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1. Introduction

In this survey we discuss the concept of concentration of measures and some applications of this idea, mainly to geometrical problems. The idea, which first appeared around the 1920s, remained almost dormant, to regain prominence only some 50 years later. At the present time, it remains at the peak of its development and at the focus of interest of a large group of mathematicians.

The phenomenon of concentration for homogeneous measures on high dimensional structures was first realized by P. Lévy on an example of the family of Euclidean spheres $\{S^n\}$. It is also clear that Lévy realized that his observation had a far more general meaning and looked for other examples.

P. Lévy's starting point was a remark made by E. Borel [Bor] (published ~1914) concerning a geometric interpretation of the law of large numbers (which I give here with a deviation estimate).

Let $C^n = [-1,1]^n$ be a cube in $\mathbb{R}^n$ with the standard Euclidean distance "dist". Then $\operatorname{diam} C^n = 2\sqrt{n}$. Consider a linear functional f, $f(x) = \sum_1^n x_i$ (i.e., $\operatorname{Ker} f = (1,\ldots,1)^{\perp}$).

Then $(1/2^n)\,\mathrm{Vol}\{x \in C^n : \mathrm{dist}(x, \mathrm{Ker} f) \geq \varepsilon\sqrt{n}\} = P\{|\frac{1}{n}\sum_1^n \xi_i| > \varepsilon | (\xi_i)_{i=1}^n$ are uniformly distributed in $[-1,1]$ independent random variables$\} \leq c_1 \exp(-c_2\varepsilon^2 n)$. Therefore, "most" of the volume of C^n is concentrated near a "small slice" (relative to the diameter).

Poincaré [Poi] also used, indirectly (at least in 1911), the following similar fact for the Euclidean sphere S^n (always equipped in our note with the rotation-invariant probability measure μ and the geodesic Euclidean distance ρ); fixed $x_0 \in S^{n+1}$; then

$$\mu\{x \in S^{n+1} : |(x, x_0)| > \varepsilon\} \leq \sqrt{\frac{\pi}{2}}\exp(-\varepsilon^2 n/2) .$$

P. Lévy [L] (around ~1919) connected this with the isoperimetric property of caps on S^n to prove the following theorem.

Theorem 1.1 (P. Lévy). *Let $A \subset S^{n+1}$ be a Borel subset, $\mu(A) \geq 1/2$ and $A_\varepsilon = \{x \in S^{n+1} : \rho(x, A) \leq \varepsilon\}$. Then $\mu(A_\varepsilon) \geq 1 - \sqrt{\pi/8}\exp(-\varepsilon^2 n/2) \to 0$ for $n \to \infty$.*

Corollary 1.2 (P. Lévy). *Let $f(x) \in C(S^n)$ be a continuous function on S^n with the modulus of continuity $\omega_f(\varepsilon)$. Let L_f define the median (Lévy mean) of $f(x)$, i.e.,*

$$\mu\{x \in S^n : f(x) \geq L_f\} \geq \tfrac{1}{2} \quad \text{and} \quad \mu\{x \in S^n : f(x) \leq L_f\} \geq \tfrac{1}{2} .$$

Then

$$\mu\{x \in S^n : |f(x) - L_f| > \omega_f(\varepsilon)\} < \sqrt{\frac{\pi}{2}}\exp(-\varepsilon^2 n/2) .$$

Interpretation (of P. Lévy): functions defined on a high dimensional Euclidean sphere with a "good" uniform continuity property (i.e., with a small local oscillation) are "nearly" constant with a high probability.

Lévy applied these facts to develop a notion and a theory of a Laplacian operator on S^∞, that is an infinite dimensional Hilbert sphere (called today the Laplace-Lévy operator). This direction of application was studied intensively by Shilov and his school, and Polishtchuk, Feller, Hida and others (see, e.g., [Hi] and a recent survey [F]). Another line of application, also noted by P. Lévy, gives descriptions of different compactifications of the Hilbert sphere – see Grinblat [G]. These applications will not be discussed in this survey.

First, we develop a general idea of concentration and then we connect it with another concept of "spectrum" and outline some applications to geometry, topology and infinite dimensional integration.

2. Lévy families; the concentration property

Let (X, ρ, μ) be a metric compact set with a metric ρ, $\operatorname{diam} X \geq 1$, and a probability measure μ. Define the concentration function $\alpha(X; \varepsilon)$ of X by

$$\alpha(X;\varepsilon) = 1 - \inf\left\{\mu(A_\varepsilon) \mid A \text{ be a Borel subset of } X,\ \mu(A) \geq \tfrac{1}{2}\right\}$$

(here $A_\varepsilon = \left\{x \in X \mid \rho(s, A) \leq \varepsilon\right\}$). Theorem 1 implies

Example 2.1: Let S^n be the Euclidean sphere equipped with the geodesic distance ρ and the rotation-invariant probability measure μ_n. Then

$$\alpha(S^{n+1};\varepsilon) \leq \sqrt{\pi/8}\exp(-\varepsilon^2 n/2) \longrightarrow 0 \quad \text{for} \quad n \to \infty$$

for any fixed $\varepsilon > 0$.

Following this example, we call a family (X_n, ρ_n, μ_n) of metric probability spaces a *Lévy family* ([Gr.M.1]) if, for any $\varepsilon > 0$, $\alpha(X_n, \varepsilon \cdot \operatorname{diam} X_n) \to 0$ for $n \to \infty$, and a *normal Lévy family* [Am.M.2] with constant $(c_1; c_2)$ if,

$$\alpha(X_n;\varepsilon) \leq c_1 \exp(-c_2\varepsilon^2 n)\ .$$

(When the factor $\operatorname{diam} X_n$ is omitted most of the examples below become normal Lévy families with their natural metric and natural enumeration.)

Let $f \in C(X)$ be a continuous function on a space X with the modulus of continuity $\omega_f(\varepsilon)$. As in Corollary 2, define a median L_f (also called a *Lévy mean*) as being a number such that $\mu\{x \in X : f(x) \geq L_f\} \geq \frac{1}{2}$ and $\mu\{x \in X : f(x) \leq L_f\} \geq \frac{1}{2}$. Then $\mu\{x : |f(x) - L_f| \leq \omega_f(\varepsilon)\} \geq 1 - 2\alpha(X, \varepsilon)$. This means that if $\alpha(X, \varepsilon)$ is small, then "most" of the measure of X is concentrated "around" one value of $f(x)$.

Comparing the above remark of E. Borel with the definition of a Lévy family, we see that the concept of a Lévy family (and especially a normal Lévy family) generalizes the concept behind the law of large numbers in two directions: a) the measures are not necessarily the product of measures (i.e., no condition of "independence") and b) any Lipschitz function on the space is considered instead of linear functionals only.

During the last 10-15 years, many new examples of Lévy families have been discovered and different techniques of estimating the concentration function have been developed. It still surprises me that (normal) Lévy families are widespread phenomena and not very rare ones. In this section, we mention three more such examples. However, we will analyze them and put them in a general framework in section 4.

Example 2.2: The family of orthogonal groups $\{SO(n)\}_{n\in\mathbb{N}}$ equipped with the Riemannian metric ρ (which is equivalent up to $\pi/2$ to the Hilbert-Schmidt operator metric) and the normalized Haar measure μ_n:

$$\alpha(SO(n);\varepsilon) \le \sqrt{\frac{\pi}{8}}\exp(-\varepsilon^2 n/8) .$$

(This follows from Gromov's isoperimetric inequality [Gr], see [Gr.M.1].)

Example 2.3: If $F_2^n = \{-1,1\}^n$ has the normalized Hamming metric

$$d(s,t) = \frac{1}{n}|\{i : s_i \neq t_i\}|$$

and the normalized counting measure μ, i.e., $\mu(A) = |A|/2^n$, then

$$\alpha(F_2^n;\varepsilon) \le \tfrac{1}{2}\exp(-2\varepsilon^2 n) .$$

(This follows from the Harper isoperimetric result [H]; see in such form [A.M.1].)

Example 2.4: The group Π_n of permutations of $\{1,\dots,n\}$ with the normalized Hamming metric

$$d(\pi_1,\pi_2) = \frac{1}{n}\big|\{i : \pi_1(i) \neq \pi_2(i)\}\big|$$

and the normalized counting measure:

$$\alpha(\Pi_n;\varepsilon) \le \exp(-\varepsilon^2 n/64)$$

(B. Maurey [Ma]).

