Astérisque

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Astérisque, tome 154-155 (1987), p. 95-113
<http://www.numdam.org/item?id=AST_1987__154-155__95_0>

EXPOSÉ n°V

THE BERNSTEIN-OSSERMAN-XAVIER THEOREMS

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I. INTRODUCTION.

The classical Bernstein theorem is the following.

THEOREM 1 (Bernstein [3]). *If $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^2-function whose graph is a minimal surface, then f is linear.*

This theorem provides a lovely example of a striking, non-trivial statement about a partial differential equation. For the condition that the graph of a C^2-function $f : U \to \mathbb{R}$, where $U \subseteq \mathbb{R}^n$ is a domain, be minimal is that f satisfy the *minimal surface equation*

$$(2) \qquad (1 + \sum_{i=1}^{n} |D_i f|^2) \sum_{j=1}^{n} D_{j,j} f - \sum_{i,j=1}^{n} (D_i f)\cdot(D_j f) D_{i,j} f = 0$$

for all $x \in U$. Here, $D_i f = \partial f/\partial x_i$ and $D_{i,j} f = D_i(D_j f)$. The Bernstein theorem asserts that when $n = 2$ the only global solutions (i.e. for which $U = \mathbb{R}^2$) are the trivial (i.e. linear) ones. This is all the more surprising because a modest amount of ingenuity supplies non-trivial local solutions. For example, it was observed classically that the functions obtained by setting $f(x_1,x_2)$ equal to $\tan^{-1}(\frac{x_2}{x_1})$ or $\cosh^{-1}(\sqrt{x_1^2+x_2^2})$ or $\log(\frac{\cos x_2}{\cos x_1})$ satisfy (1) over their domains of definition. The corresponding minimal surfaces are known as the helicoid, the catenoid and Scherk's surface, respectively.

So striking is the Bernstein theorem, that one needs scarcely say that many mathematician-hours have been devoted to generalizing it. The question of whether it generalizes to functions $f : \mathbb{R}^n \to \mathbb{R}$ or, alternatively, whether every function f satisfying (1) for all $x \in \mathbb{R}^n$ is linear, has come to be known as the *codimension-one Bernstein problem*. De Giorgi [8] settled the case $n = 3$, Almgren [2] the case $n = 4$, and Simons [19] the cases $n = 5,6,7$ of this problem affirmatively. It came, therefore, as somewhat of a surprise to the mathematical community when Bombieri, de Giorgi and Giusti [4] constructed non-linear global solutions to (1) for all $n > 7$. Both [4] and [19] made decisive use of ideas due to Fleming [9]. Subsequently, the Schoen-Simon-Yau estimates [18] have unified to some extent the treatment of the codimension-one Bernstein problem in the cases $n \leq 7$. For an account of the proof that every entire function satisfying (1) with $n \leq 5$ is linear from this latter point of view, we refer the reader to Lawson's lectures [13].

There is, of course, no reason to restrict oneself to attempting to generalize the Bernstein theorem to functions from $\mathbb{R}^k$ to $\mathbb{R}$. At the opposite extreme, one might ask whether a suitable analogue of the theorem holds for functions $f : \mathbb{R}^2 \to \mathbb{R}^k$. In this case, if we write $f = (f_1,\ldots,f_k)$, set $D_i f = (D_i f_1,\ldots,D_i f_k)$ for $i = 1,2$ and interpret $(D_i f)\cdot(D_j f)$ as the scalar product of the vectors $D_i f$ and $D_j f$, then the condition that the graph of f be a minimal surface is precisely the condition that f satisfy (1) with $n = 2$. Here, (1) is a vector equation representing a system of k scalar equations for the k-functions $f_1,\ldots,f_k$. (As in the codimension-one case, the equations (1) come from computing the Euler-Lagrange equations for the area integral - see, for example, Osserman [17,§3].) Now, if $k = 2m$ is even and if $f : \mathbb{R}^2 \to \mathbb{R}^{2m}$ is complex analytic when viewed as a function from $\mathbb{C}$ to $\mathbb{C}^n$, then an easy computation establishes that f satisfies (1). Hence, the graph of f is minimal. Alternatively, observe that the graph of a complex analytic function from $\mathbb{C}$ to $\mathbb{C}^m$ is a complex submanifold of the Kähler manifold $\mathbb{C}^{m+1}$ and, hence, minimal (see, for example, Lawson [12, p.36]). At first blush, this would seem to indicate that any attempt to generalize Bernstein's theorem to functions $f : \mathbb{R}^2 \to \mathbb{R}^k$, $k > 1$, is doomed to barrenness. Nothing could be further from the truth, for it is in precisely this case that the impulse to generalization has borne the most fruit, resulting in theorems set apart by their rich geometrical content and graced by a beauty all their own.

Nirenberg first conjectured that Bernstein's theorem might generalize to the statement that the set of positively oriented unit normals to a complete, regular, simply-connected minimal surface M in $\mathbb{R}^3$ is dense in the unit sphere unless M is a plane. Notice that this statement immediately implies Bernstein's theorem because the graph of a function $f : \mathbb{R}^2 \to \mathbb{R}$ is certainly simply-connected and the set of unit normals to M is necessarily contained in a hemisphere. Osserman proved Nirenberg's conjecture in [14] and subsequently removed the hypothesis of simple connectivity in [15]. Osserman further generalized his result in [16], proving that the set of normals to a regular, complete minimal surface $M \subset \mathbb{R}^k$, which is not a plane, cannot omit a neighbourhood of some direction. Chern and Osserman sharpened this result in [7] by exploiting the notion of the generalized Gauss map which Chern had introduced in [6]. Among other things, the latter provided a convenient set of spaces in which to speak of the density of the set of normals and allowed one to keep track of orientation.

Osserman's theorems immediately raise the question of the size of the set omitted by the normals to a complete, non-planar minimal surface. Osserman proved in [15] that, in the classical case $M \subset \mathbb{R}^3$, the set of points on the sphere omitted by the set of oriented, unit normals to M had logarithmic capacity zero. The latter result still did not exclude the possibility that the set of omitted points could be infinite, a state of affairs that persisted for ten years until Xavier showed in [21] that the set of omitted points was finite and, in fact, less than seven. At present, it is still an open question as to whether there exists a complete, minimal surface in $\mathbb{R}^3$, other than a plane, whose set of unit oriented no mals omit more than four points on the unit sphere.

