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A crossing estimate for the canonical process on a Dirichlet space and a tightness result

by Terence J. LYONS and Zheng WEIAN[1]

0. Introduction.

In this paper we introduce an extension of the Stratonovich Integral. We use this extension to establish an estimate for the number of times a Dirichlet Markov process crosses between two subsets in its state space. In turn, this enables us to show (under quite general conditions) that the measures on path space associated with the Markov processes of a family of equivalent Dirichlet forms are tight (in the topology of Meyer). Under the hypothesis that the Dirichlet forms arise from uniformly elliptic operators in Divergence form we also establish a continuity result: if the forms converge in a dominated way then the laws of the processes converge. Our method is more general than the result given here, the result stated could also be deduced using p.d.e. methods.

We also develop the extension of Stratonovich calculus to L^2 vector fields - a more systematic account should follow. In this paper we establish a maximal estimate, and also the natural result that if u is in L^2 and is divergence free (in the appropriate sense for the Dirichlet form) then the Stratonovich integral $\int_0^t u * dX_t$ is a local martingale.

Using some special linear operator Γ, Nakao [3] has introduced his definition of Stratonovich integral as an additive functional. His definition will be the same as ours if the evolution process has an initial distribution which is absolutely continuous with respect to the invariant measure. But our

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definition can be more easily extended to the case where the integrated vector fields are time-dependent.

The usual smoothness restrictions imposed on u when defining Stratonovich integrals arise from the requirement that the integral shall be a semi-martingale. As we shall explain, it seems more natural in this context that the integral should be a regular Dirichlet process and a difference of a forward and backward martingale. This is the essential point for it enables one to define a good integration theory for the class of integrands which are forward and backward predictable. Of course functions like $u(X_t)$ have just this sort of measurability property.

1. Martingale with σ-finite initial measure.

Let (Ω,Σ,μ) be a σ-finite measure space, $(\Sigma_t)_{t\in R_+}$ be a cadlag family of μ-completed sub σ-fields of Σ. We call a random process $X_t(\omega)$ <u>a square-integrable martingale with initial measure</u> μ_{Σ_0} if there is a sequence $\{B_n\}_{n=1,2,\ldots}$ so that (i) $B_n \in \Sigma_0$, (ii) $B_n \uparrow \Omega$, (iii) $\mu(B_n) < \infty$, (iv) the restriction of $X_t(\omega)$ to B_n) is a square integrable martingale in the filtration $(B_n, \Sigma\cap B_n, \Sigma_t\cap B_n, \mu(.\cap B_n)/\mu(B_n))$ for every fixed n, v) $\sup_t\int_\Omega |X_t|^2 < \infty$. We call a random process $X_t(\omega)$ <u>a local martingale with initial measure</u> μ_{Σ_0} if (i) to (iii) hold and (iv') then repeat (iv) replacing "square integrable" by "local".

Doob's inequality extends to these generalised square integrable martingales: suppose $X_0 = 0$ μ - a.e. then

$$\int_\Omega \sup_{s\le t}|X_s|^2 d\mu \le 4(\sup_{s\le t}\int_\Omega |X_s|^2 d\mu) \le 4\int_\Omega (\langle X,X\rangle_t - \langle X,X\rangle_0)d\mu \tag{1.1}$$

Here $\langle X,X\rangle_t$ is the usual bracket (that is the increasing predictable part of the Doob Meyer decomposition of X_t^2).

Let $\mathcal{X}$ be a locally compact Hausdorff space with a countable base and m be a positive Radon measure on $\mathcal{X}$ so that $\mathrm{Supp}[m] = \mathcal{X}$. Let $(\mathcal{F},\mathcal{E})$ be a Dirichlet space relative to $L^2(\mathcal{X},m)$. $\mathcal{F}$ is the domain of the Dirichlet form $\mathcal{E}$. Denote by $-A$ the unbounded positive definite self-adjoint operator associated with $\mathcal{E}$ [2].

Suppose there is a Markov process X_t on (Ω,Σ) with continuous paths in $\mathcal{X}$ let P^x be the probability measure P^x on Ω for X_t given $X_0 = x$. Suppose further that A is the generating operator of X_t in the sense that for every fixed initial distribution m' for X_0 and all $f \in \mathcal{D}(A)$,

$$N_t^f = f(X_t) - \int_0^t Af(X_s)ds - f(X_0) \tag{1.2}$$

is a (F_t)-local martingale for any σ-finite initial measure m'. (These restrictions are not too onerous, but they certainly simplify the presentation.) When we take m as the initial measure N_t^f is a square integrable martingale. Because A is self adjoint in $L^2(X,dm)$ the process

$$\bar{N}_t^f = f(X_{1-t}) - \int_0^t Af(X_{1-s})ds - f(X_1)\ , \qquad 0 \le t \le 1 \tag{1.3}$$

is a G_t-square integrable martingale where (G_t) is the natural filtration of X_{1-t}.

Remark. Even in the case when $\mathcal{X} = \mathbb{R}^d$ and $A = \frac{\partial}{\partial X_i} a^{ij} \frac{\partial}{\partial X_j}$ we will not normally impose smoothness conditions on a^{ij} and so X_t will not generally be a semi-martingale, nor will twice differentiable functions be in the domain of A.

Providing X does not explode in finite time then there is a well known relationship between the quadratic variation of the semi-martingale $f(X_t)$ and the Dirichlet norm of f

$$\int_\Omega \langle N^f,N^f\rangle - \langle N^f,N^f\rangle_0 dP^m = \int \langle \bar{N}^f,\bar{N}^f\rangle_1 - \langle \bar{N}^f,\bar{N}^f\rangle_0 dP^m = 2\mathcal{E}(f,f)\ . \tag{1.4}$$

Using this, we can extend the identity

$$f(X_t) = f(X_0) + \tfrac{1}{2}N_t^f + \tfrac{1}{2}(\overline{N}_1^f - \overline{N}_{1-t}^f) \tag{1.5}$$

to all $f \in \mathcal{F}$ by approximation argument as follows. Firstly (1.5) is clearly true for $f \in \mathcal{D}(A)$ by considering (1.2) and (1.3) in combination. Secondly, if $f \in \mathcal{F}$ one may always approximate it by $f_n \in \mathcal{D}(A)$ so that $\mathcal{E}(f_n - f, f_n - f) + \int |f_n - f|^2 dm$ converges to zero. Because of (1.4) the martingales $N_t^{f_n}$, $\overline{N}_t^{f_n}$ clearly converge in L^2 giving an expression (1.5) for $f(X_t)$ as a difference of forward and backward martingales. Note that this construction defines N^f, $\overline{N}^f$ for all $f \in \mathcal{F}$; it is not clear that they are the unique choices of martingales which satisfy (1.5).