3. First Geometrical Applications

We now turn back to an application of the Lévy example of S^n. Let $f \in C(S^n)$ be a continuous function, $w_f(\varepsilon)$ its modulus of continuity and L_f the median (Lévy mean). Let $A = \{x \in S^n : |f(x) - L_f| < w_f(\varepsilon)\}$. By Corollary 1.2, we know that $\mu(A) \ge 1 - \sqrt{\frac{\pi}{2}}e^{-\varepsilon^2 n/2}$. Fix $x_0 \in S^n$ and let ν define the Haar probability measure on $SO(n)$. Clearly

$$\nu\{T \in SO(n) : Tx_0 \in A\} = \mu(A) \ge 1 - \sqrt{\frac{\pi}{2}}e^{-\varepsilon^2 n/2} .$$

This implies that, for any finite set $\mathcal{N} \subset S^n$, $|\mathcal{N}| = N$,

$$\nu\{T \in SO(n) : T\mathcal{N} \subset A\} \ge 1 - N\sqrt{\frac{\pi}{2}}e^{-\varepsilon^2 n/2} .$$

Therefore, if $N < \sqrt{2/\pi}e^{\varepsilon^2 n/2}$, we may find a rotation T such that $T\mathcal{N}$ is a subset of A. Now, we choose $\mathcal{N}$ to be an ε-net for the unit sphere of some fixed k-dimensional subspace. This is possible for $k \simeq c\varepsilon^2 n/\log 1/\varepsilon$ (for some numerical constant $c \geq 0$). Then $T\mathcal{N}$ is again an ε-net for some great $(k-1)$-dimensional sphere and therefore we derived the following result (see [M1]):

Theorem 3.1. *There exists a universal constant $c > 0$ such that, for every integer n and $k = [c\varepsilon^2 n/\log 1/\varepsilon]$ and any continuous function $f \in C(S^n)$, there exists a great k-dimensional sphere $S^k \subset S^n$ (i.e., the unit sphere of a $(k+1)$-dimensional subspace) such that, for any $x \in S^k$*

$$|f(x) - L_f| < w_f(2\varepsilon) \ ,$$

where $w_f(\varepsilon)$ is the modulus of continuity of $f(x)$.

(Recently, Y. Gordon [Gor] removed the $\log 1/\varepsilon$ factor in the above formula for k.)

Using this theorem, we choose a function f in such a way that $f =$ Const. means a given geometric property. Then, by the theorem, we find subspaces of large dimension where this property is "almost" satisfied. (See [M4] for a number of such applications).

The estimate on dimension k in the above theorem is important and leads to numerous geometric results. As an example, we mention the famous Dvoretzky theorem [Dv] about almost euclidean sections of a convex symmetric body in $\mathbb{R}^n$. (This theorem appears in many books and surveys, so we will not discuss it here - see, i.e., [FLM], [MSch] or [P]).

In the following applications we consider functions of a few variables.

Let $W_{n,k}$ be the Stiefel manifold, i.e., the set of all indexed sets of k-orthonormal vectors in the n-dimensional euclidean space $\mathbb{R}^n$. We introduce in $W_{n,k}$ the natural Riemannian metric ρ induced by the geodesic distance ρ_g on the unit sphere S^{n-1} of $\mathbb{R}^n$: $\rho(\overline{x},\overline{y}) = \sqrt{\sum_1^k \rho_g(x_i,y_i)^2}$, and the normalized (probability) rotation-invariant measure μ (induced by the Haar measure on $O(n)$). Note that, of course, $W_{n,k}$ is a homogeneous space $O(n)/O(n-k)$ and $W_{n,1} = S^{n-1}$, $W_{n,n-1} = SO(n)$.

Example 3.2. ([M2]. [GrM1]) a. Fix $k \geq 1$; the family $\{W_{n,k}\}_{n>k}$ is a Lévy family. Moreover

$$\alpha(W_{n,k};\varepsilon) \leq \sqrt{\frac{\pi}{2}}\exp(-\varepsilon^2 n/8) \ .$$

b. Let $X_n = (S^n)^k = S^n \times \ldots \times S^n$ be the product of k spheres S^n equipped with the product measure and the natural Riemannian distance. Then

$$\alpha(X_n;\varepsilon) \le \sqrt{\pi/2}\exp(-\varepsilon^2 n/2) .$$

In the next section we will see a reason for this fact but here we outline how it can be derived from Theorem 1.1 up to different, worse constants. We demonstrate arguments in the case $W_{n,2}$.

Realize $W_{n,2}$ as a submanifold of the unit sphere S^{2n-1} of $2n$-dimensional space $\mathbb{R}^{2n}$ in the following manner: Let $\{e_k\}_1^{2n}$ be an orthonormal basis in $\mathbb{R}^{2n}$,

$$z = \sum_1^n a_k e_k + \sum_{n+1}^{2n} a_k e_k = (x;y) \in \mathbb{R}^{2n} \quad , \quad x = \sum_1^n a_k e_k \quad , \quad y = \sum_{n+1}^{2n} a_k e_k .$$

Thus, every $z \in \mathbb{R}^{2n}$ is mapped onto a pair of vectors x and y of n-dimensional space $\mathbb{R}^n$. Define two functions on the sphere $S^{2n-1} : f_1(z) = \|x\|^2 - \|y\|^2$; $f_2(z) = \langle x,y\rangle$ where

$$z = (x;y) = \sum_{k=1}^{2n} a_k e_k$$

and the scalar product

$$\langle x,y\rangle = \sum_1^n a_k \cdot a_{n+k} .$$

It is clear that the manifold $M_2 = \{z \in S^{2n-1} : f_1(z) = f_2(z) = 0\}$ realizes an embedding of $W_{n,2}$ in the sphere S^{2n-1}. Let us state the basic properties of this realization.

a) M_2 is invariant relative to a subgroup O_n of the orthogonal mappings $O(2n)$ such that $A^0 \in O_n \Longleftrightarrow A^0(x;y) = (Ax;Ay)$, where $z = (x;y) \in \mathbb{R}^{2n}$ and $A \in O(n)$.

b) The $(2n-3)$-dimensional Lebesgue measure μ on M_2 (induced by the Lebesgue measure μ_S on S^{2n-1}) is the Haar measure of the homogeneous space $M_2 (\equiv W_{n,2})$.

c) The value 0 is the Lévy mean for the functions $f_i(z)$, $i = 1,2$.

d) Let $z = (x;y) \in S^{2n-1}$ and $|\langle x,y\rangle| < \|x\| \cdot \|y\|$. Then there exists a unique point $z_0 \in M_2$ closest to z; it is connected with z by a geodesic normal to M_2. Denote the set of all such $z \in S^{2n-1}$ by $D(z_0)$; we also put $D_\varepsilon(z_0) = \{z \in D(z_0) : \rho_g(z,z_0) \le \varepsilon\}$, $D_\varepsilon T = \bigcup_{z\in T} D_\varepsilon(z)$ (where $T \subset M_2$). It is obvious that if $A^0 \in O_n$ and $A^0 z_1 = z_2$ ($z_i \in M_2$), then $A^0 D_\varepsilon(z_1) = D_\varepsilon(z_2)$.

e) Let T_i, $i = 1,2$, be arbitrary open subsets of M_2. It follows from a), b) and d) that $\mu(T_1)/\mu(T_2) = \mu_S(D_\varepsilon T_1)/\mu_S(D_\varepsilon T_2)$ for small enough $\varepsilon > 0$.

Therefore, the measure of subsets of M_2 can be evaluated from the measures of the corresponding subsets of the sphere S^{2n-1}. This allows us to apply, in the proof of Example 3.2 (in the case of $W_{n,2}$), the similar result for sphere S^{2n-1} – Theorem 1.1 – due to P. Lévy.

We now use Example 3.2 to prove the next results in exactly the same way as Example 2.1 was used to prove Theorem 3.1.

Theorem 3.3.

a. *Let $f \in C(S^n \times S^n)$, i.e., $f(x;y)$ is a continuous function of two variables x and y from the euclidean sphere S^n, and let L_f be the median of this function. Then there exist k-dimensional subspaces E_1 and E_2 for*

$$k \geq c\varepsilon^2 n/\log 1/\varepsilon$$

(c is some universal constant) such that

$$|f(x;y) - L_f| < w_f(\varepsilon)$$

for any $x \in S(E_1)$ and $y \in S(E_2)$.

b. *Let $f \in C(W_{n,2})$. Then there exists a k-dimensional subspace E for*

$$k \geq c\varepsilon^2 n/\log 1/\varepsilon$$

such that

$$|f(x;y) - L_f| < w_f(\varepsilon)$$

for any x and y from E. (Of course, we have to remember that f is defined only for x and y such that $|x| = |y| = 1$ and $x \perp y$.)

In both the above results, we see that the continuous function f defined on a high dimensional "structure" ($S^n \times S^n$ in (a.) and $W_{n,2}$ in (b.)) has the global oscillation on some substructures ($S(E_1) \times S(E_2)$ in (a.) and $W_{k,2}(E)$ in (b.)), no larger than the local oscillation $w_f(\varepsilon)$. We will discuss this phenomenon in a general form later, but now we would like to understand if Theorem 3.3.a can be improved and, in particular, if the same subspace $E = E_1 = E_2$ may be found for both variables x and y. An obvious example of the inner product $f(x,y) = (x,y)$ shows that this cannot be the case. On any two dimensional subspace the inner product $f(x,y)$ changes its values between -1 and 1, i.e., it has oscillation 2. However, as the following result shows, this example is, in some sense, the only one.

