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EXPOSE n° II

A PROOF OF DOUGLAS' THEOREM ON THE EXISTENCE OF DISC LIKE MINIMAL SURFACES SPANNING JORDAN CONTOURS ON R^n

Anthony J. TROMBA

I. INTRODUCTION.

In 1931, Jesse Douglas [1] and, simultaneously, Tibor Rado [2] solved the famous problem of Plateau, namely that every Jordan wire Γ in $\mathbb{R}^n$ bounds at least one disc type surface of least area. For this work, Douglas received the first Fields medal (along with Lars Ahlfors) at the International Congress of Mathematicians in 1936. By this time, he had shown that his methods would allow one to prove that, under certain conditions, there exists minimal surfaces of genus zero but of connectivity k spanning k Jordan contours $\Gamma_1 \ldots \Gamma_k$ in $\mathbb{R}^n$. Somewhat later, he announced and published proofs of theorems giving sufficient conditions which guarantee the existence of a minimal surface of non-zero genus spanning one or more wires in Euclidian space. The original method Douglas used in the disc case being of some historical significance deserves some description, and we shall begin with an analytical formulation of the problem.

For our purposes, we shall assume Γ to be a smooth Jordan curve in $\mathbb{R}^n$, and let $\mathcal{D} \subset \mathbb{R}^2$ be the closed unit disc. Further let $u : \mathcal{D} \to \mathbb{R}^n$ have continuous second derivatives in $\mathcal{D}$ and map $\partial\mathcal{D} = S^1$ onto Γ in a "monotonic" manner. The classical problem of Plateau asks that we minimize the area integral

$$A(u) = \int_{\mathcal{D}} \sqrt{EG - F^2}\,dxdy \tag{1}$$

where, if $u = (u^1, \ldots, u^n)$, we have $E = u_x^2 = \sum_{i=1}^{n} (\frac{\partial u^i}{\partial x})^2$, $G = u_y^2 = \sum_{i=1}^{n} (\frac{\partial u^i}{\partial y})^2$

$$F = u_x \cdot u_y = \sum_{i=1}^{n} \frac{\partial u^i}{\partial x} \cdot \frac{\partial u^i}{\partial y} .$$

The area integral is invariant under the full C^∞ diffeomorphism group of the disc, an infinite dimensional Lie group.

The Euler equations of this variational problem form a system of non-linear partial differential equations expressing the non-linear conditions that the surface have mean curvature zero (i.e., it is a minimal surface), and that it spans Γ. Let us assume for the moment that the least area solution of (1) is immersed (for n = 3 this was not proved until 1970 by Osserman-Gulliver-Alt, and for $n > 3$ it is false). With this hypothesis, Riemann, Weierstrass, H.A. Schwarz and Darboux took advantage of the presumed existence of isothermal coordinates to simplify, and, in fact, linearize the Euler equations of least area. These special coordinates amount to composing a given u with an element f of the diffeomorphism group of $\mathcal{D}$ so that the resulting map $u \circ f$, although having the same image surface as u, has its derivative in a simple form. Assuming u is already in isothermal coordinates, one has that

$$E = G , \quad F = 0 .$$

Then, one also sees that

$$A(u) = D(u)$$

where

$$D(u) = \frac{1}{2} \int_{\mathcal{D}} (E+G)\,dxdy = \frac{1}{2} \int_{\mathcal{D}} (u_x^2 + u_y^2)\,dxdy$$

is the classical Dirichlet integral.