In fact N^f can be defined for any f which is locally in $\mathcal{F}$ (the Dirichlet form is local because the process has continuous paths [F; p.114]) and its definition is pathwise; that is to say that if $f = g$ on some open set U and τ is the first time X_t quits U then N^f and N^g are equal for $t \le \tau$.

We wish to get a decomposition of X_t itself into a forward and backward local martingale like the decomposition at (1.5). We have a change of variable formula to exploit.

Let $\mathcal{X} = R^d$ and $g \in C_0^2(R^d)$. For $\{f_i\}_{i=1,\dots,d} \in \mathcal{D}(A)$, we have from Ito's formula and (1.2)-(1.3) that

$$\begin{aligned} &g(f_1(X_t), \dots, f_d(X_t)) - g(f_1(X_0, \dots, f_d(X_0)) \\ &= \tfrac{1}{2}\sum_i \int_0^t \frac{\partial}{\partial x_i} g(X_s) dN_t^{f_i} - \tfrac{1}{2}\sum_i \int_{1-t}^1 \frac{\partial}{\partial x_i} g(X_{1-s}) d\overline{N}_s^{f_i} . \end{aligned} \tag{1.6}$$

Then another approximation argument allows one to extend this to $g \in C_0^1(\mathbb{R}^d)$ and $f_i \in \mathcal{F}$. Suppose that $C_0^1(R^d) \subset \mathcal{F}$. We take a sequence of $\underline{g}_n = (g_n^i)_{i=1,\dots,d}$ and an exhaustion of $\mathbb{R}^n$ by relatively compact open sets $U_n \subset R^d$ such that $g_n^i \in C_0^1(R^d)$ and $g_n^i(x) = x^i$ on U_n . We know that for a.e. $\omega \in \Omega$, when n is sufficiently large, $X_t(\omega) \in U_n$ $(\forall t \in [0,1])$.

So that from the local nature of the definition of N^f we obtain

(1.7) $$(X_t - X_0)^i = \frac{1}{2}N_t^{x^i} - \frac{1}{2}(\overline{N}_1^{x^i} - \overline{N}_{1-t}^{x^i})$$

where x^i is the ith co-ordinate function on R^d. Putting $N^i = N^{x^i}$ and $N = (N^i)_{i=1}^d$ etc. to simplify notation we see that (1.6) becomes

(1.8) $$g(X_t) - g(X_0) = \frac{1}{2}\sum_i \int_0^t \frac{\partial}{\partial x_i} g(X_s)dN_s^i - \frac{1}{2}\sum_i \int_{1-t}^1 \frac{\partial}{\partial x_i} g(X_{1-s})d\overline{N}_s^i .$$

Observe that the integrals in (1.8) are classical forward and backward Ito integrals and present no problems of interpretation. We will use this expression f_0, X_t to express X_t as a weak solution to a stochastic differential equation in terms of Brownian motion.

So far we have considered the case when $\mathcal{X}$ is Euclidean space R^d and $M = (\Omega, \Sigma, X_t, P_x)$ is an m-symmetric diffusion process on R^d with Dirichlet form $\mathcal{E}$ on $L^2(\mathcal{X}, m)$, suppose now that it has $C_0(R^d)$ as a core. We still assume for simplicity that M admits no killing inside $\mathcal{X}$. Then from the example 5.2.1 (Fukushima), we know that if $\mathcal{E}$ is given by

(1.9) $$\mathcal{E}(u,v) = \int_{\mathcal{X}} \sum_{ij} (\nabla_i u) a_{ij} (\nabla_j v) dm ,$$

where $(a_{ij}(x))$ is measurable and positive definite, then for $g \in \mathcal{F}$ we have a pathwise version of (1.4) :

(1.10) $$\langle N^g N^g \rangle_t = \int_0^t \sum_{ij} \nabla_i g(X_s) a_{ij}(X_s) \nabla_j g(X_s) ds .$$

So that if we take N and $\overline{N}$ as in (1.7), we have

(1.11) $$d\langle N^i, N^j \rangle_t = a_{ij}(X_t)dt , \quad d\langle \overline{N}^i, \overline{N}^i \rangle_t = a_{ij}(X_{1-t})dt .$$

Take $A(x) = a_{ij}(x) = B'(x)B((x)_{ij}$ where $B(x)$ is a bounded Borel measurable matrix-valued function and $B'(x)$ is its transpose. Denote by B^{-1} its inverse. Then one may recover N as an Ito integral of two Brownian motions

(1.12) $$N_t = \int_0^t B(X_s)dW_s , \quad \overline{N}_t = \int_0^t B(X_{1-s})d\overline{W}_s ,$$

where

(1.13) $$W_t = \int_0^t B^{-1}(X_s) , \quad \overline{W}_t = \int_0^t B^{-1}(X_{1-s})d\overline{N}_s .$$

From (1.11), we can easily deduce that

$$\langle W^i, W^j\rangle_t = \delta_{ij} t\ , \quad \langle \overline{W}^i, \overline{W}^j\rangle_t = \delta_{ij} t\ .$$

So that W_t is a (F_t)-Brownian motion and $\overline{W}_t$ is a (G_t)-Brownian motion. We can write (1.7) as

$$X_t - X_0 = \frac{1}{2}\int_0^t B(X_s)dW_s - \frac{1}{2}\int_{1-t}^1 B(X_{1-s})d\overline{W}_s\ . \tag{1.14}$$

2. Crossing estimate for Dirichlet process.

If E, F are disjoint closed subsets of $\mathfrak{X}$ and X_0 is distributed like m then there is an estimate for the expected number of crossing X_t makes between E and F for $t \le 1$. This estimate depends only on $\mathcal{E}$ and will be essential to the compactness results we obtain in §3. We first establish a lemma. Let (Ω, Σ, μ) be a measure space.

LEMMA 2.1. Suppose (M_t, Σ_t) is a square-integrable martingale with σ-finite initial measure on the filtration E. Let $(S_n)_{n=1}^\infty$ be a sequence of non-negative random variables so that $S_1 \le S_2 \le \ldots$. If there is a sequence of stopping times $T_1 \le T_2 \le \ldots$ such that

$$T_{2n-1} \le S_{2n-1}\ , \quad S_{2n} \le T_{2n+1} \qquad (n = 1, 2, \ldots) \tag{2.1}$$

then

$$\sum_n \int_\Omega |M_{S_{2n}} - M_{S_{2n-1}}|^2 d\mu \le 16\int_\Omega (\langle M,M\rangle_\infty - \langle M,M\rangle_0) d\mu\ . \tag{2.2}$$

PROOF. For every fixed n, we consider the martingale $M_{t\wedge T_{2n+1}} - M_{t\wedge T_{2n-1}}$.