Theorem 3.4. *Let $f(x;y)$, x and $y \in S^{n-1}$, be a continuous function. Then there exist a continuous function $\varphi(t)$, $t \in [-1,1]$ and a k-dimensional subspace E for*

$$k \geq c\varepsilon^2 n / \log 1/\varepsilon$$

such that

$$|f(x;y) - \varphi((x,y))| < w_f(2\varepsilon)$$

for any x and y from the unit sphere $S(E)$ for the space E. (Recall that (x,y) means the inner product of x and y.)

This theorem is, indeed, a consequence of Theorem 3.3.b. For a fixed $t \in (-1,1)$, we consider a generalized Stiefel manifold $W_{n,2}(t) = \{(x;y) \subset S^{n-1} : (x,y) = t\}$ and use Theorem 3.3.b. Indeed, we consider a δ-net $\{t_j\} \subset (-1,1)$ and a family of Stiefel manifolds $W_{n,2}(t_j)$. To prove Theorem 3.4 we have to find a subspace E, as in Theorem 3.3.b, which is the same for every $W_{n,2}(t_j)$. This follows from a slight improvement of Theorem 3.3.b which states that a large measure (meaning an exponentially close to the full measure) of k-dimensional subspaces satisfy the condition of the theorem.

An important observation behind all the Theorems 3.1, 3.3 and 3.4 is that a function with a small local oscillation ("good" modulus of continuity) depends essentially (in the sense explained in precise forms in the theorems) on orbits of orthogonal transformations. We see this, in the most clear way, in Theorem 3.4: the orbit of a pair of vectors $(x;y) \subset S^{n-1}$, $x \neq \pm y$, is a generalized Stiefel manifold $W_{n,2}(t)$ for $t = (x,y)$.

We meet such an observation even more often than the phenomenon of concentration and we call it a concept of "spectrum" of functions. We discuss this in section 5.

(The subject of Theorems 3.3 and 3.4 is related to [M2] and [M3].)

4. General Methods of Analyzing Concentration Property

We return to the study of a general metric probability space (X, ρ, μ). As in section 2, most of our spaces are compact, however, in a more general framework, we typically need only a finite volume, $\mu(X) < \infty$. Clearly, the main problem in the investigation of concentration phenomenon is estimating the concentration function $\alpha(X;\varepsilon)$ (see section 2). We classify below different techniques to estimate this function.

4.A. Isoperimetric inequalities approach.

4.A.1 Riemannian case.

The following comparison theorem of Gromov ([Grl]; see also [GrM1] and [Gr2]) covers most cases of Riemannian compact manifold with positive Ricci curvature. Let X be a connected Riemannian manifold without boundary and let μ_X be its normalized Riemannian volume element. Let $R(X)$ be the Ricci curvature of X. Recall that $R(X) = \inf_\tau \mathrm{Ric}(\tau, \tau)$ where $\tau \in T(X)$ runs over all unit tangent vectors (see [Gr2] for a proof and an explanation of all notions involved).

Theorem. *Let $A \subseteq X$ be measurable and let $\varepsilon > 0$; then*

$$\mu_X(A_\varepsilon) \geq \mu_X(B_\varepsilon)$$

where B is a ball on a sphere $r \cdot S^n$ with $n = \dim X$, and r such that

$$R(X) = R(r \cdot S^n) \; (= (n-1)/r^2)$$

and $\mu_X(A) = \mu(B)$, μ being the normal on $r \cdot S^n$.

The value of $R(X)$, known in some examples (see, e.g., [CE]), together with the computation for the measure of a cap in S^n (this computation is also reflected in Theorem 1.1) lead to the following examples ([GrM1]):

4.A.2. a. Consider the family of orthogonal groups $\{SO(n)\}_{n \in \mathbb{N}}$ equipped with the Hilbert-Schmidt operator metrics and the Haar normalized measures. Then

$$\alpha(SO(n); \varepsilon) \leq \sqrt{\pi/8} \exp(-\varepsilon^2 n/8) \; .$$

(This means, in the language of section 2, that the family $\{SO(n)\}$ is a normal Lévy family with constants $c_1 = \sqrt{\pi/8}$, $c_2 = 1/8$.)

b. Similarly for each m the family $X_n = S^n \times S^n \times \ldots S^n$ $(m - times)$, $n = 1, 2, \ldots$, with the product measure and the metric

$$d(x, y) = \Big(\sum_{i=1}^{m} \rho(x_i, y_i)^2\Big) \; , \quad x = (x_1, \ldots, x_m) \; , \quad y = (y_1, \ldots, y_m) \in X_n$$

(ρ – the geodesic metric in S^n), is a normal Lévy family with constants $c_1 = \sqrt{\pi/8}$, $c_2 = 1/2$.

c. Note also that homogeneous spaces of SO_n inherit the property of being a Lévy family. Let G be a subgroup of SO_n and let $V = SO_n/G$. Let μ be the Haar measure on V and let d_n be the metric

$$d_n(t,s) = \inf\{\rho(g,h)\ ;\ \varphi g = t\ ,\ \varphi h = s\}$$

where φ is the quotient map.

Clearly $\mu(A \subseteq V) = \mu(\varphi^{-1}(A) \subseteq SO_n)$. By the definition of d_n, $\varphi^{-1}(A_\varepsilon) \supseteq (\varphi^{-1}A)_\varepsilon$. Therefore, if $\mu(A \subseteq V) \geq 1/2$, then $\mu(\varphi^{-1}(A) \subseteq SO_n) \geq 1/2$ and $\mu(A_\varepsilon) \geq \mu((\varphi^{-1}A)_\varepsilon)$. So, we have:

For $n = 1, 2, \dots$, let G_n be a subgroup of SO_n with the metric described above and with the normalized Haar measure μ_n. Then $(SO_n/G_n, d_n, \mu_n)$, $n = 1, 2, \dots$, is a normal Lévy family with constants $c_1 = \sqrt{\pi/8}$ and $c_2 = 1/8$.

d. As a consequence, we see that any family of Stiefel manifolds, which we introduced in section 3, $\{W_{n,k_n}\}_{n=1}^{\infty}$ with $1 \leq k_n \leq n$, $n = 1, 2, \dots$, is a normal Lévy family with constants $c_1 = \sqrt{\pi/8}$ and $c_2 = 1/8$.

e. Recall that the Grassman manifold $G_{n,k}$, $1 \leq k \leq n$, is a metric space of all k-dimensional subspaces of $\mathbb{R}^n$ with the distance being the Hausdorff distance between the unit spheres of the subspaces ξ and η: $\rho(\xi, \eta) = \sup\{\rho(x, S^{n-1} \cap \xi) \mid x \in S^{n-1} \cap \eta\}$. Clearly, $G_{n,k}$ is a homogeneous space of $O(n)$ and let μ be the normalized rotation invariant (Haar) measure on $G_{n,k}$. Then, again $(G_{n,k}, \rho, \mu)$ is a normal Lévy family with constants $c_1 = \sqrt{\pi/8}$ and $c_2 = 1/8$.

f. Let $V_{n,k} = \{(\xi; x) \mid \xi \in G_{n,k} \text{ and } x \in S^{n-1} \cap \xi\}$ be the canonical sphere bundle over the Grassman manifold $G_{n,k}$. $V_{n,k}$ is a homogeneous space of $O(n)$ and therefore we may define a natural metric and the normalized Haar measure. Again, any family $\{V_{n,k_n}\ ,\ 1 \leq k_n < n\}_{n \geq 1}$ is a normal Lévy family with the same constants $c_1 = \sqrt{\pi/8}$ and $c_2 = 1/8$.

g. Let $T_n = (S^1)^n$ be an n-dimensional torus equipped with a Riemannian metric and the Haar probability measure. Then there exist universal constants c_1 and $c_2 > 0$ such that

$$\alpha(T_n, \varepsilon) \leq c_1 \exp(-c_2 \varepsilon^2)\ .$$

(Note, that Diam. $T_n = \pi\sqrt{n}$ and therefore $\varepsilon > 0$ runs up to $\approx \sqrt{n}$.)

The same estimate on the concentration function is true also, of course, for a cube $C^n = [0,1]^n$, equipped with the euclidean metric and the Lebesgue measure.

The result may be derived from Theorem 4.A.1 (although $R(T_n) = 0$). However, the easiest way to prove it, as noted by G. Pisier, is to use the result 4.A.3 below, for the canonical

Gaussian measure γ on $\mathbb{R}^n$. Indeed, there exists a Lipschiptz-1 measure preserving map $(\mathbb{R}^n, \gamma) \to (C^n, \mathrm{Vol})$. Then, clearly $\alpha(C^n; \varepsilon) \leq \alpha\big((\mathbb{R}^n, \gamma); \varepsilon\big)$.

4.A.3 Isoperimetric approach, probabilistic case.