If u is not in isothermal coordinates, then

$$A(u) \leqslant \int_{\mathcal{D}} \sqrt{EG}\, dxdy \leqslant \frac{1}{2} \int_{\mathcal{D}} (E + G)\,dxdy = D(u) .$$

It is not difficult to see that

$$A(u) = D(u)$$

if and only if $E = G$, $F = 0$. If u is assumed to be in isothermal coordinates for the image surface S, then the Euler equations for the variational problem $A(u)$ or $D(u)$ become the linear system

$$\Delta u = 0$$

or

$$\Delta u^i = 0 , \; i = 1,\dots,n .$$

Thus, one is led to Plateau's definition of a minimal surface of disc type as a map $u : \mathcal{D} \to \mathbb{R}^n$ for which we have Laplace's equation

$$\Delta u = 0 , \tag{2}$$

and the non-linear conditions

$$E = G , \quad F = 0 \tag{3}$$

with $u : S^1 \to \Gamma$ a homeomorphism If u is a solution to (2) and (3), we shall denote by $\tilde{u}^j$ the harmonic conjugate of each u^i, and by f_j the holomorphic map $f_j = u^j + i\tilde{u}^j$, $i = \sqrt{-1}$ and by F, the holomorphic map of $\mathcal{D}$ into $\mathbb{C}^n$, $F(z) = (f_1(z),\dots,f_n(z))$, $z = x + iy$. Condition (3) may then be expressed as

$$F'(z)^2 = \sum_{j=1}^{n} f_j'(z)^2 = (E-G) + 2iF = 0 \ .$$

A complete existence proof for the solution of (2) and (3) evaded researchers until the work of Douglas and Rado.

Rado based his solution on a first approximation to a lower bound of the integral $A(u)$ by means of polyhedral surfaces, and then by mapping these surfaces conformally onto the unit disc. Douglas, however, avoided using conformal mapping at all, and was able to obtain the classical Riemann mapping theorem as a consequence of his existence theorem.

The relation $A(u) \leqslant D(u)$, the fact that $A(u) = D(u)$ for isothermal maps u, and the analog with geodesics between a length functional and an energy functional makes it plausible that minima of D would be minima of A. This is, in fact, the case.

However, in his prize winning paper Douglas did not use the Dirichlet functional D but another functional H which is now called the Douglas functional. Since any harmonic vector $u : \mathcal{D} \to \mathbb{R}^n$ is determined by its boundary values, Douglas was able to represent the Dirichlet functional as a functional depending only on the boundary values of u, and he obtained the expression

$$H(u) = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{[u(\alpha)-u(\beta)]^2}{4 \sin^2 \frac{(\alpha-\beta)}{2}} \, d\alpha \, d\beta \ .$$

The reader should be warned that the boundary conditions on u are not Dirichlet conditions but rather the far more complicated non-linear condition that $u : \partial\mathcal{D} \to \Gamma$. Moreover, one should note that the expression $H(u)$ transforms a variational problem from one that involves derivatives to one that does not. It was this brilliant device combined with two other important facts that led Douglas to his solution. The first is that once a three point condition was imposed on the space of admissible u's (u takes three points Q_1, Q_2, Q_3 on $\partial\mathcal{D}$ to three fixed points on Γ) Douglas was able to call upon a result of Frechet (published in his thesis) from which it follows that the space of u's with $H(u)$ uniformly bounded (now thought of as boundary mappings) was a compact set in the L^∞ topology. Secondly, if (u_n) is a sequence which converges in this topology to u with $H(u_n) \leqslant M$ for all n, then

$$H(u) \leqslant \liminf_{n\to\infty} H(u_n) \ .$$

This immediately implies that a minimum u of H (and hence of D) exists. He was further able to show that such a minimum u satisfied conditions (2) and (3) when thought of as a harmonic map of the disc. Thus, if Douglas was to use the ideas of Frechet, a boundary functional H was an absolute necessity.

In 1936 Courant succeeded in what this author believes was a great simplification of Douglas' original 1931 paper. He gave a simple and direct proof of the existence of a disc type minimal surface spanning a Jordan curve in $\mathbb{R}^n$. Courant was able to use directly Dirichlet's functional instead of the Douglas functional. Compactness of a minimizing sequence satisfying a three point condition then followed, not from Frechet, but directly from the now famous Courant-Lebesgue lemma. The purpose of this paper is to present a complete proof of Douglas' result.