Then

$$|M_{S_{2n}} - M_{S_{2n-1}}|^2 \le 2|M_{S_{2n}} - M_{T_{2n-1}}|^2 + 2|M_{S_{2n-1}} - M_{T_{2n-1}}|^2$$

$$\le 4\Big(\sup_{T_{2n-1}\le t\le T_{2n+1}} |M_{t\wedge T_{2n+1}} - M_{t\wedge T_{2n-1}}|^2\ .\Big)$$

From (1.1.) ,

$$\int_\Omega |M_{S_{2n}} - M_{S_{2n-1}}|^2 d\mu \le 16\int_\Omega (\langle M,M\rangle_{T_{2n+1}} \langle M,M\rangle_{T_{2n-1}})^{d\mu}\ .$$

Summing over all n on the both sides of above inequality, we obtain the conclusion.

Now we return to the Dirichlet space $(\mathcal{F},\mathcal{E})$ relative to $L^2(\mathcal{X},m)$. Let F and G be two disjoint closed sets of $\mathcal{X}$. We define

$Q^{(F,G)} = \inf\{\mathcal{E}(u,u) \mid u \in \mathcal{F}, \quad u(x) = 0 \text{ if } x \in F \text{ and } u(x) = 1 \text{ when } x \in G\}$.

We always suppose that the set over which the infimum is taken contains an element.

Let h, f, g be functions from a linearly ordered set I into $\overline{R}$, with $f < g$. The number $N(I,h,f,g)$ of upcrossings of $[f,g]$ by h is defined as the supremum of the values of the positive integers k for which there exists $t_1 \dots t_{2k}$ in I satisfying $h(t_j) \le f(t_j)$ when j is odd and $g(t_j) \le h(t_j)$ when j is even.

We give now an estimate for number of crossings between F and G of the continuous Markov process X_t associated to $\mathcal{E}$. Let η be a function from $\mathcal{X}$ to $[0,1]$ satisfying $\eta(x) = 0$ if $x \in F$ and $\eta(x) = 1$ if $x \in G$. We define $N(I,X.,F,G) = N(I,\eta(X.),0,1)$ whenever I is a subset of the positive real numbers.

THEOREM 2.2. <u>We have</u>

$$\int_\Omega N([0,1],X.,F,G)dP^m \le 34Q^{(F,G)} .$$

(<u>Here</u> P^m <u>is the law of</u> $X.$ <u>started with</u> X_0 <u>distributed as</u> m – <u>the reversible and stationary measure</u>.)

PROOF. Set

$T_1 = \inf\{t \ge 0;\ X_t \in F\} \wedge 1, \quad T_2 = \inf\{t \ge T_1;\ X_t \in G\} \wedge 1,$

$T_3 = \inf\{t \ge T_2;\ X_t \in F\} \wedge 1, \ \dots$

$\underline{T}_1 = \inf\{t \ge 0;\ X_{1-t} \in G\} \wedge 1, \quad \underline{T}_2 = \inf\{t \ge \underline{T}_1;\ X_{1-t} \in F\} \wedge 1,$

$\underline{T}_3 = \inf\{t \ge \underline{T}_2;\ X_{1-t} \in G\} \wedge, \ \dots .$

Then excepting the first and last intervals one always has that if $T_i < T_{i+1}$ then there is exactly one j such that $\underline{T}_j \in (T_i, T_{i+1})$. Denote by $J(\omega)$ the largest j such that $\eta(X_{T_{2j}}) - \eta(_{T_{2j-1}}) = 1$. Then

$$N([0,1],X.,F,G) \le \sum_{j=1}^{J} |\eta(X_{T_{2j}}) - \eta(X_{T_{2j-1}})|^2 .$$

Now we suppose $\eta = f \in \mathcal{F}$, by (1.5).

$$(2.3) \quad N([0,1],X.,F,G) \le (\sum_{j=1}^{J} |N^f_{T_{2j}} - N^f_{T_{2j-1}}|^2 + \sum_{j=1}^{J} |\overline{N}^f_{T_{2j}} - \overline{N}^f_{T_{2j-1}}|^2 .$$

It is easily deduced from (1.4) that

$$\int_\Omega \sum_{j=1}^{J} |N^f_{T_{2j}} - N^f_{T_{2j-1}}|^2 dP^m \le 2\mathcal{E}(f\ f) .$$

Now the only thing that remains is to estimate the second term of (2.3). Denote

$$S_1 = (1-T_{2j}) , \quad S_2 = (1-T_{2J-1}) , \quad S_3 = (1-T_{2j-2}) , \ \dots ,$$

then

$$\underline{T}_{2n-1} \le S_{2n-1} , \quad S_{2n} \le \underline{T}_{2n+1} \qquad (n = 1,2,\dots,2J) .$$

We apply now lemma (2.1) to the second term of (2.3) and deduce from (1.4) the conclusion.

3. Some Tightness Results for Laws of Canonical Processes Associated to Dirichlet Forms.

This section has two main results. The first (Theorem 3.2) exploits the crossing estimate of §2 to obtain a tightness criteria for the measures on path space associated with uniformly equivalent Dirichlet forms - this tightness criteria is expressed in terms of the Meyer topology on path space rather than the more usual Skorohod topology. Although this is a weaker topology than the usual one it seems peculiarly well adapted to the problem at hand. The methods would easily extend to simple infinite dimensional problems.

The second theorem (3.4) is concerned with convergence. Suppose the forms converge and the processes converge then we give hypotheses which guarantee that the process associated with the limiting form is the limiting process. It is clear that a tightness result is not very useful without a theorem like this one.

Denote by $D[0,1]$ the space of all cadlag functions defined on $[0,1]$ taking values in $\mathfrak{X}$. P.A. Meyer introduced a tightness criterion for probability measures on $D[0,1]$ in a topology which is weaker than Skorohod's one. Our method for proving tightness is the same as that utilized in [4].

Let $\mathfrak{X}_\Delta = \mathfrak{X} \cup \Delta$ be the one-point compactification of $\mathfrak{X}$. When $\mathfrak{X}$ is already compact, Δ is regarded as an isolated point. For any subset $A \subset \mathfrak{X}$, $A \cup \Delta$ is endowed with the relative topology as a subspace of $\mathfrak{X}_\Delta$.

Let λ be the Lebesgue measure on $[0,1]$. Let $w(t)$ be a $\mathfrak{X}_\Delta$ valued Borel function on $[0,1]$. By definition, the <u>pseudo-path</u> of w is a probability law on $[0,1] \times \mathfrak{X}_\Delta$: the image measure of λ under the mapping $t \mapsto (t,w(t))$. We denote by $\mathcal{P}$ the mapping which associated to a path w its pseudo-path: $\mathcal{P}$ identifies two paths if and only if they are equal λ a.e.. In particular, $\mathcal{P}$ is 1-1 on D, and provides us with an imbedding of D into the compact space $\overline{\mathcal{P}}$ of all probability laws on the compact space $[0,1] \times \mathfrak{X}_\Delta$. Meyer gave the name of <u>pseudo-path topology</u> to the induced topology on D. Denote by $D_\Delta[0,1]$ the space of all cadlag functions defined on $[0,1]$ taking values in $\mathfrak{X}_\Delta$.