The isoperimetric problem for the euclidean space $\mathbb{R}^n$, equipped with the canonical Gaussian measure γ, was solved by C. Borell [Bo]. He used the isoperimetric property of caps in S^n, i.e., Theorem 1.1 of P. Lévy, and the following useful observation of H. Poincare: a Gaussian measure in $\mathbb{R}^n$ may be viewed as a limit when $N \to \infty$ of measures induced by the orthogonal projection $\mathbb{R}^N \to \mathbb{R}^n$ from the rotation-invariant probability measure on $\sqrt{N} S^{N-1}$. Therefore,

$$\alpha\big((\mathbb{R}^n, \gamma); \varepsilon\big) \leq \frac{1}{\sqrt{2\pi}} \int_{\varepsilon}^{\infty} e^{-t^2/2} dt \approx \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2} .$$

The direct simple proof of this result with the slightly worse constants due to Maurey and Pisier may be found in [P] or [MSch], Appendix 5.

Note that, in the language of section 2, the above estimate on the concentration function means that the family $(\mathbb{R}^n, \gamma)$ is a normal Lévy family although $\mathbb{R}^n$ is not a compact space (even with the infinite diameter).

4.A.4 Isoperimetric approach; discrete examples.

We have already mentioned that Example 2.3 is a direct consequence of the solution of the isoperimetric problem in F_2^n by Harper. For the really simple proof of this fact, see [FF]. A far reaching generalization of this isoperimetric problem was considered by Wang-Wang [WW] who used a discrete analog of the classical Steiner symmetrization.

4.A.5 Convex compact sets in $\mathbb{R}^n$.

Consider a fixed convex compact body $K \subset \mathbb{R}^n$ equipped with a euclidean metric ρ and the volume probability measure μ_{Vol} (i.e., $\mu_{\mathrm{Vol}}(A) = \mathrm{Vol}(A \cap K)/\,\mathrm{Vol}\,K$). We know how to estimate the concentration function $\alpha(K; \varepsilon)$ for some special examples (like $K = D_n$ – the euclidean ball or $K = C^n$ – the n-dimensional unit cube). However, in a general case, one knows only how to estimate a "restricted" concentration function (by restricting infimum on convex symmetric subsets of $\mathbb{R}^n$):

Define

$\tilde{\alpha}(K; \varepsilon) = 1 - \inf\big\{\mu_{\mathrm{Vol}}(A_\varepsilon) \mid A$ be a convex symmetric subset of $\mathbb{R}^n$ such that $\mu_{\mathrm{Vol}}(A) \geq 1/2\big\}$.

The following theorem is a reformulation of C. Borell's [Bo] result in our language (see [MSch] App. III).

Theorem. *There exist universal constants c_1 and $c_2 > 0$ such that*

$$\widetilde{\alpha}(K;\varepsilon) \le c_1 \exp(-c_2\varepsilon) .$$

(Note, that in many natural examples ε varies up to $\sqrt{n}$.)

Indeed, Borell proves a more general statement which involves probability measures on $\mathbb{R}^n$ satisfying the log-concavity condition: $\mu\big(\lambda A + (1-\lambda)B\big) \ge \mu(A)^\lambda \mu(B)^{1-\lambda}$. The Brunn-Minkowski inequality allows us to apply this general fact to μ_{Vol}. Because the Brunn-Minkowski inequality is, indeed, an isoperimetric type inequality in $\mathbb{R}^n$, we see again that isoperimetric inequalities imply the concentration phenomenon for convex sets in $\mathbb{R}^n$.

4.A.6. *The unit spheres of uniformly convex* finite dimensional Banach spaces have a good concentration property as was shown in [GrM3]. More precisely:

Let a normed space $X = (\mathbb{R}^{n+1}, \|\cdot\|)$ have, for a fixed $\varepsilon > 0$, *the modulus of convexity* at least $\delta(\varepsilon) > 0$. It means that for every two points x, y in X, $\|x\| = \|y\| = 1$ and $\|x-y\| \ge \varepsilon$,

$$1 - \frac{\|x+y\|}{2} \ge \delta(\varepsilon) .$$

Also let $\delta(\varepsilon)$ be a monotone (increasing) function. Let $K = \{x \in \mathbb{R}^{n+1} \mid \|x\| \le 1\}$ be the unit ball of X and $S(X) = \{x \mid \|x\| = 1\}$ the unit sphere. The standard $(n+1)$-dimensional volume on $\mathbb{R}^{n+1}$ induces the probability measure μ on $S(X)$: for any Borel set $A \subset S(X)$,

$$\mu(A) \overset{\text{def}}{=} \mathrm{Vol}_{n+1}(\cup tA,\ 0 \le t \le 1)/\,\mathrm{Vol}\,K .$$

Theorem ([GrM3]). *Let $\delta_X(\varepsilon)$ be the modulus of convexity of a normed $(n+1)$-dimensional space X and μ be the above probability measure on $S(X)$. Define $a(\varepsilon) = \delta\big((\varepsilon/8) - \theta_n\big)$ and $\delta(\theta_n/4) = 1 - (1/2)^{1/(n-1)} \simeq (\ln 2)/(n-1)$. Then, for every Borel set $A \subset S(X)$, $\mu(A) \ge 1/2$, and for every $\varepsilon > 0$*

$$\mu(A_\varepsilon) \ge 1 - e^{-a(\varepsilon)n}$$

where $A_\varepsilon = \big\{x \in S(X)\ ;\ \rho(x, A_\varepsilon) \le \varepsilon\big\}$ and $\rho(x,y) = \|x-y\|$. (This means that the concentration function $\alpha\big(S(X);\varepsilon\big) \le \exp\big(-a(\varepsilon)n\big)$.)

4.B Probabilistic Methods.

A concept of concentration was started, as we noted in section 1, from a probabilistic observation and the main interpretation is also probabilistic. So, it is only natural that some probabilistic methods may be useful in proving a concentration property in some examples.

The most developed technique here is a martingale approach which was started by B. Maurey on Example 2.4 [Ma] and became a developed method after a series of results of G. Schechtman ([Sch1], [Sch2]; see also [MSch], Chapters 7 and 8)

4.B.1 Martingale approach.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and let $f \in L_1(\Omega, \mathcal{F}, P)$. Then $\mu(A) = \int_A f dP$, $A \in \mathcal{G}$, defines a measure on $\mathcal{G}$ which is absolutely continuous with respect to $P \mid \mathcal{G}$. Consequently, by the Radon-Nikodym Theorem, there exists a unique $h \in L_1(\Omega, \mathcal{G}, P)$ such that $\int_A h dP = \int_A f dP$ for all $A \in \mathcal{G}$. This h is called the *conditional expectation* of f with respect to $\mathcal{G}$ and denoted $h = E(f \mid \mathcal{G})$.

Given a sequence $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots \subseteq \mathcal{F}$ of σ-algebras, a sequence $f_1, f_2, \ldots$ of functions $f_i \in L_1(\Omega, \mathcal{F}_i, P)$ is said to be a *martingale* with respect to $\{\mathcal{F}_i\}_{i=1}^{\infty}$ if $E(f_i \mid \mathcal{F}_{i-1}) = f_{i-1}$ for $i = 2, 3, \ldots$

The following special case is typical for most of the applications: Ω is a finite set, P is the normalized counting measure $P(A) = |A|/|\Omega|$; $\{\Omega_i\}_{i=1}^{k}$ is a sequence of partitions of Ω each of which refines the previous one; $\mathcal{F}_i$ is the algebra generated by Ω_i. For a function f on Ω, $E(f \mid \mathcal{F}_i)$ is simply the function which is constant on atoms of Ω_i, the constant on each atom is the average value of f on this atom.

The following deviation inequality (see, e.g., [St.] for this and similar inequalities)plays a central role in a martingale approach to the concentration phenomenon. It estimates from above the probability of large deviation of a function from its expectation.

Lemma. *Let $f \in L_\infty(\Omega, \mathcal{F}, P)$, $\{\phi, \Omega\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \ldots \subseteq \mathcal{F}_n = \mathcal{F}$, and put $d_i = E(f \mid \mathcal{F}_i) - E(f \mid \mathcal{F}_{i-1})$, $i = 1, \ldots, n$. Then, for all $\varepsilon \geq 0$, $P(|f - Ef| \geq \varepsilon) \leq 2\exp(-\varepsilon^2/4\sum_{i=1}^{n} \|d_i\|_\infty^2)$.*

In examples, a "right" chain $\{\mathcal{F}_i\}$ of σ-algebras gives a "right" algebraic organization of a computation. In such a way Examples 2.3 and 2.4 can be proved. We formulate one more result which contains both these examples.

Given a compact metric group G with a translation invariant metric d (i.e., $d(g,h) = d(rg, rh) = d(gr, hr)$ for all $g, h, r \in G$) and a closed subgroup H. One can define a natural metric $\overline{d}$ on G/H by

$$\overline{d}(rH, sH) = d(r, sH) = d(s^{-1}r, H) \ .$$

The translation invariance of d implies that this is actually a metric and that $d(r, sH)$ does not depend on the representative r of rH.

Theorem. (see, e.g., [M.Sch], Chapter 7). *Let G be a group, compact with respect to a translation invariant metric d and let μ be the normalized Haar measure on G. Let $G = G_0 \supseteq G_1 \supseteq \ldots \supseteq G_n = \{1\}$ be a decreasing sequence of closed subgroups of G. Let a_k be the diameter of G_{k-1}/G_k, $k = 1, \ldots, n$. Then the concentration function*

$$\alpha(G;\varepsilon) \leq 2\exp\big(-\varepsilon^2/16\sum_{k=1}^{n} a_k^2\big) .$$

(We have, in the case of Example 2.4, $G = \pi_n$, $G_k = \pi_{n-k}$, $a_k = 2/n$, and we derive the result.)