II. THE EXISTENCE RESULT.

Let $\Gamma = \alpha(s')$, $\alpha : s' \to \mathbb{R}^n$ a smoothly embedded curve and $\mathcal{D}$ the unit disc in the plane. Let N_Γ be all those continuous maps $w : \mathcal{D} \to \mathbb{R}^n$ with finite Dirichlet's integral with $w|\partial\mathcal{D}$ homotopic to α and let $N_\Gamma \subset N_\Gamma^*$ be those w which map $\partial\mathcal{D} = S^1$ "monotonically" onto Γ. Let $E : N_\Gamma \to \mathbb{R}$ now denote Dirichlet's functional : if $u : \mathcal{D} \to \mathbb{R}^n$, $u = (u^1, \ldots, u^n)$

$$E(u) = \sum_{i=1}^{n} \int_{\mathcal{D}} \nabla u^i . \nabla u^i$$

Then Douglas proved

MAIN THEOREM. There exists an absolute minimum $u \in N_\Gamma$ for Dirichlet's functional. Moreover, on the interior[2] of the disc, u satisfies

$$(4) \qquad u'(z) \equiv 0 = \sum_{j=1}^{n} \left(\frac{\partial u^j}{\partial x} - i \frac{\partial u^j}{\partial y}\right)^2 ,$$

i.e., u is a minimal surface.

Thus, Douglas' famous 1931 result consists of two basic parts. The first was to show that Dirichlet's integral had an absolute minimum in the space N_Γ, and the second was to show that relation (4) held, and both parts were not trivial. If we know a priori that the minimum u is of Sobolev class $H^{5/2}(\mathcal{D}, \mathbb{R}^n)$, then a simple proof shows that (4) holds in the case Γ is a smoothly embedded wire. Briefly, write u in polar coordinates $u(r,\theta)$, $u(\theta) = u(1,\theta)$. Then, u a minimum[3] implies that $dE(u) = 0$ and so

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[2] $u'(z)$ is not defined on S^1.

[3] It is not completely obvious that if u is a minimum among monotonic maps that $dE(u) = 0$. This requires additional arguments

$$0 = dE(u)[h] = \int_{\mathcal{D}} (\Delta u)h + \int_{S^1} \frac{\partial u}{\partial r}.h\, d\theta$$

for all $h \in T_u N_\Gamma^*$. Thus, $\Delta u = 0$ and $\int_{S^1} \frac{\partial u}{\partial r}.h\, d\theta = 0$ for all $h : \mathcal{D} \to \mathbb{R}^n$ of class H^1 on S^1 and $h(\theta) \in T_{u(\theta)}\Gamma$. If we set $h(\theta) = \mu.\frac{\partial u}{\partial \theta}$, $\mu : S^1 \to \mathbb{R}$ continuous,we see that

$$\int_{S^1} \mu(\theta) \frac{\partial u}{\partial r} . \frac{\partial u}{\partial \theta} d\theta = 0$$

for all such μ, where . denotes the $\mathbb{R}^n$ inner product. Consequently $u_r.u_\theta = \frac{\partial u}{\partial r} . \frac{\partial u}{\partial \theta} = 0$ on S^1. Since u is assumed to be of class $H^{5/2}$, $\theta \to \frac{\partial u}{\partial r} . \frac{\partial u}{\partial \theta}$ is continuous on S^1, and therefore

$$z^2u'(z)^2 = (u_r - iu_\theta)^2 = \sum_{j=1}^{n} (\frac{\partial u^j}{\partial r} - i\frac{\partial u^j}{\partial \theta})^2$$

$$= (u_r.u_r - u_\theta.u_\theta) - 2i(u_r.u_\theta)$$

is a holomorphic function on the open disc which is continuous up to the boundary. Moreover,the minimality of u implies that the imaginary part of $z \to z^2u'(z)^2$ vanishes on the boundary of the disc implying that this function is constant (use the Schwarz reflection principle). However,$z \to z^2u'(z)^2$ has a zero at 0 which implies that $u'(z)^2 \equiv 0$,and thus u is a minimal surface.