In what follows, we give an exposition of this set of ideas including a complete proof of Xavier's result. The author has relied heavily on the excellent expositions of Lawson [12] and Osserman [17], to which the reader is referred for related results. The author is grateful to C. Margerin, B. Lawson, and D. Hoffman for comments on the subject matter of this manuscript. He thanks I.H.E.S. for its support and hospitality during the preparation of this paper. He was also supported by a NATO grant from the Natural Sciences and Engineering Council of Canada and a grant from the National Science Foundation.

II. THE GENERALIZED GAUSS MAP.

We begin with some general remarks on immersions of surfaces into $\mathbb{R}^n$. We then introduce the notion of the generalized Gauss map. The latter is a natural global object associated with an immersed surface and, as much, affords not only a convenient technical tool, but a genuine mean to understanding the nature of the surface.

Let $\psi : M \to \mathbb{R}^n$ be an immersion on an oriented, two-dimensional C^k manifold with $k \geq 2$. We take the metric on M to be that induced from $\mathbb{R}^n$ by ψ.

Choose an oriented atlas on M representing the given C^k structure. Then, each point $p \in M$ possesses a chart (O,h,U) consisting of a neighbourhood O of

p in M and a homeomorphism $h : O \to U$, where U is a domain in $\mathbb{R}^2$. The theorem about the existence of isothermal parameters (see, for example, [5]) asserts that if $k \geq 2$, then each point p possesses a chart, again denoted (O,h,U), compatible with the original atlas with the further property that $\psi = \Psi \circ h^{-1}$ is conformal as a map $U \to \mathbb{R}^n$. Given two charts (O_i,h_i,U_i) which are C^k compatible with the original atlas on M and such that $\psi_i = \Psi \circ h_i : U_i \to \mathbb{R}^n$, $i = 1,2$, are conformal, it is evident that the transition functions $h_i \circ h_j^{-1}$, $1 \leq i,j \leq 2$ are conformal or anticonformal where defined. Since M is oriented, these latter two remarks mean that we can replace our original C^k-atlas on M with a compatible atlas $A = \{(O_\alpha,h_\alpha,U_\alpha)\}$, where α runs over some index set, which is i) conformal (in the sense that the transition functions $h_\alpha \circ h_\beta^{-1}$ are conformal where defined) and such that ii) $\psi_\alpha = \Psi \circ h_\alpha^{-1} : U_\alpha \to \mathbb{R}^n$ is conformal for every α. If we view the U_α as domains in $\mathbb{C}$, then it is easy to see that the transition functions are in fact holomorphic and we have, therefore, established the following.

PROPOSITION 3. <u>Let $\Psi : M \to \mathbb{R}^n$ be an immersion of an oriented, two-dimensional C^k manifold M into $\mathbb{R}^n$. If $k \geq 2$ and M is given the Riemannian metric induced by Ψ, then there exists a complex-analytic structure A on M which is C^k compatible with the original differentiable structure on M and with respect to which M is a Riemann surface and Ψ a conformal map.</u>

Let A be as above and fix a chart $(O,h,U) \in A$. The induced metric is $ds^2 = \sum_{i,j=1}^{2} g_{ij}dx_i dx_j$ where $g_{ij} = (D_i\psi)\cdot(D_j\psi)$ and $\psi = \Psi \circ h^{-1}$. The map ψ is conformal if and only if $ds^2 = \lambda(dx_1^2+dx_2^2)$ where $\lambda = \lambda(x_1,x_2) > 0$; that is, if and only if $g_{11} = g_{22} = \lambda$, and $g_{12} = 0$. Now, view U as a domain in $\mathbb{C}$, and set $z = x_1+ix_2$ and $D = \frac{1}{2}(D_1-iD_2)$. The conditions on the g_{ij} can be rewritten as conditions on the map $\varphi \equiv D\psi : U \to \mathbb{C}^n$ as follows.

PROPOSITION 4. <u>Let $\Psi : M \to \mathbb{R}^n$ and A be as in proposition 1. If (O,h,U) is any chart in A, then the function $\varphi \equiv D\psi : U \to \mathbb{C}^n$, where $\psi = \Psi \circ h^{-1}$, satisfies</u>

4.i) $\qquad \varphi\cdot\varphi \equiv 0$, <u>and</u>

4.ii) $\qquad 2|\varphi|^2 = \lambda > 0$

<u>where the induced metric is given by</u>

4.iii) $\qquad ds^2 = \lambda|dz|^2$.

Condition 4.ii) merely expresses the fact that $\psi : U \to \mathbb{R}^n$ is conformal and condition

4.iii) the fact that Ψ is a <u>regular</u> immersion.

Starting with the immersion $\Psi : M \to \mathbb{R}^n$, we get a complex structure $A : \{(O_\alpha, h_\alpha, U_\alpha)\}$ on M and, hence, a family $\{\varphi_\alpha : U_\alpha \to \mathbb{C}^n\}$ of maps by setting $\psi_\alpha = \Psi \circ h_\alpha^{-1}$ and $\varphi_\alpha = D\psi_\alpha$. Each φ_α satisfies equations 4ii) and 4.iii). Since this family was manufactured out of a global object, namely Ψ, it is natural to ask whether the family $\{\varphi_\alpha\}$ represents the local data corresponding to some global object defined independently of a particular set of coordinates. This is, indeed, the case and we shall see below (Proposition 6) that the appropriate global object is the Gauss map.

Given an immersion $\Psi : M \to \mathbb{R}^n$ as in Proposition 1, define the <u>generalized Gauss map</u> G to be the map which assigns to each point $p \in M$, the oriented tangent space T_pM to M at p viewed as a linear subspace of $\mathbb{R}^n$. Thus, G is a map from M to $G_o(2,n)$ where $G_o(2,n)$ denotes the Grassmannian space of oriented, two-dimensional subspaces of $\mathbb{R}^n$. The map G obviously does not depend on the choice of coordinates on M.