4.B.2. The empirical distribution method; Khinchine-type inequalities.

A very successful use of concentration phenomenon through an empirical distribution approach was demonstrated by G. Schechtman [Sch3] and further developed in [BLM]. We outline some remarks from [BLM]. Let (Ω, μ) be a probability space and let $\Phi(t)$ be a convex increasing function on $[0, \infty)$ such that $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$. We denote by $L_\Phi(\mu)$ the (Orlicz) space of all measurable real-valued functions f on Ω such that $\int_\Omega \Phi(|f|/\lambda)d\mu < \infty$ for some $\lambda > 0$ and put

$$\|f\|_{L_\Phi(\mu)} = \inf\big\{\lambda > 0 \; ; \; \int_\Omega \Phi(|f|/\lambda)d\mu \leq 1\big\} .$$

Of course the functions t^p give rise to the $L_p(\mu)$ spaces.

We are concerned here with two more functions

$$\psi_1(t) = e^t - 1 \quad , \quad \psi_2(t) = e^{t^2} - 1 \qquad t \geq 0 .$$

Lemma. *Let $\{g_j\}_{j=1}^N$ be independent random variables with mean 0 on some probability space (Ω, μ). Assume that $\|g_j\|_{L_{\psi_1}(\mu)} \leq M$ for some constant M and every $1 \leq j \leq N$. Then, for $0 < \varepsilon < 4M$,*

$$\text{Prob}\left\{\left|\sum_{j=1}^{N} g_j\right| \geq \varepsilon N\right\} \leq 2\exp(-\varepsilon^2 N/16M^2) .$$

If, in addition, $\|g_j\|_{L_{\psi_2}(\mu)} \leq M$ then there exist universal constants $c_1 > 0$ and $c_2 > 0$ such that, for any $\varepsilon > 0$,

$$\text{Prob}\left\{\left|\sum_{j=1}^{N} g_j\right| \geq \varepsilon N\right\} \leq c_1\exp(-c_2\varepsilon^2 N/M^2) .$$

When one usually applies this lemma, we have a function $f(t)$ defined on Ω and we are looking for an approximation of an integral

$$\frac{1}{N}\sum_{1}^{N}|f(t_i)| \simeq \int_{\Omega}|f(t)|d\mu(f) = \|f\|_{L_1(\mu)} . \tag{4.1}$$

Indeed, we need to have a good estimate on a measure of such a $\bar{t} = (t_i)_{i=1}^N \in \Omega^N$ that (4.1) will hold up to some $\varepsilon > 0$. Then we introduce a random variable defined by a function $g(t) = |f(t)| - \|f\|_{L_1(\mu)}$ and the use of the lemma will now depend on an inequality of the type

$$\|f\|_{L_{\psi_1}} \le C\|f\|_{L_1} \qquad (\text{or } \|f\|_{L_{\psi_2}} \le C\|f\|_{L_1}) .$$

Note that these inequalities are equivalent to the following ones:

$$\|f\|_{L_p} \le C \cdot p \cdot \|f\|_{L_1} \qquad (\text{respectively } \|f\|_{L_p} \le C\sqrt{p}\|f\|_{L_1})$$

for all $p \ge 1$ and a universal $C > 0$. We call such inequalities Khinchine-type inequalities because of the following simplest of examples due to Khinchine: $1 \le p < \infty$, $\forall \lambda_i \in \mathbb{R}$,

$$\Big(\operatorname*{Ave}_{\varepsilon=\pm 1}\Big|\sum_{1}^{n}\varepsilon_i\lambda_i\Big|^p\Big)^{1/p} \le C \cdot \sqrt{p}\operatorname*{Ave}_{\pm}\Big|\sum_{1}^{n}\pm\lambda_i\Big|$$

(C is a universal constant). Such inequalities are a much studied subject. They are known in different situations by different names (Kahane inequality; Landau-Shepp-Fernique Theorem; Marcus-Pisier inequality – see, e.g., [MaP]; in the case of linear functionals f on a convex compact body in $\mathbb{R}^n$ – see [GrM2]).

4.C The Laplacian operator approach.

A unified approach to estimate the concentration function $\alpha(X;\varepsilon)$ of a metric probability space (X,ρ,μ) involves the first non-trivial eigenvalue of Laplacian-type operators on X. We consider separately a smooth (Riemannian) case and a discrete case.

4.C.1 Riemannian case.

Consider a *compact connected Riemannian manifold* M with μ being the *normalized Riemannian volume element* of M. Then the Laplacian $-\Delta$ on M has its spectrum consisting of eigenvalues $0 = \lambda_0 < \lambda_1(M) \le \lambda_2(M) \ldots$. The first non zero eigenvalue λ_1 may be represented by the min-max principle as the largest constant such that

$$\lambda_1\|f\|_{L_2}^2 \le (-\Delta f, f) = \int_M |\nabla f|^2 d\mu$$

for every "sufficiently smooth" function f on M such that $\int_M f = 0$. This inequality contains an estimate on the concentration function $\alpha(M;\varepsilon)$:

Theorem ([GrM1]). *Let (M, ρ, μ) be a compact connected Riemannian manifold (μ the normalized Riemannian volume). Let $A \subseteq M$ with $a = \mu(A) > 0$. Then, for all $\varepsilon > 0$,*

$$\mu(A_\varepsilon) \geq 1 - (1 - a^2) \exp\left(- \varepsilon \sqrt{\lambda_1} \log(1 + a)\right) . \tag{4.2}$$

Therefore,

$$\alpha(M, \varepsilon) \leq \frac{3}{4} \exp\left(- \varepsilon \sqrt{\lambda_1(M)} \log 3/2\right) .$$

When we apply this theorem to Examples 4.A.2, we see that we indeed obtain worse results than using Theorem 4.A.1. (Note that, e.g., $\lambda_1(S^n) = n$ and $\lambda_1(T_n) = 1$.) However, in general,the above theorem cannot be essentially improved. We will show this in the next discrete case. The examples to come have, in some sense, hyperbolic type, as opposed to Examples 4.A.2 which are, in a sense, "elliptic". Non-exactness of (4.2) in all cases of 4.A.2 may be related to this difference. I don't yet have a correct understanding of this distinction.

4.C.2 Discrete case.

A generalization of 4.C.1 to graphs was realized in [AlM1]. We give below (section 4.C.2 and 4.C.3) a short account of these results.

Let $G = (V, E)$ be a connected graph on $|V| = n$ vertices. We equip V with the counting probability measure $\mu(A \subset V) = |A|/|V|$ and with the path metric: $\rho(x, y) = \{$the smallest number of edges in a path which joins vertices x and $y\}$. Let D be an orientation of G. Let $C = C_D = (e, v)_{v \in E, v \in V}$ be the incidence matrix of D, i.e., a matrix with $|E|$ rows indexed by the edges of D and $|V|$ columns indexed by the vertices of D in which $c_{e,v} = 1$ if v is the head of e, -1 if v is the tail of e, and 0 otherwise.

Define $Q = Q_G = C^T \cdot C$. Then $Q = \operatorname{diag}\left(d(v)\right)_{v \in V} - A_G$, where $d(v)$ is the degree of the vertex $v \in V$ and A_G is the adjacency matrix of G. Therefore Q is independent of the orientation D of G.

Let $L^2(V)$ ($L^2(E)$) denote the space of real valued functions on V (on E) with the usual scalar product (f, g) and the usual norm $\|f\| = \sqrt{(f, f)}$ induced by it.

Consider the quadratic form (Qf, f) defined on $f \in L^2(V)$. Then

$$(Qf, f) = (Cf, Cf) = \sum_{e \in E} \left(f(e^+) - f(e^-)\right)^2$$

where e^+ and e^- denote the head and the tail of the edge e of D. Since G is connected we conclude that $(Qf, f) \geq 0$ for all $f \in L^2(V)$ and equality holds iff f is a constant. Let

$0 = \lambda_0 < \lambda_1 = \lambda_1(G) \le \lambda_2 \le \cdots \le \lambda_{n-1}$ be the eigenvalues of Q, each appearing in accordance with its multiplicity.

It is helpful to consider the operator $A : L^2(V) \to L^2(V)$ as the (minus) Laplace operator for the graph $G = (V, E)$. Then the analog of Theorem 4.C.1 may be shown for such a discrete Laplace operator.

Theorem ([Al.M]). *Let $d = \max\{d(v) \mid v \in V\}$ be the maximum degree of a vertex of the graph G. Then (using the above notations) the concentration function of (V, ρ, μ) is estimated by*

$$\alpha(V; \varepsilon) \le \tfrac{1}{2} \exp\{-\varepsilon\sqrt{\lambda_1/2d} \log 2\} .$$

(Note that ε is an integer in this case and ranges from 1 to the diameter of V.)