Douglas and Courant could not employ such an easy proof since for them $\frac{\partial u}{\partial \theta}$, $\frac{\partial u}{\partial r}$ were <u>not</u> defined up to the boundary of the disc, complicating the situation considerably. Due to the important breakthrough on regularity in 1968 by S. Hildebrandt,we now know that minimal surfaces are as smooth as the boundary wire. However,his method needed the conformality relation $u'(z)^2 \equiv 0$ in the interior of the disc. Thus,we cannot assume that a minimum is smooth up to the boundary without proving that the conformality relation (3) holds on the interior.

We shall now give our own proof of (3).

THEOREM A.-<u>Let</u> $u \in N_\Gamma$ <u>be a minimum for Dirichlet's functional</u> $E : N_\Gamma \to \mathbb{R}$. <u>Then</u> u <u>satisfies</u> $u'(z)^2 \equiv 0$ <u>on the interior of the disc</u>.

<u>Proof</u>. Our goal will be to show that the holomorphic (a minimum u must be harmonic) quadratic differential $u'(z)^2dz^2$ defined on the interior of the disc reflects across the boundary to a C^∞ holomorphic quadratic differential on the sphere. So let $z_o \in \partial\mathcal{D}$,and consider a neighborhood U of z_o conformally equivalent to the upper half $\mathcal{D}^+$ of the unit disc via $w \to \phi(w) = z$, $w \in \mathcal{D}^+$, $\phi(0) = z_o$ and with $U \cap \partial\mathcal{D}$

being taken diffeomorphically onto $(-1,1) \times \{0\}$.

Now, by the invariance of Dirichlet's integral under conformal transformations, we have

$$2E(v) = \sum_{j=1}^{n} \int_{\mathcal{D}} \nabla v^j . \nabla v^j = \sum_{j=1}^{n} \int_{\mathcal{D}-U} \nabla v^j . \nabla v^j + \sum_{j=1}^{n} \int_{U} \nabla v^j . \nabla v^j$$

$$= \sum_{j=1}^{n} \int_{\mathcal{D}-U} \nabla v^j . \nabla v^j + \sum_{j=1}^{n} \int_{\mathcal{D}^+} \nabla \tilde{v}^j . \nabla \tilde{v}^j$$

where $\tilde{v}^j(w) = v^i(\emptyset(w))$, $w = x + iy$. For ε small define a one parameter curve $u(\varepsilon) : D \to \mathbb{R}^n$, $u(\varepsilon) : S^1 \to \Gamma$ monotonic, and $u(0) = u$ by taking new coordinates

$$x = x' + \varepsilon\rho(x',y')$$
$$y = y'$$

where ρ is an arbitrary C^2 function on $\mathcal{D}^+$ with the restriction that ρ vanishes outside $\mathcal{D}^+_{3/4}$, the upper disc of radius 3/4 . Denote this correspondence by $(x',y') \mapsto s_\varepsilon(x',y') = (x,y)$ or $w' \mapsto s_\varepsilon(w') = w$. Set

$$u(\varepsilon) = u \quad \text{outside} \quad U$$

and

$$\tilde{u}(\varepsilon)(w') = \tilde{u}(w)$$

for $w,w' \in \mathcal{D}^+$.

So

$$u(\varepsilon)(z) = \begin{cases} u(z) & \text{if} \quad z \in \mathcal{D}-U \\ \tilde{u}(S_\varepsilon(w')), & w' \in \mathcal{D}^+ \end{cases} .$$

Then

$$2E(u(\varepsilon)) = \sum_{j=1}^{n} \int_{\mathcal{D}-U} \nabla u^j . \nabla u^j dxdy + \sum_{j=1}^{n} \int_{\mathcal{D}^+} \nabla \tilde{u}^j(\varepsilon) . \nabla \tilde{u}^k(\varepsilon) dx'dy'$$

$$= \sum_{j=1}^{n} \int_{\mathcal{D}-U} \nabla u^j . \nabla u^j dxdy + \sum_{j=1}^{n} \int_{\mathcal{D}^+} [(\tilde{u}^j_{x'}(\varepsilon))^2 + (\tilde{u}^j_{y'}(\varepsilon))^2] dx'dy' .$$