Let $\mathcal{R}$ be the set of all pairs of relatively compact open sets (E,F) with disjoint closures in $\mathfrak{X}_\Delta$. Let τ be a finite subdivision on $[0,1]$

$$\tau : 0 = t_0 < t_1 < \dots t_n = 1 .$$

We define for $\mu \in \overline{\mathcal{P}}$ a positive integer $N^{EF}(\mu)$ by the following condition: $N^{EF}(\mu) \geq k$ if and only if there exist elements of τ denoted as follows

$$0 \leq t_{i_1} < t_{i_1'} < t_{i_2} < t_{i_2'} \dots < t_{i_k} < t_{i_k'} < 1$$

such that μ charges (i.e. gives strictly positive measure to) each one of the open sets in $[0,1]$

$$(t_{i_1}, t_{i_1'}) \times E\ ,\quad (t_{i_2}, t_{i_2}) \times F\ ,\quad (tt_{i_3}, t_{i_3}) \times E\ ,\ \ldots\ .$$

The sets $\{\mu : N_\tau^{EF}(\mu) \geq k\} = \{\mu : N_\tau^{EF}(\mu) > k-1\}$ are open in $\overline{\mathcal{F}}$, so that N_τ^{EF} is a l.s.c. function, and the same is true for the function

$$N^{EF} = \sup_\tau N_\tau^{EF}\ .$$

We now formulate the analogous theorem to that in [4]. Let $(\mathcal{F}_n, \mathcal{E}_n)$ be a sequence of Dirichlet spaces relative to $L^2(\mathfrak{X}, m_n)$. We denote by $P^{\mathcal{E}_n,\mu_n}$ the probability measure on $D[0,1]$ induced by the Markov process associated to $\mathcal{E}_n$ with an initial probability measure μ_n (not necessarily be equal to m_n).

THEOREM 3.1 Let μ_n be absolutely continuous with respect to m_n and $\sup_{x,n} d\mu_n/dm_n < \infty$. If for every pair $(E,F) \in \mathfrak{K}$,

(3.1) $$\sup_n C^{(\overline{E},\overline{F})}(\mathcal{E}_N) < \infty\ .,$$

then there exists a subsequence n' of n such that $(P^{\mathcal{E}_{n'},\mu_{n'}})$ which converges weakly on $D_\Delta[0,1]$ to a law P.

PROOF. There exists a subsequence $(P^{\mathcal{E}_{n'},\mu_{n'}})$ which converges weakly on $\overline{\mathcal{F}}$ to some law P on $\overline{\mathcal{F}}$. We are going to prove that P is carried by $D_\Delta[0,1]$. From the Theorem 2 of [1], it is sufficient to prove that

(3.1') $$\forall (F,G) \in \mathfrak{K}\ ,\quad N^{FG} < \infty \qquad \text{a.s.}[P]\ .$$

But (3.1') is just a corollary to theorem 2.2 and (3.1).

Now we suppose that $\mathfrak{X} = R^d$, and the Dirichlet forms are generated by

(3.2) $$\mathcal{E}_n(f,g) = \int_{R^d} (\nabla f(x))^t A_n(x)(\nabla g(x))\,dm_n\ ,\qquad \forall f,\ g \in C_0^1(R^d)$$

where $A_n(x)$ is a bounded positive-definite matrix on R^d. In this case, $C_0^1(R^d)$ is a common core for all $\mathcal{E}_n$.

We have corollary to Theorem 3.1.

THEOREM 3.2. Let m_n be a sequence of Radon measure and let $\mathcal{E}_n$ be a sequence of Dirichlet forms generated by (3.2). Suppose that there is a Radon measure , such that for all $f \in C_0(R^d)$

$$\int_{R^d} f(x)m_n(dx) \longrightarrow \int_{R^d} f(x)m(dx) , \tag{3.3}$$

and suppose that $A_n(x)$ are uniformly bounded. Then we may choose a subsequence n' of the n and an exhaustion of R^d by increasing bounded open sets O_j such that for all bounded continuous functions g on D_Δ (with the pseudo-path topology) we have the following that if we abbreviate the measure $P_{n'}$ starting with X_0 distributed like $m_{n'}|_{O_j}$ as $P_j^{n'}$ then

$$P_j^{n'} g \longrightarrow Q_j(g) \tag{3.4}$$

for some finite measure Q_j on D .

Now we consider questions of continuity and suppose that there is a positive definite matrix $A(x)$ (bounded above and away from zero) such that

$$A_n(x) \longrightarrow A(x) , \quad \forall x \in R^d . \tag{3.5}$$

Applying 3.2, it is easy to obtain limit processes for subsequences $A_{n'}$; it is not clear at all that the limit law is independent of the subsequence and corresponds to the law one obtains by generating the Markov process associated to A .

We will discuss the behaviour of the processes corresponding to the limit laws P on $D_\Delta[0,1]$.

Suppose by fixing a subsequence that the measures P_j^k converge to some measure Q_j . Let the probability measure on path space arising from the Dirichlet process associated to A and started on O_j with law $m|_{O_j}$ be denoted by P_j . We will outline a proof that the resolvant of P_j^k converges to that of P_j . In turn this implies that the semi-groups converge in the strong operator topology and thus the finite dimensional distributions of the P_j^k converge to those of P_j (in the usual weak* toplogy). Because the P_j^k converge to Q_j in the topology of Meyer we also know that for Lebesgue almost all finite sequences $0 \le t_1 < t_2 < \ldots < t_n \le 1$ the finite dimensional distributions of P_j^k converge to those of Q_j. Because all the measures are supported on $D_\Delta[0,1]$ this allows us to identify the limit P_j as Q_j.