The space $G_o(2,n)$ can be naturally identified with another space which carries a canonical complex structure. To see how this works, observe that an oriented plane $T \subset \mathbb{R}^n$ can be viewed as a complex line in $\mathbb{C}^n$. Choose $\{u,v\}$, where $u,v \in \mathbb{R}^n$, to be an oriented orthonormal basis of T and defined $\lambda(T)$ to be the complex line spanned by the vector $u-iv \in \mathbb{C}^n$. If $\{u',v'\}$ is any other oriented orthonormal basis of T, then $e^{i\theta}(u-iv) = u'-iv'$ for some $0 \leq \theta < 2\pi$. Thus, the map $T \to \lambda(T)$ is well-defined and induces a map $\kappa : G_o(2,n) \to \mathbb{C}P^{n-1}$.

The map κ is clearly injective. It is not surjective since any line in $\mathbb{C}^n$ which is the image of a plane in $\mathbb{R}^n$ must be spanned by a vector $w \in \mathbb{C}^n$ of the form $w = u-iv$, where u and v are orthogonal vectors in $\mathbb{R}^n$. In other words, if $w = (w_1,\dots,w_n)$, the components must satisfy the equation $w\cdot w = w_1^2+\dots+w_n^2 = 0$. Viewing $(w_1,\dots,w_n)$ as homogeneous coordinates on $\mathbb{C}P^{n-1}$, we see that the image of k lies on the hypersurface $Q_{n-2} \subset \mathbb{C}P^{n-1}$ defined by the equation $w_1^2+\dots+w_n^2 = 0$. It is easy to check that the map $k : G_o(2,n) \to Q_{n-2}$ is bijective.

In what follows, if $w \in \mathbb{C}^n - 0$, we let $[w]$ denote the point in $\mathbb{C}P^{n-1}$ with homogeneous coordinates $w = (w_1,\dots,w_n)$. Thus, if $T \in G_o(2,n)$ and u,v is any oriented orthogonal basis with $\|u\| = \|v\|$, we have

(5) $$k(T) = [u-iv] \quad .$$

Since Q_{n-2} inherits a natural complex structure as a submanifold of $\mathbb{C}P^{n-1}$, it will be convenient to work with the following definition.

Definition. If $\Psi : M \to \mathbb{R}^n$ is an immersion (M as in Proposition 1), then the (generalized) Gauss map $\Phi : M \to Q_{n-2}$ is the map $\Phi \equiv \kappa \circ G$.

Notice that Φ depends only on the immersion $\Psi : M \to \mathbb{R}^n$ and is defined independently of any coordinate system. However, Φ has a particularly nice representation with respect to any chart (O,h,U) belonging to the complex structure A given by proposition 1. This is so because the fact that $\psi : U \to \mathbb{R}^n$ is conformal means that, for any $\zeta \in U$, the vectors $D_1\psi(\zeta)$ and $D_2\psi(\zeta)$ are of equal length and form an oriented orthogonal basis of $T_{h^{-1}(\zeta)}M$. Recall that we have set $\varphi = \frac{1}{2}(D_1\psi - iD_2\psi)$. Comparing with (5) gives the following result.

PROPOSITION 6 . Let $\Psi : M \to \mathbb{R}^n$ be an immersion and $\Phi : M \to Q_{n-2}$ the associated Gauss map. Let Ψ and A be as in Proposition 1, and suppose that (O,h,U) is any chart in a . If $\phi \equiv \Phi \circ h^{-1} : U \to Q_{n-2}$ denotes the local representative of Φ , then $\phi = [\varphi]$ (where $\varphi = D\psi$ and $\psi = \Psi \circ h^{-1}$) .

Thus, the family of maps $\{\varphi_\alpha : U_\alpha \to \mathbb{C}^n\}$, defined earlier, just represents the Gauss map Φ with respect to the conformal atlas A on M and homogeneous coordinates on $\mathbb{C}P^{n-1}$.

Now,let us make the additional assumption that $\Psi : M \to \mathbb{R}^n$ is a minimal immersion. Let A be the complex structure given by Proposition 1. Then Ψ is minimal if any only if, for every chart (O,h,U) in A , the map $\psi = \Psi \circ h^{-1}$ is harmonic (see [12, p.13] or [17, p.29]). Since the Laplace-Beltrami operator is $\frac{1}{\lambda}\overline{D}(D\psi)$ with respect to the induce metric 4.iii), the map ψ is minimal if and only if $\overline{D}(D\psi) = 0$ and, hence, if and only if $\varphi = D\psi$ is analytic. Applying Proposition 6 and the fact that the map $\mathbb{C}^n - 0 \to \mathbb{C}P^{n-1}$ given by $z \to [z]$ is holomorphic clearly establishes the following.

PROPOSITION 7 . If $\Psi : M \to \mathbb{R}^n$ is a minimal immersion, then the Gauss map $\Phi : M \to Q_{n-2}$ is holomorphic where M is given the complex structure that makes Ψ conformal with respect to the metric on M induced by Ψ and Q_{n-2} the complex structure it inherits as a complex submanifold of $\mathbb{C}P^{n-1}$.

The converse of Proposition 7 also holds and is a pleasant exercise using the local representation of Φ (Proposition 6) and the definition of the complex structure on $\mathbb{C}P^{n-1}$. For details, see [11, p.9].

In view of Proposition 6 and the definition of the operator D , Proposition 7 is just a more global reformulation of the classical result that an immersed surface S in $\mathbb{R}^n$ is minimal if and only if the restrictions of the coordinate functions on $\mathbb{R}^n$ to S are harmonic functions of any local isothermal parameters on S.

In the literature, the conjugate $\overline{\Phi} = \overline{\kappa}\circ G$ of Φ is often referred to as the (generalized) Gauss map. This is because $G_o(2,n)$ has a natural Riemannian structure as a symmetric space (see [6]) with respect to which the map $\overline{\kappa}$ is conformal. However, the price paid for preferring a conformal to an anticonformal identification of $G_o(2,n)$ with Q_{n-2} is an antiholomorphic Gauss map.

III. OSSERMAN'S THEOREM.

We now want to prove Osserman's generalization of Bernstein's theorem.

THEOREM 8 (Osserman [16]). If the normals to a complete minimal surface in $\mathbb{R}^n$ omit a neighbourhood of some direction, then the surface is a plane.

The proof will proceed in several steps. We retain the notation of parts I and II.

