T(2^{-k},c_1 k^{-1} 2^{-k}) < c_1 k^{-1} 2^{-k}\}$$

$$= P\{\sup_{0\le s\le c_1 k^{-1} 2^{-k}} |X_s| > 2^{-k}\} = O(\tfrac{1}{k})$$

using Lemma 1.3(b).

This implies that a.s.

$$0 < c_2 \le \liminf_{a\downarrow 0} \frac{T_1(a,a)+T_2(a,a)}{\varphi(2a)} \le c_3 < +\infty,$$

and an application of a suitable zero-one law tells us that, for some constant $c > 0$

$$\liminf_{a\downarrow 0} \frac{T_1(a,a)+T_2(a,a)}{\varphi(2a)} = c \quad \text{a.s.,}$$

Our usual method of applying Lemma 1.4 gives

$$\varphi\text{-}p(G[0,1]) \ge c_4 > 0 \quad \text{a.s.} \tag{9}$$

$$\varphi\text{-}p(G(H)) \le c_5 < +\infty \quad \text{a.s.} \tag{10}$$

where H are the good points in [0,1], that is those t for which

$$\liminf_{r\downarrow 0} \frac{\mu(B(X_t,r))}{\varphi(2r)} \ge \frac{1}{2}\,c.$$

The method used up to date to deal with the bad points does not work, so we use a different idea. We try to pick up most of the bad points by a deterministic lattice and then add all the points not close to these. Let

$$t_{i,k} = i\,k^{-3/2}2^{-k};\; i = 0,1,\cdots,2^k k^{3/2}.$$

For $t = t_{i,k}$, we say that the semi-dyadic cube $S_{i,k}$ of side 2^{-k} is bad if $X_{t_{i,k}}$ is in the center cube of side 2^{-k-2} and

$$\frac{\mu(B(X_t, 2^{-k}))}{\varphi(2^{-k+1})} < \frac{1}{4}\, c. \tag{11}$$

We add in the bounded number of semi-dyadic cubes of side 2^{-k} whose centers are contiguous to those of a bad $S_{i,k}$. Let N_k be the total number of such cubes, and note that

$$E(N_k) = k^{3/2} 2^k \cdot ck^{-2} \tag{12}$$

since the probability of the event (11) is $O(k^{-2})$. Let M_k be the number of cubes entered by G_t which are not within 2^{-k-2} of one of the points $(t_{i,k}, X(t_{i,k}))$. Since Lemma 6.1 of [7] tells us that the expected number of dyadic cubes entered is $O(k2^k)$, we can restart the process at the hitting time τ of a new cube to give, for $t_{i-1,k} \leq \tau < t_{i,k}$

$$E(M_k) < ck2^k\, P\{|X_\tau - X(t_{i,k})| > 2^{-k-2}\}.$$

Since $t_{i,k} - \tau \leq k^{-3/2} 2^{-k}$, Lemma 1.3(a) gives

$$E(M_k) \leq ck2^k k^{-3/2}. \tag{13}$$

If we use all the $(N_k + M_k)$ dyadic cubes we have now counted, we will certainly have included any bad cubes. Hence

$$\begin{aligned} E\ \varphi\text{-}P^{**}(G(H^c)) &\leq c \sum_{k=k_0}^{\infty} E(M_k + N_k)\varphi(2^{-k}) \\ &\leq c' \sum_{k=k_0}^{\infty} k^{-3/2}, \end{aligned}$$

by (12) and (13). If we let $k_0 \to \infty$, we see that

$$E\ \varphi\text{-}p^{**}(G(H^c)) \leq E\ \varphi\text{-}P^{**} G(H^c) = 0,$$

so

$$\varphi\text{-}p^{**}G(H^c) = 0,$$

implying

$$\varphi\text{-}p(G(H^c)) = 0,$$

and (10) now gives

$$\varphi\text{-}p(G[0,1]) \le c_5.$$

The argument first used in [17] now shows that there is a positive constant c_1 such that a.s.