Suppose $m_k(dx) = q_k(x)dx$ and $m(dx) = q(x)dx$ are the respective measures determining a sequence of L^2 spaces and $A^{(k)}(x)$ are a sequence of positive definite matrix valued functions such that $q_k A^{(k)}$ is monotone increasing (or decreasing) to qA. And such that both q_k and $A^{(k)}$ are uniformly bounded (in x and k) above and below (as functions and as forms) by strictly positive constants. Formally the self adjoint operator associated to the form

$$\varepsilon_k(f,g) = +\int_{R^d} \frac{\partial}{\partial x_i} f \, A_{ij}^{(k)} \, \frac{\partial}{\partial x_j} g \, q_k(x)dx$$

and Hilbert space $L^2(q_k dx)$ is

$$L^{(k)} = \frac{1}{q_k} \frac{\partial}{\partial x_i} q_k A_{ij}^{(k)} \frac{\partial}{\partial x_j} .$$

If $R_\lambda^{(k)}$ is the resolvent operator of $L^{(k)}$ on $L^2(q_k dx)$ then because we may view $L^2(qdx)$ as the same space as $L^2(q_k dx)$ but with an equivalent norm we will also think of $R_\lambda^{(k)}$ as an operator on $L^2(qdx)$. Under the above hypotheses for all $f \in L^2(qdx)$ one has $R_\lambda^{(k)} f \to R_\lambda f$ in L^2 norm. We do not prove this fully here as our proof is a little awkward. To give the idea we mention the cases (i) where the form ε^k is kept fixed so $q_k A_k = qA$ but the Hilbert space norm is changing, and (ii) where q_k is kept fixed but the forms vary.

The case (ii) is completely standard operator theory (see p.459 of Kato or pp.117, 118 E.B. Davies). To treat case (i) it seems we need to exploit the fact that the forms are Dirichlet. In this case the operators $L^{(k)}$ are just $\frac{1}{q^k}\frac{\partial}{\partial x_i} qA\frac{\partial}{\partial x_j}$, we may relate the resolvent of $L^{(k)}$ to the process, X_t connected with $\mathcal{E}$, $L^2(qdx)$, (through the Feynman-Kac formula) we have an explicit formula for $R^k_\lambda f$:

$$(R^k_\lambda f)(x) = E^x\left(\int_0^\infty f(X_t)e^{-\lambda\int_0^t \frac{q_k(X_s)}{q(X_s)}ds} \frac{q_k(X_t)}{q(X_t)}dt\right) .$$

At least for $f \in C_0(R^d)$ the above integrals are well defined. The uniform estimates on $\frac{q_k(X_s)}{q(X_s)}$ from above and below allows the convergence to be deduced from an application of the dominated convergence theorem observing that one can dominate the integral for $R^k_\lambda f$ by that for $\frac{1}{\epsilon}R_{\epsilon\lambda}|f|$ for small ϵ . An application of the closed graph theorem completes this case ofthe argument.

Return to our earlier considerations about the convergence of P_j to Q_j . Let f,g be a pair of functions in $C_o(R^d)$ with the support of f in O_j . Then one has

$$\int g(P^k_t f)m_k dx = \int g(P_{j,t}^{\ k} f)m_k dx$$

in the "note added in proof" of Meyer and Zheng [4] this converges to

$$E^{Q_j}(g(X_0)f(X_\tau)) = \int g\underline{P}_t f \, gdx$$

for all $g \in L^2(q\ dx)$ supported on O_j and for all j .

Define $\underline{R}_\lambda f$ by

$$\underline{R}_\lambda f = \int_0^\infty e^{-\lambda t}\underline{P}_t f\ dt .$$

Now $\{R^k_\lambda f\}_k$ converges in $L_2(m)$-norm to $R_\lambda f$. But from (3.12) and the definition of $\underline{R}_\lambda f$, we know that $\underline{R}_\lambda$ is the weak limit of R^k_λ . Thus, from the uniqueness of weak limit, we deduce that $R_\lambda f = \underline{R}_\lambda f$ a.e.[m] . So that $P_t f = \underline{P}_t f$ a.e.[m] amd $P^k_t f$ converge weakly to $P_t f$. Moreover, since P_t is symmetric,

$$\text{(3.20)} \qquad \int_{R^d}(P^k_t f)^2 dm = \int_{R^d}(P^k_t f)^2 dm_k + \int_{R^d}(P^k_t f)(q-q_k)dx$$

$$= \int_{R^d} fP^k_{2t} f dm_k + \int_{R^d}(P^k_t f)^2(q-q_k)dx$$

and as P_t^k is a contraction and $1 - \frac{q_k}{q}$ is bounded and converges to zero pointwise one has in the limit as k goes to infinity

$$\int_{R^d} fP_{2t} f dm = \int_{R^d} (P_t f)^2 dm .$$

So that we have weak convergence and convergence of the norms; thus the $P_t^k f$ converge in L_2-norm to $\underline{P}_t f$.

Now we approach our target. To prove that the finite dimensional distributions converge. We take the sub-sequence P^k appearing in theorem 3.2; we will show that for any given bounded measurable functions $f_0, \ldots, f_n$ and almost every $0 = s_0 < s_1 < \ldots < s_n \leq 1$

(3.22)

$$E^{Q_j}[f_0(X_{s_0})\ldots f_n(X_{s_n})] = \int I_{0_j} f_0 P_{s_0}(f_1 P_{s_1 - s_0}(\ldots f_{n-1} P_{s_n - s_{n-1}} f_n) dm .$$

From lemma 5 of Meyer and Zheng [4], almost all finite dimensional distributions of $\{P^{n_k}\}_k$ converge. Thus to prove (3.22) it is sufficient to show that

$$\int I_{0_j} f_0 P_{s_0}^{n_k}(f_1 P_{s_1 - s_0}^{n_k}(\ldots f_{n-1} P_{s_n - s_{n-1}}^{n_k} f_n) dm_{n_k} \xrightarrow{k \to \infty}$$

(3.23)

$$\int I_{0_j} f_0 P_{s_0}(f_1 P_{s_1 - s_0}(\ldots f_{n-1} P_{s_n - s_{n-1}} f_n) dm .$$

But (3.23) follows from a repeated use of the following

(3.24) PROPOSITION. Let $\{g_k\}_{k=1,\ldots,\infty} \in L_2(m)$ such that g_k converge to g_∞ in $L_2(m)$-norm. Then

$$\int_{R^d} (P^{n_k} g_k - P_t g_\infty)^2 dm_k \xrightarrow{k \to \infty} 0$$

and thus

$$\int_{R^d} g_1 P_t^{n_k} g_k dm_k \xrightarrow{k \to \infty} \int_{R^d} g_1 P_t g_\infty dm .$$

PROOF. $\int_{R^d}(P_t^{n_k}g_k-P_t g_\infty)^2dm \le 2\int_{R^d}(P_t g_\infty-P_t^{n_k}g_\infty)^2dm + 2\int_{R^d}(P_t^{n_k}g_\infty-P_t^{n_k}g_k)^2dm_k$

$$+\ 2\int_{R^d}(P_t^{n_k}g_\infty-P_t^{n_k}g_k)^2(q-q_k)dX$$

and all the right hand terms go to zero as K tends to ∞.

Also

$$\int_{R^d} g_1P_t^{n_k}g_k dm_k - \int_{R^d} g_1P_t g_\infty dm$$

$$\le \int_{R^d} g_1P_t^{n_k}g_k|q_k-q|dX + \int_{R^d} g_1P_t^{n_k}g_k dm - \int_{R^d} g_1P_t g_\infty dm$$

which tends to zero.