α) We first interpret the hypothesis of Osserman's theorem in terms of the Gauss map. Suppose that the surface is given by a minimal immersion $\Psi : M \to \mathbb{R}^n$, and let A be the complex structure on M given by proposition 1. Fix $p \in M$ and a chart (O,h,U) in A such that $p \in O$. A vector $v \in \mathbb{R}^n$ is normal to M at p if and only if $v\cdot D_1\psi = v\cdot D_2\psi = 0$ (where $\psi = \Psi\circ h^{-1}$) or, equivalently, $v\cdot D\psi = 0$, the partials being evaluated at $h(p)$. By proposition 6, $v = (v_1,\dots,v_n)$ is normal to T_pM if and only if $\Phi(p)$ satisfies the equation $v_1w_1+\dots+v_nw_n = 0$ where $w_1,\dots,w_n$ are homogeneous coordinates on $\mathbb{C}P^{n-1}$. Thus, $v = (v_1,\dots,v_n)$ is normal to M at some point if and only if the image of Φ meets the hyperplane $v_1w_1+\dots+v_nw_n = 0$.

On the other hand, to say that the normals to M omit a neighbourhood of some direction v is to say that the cosine of the angle between any tangent vector w to M and v is uniformly bounded away from zero by some $\varepsilon > 0$:

$$\frac{|v\cdot w|}{\|v\|\ \|w\|} \geq \varepsilon \geq 0 . \tag{9}$$

If the above holds for every $w \in \mathbb{R}^n$ tangent to M, it certainly holds for every $w = \frac{1}{2}(u-iu') \in \mathbb{C}^n$ where $u,u' \in \mathbb{R}^n$ are orthogonal and both tangent to M, and hence, for every $w \in \mathbb{C}^n$ such that $[w] = \Phi(p)$ for some $p \in M$. Thus, the hypothesis of Osserman's theorem can be rephrased as :

(*) <u>There exists $v \in \mathbb{R}^n$ and $\epsilon > 0$ such that (9) holds for any $w \in \mathbb{C}^n$ such that $w = (w_1, \ldots, w_n)$ are homogeneous coordinates for some point in the image $\Phi(M)$ of the Gauss map</u>.

β) Now, observe that we can pass to the universal cover $\pi : \tilde{M} \to M$ where $\tilde{M}$ is simply connected and π a local homeomorphism. The complex structure on M making $\Psi : M \to \mathbb{R}^n$ a conformal map lifts via π to a complex structure on $\tilde{M}$ making $\tilde{\Psi} \equiv \Psi \circ \pi : \tilde{M} \to \mathbb{R}^n$ a conformal immersion. Moreover, $\tilde{\Psi}$ is minimal if and only if $\Psi : M \to \mathbb{R}^n$ is minimal. It is clear from proposition 6 that the image of the Gauss map $\tilde{\Phi}$ coincides with the image of Φ and, hence, that $\tilde{\Phi}$ satisfies hypothesis (*) on $\tilde{M}$ if and only if Φ satisfies (*) on M. Finally, it is easy to check that the induced metric on $\tilde{M}$ is complete if and only if that on M is complete.

γ) By the Koebe uniformization theorem (see, for example, [1]), $\tilde{M}$ is conformally equivalent to either the unit sphere S^2, the open disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, or the complex plane $\mathbb{C}$. The first possibility is ruled out because a minimal surface cannot be compact. Thus, upon passing to the universal cover we obtain a minimal (conformal) immersion a) $\tilde{\Psi} : \Delta \to \mathbb{R}^n$ or b) $\tilde{\Psi} : \mathbb{C} \to \mathbb{R}^n$. In both cases, we have a global parametrization.

Our strategy will be to show that the theorem holds in case b) (step δ) and to show that case a) cannot arise, because it contradicts completeness (step ε).

δ) Suppose that passing to the universal cover gives a minimal (conformal) immersion $\tilde{\Psi} : \mathbb{C} \to \mathbb{R}^n$. We have the Gauss map $\tilde{\Phi}(z) = [\tilde{\varphi}(z)]$, where $\tilde{\varphi} = D\tilde{\Psi}$, defined for every $z \in \mathbb{C}$. Set $\tilde{\varphi} = (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_n)$. Choose a unit vector v and a number ϵ satisfying hypothesis (*). For each k, $1 \leq k \leq n$, the entire function $\tilde{\varphi}_k/(v \cdot \tilde{\varphi})$ satisfies the inequality $|\tilde{\varphi}_k|/|v \cdot \tilde{\varphi}| \leq |\tilde{\varphi}|/|v \cdot \tilde{\varphi}| \leq 1/\epsilon$ and is, therefore, a constant, say c_k, by Liouville's theorem. Thus

$$\tilde{\varphi}(z) = (v \cdot \tilde{\varphi}(z))(c_1, \ldots, c_n)$$

for all $z \in \mathbb{C}$. Taking the image in $\mathbb{CP}^{n-1}$, we get $\tilde{\Phi}(z) = [c_1, \ldots, c_n]$ for all $z \in \mathbb{C}$. Thus, the Gauss map $\tilde{\Phi}$ is constant and, hence, the Gauss map Φ of $\Psi : M \to \mathbb{R}^n$ is constant. Thus, the image of M (under Ψ) has the same tangent plane everywhere. Since M is complete, $\Psi(M)$ must be a plane.

ε) We now show that passing to the universal cover cannot give a minimal immersion $\tilde{\Psi} : \Delta \to \mathbb{R}^n$ and, so, complete the proof of Osserman's theorem. We formulate the main point as a lemma.

LEMMA 10. If $\Psi : \Delta \to \mathbb{R}^n$ is a minimal, conformal immersion and if the Gauss map Φ satisfies (*), then the metric on Δ is incomplete.

Proof of lemma 1. We show that the metric is incomplete by finding a path α joining the origin of Δ to a point on the boundary $\partial\Delta$ for which $\int_\alpha ds < \infty$, where ds is the arclength with respect to the metric.