4.C.3. The most important examples of the use of Theorem 4.C.2 are the so called Cayley graphs. Let V be a finite group and $S \subset V$ some set of generators. Assume $S = S^{-1}$ and the identity $e \notin S$. We join v and u from V by an edge if $u = s^{-1}v$ for $s \in S$. Then a path distance in such a graph, $G = (V; E)$, is the word distance in V induced by S. The degree d of G is equal to $|S|$. Consider $L_2(V)$ and let π be the left regular representation of V: $\pi(t)f(v) = f(t^{-1}v)$. So if $\{e_t\}_{t \in V}$ is the natural basis of L_2 then $\pi(t)e_v = e_{tv}$ and the Lapalcian

$$Q = |A| \cdot Id - \sum_{s \in S} \pi(s) .$$

This is a self-adjoint operator (note that $S = S^{-1}$) and $\lambda_1(Q) = \{|S|$ – the second largest positive eigenvalue of $A(G) = \sum_{s \in S} \pi(s)\} \ge |S| - \|A|_{L_2^0}\|$ where $L_2^0 \oplus \{\text{Const.}\} = L_2(V)$.

To use Theorem 4.C.2 for a construction of a Lévy family of graphs, we have to find natural families of groups V_i, $|V| = n_i \to \infty$, and generators $S_i \subset V_i$, $|S_i| = d_i \le$ Const., such that $\|A(G_i)|_{L_2^0}\| \le |S_i| - \varepsilon$ for some positive fixed ε. This brings us to the well-known – in Representation Theory – T-property of Kazhdan [K1] which gives us a number of most interesting examples of Lévy families.

Remark. If $|V| = n$ and $\mu(A) < \frac{1}{n}$ then $\mu(A) = 0$. This helps us to estimate $\operatorname{diam} G \le 2\left[\sqrt{2d/\lambda_1} \log_2 n\right]$. So, if $\lambda_1(G) \ge c > 0$ and G is, say, 4-regular, then $\operatorname{diam} G \le C \log_2 n$ and, of course, $\operatorname{diam} G \ge c_1 \log_2 n$. This shows that (4.2) cannot be improved for these series of examples.

Example. Fix $k \geq 2$ and consider a group of matrices $SL_k(\mathbb{Z})$ (of determinant 1 and integer entries). Fix a set of generators $S = \{s_1, s_2, s_1^{-1}, s_2^{-2}\}$ (such a set exists), i.e., $|S| = d = 4$. Let $\mathbb{Z}_p$ be the ring mod $p \geq 2$ and let

$$\varphi_p : SL_k(\mathbb{Z}) \longrightarrow SL_k(\mathbb{Z}_p) \qquad (\stackrel{\text{def}}{\equiv} V_p)$$

be the natural surjection. Then $\varphi_p S = S_p$ are generators of V_p which define a word-metric ρ_p. Then there exists a numerical constant $c > 0$ such that the concentration function

$$\alpha(V_p; t) \leq \tfrac{1}{2} \exp(-ct)$$

(the bound does not depend on p). Note also that $n = |SL_k(\mathbb{Z}_p)| = p^{k^2-1}$ and $\operatorname{Diam} V_p \simeq c \log n \simeq c(k) \log p$. So, taking t being a small portion of diameter $t = \delta \log p$, we have a very strong concentration phenomenon for a family $\{V_p\}$.

Warning. As D. Kazhdan has proved in [K1], $SL_k(\mathbb{Z})$ has the T-property only for $k \geq 3$; however the above construction works (as was also shown by Kazhdan) even for $k = 2$.

Remarks:. The Lévy families of graphs constructed above are examples of what is known in Combinatorics as families of linear expanders. Expanders, which are the subject of an extensive literature, have numerous applications in theoretical computer science, including design of various sorting and interconnection networks. See [Al] and its references for more details.

4.C.4. As Theorems 4.C.1 and 4.C.2 show, the concentration property on a Riemannian manifold (or a graph) depends on a λ_1 of a suitable Laplacian. Therefore, all available methods for estimation λ_1 from below may be applied for estimation of the concentration function. This brings us to different types of mathematics. I will give one example, elaborated to me by D. Kazhdan [K2], which uses a modern knowledge of semisimple Lie groups.

Example. Consider the group $SL_n(\mathbb{R})$ equipped with the standard Riemannian metric (i.e., infinitesimal Hilbert-Schmidt metric) and a Haar measure. Let Γ be any lattice of $SL_n(\mathbb{R})$ (for example $\Gamma = SL_n(\mathbb{Z})$). Consider the Riemannian manifold (of a finite volume) $M = \Gamma \backslash^{SL_n(\mathbb{R})}$. Then $\lambda_1 \geq cn^3$ (for some universal constant $c > 0$). Therefore

$$\alpha(M; \varepsilon) \lesssim \exp(-c\varepsilon n^{3/2}) \ .$$

5. The Concept of a Spectrum

We have already noted in section 3 that the concept of concentration is often applied in geometry through another concept of "spectrum" of uniformly continuous functions on high dimensional structures.

It will be easier to emphasize the main idea in an infinite dimensional language.

5.1. Let X be an infinite dimensional Banach space, $S = S(X) = \{x \in X \mid \|x\| = 1\}$ and let $UC(S)$ be the space of all uniformly continuous real valued functions on S. If $f \in UC(S)$ then $a \in I\!R$ belongs to *the spectrum* $\mathcal{S}(f)$ of the function f iff for every $\varepsilon > 0$ and every $n \in I\!N$ there exists a subspace $E \hookrightarrow X$, $\dim E = n$ and $|f(x) - a| < \varepsilon$ for any $x \in S \cap E$.

Theorem [M6]. *For every $f \in UC(S)$, the spectrum $\mathcal{S}(f) \neq \emptyset$.*

Remarks. 1. In the case of $X = H$ being an infinite dimensional Hilbert space, this is an immediate consequence of Theorem 3.1.

2. Some applications of this theorem to Geometry of Normed Spaces, see [M4].

We describe a few more examples in the same spirit. Let $S^\infty = S(\ell_2)$.

5.2. Let $W_{\infty,2} = \{(x; y) \mid x \text{ and } y \in S^\infty \text{ and } x \perp y\}$ be a 2-Stiefel manifold; then we may define, in a similar way, the spectrum $\mathcal{S}(f)$ for $f \in UC(W_{\infty,2})$: $a \in I\!R$ belongs to *the spectrum* $\mathcal{S}(f)$ of the function f iff for every $\varepsilon > 0$ and every $n \in I\!N$ there exists a subspace $E \hookrightarrow \ell_2$, $\dim E = n$ and $|f(x, y) - a| < \varepsilon$ for any $x \in S^\infty \cap E$ and $y \in S^\infty \cap E$.

Theorem [M3]. *For every $f \in UC(W_{\infty,2})$, the spectrum $\mathcal{S}(f) \neq \emptyset$.*

(Clearly, this is a consequence of Theorem 3.3.b.)

Similarly, we could define a spectrum of a uniformly continuous function of two variables $f \in UC(S^\infty \times S^\infty)$ and the statement $\mathcal{S}(f) \neq \emptyset$ would be, in this case, an interpretation of Theorem 3.3.a.

In the definition of the spectrum of functions defined on $W_{\infty,2}$ (and $S^\infty \times S^\infty$), we did not emphasize the role of groups which act on these manifolds. However, the fact that these were homogeneous spaces, i.e., suitable groups act transitively, was crucially important when we defined a point of the spectrum as a number.

5.3. If, for example, we consider in the second case of $S^\infty \times S^\infty$ an action in $S^\infty \times S^\infty$ of another group $G = \{(T; T)\ ,\ T \in U(S^\infty)$ – a unitary operator in $\ell_2\}$, then G does not act transitively on $S^\infty \times S^\infty$. Therefore, the spectrum $\mathcal{S}(G; f)$ of a uniformly continuous function $f \in UC(S^\infty \times S^\infty)$ with respect to the action G consists in this case of continuous functions

$\varphi(t)$ of a parameter $t \in [-1,1]$ which determines different orbits of the action G. This is an interpretation of Theorem 3.4.

The main observation behind the notion of spectrum is that uniformly continuous functions on infinite dimensional G-spaces depend "essentially" only on orbits of an action G. Of course we may interpret differently the word "essentially". An interpretation by measure will bring us back to the Lévy family-notion. However, considering substructures (say, linear subspaces in 5.1-5.2) where a function is "almost" constant we come to the concept of spectrum. (This notion is discussed in [M3], [M4] and, in a more general context of G-spaces, in [GrM1] – see 5.6 below. For the case of Riemannian manifolds – see Gromov [Gr3].) Note that the well known Ramsey type theorems in Combinatorics are very close in spirit to the discussed notion of spectrum.