Now we establish a theorem.

Let $\mathcal{E}_n$, $\mathcal{E}$ be determined by the A_n, A and $m_n = q_n dx$, $m = qdx$ as in (3.2). Suppose there is a number $c \in (0,\infty)$ such that

$$c > q_nA_n \ge q_{n+1}A_{n+1} > \frac{1}{c}$$

in the sense of forms and q_nA_n converges to qA. Assume also that $c > q_n > \frac{1}{c}$ for all n and q_n converges to q pointwise and let $m_n(dx) = q_n(x)dx$ etc.

Fix a positive function $p \in C_0(R^d)$. Let X^n be the process associated to $(\mathcal{E}_n m_n)$ started with law $p(x)m_n(dx)$ and N^n be its local martingale part.

THEOREM 3.3. <u>The pairs</u> (X^n,N^n) <u>converge jointly in distribuion to</u> (X,N) <u>in the topology of Meyer</u>.

PROOF. From Theorem 3.1 and (3.22), we know the convergence in distribution of X^n to X. Without losing generality, we may suppose that $\int_{R^d} pdm_n = \int_{R^d} pdm = 1$ in order to make the laws probability measures. Then, from Skorohod's theorem, we can suppose that X^n and X are processes on the same probability space (Ω,Σ,P) such that for a.e. fixed $\omega \in \Omega$, $X^n_\cdot(\omega)$ converges in measure to $X_\cdot(\omega)$ for Lebesgue measure on $[0,1]$ (see "note added in proof" (Meyer and Zheng)).

For every $u \in \mathfrak{D}(\mathcal{E})$, there exists a sequence of $u^k \in C^1_0(R^d)$ such that $\mathcal{E}_1(u-u^k,u-u^k) \longrightarrow 0$ because $C^1_0(R^d)$ is a core.

Clearly $u^k(X^n_t)$ converges in probability to $u^k(X_t)$ as n goes to ∞ for $t \in H \subset \mathbb{R}$ where H is a set of full Lebesgue measure. Because $\mathcal{E}_1(u-u^k,u-u^k) \geq c\, \mathbb{E}(|u(x^n_t) - u^k(X^n_t)|^2)$ independent of n it follows that $u(X^n_t)$ also converges in probability to $u(X_t)$ as n goes to ∞ .

Take $f \in C_0(R^d)$. Let G^n_α and (G_α) be resolvents associated to $\mathcal{E}_n$ and $\mathcal{E}$ respectively and $u = G_1 f$, $u^n = G^n_1 f$. Then from ((5.2.22), Fukushima),

$$u(X_t) - u(X_0) = \int_0^t (u(X_s) - f(X_s))dx + N^u_t ,$$

$$u^n(X^n_t) - u^n(X^n_0) = \int_0^t (u^n(X^n_s) - f(X^n_s))dx + (N^n_t)^u .$$

Since G^n_1 converge in $L_2(m)$-norm to $G_1 f$ and since $C_0(R^d)$ is "uniformly" dense in the set of all bounded functions which belong to $L_2(m) \cap (\cap_n L_x(m_n))$,

$$\sup_{t\in[0,1]} \left|\int_0^t (u^n(X^n_s)-f(X^n_s))ds - \int_0^t (u(X_s)-f(X_s))ds\right| \xrightarrow{n\to\infty} 0 \quad \text{in } L^2 .$$

So that $(N^n_\cdot)^u$ converge in measure to $N^u_\cdot$. Since $G^n_1(C_c(R^d))$ is dense in $\mathfrak{D}(\mathcal{E}^n)$ and $R_1(C_0(R^d))$ is dense in $\mathfrak{D}(\mathcal{E})$, we deduce that $(N^n_\cdot)^f$ converge in measure to N^f for all $f \in C_0(R^d)$.

Now, we repeat the procedure from (1.6) to (1.7), and deduce that $N^n_\cdot$ converge in Lebesgue measure to $N_\cdot$, that is our conclusion of the theorem.

4. Stratonovich Integration.

In (1.8), we obtained an Ito formula for Dirichlet process. It suggests to us a definition of Stratonovich integration for this kind of process.

Consider again the case where $\mathfrak{X}$ is Euclidean space R^d and $M = (\Omega,\Sigma,X_t,P_x)$ is an m-symmetric diffusion process on $\mathfrak{X}$ whose Dirichlet form $\mathcal{E}$ on $L^2(\mathfrak{X};m)$ possesses $C_0^1(R^d)$ as its core. We assume for simplicity that M admits no killing or explosion inside $\mathfrak{X}$. Suppose also that $\mathfrak{X}$ is given by (1.9).

Now let U^n be a sequence of bounded open sets such that $U^n \uparrow R^d$ and suppose $v(x) = (v^i(x))$ is a vector field (without any smoothness condition!) so that

$$\int \sum_{ij} v^i(x)a_{ij}(x)v^j(x)I_{U^n}(x)dm < \infty , \quad \forall n . \tag{4.1}$$

Then it is easy to verify that if we denote

$$T_n = \inf_t\{t,X_tU_n\} \quad \text{and} \quad \overline{T} = \inf\{t,X_{1-t} \notin U_n\} , \tag{4.2}$$

we will have

$$\sum_i E[\int_0^{T_n}(v^i(X_s))^2 d\langle N^i,N^i\rangle_s] < \infty , \tag{4.3}$$

$$\sum_i E[\int_0^{\overline{T}_n}(v^i(X_{1-s}))^2 d\langle N^i,,N^i\rangle_s] < \infty . \tag{4.4}$$

Because $T_n,\overline{T}_n$ increase to infinity we can define <u>Stratonovich integral</u> of $v(X_t)$ with respect to X (and implicitly the usual Riemannian metric) as follows:

$$\int_0^t v(X_s)*dX_s = \frac{1}{2}\sum_i\int_0^t v^i(x_s)dN_s^i - \frac{1}{2}\sum_i\int_{1-t}^1 v^i(X_{1-s})d\overline{N}_s^i , \quad 0 \le t \le 1 \tag{4.5}$$

Thus, (1.8) becomes

$$g(X_t) - g(X_0) = \int_0^t \nabla g(X_s)*dX_s . \tag{4.6}$$

Note that to simplify matters we have restricted ourselves to $\mathfrak{X} = \mathbb{R}^d$; if we let $\mathfrak{X}$ be a smooth manifold and let v transform as a 1-form then the definition of $\int_0^t v(X_s) * dX_s$ is intrinsic to the manifold. In particular we have defined a meaningful stochastic integral of <u>any</u> L^2 1-form against the Brownian motion on a manifold. The price one pays for the extension from smooth to L^2 forms is that the integral is not a semi-martingale but one of these forward and backward martingale processes.