By proposition 6, the Gauss map Φ is given by $\Phi(z) = [\varphi(z)]$, for all $z \in \Delta$, where $\varphi \equiv D\Psi$. By proposition 3, $ds^2 = \lambda|dz|^2$ where $2|\varphi|^2 = \lambda$. By hypothesis (*), there exists a unit vector v and an $\varepsilon > 0$ such that $|v\cdot\varphi|/\|\varphi\| \geq \varepsilon > 0$ at every $z \in \Delta$. Thus, for any path in Δ, we have

$$\int_\alpha ds = \int_\alpha \sqrt{\lambda}\,|dz| = \sqrt{2}\int_\alpha |\varphi|\;|dz| \leq \frac{\sqrt{2}}{\varepsilon}\int_\alpha |v\cdot\varphi|\;|dz| \;.$$

Define $w = f(z) = \int_0^z v\cdot\varphi(\zeta)d\zeta$. The function w is holomorphic, non-constant, and $f'(z) \neq 0$ for any $z \in \Delta$ (because $v\cdot\varphi(z) \neq 0$ for any $z \in \Delta$).

Thus, we have an inverse function $z = h(w)$ defined in a disk about $w = 0$. Let R be the radius of the largest such disk. Because $|h(w)| < 1$, we have $R < \infty$ (otherwise, Liouville's theorem would force h to be a constant). Thus, there exists a point w_o with $|w_o| = R$ with the property that h cannot be extended throughout a neighbourhood of w_o. Let $L = \{tw_o : 0 \leq t < 1\}$ and set $\alpha = h(L)$. Then, α is an analytic curve in Δ beginning at 0 and going to $\partial\Delta$ (if α does not go to $\partial\Delta$, there would exist a sequence of points on α converging to a point $z_o \in \Delta$ for which $w_o = f(z_o)$; but, since $f'(z_o) \neq 0$, h would extend throughout a neighbourhood of w_o). Moreover,

$$\int_\alpha |v\cdot\varphi|\,|dz| = \int_\alpha \left|\frac{dw}{dz}\right|\,|dz| = \int_L |dw| = R \;.$$

Thus, α has length less than or equal to $\sqrt{2}R/\varepsilon$. Since this is finite, the metric on Δ is not complete and the lemma is proved.

Thus, if we start with a minimal immersion $\Psi : M \to \mathbb{R}^n$ satisfying (*), and if passing to the universal cover gives $\tilde{\Psi} : \Delta \to \mathbb{R}^n$ where Δ is conformally equivalent to $\tilde{M}$, then the metric on $\tilde{M}$ is not complete. Thus, the metric on M cannot be complete and this establishes Osserman's theorem.■

REMARK. Notice that the statement we have given of Osserman's theorem does not immediately imply Bernstein's theorem when specialized to $n = 3$. For it does not rule out, as possible candidates for minimal surfaces, graphs of functions with the property that the set of all oriented (say "upwards") unit normals is dense in a hemisphere. The set of all unit normals (that is, both "upwards"- and "downwards"-pointing normals) to such a graph would be dense in the sphere. Fortunately, we have actually proved somewhat more than we stated. To see this, observe that we never used the fact that $v \in \mathbb{R}^n$ in hypothesis (*). All details of the proof carry through without change upon assuming that $v \in \mathbb{C}^n$. This remark, together with the comments in step α) of the proof, allow us to rephrase what we have proved as follows.

THEOREM 11. (Osserman-Chern[7]). If $\Psi : M \to \mathbb{R}^n$ is a complete minimal surface which is not a plane, then the Gauss map Φ meets a dense set of hyperplanes.

Because the Gauss map involves oriented tangent spaces, this latter statement does imply the Bernstein theorem when $n = 3$. In fact, it is easy to see that when $n = 3$, it actually implies that the image of the Gauss map is dense in the conic $Q_1 \subset \mathbb{C}P^2$. We have already seen that there is a continuous bijection between Q_1 and $G_o(2,3)$ and, since there are obviously continuous bijections between $G_o(2,3)$, the set $G_o(1,3)$ of oriented lines or normals in $\mathbb{R}^3$, and the unit sphere S^2, we conclude that the set of oriented normals to a complete minimal surface in $\mathbb{R}^3$ is dense in S^2. This clearly implies the Bernstein theorem.

IV. THE CLASSICAL GAUSS MAP.

We now turn to the question of the size of the set omitted by the normals to a complete, minimal surface. We restrict our attention to surfaces immersed in $\mathbb{R}^3$. As always, if $\Psi : M \to \mathbb{R}^3$ is the immersion, we suppose that M is an oriented C^k manifold, $k \geq 2$, endowed with the Riemannian metric induced by Ψ. In the case of immersions in $\mathbb{R}^3$, it is more convenient to define the Gauss map directly in terms of the normals to the surface.

Definition 12. Let $\Psi : M \to \mathbb{R}^3$ be an immersion. If S^2 denotes the unit sphere in $\mathbb{R}^3$, then the classical Gauss map $N : M \to S^2$ is the map which assigns to each $p \in M$ the oriented unit normal to M at p, viewed as an element of S^2.

For future use, we want to establish some of the properties of the classical Gauss map N. We do this by first establishing a relationship between N and the generalized Gauss map Φ.

Suppose that $w = (w_1, w_2, w_3)$ are homogeneous coordinates of a point on $Q_1 \subset CP^2$. Then, we have seen that $w = u - iv$ where $u, v \in \mathbb{R}^3$ are orthogonal and

and $\|u\| = \|v\|$. The unit oriented normal to the plane spanned by $\{u,v\}$ is just $\frac{u\times v}{\|u\times v\|}$, where $u \times v$ denotes the usual vector product of u and v in $\mathbb{R}^3$. Since u and v are orthogonal, $\|u\times v\| = \|u\|\,\|v\|$. Since $\|w\|^2 = \|u\|^2 + \|v\|^2$ and $\|u\| = \|v\|$, we have $\|u\times v\| = \|w\|^2/2$. On the other hand, $u \times v = \mathrm{Re}(w) \times (-\mathrm{Im}(w)) = \frac{i}{4}(w+\overline{w}) \times (w-\overline{w}) = \frac{i}{4}((\overline{w} \times w) - (w \times \overline{w})) = -\frac{i}{2}(w \times \overline{w})$, where we have formally extended the vector product on $\mathbb{R}^3$ to $\mathbb{C}^3$. Thus, if $w = (w_1, w_2, w_3)$ are homogeneous coordinates of points on Q_1 and S^2 is the unit sphere centered at the origin in $\mathbb{R}^3$, define the map $\nu : Q_1 \to S^2$ by

$$\text{(13)} \qquad \nu : w \to -\frac{i(w\times\overline{w})}{\|w\|^2} .$$

(Observe that $\|w \times \overline{w}\| = 2\|u \times v\| = \|w\|^2$ because $w \in Q_1$, so that $\nu(w) \in S^2$; moreover, $\nu(w) = \nu(cw)$ for any $c \in \mathbb{C}$, so that ν does not depend on the choice of homogeneous coordinates for a point).