5.4. The next example (from [GrM1]) will be used in topological applications. Define $G_k(H)$ the Grassmann manifold of k-dimensional subspaces of an infinite dimensional Hilbert space H and let $S^\infty = S(H)$ be the unit sphere of H (i.e., $S^\infty = \{x \in H \mid |x| = 1\}$. Consider $V_{\infty,k} = \{(\xi,x) \mid \xi \in G_k(H)\ ,\ x \in \xi \cap S^\infty\}$ – the canonical sphere bundle over Grassmannian. Let $f : V_{\infty,k} \to \mathbb{R}$ be a uniformly continuous function ($f \in UC(V_{\infty,k})$). We say that $a \in \mathbb{R}$ is from the *spectrum* $\mathcal{S}(f)$ iff for every $n \in \mathbb{N}$, $n > k$, and any $\varepsilon > 0$ there exists an n-dimensional subspace $E \hookrightarrow H$ such that

$$|f(\xi;x) - a| < \varepsilon$$

for every $(\xi;x) \in V_{n,k}(E)$, i.e., for every k-dimensional subspace $\xi \hookrightarrow E$ and any $x \in \xi$, $|x| = 1$. Again, we have

Theorem. *For every $f \in UC(V_{\infty,k})$, the spectrum $\mathcal{S}(f) \neq \emptyset$.*

This result follows from a concentration propery of a family $\{V_{n,k}\}_{n>k}$ – see example 4.A.2f.

5.5 G-spaces. Consider a metric space X with a uniformly continuous action of a group G. Let $G = \bigcup_{n>1}^{\infty} G_n$ where $G_n \subseteq G_{n+1}$ are subgroups of G. Following [GrM1] we call G-space $(X;G)$ *Lévy* if there is a sequence of G_n-invariant probability measures μ_n on X such that $\{Y_n = X, \mu_n\}$ is a Lévy family. Assume in addition that X is the closure of $\bigcup_{n\geq 1} X_n$, $X_n \subset X$, μ_n is supported by X_n and G_n acts transitively on X_n. To describe a few typical examples, we consider an infinite dimensional Hilbert space H with an orthonormal basis $\{e_i\}_{i=1}^\infty$. Let the orthogonal group $SO(n)$ be realized as unitary operators on H which are the identity on span$\{e_i\}_{i>n}$. Then $SO(n) \subset SO(n+1)$, $n \in \mathbb{N}$. Define $G = SO(\infty) \stackrel{\text{def}}{=} \bigcup_{n\geq 1} SO(n)$ equipped with the Hilbert-Schmidt operator metric. In all the following examples $G_n = SO(n)$ and $G = SO(\infty)$.

a. $X = S^\infty$ and $X_n = S(\text{span}\{e_i\}_1^n)(= S^{n-1})$

b. $X = W_k(H)$ – k-Stiefel manifold (i.e., the manifold of all indexed sets of k orthonormal vectors from H with the metric described in section 3); $X_n = W_k(\text{span}\{e_i\}_{i=1}^n)$ $(\equiv W_{n,k})$.

c. $X = G_k(H)$ – the Grassmann manifold of k-dimensional subspaces of H; $X_n = G_{n,k} = G_{n,k}(\text{span}\{e_i\}_{i=1}^n)$.

5.6 Spectrum of functions on G-spaces. Let $f : X \to \mathbb{R}$ be a uniformly continuous function on X, i.e., $f \in UC(X)$. We say that a number $a \in \mathbb{R}$ is from *the spectrum* of f iff, for every $\varepsilon > 0$ and for every n, there exists $g \in G$ such that $|f(x) - a| < \varepsilon$ for every $x \in gX_n$.

Theorem ([GrM1]). *If $(X; G)$ is a Lévy G-space and $\{G_n\}_{n\geq 1}$ are compact groups, then for any bounded uniformly continuous function $f : X \to \mathbb{R}$ the spectrum of f is not empty.*

5.7. Some *generalizations* to a *Riemannian case* were shown by Gromov ([Gr3], section 9). We give an example of such a result.

Let V be a complete Riemannian manifold of dimension $\geq n$ and let a locally compact group G act isometrically on V. Consider a Lipschitz map $f : V \to X$ onto an m-dimensional subset of Riemannian manifold X. We denote

$$\text{Lip}(f) = \sup\left\{\text{dist}(f(t_1), f(t_2))/\text{dist}(t_1, t_2)\right\}$$

the Lipschitz constant of the map f (= Dil$f$ in Gromov's terminology).

Theorem ([Gr3],section 9.3). *Let either the group G be amenable or the function f be invariant under a discrete subgroup $\Gamma \subset G$ for which the quotient V/Γ has subexponential growth.* Then, for every $(q+1)$-points set $K \subset V$, there exists $g \in G$ such that we have, for a translate $K_0 = gK$,*

$$\text{Osc} f\big|_{K_0} \leq \text{Lip}(f) \cdot (\text{Diam}\, K)\sqrt{qm/n}\ .$$

(Therefore, if the dimension n of V is much larger than the dimension m of the image Im f, *then oscillation of f on some translate of K is very small.)*

* We say that a Riemannian manifold M (in our case V/Γ) has subexponential growth if the function $\varphi(\rho) = \text{Vol}(*)_\rho$ has subexponential growth. (Here $(*)_\rho$ means the ρ-neighborhood of the point $(*)$.)

5.8 Historical notes. The concept of spectrum first appeared in [M5] and was later developed in [M6] and [M3]. It was called, at that stage, (*)-spectrum (or finite dimensional spectrum) and the term "spectrum" was kept for an infinite-dimensional analog. However, it was (and *is*) a problem if any uniformly continuous function $f(x)$ on $S^\infty = S(H)$ has a non-empty infinite-dimensional spectrum. This means:

> Does there exist $a \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists an *infinite-dimensional* subspace E of a Hilbert space H, such that $|f(x) - a| < \varepsilon$ for any $x \in S^\infty \cup E$?

(In a slightly different but equivalent form it is known today as the "distortion problem" for ℓ_2) Because this main fact of the theory could not be proved, a term (infinite-dimensional) "spectrum" became obsolete and then, with time, we dropped (*) and started to define "spectrum" as it is defined in this paper.

It is also curious to note a geometric problem, the solution of which first pointed towards the concept of spectrum and made use of the Lévy concentration property of spheres.

Let X be an infinite dimensional Banach space and $x \in S(X)$, i.e., $\|x\| = 1$. Is it true that for any $n \in \mathbb{N}$ and $\lambda > 0$

$$\inf_{\substack{E_n \text{ a subspace} \\ \dim E_n = n}} \sup\{\|x+y\| \mid y \in E_n\ ,\ \|y\| = \lambda\} \le \sup_{\substack{E_n \hookrightarrow X \\ \dim E_n = n}} \inf\{\|x+y\| \mid y \in E_n\ ,\ \|y\| = \lambda\}\ ?$$

The positive answer [M7] was obtained using the following lemma:

Lemma. *For any $x \in S(X)$, any $\lambda > 0$, $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists an n-dimensional subspace E_n such that a function $f(y) = \|x+y\|$ has an oscillation* $\operatorname{osc}\{f(y) \mid y \in E_n\ ,\ \|y\| = \lambda\} < \varepsilon$.

Clearly, the Lemma follows immediatly from Theorem 5.1.

6. Applications to Topology and Fixed Point Theorems

Results of this section were originated in [GrM1]. We start with an interpretation of Theorem 5.4 in the case $S(f) = \{0\}$.

6.1 Corollary. *Let $f : V_{\infty,k} \to \mathbb{R}$ be a uniformly continuous function. If for every k-dimensional subspace $\xi \hookrightarrow H$ there exists $x_\xi \in \xi$, $|x_\xi| = 1$, such that $f(\xi, x_\xi) = 0$ then for any $\varepsilon > 0$ there exists a subspace $\xi_\varepsilon \in G_k(H)$ such that $|f(\xi_\varepsilon, x)| < \varepsilon$ for every $x \in \xi_\varepsilon$, $|x| = 1$.*

Clearly, this statement has a topological spirit. Indeed, nonexistence of a uniformly continuous section $\varphi : G_k(H) \to V_{\infty,k}$ is an immediate consequence:

If such a uniformly map φ exists then consider $M_1 = \{(\xi, x = \varphi(\xi))\}_{\xi \in G_k(H)} \subset V_{\infty,k}$ and $M_2 = \{(\xi, -\varphi(\xi))\}_\xi \subset V_{\infty,k}$; the distance between sets M_1 and M_2 is positive and therefore, by Urysohn's Theorem, there exists a uniformly continuous function, such that $f(M_1) = 0$ and $f(M_2) = 1$ which contradicts Corollary 6.1.

Of course, it is well known that even a continuous section does not exist in this example. However, such an approach leads to a general fact which we describe next.

6.2 The bundle sections. Let $(X;G)$ be a Lévy G-space and Y a compact G-space. If G acts freely on X we have a natural fibration $(X \times Y)/G \to X/G$ with fiber Y (use a diagonal action in $X \times Y : (x;y)g = (xg, g^{-1}y)$). Let G act equicontinuously on X. Then

Theorem. *The uniformly continuous section $X/G \to (X \times Y)/G$ exists only in the trivial case of existence of a common fixed point of G on Y.*

In the example described in 6.1, we have $X = W_k(H)$, $G = O(k)$ – the orthogonal group and $Y = S^{k-1}$. As it is clear from the Theorem, we could take instead of fiber S^{k-1} any compact $O(k)$-space Y where $O(k)$ acts without common fixed point.