We discuss now the convergence in distribution of stochastic integrals and partially extend theorem 3.3.

Suppose Q is a measure on R^d such that $Q(R^d) < \infty$ and $Q(B) \le m(B) \quad (\forall B \in \mathcal{B}(R^d))$.

LEMMA 4.1. <u>Under the hypothesis of theorem 3.3. if</u> v <u>is a smooth vector field of compact support</u> $\sum_i \int_0^t v^i(X_s^n) dN_s^{n,i}$ <u>under the law</u> $P_n^{\mathcal{E},Q}$ <u>converges in distribution to</u> $\sum_i \int_0^t v^i(X_s) dN_s^i$ <u>under the law</u> $P^{\mathcal{E},Q}$ <u>(in the topology of Meyer).</u>

PROOF. Because of the compact support of v the martingales $\sum_i \int_0^t v^i(X_s^n) dN_s^{n,i}$ for $n \in 1, \ldots \infty$ are uniformly bounded in L^2 and so from Theorem 4 of [ ] Meyer and Zheng, their laws are tight. Thus, to prove the lemma it is sufficient to show that the finite dimensional distributions of $\left(\sum_i \int_0^t v^i(X_s^n) dN_s^{n,i}\right)_{t \in H}$ converge to those of $\left(\sum_i \int_0^t v^i(X_s) dN_s^i\right)_{t \in H}$ where H is a set of full Lebesgue measure.

Let $\tau = \{t_0 < t_1 < \ldots < t_k = t\}$ be a partition of $[0,1]$ and suppose $\tau \subset H$. Let s be in $[0,t]$ and define

$$X_s^\tau = X_{t_j} \quad \text{if} \quad t_j \le s < t_{j+1}$$

and define $X^{n,\tau}$ in a similar way. Then

$$E\left(\left(\sum_i \int_0^t [v^i(X_s^\tau) - v^i(X_s)] dN_s^i\right)^2\right)$$

$$= 2\mathbb{E}\left(\sum_{ij} \int_0^t (v^i(X_s^\tau) - v^i(X_s))(v^j(X_s^\tau) - v^j(X_s)) a_{ij}(X_s) ds\right)$$

$$\le c\, \mathbb{E}\left(\sum_i \int_0^t (v^i(X_s^\tau) - v^i(X_s))^2 ds\right) \quad \text{for some } c > 0 . \tag{4.7}$$

Of course we have a similar estimate replacing X with X^n in the above. The constant c may be chosen to be independent of n because of our hypotheses.

Since v^i are smooth we have

$$E([v^i(x_s^\tau)-v^i(X_s)]^2) = E([v^i(X_{t_j})-v^i(X_s)]^2) \le (s-t_j)\sum_i \mathcal{E}(v^i,v^i)$$

and this is of order $\frac{1}{|\tau|}$ where $|\tau| = \max_j |t_{j+1} - t_j|$. Thus we have uniform convergence of 4.7 to zero independent of which $X^{(n)}$ we apply the estimate to as $|\tau| \to 0$. Suppose that g is a bounded and continuous function on $\mathbb{R}^{r_0}$ some r_0 . Let $s_1, \ldots, s_{r_0} \in H$ be any fixed sequence. We wish to prove that

$$\lim_{n\to 0} E(g((\sum\int_0^{s_r} v^i(X_s^n)dN_s^{n,i})_{r=1}^{r_0})) = E(g((\sum_i\int_0^{s_r} v^i(X_s)dN_s^{n,i})_{r=1}^{r=r_0}) .$$

Because of our uniform approximation it suffices to prove this with $X^{\tau,n}$ substituted for X^n . But in this case the problem reduces to one concerning only the convergence of $N^{n,i}$ to N^i at the times $\{t_j\} \cup \{s_j\}$. This was the situation dealt with in theorem 3.3.

THEOREM 4.2. Under the hypothesis of theorem 3.3, if $v(x)$ is a vector field which satisfies (4.4) approximated by smooth fields v_k of compact support such that

$$\int\sum_{ij}(v^i(x)-v_k^i(x))a_{ij}(x)(v^j(x)-v_k^j(x))q(x)dx \xrightarrow{k\to\infty} 0 .$$

Then $\sum_i\int_0^t v^i(X_s)*dX_s^{n,i}$ converge in distribution (in the Topology of Meyer) to $\sum_i\int_0^t v^i(X_s)*dX_s^i$ given that X, X^n all start with distribution Q .

PROOF. Let g be a continuous function on $D[0,1]$, and let $f \in C_0^1(R^d)$. We consider the processes $(X_t^n,N_t^n,\overline{N}_{1-t}^n)$. Let $h : R^d \to R$ by $(h/x)_i = x_i$

if $|x_i| < c = \pm c$ if $\pm c \lessgtr x$. Then

$$E[g(X^n,N^n,\overline{N}_{1-.})f(\frac{1}{\varepsilon}\int_{1-\varepsilon}^{1} h^c(X_s^n)ds)]$$

(4.10)

$$\xrightarrow{n\to\infty} E[g(x,N,\overline{N}_{1-.})f(\frac{1}{\varepsilon}\int_{1-\varepsilon}^{1} h^c(X_s)ds)]$$

because the above functions on path space are continuous for the topology of Meyer. However

$$E[f(\frac{1}{\varepsilon}\int_{1-\varepsilon}^{1} h^c(X_s^n)ds)-f(h^c(X_1^n)))^2]\xrightarrow{\varepsilon\to 0} 0 \text{ , uniformly in } n \text{ .}$$

Choosing c so that $[-c,c]^d$ contains the support of f we obtain

$$\text{(4.11)} \qquad E[g(X^n,N^n,\overline{N}_{1-.}^n)f(x_1)] \xrightarrow{n\to\infty} E[g(XN,\overline{N}_{1-.})f(X_1)] \text{ .}$$

On the other hand, if $Q_k = h_k(x)m(dx)$ where $0 \le h_k \uparrow 1$ a.e. $[m]$ and $h_k \in C_0^1(R^d)$, then

$$\operatorname*{Max}_n |E^{\varepsilon_n,Q_k}(g(X^n,N^n,\overline{N}_{1-.}^n)P(X_1) - E^{\varepsilon_n,m} g(X^n,N^n,\overline{N}_{1-.}^n)P(X_1)| \xrightarrow{k\to\infty} 0 \text{ .}$$

So that from (4.11),

$$E^{\varepsilon_n,m}[g(X^n,N^n,\overline{N}_{1-.}^n)f(X_1)] \xrightarrow{n\to\infty} E^{\varepsilon,m}[g(X,N,\overline{N}_{1-.})f(X_1)] \text{ .}$$

Denote $Q' = f(x)m(dx)$, then the time-reversed processes X_{1-t}^n under the laws $P^{\varepsilon_n,Q'}$ converge. From the definition of Stratonovich integration (4.5), applying lemma 4.1 to $\sum_i\int_0^t v_k^i(X_s)dN_s^i$ and $\sum_i\int_{1-t}^1 v_k^i(X_{1-s})d\overline{N}_s^i$, we obtain the conclusion of this theorem for each $v_k \in C_0^1(R^d)$, then let $k \to \infty$, we obtain the theorem in the general case.