Note also that if $\Phi(p) = [w]$ for some $p \in M$ and if we write $w = u-iv$ as above, then u and v can be thought of as orthogonal vectors in T_pM. Thus, $\nu(w)$ is the unit normal to M at p. This remark clearly proves the following.

PROPOSITION 14. <u>If</u> $\Psi : M \to \mathbb{R}^3$ <u>is an immersion, then the classical Gauss map</u> $N : M \to S^2$ <u>is related to the generalized Gauss map</u> $\Phi : M \to Q_1$ <u>by the formula</u>

$$\text{(15)} \qquad N = \nu\circ\Phi$$

<u>where</u> $\nu : Q_1 \to S^2$ <u>is the map defined by</u> (13).

Relation (15) holds independently of the choice of local coordinates on M. In conjunction with proposition 6, it implies that N has a nice expression in terms of the complex structure given by proposition 1.

PROPOSITION 16. <u>Let</u> $\Psi : M \to \mathbb{R}^3$ <u>be an immersion and let</u> A <u>be the complex structure on</u> M <u>given by proposition 1. If</u> (O,h,U) <u>is any chart in</u> A, <u>then</u>

$$\text{(17)} \qquad n = N\circ h^{-1} = \frac{-i(\varphi\times\overline{\varphi})}{\|\varphi\|^2}$$

<u>where</u> $[\varphi] = \Phi\circ h^{-1}$.

Now, suppose that Ψ is minimal, so that Φ is holomorphic. We want to show that N is holomorphic. The reader may enjoy proving this directly by showing that ν is holomorphic, where the complex structure on S^2 is the standard one given by stereographic projection onto the complex plane. We follow a slightly different approach which will allow us to introduce the very useful Weierstrass representation of a minimal surface.

PROPOSITION 18. Let U be a domain in $\mathbb{C}$. If g is an arbitrary meromorphic function in U and f an analytic function in U, which has a zero of order $2m$ at each point where g has a pole of order m, then the functions

$$(19) \qquad \varphi_1 = \frac{1}{2} f(1-g^2) \ , \ \varphi_2 = \frac{i}{2} f(1+g^2) \ , \ \varphi_3 = fg$$

will be analytic in U and satisfy $\varphi\cdot\varphi = 0$ where $\varphi = (\varphi_1, \varphi_2, \varphi_3)$. Conversely, every triple $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ of analytic functions on U, satisfying $\varphi\cdot\varphi = 0$, can be represented in the form (19) except for $\varphi_1 = i\varphi_2$, $\varphi_3 \equiv 0$.

Proof. That the functions (19) satisfy $\varphi\cdot\varphi = 0$ is a straightforward computation. Conversely, given f and g one sets

$$(20) \qquad f = \varphi_1 - i\varphi_2 \ , \quad g = \frac{\varphi_3}{\varphi_1 - i\varphi_2} \ .$$

A short computation yields the relation $\varphi_1 + i\varphi_2 = -\dfrac{\varphi_3^2}{\varphi_1 - i\varphi_2} = -fg^2$, which implies (19) and shows the necessity of the condition on the zeroes of f and the poles of g. The relation (20) fails only when $\varphi_1 \equiv i\varphi_2$, in which case $\varphi_3 \equiv 0$. This completes the proof. ∎

In practice, the exceptional case $\varphi_3 \equiv 0$ of the above can always be avoided. For ,the functions $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ will always arise as the local expression of Φ with respect to a chart $(0,h,U)$ of the complex structure A on M making $\Psi : M \to \mathbb{R}^3$ a conformal map. If $\varphi_3 \equiv 0$, then $\Psi(0) \subseteq \mathbb{R}^3$ lies on the plane $x_3 = 0$. This can easily be avoided by rotating the coordinates in $\mathbb{R}^3$.

Proposition 18 means that for any choice of f and g as above, we can think of $\alpha : \zeta \to (f(\zeta), g(\zeta)) \to (\varphi_1(\zeta), \varphi_2(\zeta), \varphi_3(\zeta))$ as giving an (analytic) parametrization of Q_1. Since, again by proposition 18,every analytic parametrization of Q_1 (with one exception, which is easily handled separately) can be written in this form, to show that n is holomorphic when φ is, it suffices to show that for any α, as above, the composition $\mathbb{C} \overset{\alpha}{\to} Q_1 \overset{\nu}{\to} S^2 \overset{\sigma}{\to} \mathbb{C}$ is meromorphic. Here, $\sigma : S^2 \to \mathbb{C}$ is stereographic projection from a point, say $(0,0,1)$, of S^2 and we are using the elementary fact that the standard complex structure on S^2 is defined so that the analytic functions on S^2 are precisely those that correspond to meromorphic functions on $\mathbb{C} \cup \{\infty\}$ under stereographic projection.

If (a_1,a_2,a_3) is a point on S^2, then an easy computation shows that the image of (a_1,a_2,a_3) under stereographic projection σ from $(0,0,1)$ to the plane $x_3 = 0$ with complex coordinate $z = x_1+ix_2$ is given by

(21) $$\sigma(a_1,a_2,a_3) = \frac{a_1+ia_2}{1-a_3} .$$

We now compute $\sigma\circ\nu\circ\alpha$. By (10), we have $-i(\varphi\times\bar{\varphi}) = |f|^2(1+|g|^2)(\mathrm{Re}(g),\mathrm{Im}(g),\frac{1}{2}(|g|^2-1))$ and $\|\varphi\|^2 = \frac{1}{2}|f|^2(1+|g|^2)^2$. Taking the quotient gives an expression for $\nu\circ\alpha$. Using (12), we get $\sigma\circ\nu\circ\alpha(\zeta) = g(\zeta)$. This, together with proposition 2, implies the following.