6.3 Fixed point theorems. The next general fact shows that a concentration property of a space puts a limitation on an equicontinuous action in this space.

6.3.a Theorem. *Let (X,G) be a Lévy G-space, X-compact and G act equicontinuously on X. Then there is $x \in X$ which is fixed under the action of G.*

In a proof of the next, more concrete, fact the concentration property of the family of sphere $\{S^n\}$ is used together with the spectral theorem for a unitary operator:

6.3.b Theorem. *Let K be a Hausdorff compact space and $A_i : K \to K$ – continuous maps. If there exist a uniformly continuous map $\varphi : S^\infty \to K$ and pair-wise commutative unitary operators $U_i : S^\infty \to S^\infty$ such that $\varphi U_i = A_i \varphi$ (i.e., we have commutative diagrams) then there exists a common fixed point $x_0 = A_i x_0$ (i's run through any set of indices).*

To describe the main idea behing this kind of application of Lévy families, we introduce an *infinite dimensional analog* of the concentration property.

6.4 Concentration property of infinite dimensional G-space. Let M be a metric space (with a metric ρ) and let G be a family of uniformly equicontinuous maps from M to M. This means that there exists a continuous function $\omega(\varepsilon)$ (defined for $\varepsilon \geq 0$), $\omega(0) = 0$, such that $\rho(gx, gy) \leq \omega\big(\rho(x,y)\big)$ for any $x, y \in M$ and any $g \in G$.

We say that a subset $A \subset M$ is *essential* (with respect to the action of G) iff, for every $\varepsilon > 0$ and every finite subset $\{g_1, \dots, g_n\} \subset G$,

$$\bigcap_{i=1}^{n} g_i A_\varepsilon \neq \emptyset$$

where $A_\varepsilon = \big\{x \in M : \rho(x, A) \leq \varepsilon\big\}$.

The idea behing the notion of "essential" set is, clearly, some kind of substitution for sets of full (or "almost" full) measures in the finite dimensional case. Note, that in all our applications we used sets of "almost" full measures – "random" subsets. So, essential sets are, indeed, "random" subsets of infinite dimensional manifolds.

Definition. The pair (M, G) has the *property of concentration* iff for every finite covering $M = \bigcup\limits_{1}^{N} M_i$, $M_i \subset M$, there exists M_{i_0} which is essential (for the action of G).

A map $\varphi : M \to K$ from a metric space M to a compact K is called uniformly continuous if for any closed subset $A \subset K$ and any open neighborhood $O(A)$ there exists $\varepsilon > 0$ such that $\varphi\big([\varphi^{-1}(A)]_\varepsilon\big) \subset O(A)$.

Theorem. *Consider a pair (M, G) with the property of concentration and a compact K. Let $\varphi : M \to K$ and $g_\alpha : M \to M$, $\{g_\alpha\}_{\alpha \in \Lambda} \subset G$, be uniformly continuous maps and let continuous maps $\{A_\alpha : K \to K\}_{\alpha \in \Lambda}$ be such that $\varphi g_\alpha = A_\alpha \varphi$ for $\alpha \in \Lambda$. Then there exists $x_0 \in K$ such that $A_\alpha x_0 = x_0$ for every $\alpha \in \Lambda$.*

To apply this theorem, we have to have a developed method to check that a pair (M, G) has a concentration property. Unfortunately, the only known way to check a concentration property is through concentration of a measure phenomenon. In such a way we establish the following examples.

Examples.

A. Let H be an infinite dimensional Hilbert space and $\{e_i\}_{i=1}^{\infty}$ an orthonormal basis of H. Let, as in 5.5, the orthogonal group $SO(n)$ be realized as unitary operators on H which are the identity on $\operatorname{span}\{e_i\}_{i>n}$. Then $SO(n) \subset SO(n+1)$, $n \in \mathbb{N}$. We consider $M = G = SO(\infty) \overset{\text{def}}{=} \bigcup\limits_{n \geq 1} SO(n)$ with the Hilbert-Schmidt operator metric ρ on M. Then:

$$(M, G) \quad \textit{has the concentration property}\ .$$

B. Let $S^\infty = \{x \in H\ ;\ |x| = 1\}$ be the unit sphere of the Hilbert space H with the standard euclidean distance. Then

$$(S^\infty, G = SO(\infty)) \quad \textit{has the concentration property .}$$

C. Let u be any unitary operator on H and $G = \{u^n\}_{n=-\infty}^{\infty}$. Then

$$(S^\infty, G) \quad \textit{has the concentration property .}$$

D. Let $\mathcal{M}$ be a family of pairwise commuting unitary operators in H. Then

$$(S^\infty, \mathcal{M}) \quad \textit{has the concentration property .}$$

Indeed every Lévy family of spaces corresponds to some space with the concentration property in the above sense. However, we don't know any answer for the following question.

Problem. Let U be the group of *all* unitary operators $H \to H$, $\dim H = \infty$. Does (S^∞, U) have the concentration property?

More details on the subject of this subsection and proofs may be found in [M8] where a finite dimensional version of fixed point theorems 6.3 and of this subsection are also discussed.

7. Averages of Uniformly Continuous Functions on Infinite Dimensional Manifolds

In this final section, I would like to show how the notion of spectrum, developed in sections 3 and 5, can be used for the original Lévy purpose – to define an Average of a uniformly continuous function $f(x)$ on S^∞.

Let $\bar{e} = \{e_i\}_1^\infty$ be an orthonormal basis of H, $E_n = \text{span}\{e_i\}_1^n$ and the unit sphere $S(E_n) = S^\infty \cap E_n = S^{n-1}$. Define $L_f(n) = L_{f|_{S^{n-1}}}$ the Lévy mean of $f(x)$ restricted on S^{n-1}.

If there exists $\lim L_f(x) \overset{\text{def}}{=} L_f(\bar{e})$ we may consider it to be an average of the function f on S^∞. Clearly, it depends on the basis $\bar{e}$.

Important remark. Let $f \in UC(S^\infty)$, i.e., f is a uniformly continuous function defined on S^∞. Then $L_f(\bar{e})$ exists for any orthonormal basis $\bar{e}$ iff the spectrum $\mathcal{S}(f)$ is a single point: $\mathcal{S}(f) = \{a\}$. Clearly, in this case, that limit is independent of a basis $\bar{e}$ and $L_f(\bar{e}) = a = \text{Ave}\{f(x) \mid x \in S^\infty\}$. We may describe this observation in other terms. Fix an orthonormal basis $\bar{e}_0$; then any other orthonormal basis $\bar{e}$ is the unitary rotation of $\bar{e}_0 : \bar{e} = T\bar{e}_0$ for a unitary operator T. Therefore, if we want the average $L_f(\bar{e}_0)$ to be invariant with respect to any unitary transformation (i.e., $L_f(\bar{e}_0) = L_{f\circ T}(\bar{e}_0) = L_f(T\bar{e}_0)$), then, automatically, $S(f) = \{a\}$ and $L_f = a$.

We see that the algebra of uniformly continuous functions on S^∞ having the spectrum consisting of a single value coincides with the class of functions having an invariant average with respect to any unitary rotation. So, the following criterion on a function to have a single-valued spectrum may be useful.

Proposition. *Let $f \in UC(S^\infty)$; $S(f) = \{a\}$ iff for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for any n-dimensional subspace E,*

$$\inf\{|f(x) - a| : x \in E \cap S^\infty\} < \varepsilon .$$

In a more general framework, we may be interested in a notion of an average of a function invariant with respect to a subgroup G of the unitary group $U(H)$ of a space H. Then, similarly, the spectrum on orbits of the action of G has to be a single point. For example, let $H = H_1 \oplus H_2$, and $i : H_1 \to H_2$ an isometry. Define groups

$$G_1 = \{(A; B) \mid A \in U(H_1) \ , \ B \in U(H_2)\} ,$$

i.e., $T \in G_1$ iff $T(x_1; x_2) = (Ax_1; Bx_2)$ where $x_k \in H_k$, $k = 1, 2$, and

$$G_2 = \{(A; iA) \mid A \in U(H_1)\} .$$

Then the orbits of the action G_1 are $r_1 S^\infty \times r_2 S^\infty$-manifolds ($r_1^2 + r_2^2 = \|x\|^2$ where $x = (x_1; x_2) \in H$) and the spectrum from 5.2 has to be used. The orbits of the action G_2 are $W_{\infty,2}$-manifolds and again 5.2 has to be used.

If we consider a function of two variables $f(x, y)$, $x \in S^\infty$, $y \in S^\infty$, then a unitary tranformation preserves the inner probuct $t = (x, y)$ and the orbits are generalized Stiefel manifolds, as in Theorem 3.4 and 5.3.

In all of these cases there exists an easy criterion on functions to have the spectrum consisting of a single value, similar to the above proposition.

More information on this subject and a few concrete integral formulas may be found in [M2], [M3].

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