$$\varphi\text{-}p(G[0,t]) = c_1 t \quad \text{for all } t > 0,$$

and this implies, for every Borel set $E \subset \mathbb{R}^+$ that

$$\varphi\text{-}p(G(E)) = c_1 |E|.$$

(b) Clearly, the total time spent in $B(X_t, r)$ is at least $T(\frac{1}{2} r, \frac{1}{2} r)$ so we deduce

$$c_6 \le \liminf_{r\downarrow 0} \frac{T_1(\frac{1}{2} r, \frac{1}{2} r) + T_2(\frac{1}{2} r, \frac{1}{2} r)}{\varphi(2r)} \le \liminf_{r\downarrow 0} \frac{T_1(r) + T_2(r)}{\varphi(2r)}$$

$$\le \limsup_{a\downarrow 0} \frac{T_1(r)}{\varphi(2r)} + \limsup_{a\downarrow 0} \frac{T_2(r)}{\varphi(2r)} = 2c_7$$

by Section 4 of [8]. This means that we again have

$$\liminf \frac{\mu\, B(X_t, r)}{\varphi(2r)} = c \quad \text{a.s.}$$

for each t in (0,1). We can repeat the density argument Lemma 1.4 and again use the bad point argument of (a) since X_{t_0} is less likely to be a bad point for the range than for the graph.

References

1. M. T. Barlow. Continuity of local times for Lévy processes. Zeitschrift f. Wahrsch. 69 (1985), 23-35.

2. M. T. Barlow. Necessary and sufficient conditions for the continuity of local time of Lévy processes. To appear, Annals of Probability.

3. M. Jain and W. E. Pruitt. The correct measure function for the graph of a transient stable process. Zeitschrift f. Wahrsch. 9 (1968), 131-138.

4. J. F. LeGall and S. J. Taylor. The packing measure of planar Brownian motion. Seminar on Stochastic Processes, Birkhauser (1986), 139-148.

5. P. Lévy. Le mésure de Hausdorff de la courbe du mouvement brownien. Giovn Ist Ital. Attuari 16 (1953), 1-37.

6. W. E. Pruitt and S. J. Taylor. The potential kernel and hitting probabilities for the general stable process in $\mathbb{R}^N$. Trans. Amer. Math. Soc. 146 (1969), 299-321.

7. W. E. Pruitt and S. J. Taylor. Sample path properties of processes with stable components. Zeitschrift f. Wahrsch. 12 (1969), 267-289.

8. W. E. Pruitt and S. J. Taylor. Hausdorff measure properties of the asymmetric Cauchy processes. Annals Prob. 5 (1977), 608-615.

9. W. E. Pruitt and S. J. Taylor. The packing dimension of the sample path of a Lévy process. In preparation.

10. X. S. Raymond and C. Tricot. Packing regularity of sets in n-space. Preprint.

11. F. Rezakhanlou. The packing measure of the graph and level sets of continuous functions. Preprint.

12. S. J. Taylor. Sample path properties of a transient-stable process. J. Math. Mechanics 16 (1967), 1229-1246.

13. S. J. Taylor. The use of packing measure in the analysis of random sets. Stochastic Processes and their Applications. Springer Lecture Notes 1203 (1985), 214-222.

14. S. J. Taylor. The measure theory of random fractals. Math. Proc. Camb. Phil. Soc. 100 (1986), 383-406.

15. S. J. Taylor and C. Tricot. Packing measure and its evaluation for a Brownian path. Trans. Amer. Math. Soc. 288 (1985), 679-699.

16. S. J. Taylor and C. Tricot. The packing measure of rectifiable subsets of the plane. Math. Proc. Camb. Phil. Soc. 99 (1986), 285-296.

17. S. J. Taylor and J. G. Wendel. The exact Hausdorff measure of the zero set of a stable process. Zeitschrift f. Wahrsch. 6 (1966), 170-180.

Fraydoun Rezakhanlou
Courant Institute
251 Mercer Street
New York, NY 10012
USA

S. James Taylor
Department of Mathematics
Math/Astronomy Building
University of Virginia
Charlottesville, VA 22903
USA