PROPOSITION 22. Let $\Psi : M \to \mathbb{R}^3$ be a minimal immersion and let (O,h,U) be a chart of the complex structure A given by proposition 1. Define f and g as in (20) (where $[\varphi] = \Phi\circ h^{-1}$). Then

(23) $$\sigma\circ n = g$$

where $n = N\circ h^{-1}$ and $\sigma : S^2 \to C$ is stereographic projection from the point $(0,0,1) \in S^2$. The induced metric is given by $ds^2 = \lambda dz^2$ where

(24) $$\lambda = |f|^2(1+|g|^2)^2 .$$

Proposition 22 and the remarks following proposition 18 allow us to assert the following

PROPOSITION 24. If $\Psi : M \to \mathbb{R}^3$ is a minimal immersion, then the classical Gauss map $N:M \to S^2$ is holomorphic where M is given the complex structure making Ψ conformal and S^2 the usual complex structure induced from $\mathbb{C}$ by stereographic projection.

As in proposition 7, we have obtained a holomorphic (instead of antiholomorphic) Gauss map at the price of an anticonformal identification. Here, the map σ, defined by (21), is anticonformal with respect to the Riemannian structure on S^2 induced from $\mathbb{R}^3$.

We remark that propositions 18 and 22 are exceedingly valuable in the (classical) theory of minimal surfaces. For example, using the fact that a simply connected minimal surface in $\mathbb{R}^3$ can be reconstructed from its Gauss map, we can find many examples of minimal surfaces by choosing f and g appropriately. For details, see [12] and [17].

V. XAVIER'S THEOREM.

We now prove Xavier's theorem.

THEOREM 25. (Xavier[21]). Let $\Psi : M \to \mathbb{R}^3$ be a regular minimal immersion whose image is not a plane. If the induced metric on M is complete, then the image of the Gauss map $N : M \to S^2$ cannot omit seven points or more.

The proof follows the same general outline as the proof of Osserman's theorem. As in steps β) and γ) of the latter, we pass to the universal cover $\pi : \widetilde{M} \to M$ to get a minimal immersion $\widetilde{\Psi} : \mathbb{C} \to \mathbb{R}^3$ or $\widetilde{\Psi} : \Delta \to \mathbb{R}^3$, according to whether $\widetilde{M}$ is conformally equivalent to $\mathbb{C}$ or Δ. We show that the theorem holds in the former case, and that the latter case cannot arise because it contradicts completeness. Again, we break the proof into several steps.

α) We now show that the theorem holds when $\widetilde{M}$ is conformally equivalent to $\mathbb{C}$. By the remarks in step β) of the proof of Osserman's theorem, this will clearly follow from the lemma below.

LEMMA 6. If $\Psi : \mathbb{C} \to \mathbb{R}^3$ is a minimal conformal immersion and if the induced metric on $\mathbb{C}$ is complete, then the image of the Gauss map N omits at most two points, unless the image of Ψ is a plane.

Proof. Set $\varphi = D\Psi$. If $\varphi_1 \equiv i\varphi_2$, $\varphi_3 \equiv 0$ where $\varphi = (\varphi_1, \varphi_2, \varphi_3)$, then the x_3 coordinate on $\mathbb{R}^3$ is constant on $\Psi(\mathbb{C})$ and, by completeness, the image of Ψ must be a plane. Otherwise, define g as in (20). The function g is meromorphic and defined on the entire plane. By Picard's theorem, g omits at most two points. Since $g = \sigma n$ (proposition 22), the same is true of n and this completes the proof. ∎

β) In view of the above, we can restrict our attention to the case when $\widetilde{M}$ is conformally equivalent to the unit disk Δ. Since the image of the Gauss map on the universal cover Δ coincides with the image of the Gauss map on M, and since M is complete if and only if Δ is complete, we may as well assume that $M = \Delta$ and let $\psi : \Delta \to \mathbb{R}^3$ be the minimal conformal immersion. We then have globally parametrized Gauss maps $\Phi = \phi = [\varphi]$ and $N = n$ on Δ. We define g as in (20). The formula $g = \sigma n$ (proposition 22) shows that the poles of g occur exactly where n takes the value $(0,0,1)$. Since we subsequently assume that n omits seven points, we can assume by rotating coordinates in $\mathbb{R}^3$, that (0,0,1) is one of the omitted points and, hence, that g is analytic on Δ (and omits six points on $\mathbb{C}$) . Thus to prove Xavier's theorem, we need only establish the following.

LEMMA 27. Let $\Psi : \Delta \to \mathbb{R}^3$ be a minimal conformal immersion and suppose that $g = n$ is analytic on Δ and omits six distinct points of $\mathbb{C}$. Then the induced metric on Δ is not complete.

γ) To prove lemma 27, we will need some device which allows us to determine when a metric on Δ is incomplete. In the proof of Osserman's theorem, we used the fact that completeness implies that the arclengths of curves starting in D and ending at ∂D are infinite, and we produced a curve of this sort for which the arclength was finite. To prove Xavier's result, we use a more delicate criterion, due to Yau, which states that completeness implies that the result of integrating certain functions over the entire space (in our case, Δ) is infinite and we produce a function of the required type whose integral is finite. Yau's result is the following (we state it without proof).

LEMMA 28 (Yau [22, theorem 1]). Let M be a complete Riemannian manifold with infinite volume and let u be a non-negative function with the property that log u is harmonic almost everywhere. Then

$$\int_M u^p = \infty$$

for every $p > 0$. That is, $u \notin L^p(M)$ for any $p > 0$.

The hypothesis of infinite volume is made merely to avoid having to treat constant functions separately. Notice that any simply-connected, complete minimal surface in $\mathbb{R}^3$ is a simply connected, complete surface of non-positive curvature in $\mathbb{R}^3$ and, hence, certainly has infinite area.

δ) We now want to use the hypotheses of lemma 27 to construct a function u which satisfies the hypotheses of Yau's theorem (lemma 28, but which is such that $u \in L^p(\Delta)$ for some $p > 0$. This will show that the metric on Δ is incomplete.