5. Vector field and supermartingale.

Now, let v be a vector field satisfying the hypothesis of theorem 4.2. Suppose U is an open set of R^d such that for all non-negative function $h \in C_0^1(U)$,

$$\text{(5.1)} \qquad \int_{R^d} \nabla h(x)A(x)v(x)q(x)dx \ge 0 \text{ , where } A(x) - (a_{ij}(x)) \text{ .}$$

We will prove that such a field integrates to a local supermartingale for X until its first exit from U.

Let

$$H_t^{n,v_k} = \int_0^t h(X_s^n) \cdot [v(X_s^n)_* dX_s^n] = \frac{1}{2}\sum_i \int_0^t h(X_s^n) v^i(X_s^n) dN_s^{n,i} - \frac{1}{2}\sum_i \int_{1-t}^1 h(X_s^n) v^i(X_s^n) d\bar{N}_s^{n,i}$$

(5.2)

$$- \frac{1}{2}\sum_{ij} \int_0^t \nabla_j h(X_s^n) a_{ij}^n (X_s^n) v^i(X_s^n) ds .$$

So formally H is the Ito integral of h against $v * dX_s$.

We can easily verify that

$$\int_{R^d} |H_t^{n,v_k} - H_t^{n,v}|^2 \, m(dx) \xrightarrow{k\to\infty} 0 \tag{5.3}$$

where v_k are the approximating smooth fields.

Let Q be the measure in lemma 4.1. Since

$$E^{\mathcal{E}_n, Q}\left[\int_0^1 \left|\sum_{ij} \nabla_j h(X_s^n) a_{ij}^n (X_s^n) v^i(X_s^n)\right|^2 ds\right]$$

are uniformly bounded with respect to n, if a_{ij}^m are continuous, then from lemma 4 of (Zheng [1]), we know

$$\int_0^t \sum_{ij} \nabla_j h(X_s^n) a_{ij}^n (X_s^n) v^i(X_s^n) ds$$

converges in distribution to

$$\int_0^t \sum_{ij} \nabla_j h(X_s) a_{ij}(X_s) v^i(X_s) ds \quad \text{as } n \text{ tends to } \infty .$$

So that we obtain

LEMMA 5.1 <u>Under the hypothesis of theorem</u> 3.3 <u>the following convergence holds in sense of distribution</u>

$$\int_0^t \sum_{ij} \nabla_j h(X_s^n) a_{ij}^n (X_s^n) v^i(X_s^n) ds \longrightarrow \int_0^t \sum_{ij} \nabla_j h(X_s) a_{ij}(X_s) v^i(X_s) ds . \tag{5.4}$$

To consider separately the convergence of every term of the right of equality (5.2), we get

LEMMA 5.2. Under the hypothesis of theorem 3.3, the following convergence holds in sense of distribution:

$$(5.5)\qquad \int_0^t h(X_s^n)\ .\ [v(X_s^n)_* dX_s^n] \xrightarrow{n\to\infty} \int_0^t h(X_s)\ .\ [v(X_s)_* dX_s]\ .$$

PROOF. By theorem 4.2 and lemma 5.1, we obtain (5.5).

Now we establish a theorem. Let $m = q(x)dx$.

THEOREM 5.3. Let Dirichlet form $\mathcal{E}$ be generated by

$$\mathcal{E}(u,v) = \int_{R^d}\sum_{ij}\nabla_i u(x) a_{ij}(x)\nabla_j v(x) q(x)dx\ ,\quad u,\ v \in C_c^1(R^d)\ ,$$

where a_{ij} and q are Borel functions satisfying

$$(5.6)\qquad 0 < \frac{1}{c} < \sum_{ij} w_i a_{ij}(x) w_j < C < \infty\ ,\quad \forall \text{ unit vectors } w = (w_j)\ ,$$

and

$$(5.7)\qquad 0 < \frac{1}{c} < q(x) < C < \infty\ ,\quad \forall\, x \in R^d \text{ and some fixed } c > 1\ .$$

Suppose v is a vector field satisfying (5.1), then $\int_0^t v(X_s)_* dX_s$ is a supermartingale in the random open set $B = \{(\omega,t);\ X_t(\omega) \in U\}$.

PROOF. It is well known that in the case when a_{ij} and q are smooth, the conclusion is true. We can approximate a_{ij} and q by a sequence of (a_{ij}^n) and (q^n) such that $q^n(a_{ij}^n) \uparrow q(a_{ij})$ and that (a_{ij}^n) and q^n are simple functions. But for every simple function (a_{ij}^n) and q^n , we can always approximate them by $(a_{ij}^{n,m})$ and $(q^{n,m})$ such that $q^{n,m}(a_{ij}^{n,m}) \downarrow q^n(a_{ij}^n)$ and such that they are given by

$$(a_{ij}^{n,m}) = \sum_k J^{n,m,k}\, I_{B_k} \quad\text{and}\quad q^{n,m} = \sum_k r^{n.m.k}\, I_{B_k}$$

where $J^{n,m,k}$ is constant positive definite matrix, $q^{n,m,k}$ is constant and B_k are open sets.

Denote $v_n = (a_{ij}^n)^{-1}(a_{ij})q_n^{-1}qv$, then we can easily see that for $h \in C_c^1(U)$,

$$(5.10)\qquad \int_{R^d}\nabla h(x)A^n(x)v_n(x)q_n(x)dx \ge 0\ ,\ \text{where}\ A^n(x) = (a_{ij}^n(x))\ .$$

Without losing generality, we can suppose that U is bounded; then v_n

converge to v in L_2-norm uniformly with respect to all m_n and m . If $\int_0^t v_n(X_s^n)_* dX_s^n$ is a supermartingale in random open sets $B^n = \{(\omega,t);\ X_t^n(\omega) \in U\}$ by lemma 5.2, we have the convergence relation (5.5) for all $h \in C_c^1(U)$. Since the processes appeared on the left side of (5.5) are supermartingales, so are those appearing on the right. Thus, $\int_0^t v(X_s)_* dX_s$ is a supermartingale in B . Thus, we have reduced the problem to the smooth case by repeating the above approximation procedure and convergence argument. Since the theorem is true in smooth case, we obtain the conclusion.

REMARK. Readers could find some properties of supermartingale in random open sets in (Zheng [2]).

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