Let $\Psi : \Delta \to \mathbb{R}^3$ be as in lemma 27 and construct f and g by formula (20) (where $\varphi = (\varphi_1,\varphi_2,\varphi_3) = D\Psi$) . Let $a_1,\dots,a_6$ be six distinct complex numbers omitted by the image of g (on Δ) . By proposition 18, $|f| > 0$ on Δ . Set

$$h = f^{-2/p} g' \prod_{i=1}^{6} (g-a_i)^{-\alpha} \qquad (29)$$

where α is any number of the form $1-1/k$, $k \in \mathbb{Z}$, such that $\frac{10}{11} \le \alpha < 1$ and $p = 5/(6\alpha)$. In other words,

$$p = \frac{5k}{6k-1}, \quad k \ge 11, \quad k \in \mathbb{Z} \quad . \tag{30}$$

Set $u = |h|$.

Notice that u satisfies the conditions of lemma 28. For the induced metric on Δ is given by $\lambda|dz|^2$ where λ is as in (24). The Laplace-Beltrami operator with respect to this metric is $\frac{1}{\lambda}\bar{D}D$ and, so, the function $u = |h|$ is harmonic almost everywhere (there can be a discrete set where g' vanishes). Thus, to prove lemma 27 and, hence, Xavier's theorem, it suffices to establish the following claim.

CLAIM 31 *Let* $u = |h|$ *where* h *is as in (15). Then* $u \in L^p(\Delta)$ *where* p *is as in* (30).

ε) Before proving the claim, we establish two estimates which we will need later. In order to avoid confusion, we write $L^p(\Delta_{st})$ in lemmas 5 and 6, below, to indicate that we are considering the unit disk with its standard, incomplete metric.

LEMMA 32. *If* v *is a holomorphic function in the unit disk* Δ *which omits two values, then there is a constant* C *such that*

$$\frac{|v'|}{1+|v|^2} \le \frac{C}{1-|z|^2} \tag{33}$$

for all z *in* Δ. *In particular,* $\frac{v'}{1+|v|^2} \in L^p(\Delta_{st})$ *for all* $0 < p < 1$.

Proof. A function v on Δ is said to be *normal* if the family $\{v(\theta(z))\}$, where θ ranges over all conformal transformations of Δ to itself, is normal in Montel's sense. Any function omitting two values is normal in this sense (by p.169 of [10]), and any normal function satisfies an estimate of the form (33) by theorem 6.5 of [10].

LEMMA 34. *Let* w *be a holomorphic function on* Δ *which omits two values, one of which is* 0. *If* k *is any positive integer and* $\alpha = 1-1/k$, *then*

$$\frac{|w'|}{|w|^{\alpha}+|w|^{2-\alpha}} \in L^p(\Delta_{st})$$

for every p *with* $0 < p < 1$.

Proof. Apply the estimate (33) with $v = w^{1/k}$.

φ) With these two estimates in hand, we can now prove the claim and, hence, Xavier's theorem. Since the area element is $\lambda dxdy$, we must show that

$$\int_\Delta \frac{|g'|^p(1+|g|^2)^2}{\prod_{i=1}^{6}|g-a_i|^{p\alpha}}\, dxdy < \infty \quad .$$

Denote the integrand by H. Let $\Delta_j = \{z \in \Delta \mid |g(z)-a_j| \le \varepsilon\}$ where $0 < \varepsilon < \frac{1}{4} \min\limits_{1\le i<k\le 6} |a_i-a_k|$. Let $\Delta' = \Delta \setminus \bigcup\limits_{j=1}^{6} \Delta_j$. Then

$$\int_\Delta Hdxdy = \sum_{j=1}^{6} \int_{\Delta_j} Hdxdy + \int_{\Delta'} Hdxdy \quad .$$

On Δ_j, we have the estimate

$$H \le C\left(\frac{|g'|^p}{|g-a_j|^{p\alpha}}\right) .$$

We may also assume that $\varepsilon < 1$ so that

$$|g-a_j|^\alpha \ge |g-a_j|^{2-\alpha} \quad .$$

Adding $|g-a_j|^\alpha$ to both sides and taking p^{th} powers yields $2^p|g-a_j|^{p\alpha} \ge (|g-a_j|^\alpha + |g-a_j|^{2-\alpha})^p$ from which we obtain

$$\frac{|g'|^p}{|g-a_j|^{p\alpha}} \le 2^p \frac{|g'|^p}{(|g-a_j|^\alpha+|g-a_j|^{2-\alpha})^p} \quad .$$

By lemma 34, the integral is bounded and hence, certainly, $\int_{\Delta_j} Hdxdy < \infty$.

It remains to bound the integral of H over Δ' .

Notice that since $g-a_j$ is bounded away from zero on Δ' for $1 \le j \le 6$, the quotient $|g-a_j|/|g-a_6|$ is bounded away from zero on Δ' for $1 \le j \le 6$. Note also that $|g|/|g-a_6|$ is bounded from above on Δ' . It follows that on Δ' ,

$$H \le C \frac{|g'|^p}{|g-a_6|^{6p\alpha-4}} = C \frac{|g'|^p}{|g-a_6|}$$

for some constant c . (The equality results from the fact that $6p\alpha = 5$).

If, in addition, we choose $\alpha = 1 - \frac{1}{k} \geq \frac{10}{11}$, we have $(2-\alpha)p = \frac{5}{6\alpha}(2-\alpha) \leq 1$. Using again the fact that $|g-a_6|$ is bounded away from zero on Δ' , we conclude that $|g-a_6| \geq C'(|g-a_6|^{\alpha}+|g-a_6|^{2-\alpha})^p$. Thus

$$H \leq C'' \frac{|g'|^p}{(|g-a_6|^{\alpha}+|g-a_6|^{2-\alpha})^p} .$$

Apply lemma 34,as above, to conclude that

$$\int_{\Delta'} H < \infty .$$

This completes the proof of the claim and, hence, of Xavier's theorem. ∎

In closing, we remark that a result of Voss ([20], or see [17, p.72]) asserts that, for any set of k points in S^2 , where $1 \leq k \leq 4$, there is a complete minimal surface in $\mathbb{R}^3$ whose Gauss map omits exactly these points. There is no known example of a minimal surface whose Gauss map omits five or six points and it would be interesting to know whether such exists. Indeed, until this question is resolved, there can be really no satisfactory conclusion to this account.

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