By the chain rule and the change of variables formula, we find that this is equal to

$$\sum_{j=1}^{n} \int_{\mathcal{D}-U} \nabla u^j . \nabla u^j dx\, dy + \sum_{j=1}^{n} \int_{\mathcal{D}^+} [\tilde{u}^j_x(\varepsilon)\{1 + \varepsilon\rho_{x'}\}]^2 \frac{dxdy}{1 + \varepsilon\rho_{x'}}$$

$$+ \sum_{j=1}^{n} \int_{\mathcal{D}^+} [\tilde{u}^j_x(\varepsilon)\{\varepsilon\rho_y\} + \tilde{u}^j_y(\varepsilon)]^2 \frac{dxdy}{1 + \varepsilon\rho_{x'}} .$$

Using the fact that $\tilde{u}^j_x(\varepsilon) = \tilde{u}^j_x$ and $\tilde{u}^j_y(\varepsilon) = u^j_y$, the above is seen to equal

$$\sum_{j=1}^{n} \int_{\mathcal{D}-U} \nabla u^j . \nabla u^j dxdy + \sum_{j=1}^{n} \int_{\mathcal{D}^+} (\tilde{u}^j_x)^2 (1 + \varepsilon\rho_{x'}) dxdy$$

$$+ \varepsilon^2 \int_{\mathcal{D}^+} \rho_y^2 \{\tilde{u}^j_x\}^2 \frac{dxdy}{1 + \varepsilon\rho_{x'}} + \int_{\mathcal{D}^+} \{\tilde{u}^j_y\}^2 \frac{dxdy}{1 + \varepsilon\rho_{x'}}$$

$$+ 2 \varepsilon \int_{\mathcal{D}^+} \tilde{u}^j_x \tilde{u}^j_y \frac{\rho_y}{1 + \varepsilon\rho_{x'}} dxdy .$$

Expanding $\frac{1}{1 + \varepsilon\rho_{x'}}$, we get that this is equal to

$$\sum_{j=1}^{n} \int_{\mathcal{D}-U} \nabla u^j . \nabla u^j dxdy + \sum_{j=1}^{n} \int_{\mathcal{D}^+} \nabla\tilde{u}^j . \nabla\tilde{u}^j dxdy$$

$$+ \varepsilon \sum_{j=1}^{n} \int_{\mathcal{D}^+} [(\tilde{u}^j_x)^2 \rho_{x'} - (\tilde{u}^j_y)^2 \rho_{x'}] dxdy$$

$$+ 2\varepsilon \sum_{j=1}^{n} \int_{\mathcal{D}^+} \tilde{u}^j_x \tilde{u}^j_y \rho_y dxdy + \varepsilon^2 Q$$

with Q remaining bounded as $\varepsilon \to 0$.

The sum of the first two terms is just $2E(u) = \sum_{j=1}^{n} \int_{\mathcal{D}} \nabla u^j . \nabla u^j dx\, dy$ which is constant (in ε). Assuming u is a minimum

$$\frac{d}{d\varepsilon} 2E(u(\varepsilon)) \Big|_{\varepsilon=0} = 0 .$$

Since $x' \to x$ as $\varepsilon \to 0$, this clearly implies that

$$(6) \qquad \sum_{j=1}^{n} \int_{\mathcal{D}^+} [(\tilde{u}^j_x)^2 - (\tilde{u}^j_y)^2] \rho_x dxdy + 2 \sum_{j=1}^{n} \int_{\mathcal{D}^+} \tilde{u}^j_x \tilde{u}^j_y \rho_y dxdy = 0$$

for all ρ whose support is in $\mathcal{D}^+_{3/4}$. We can rewrite equation (6) in the more convenient form

$$(7) \qquad \mathcal{R}e \int_{\mathcal{D}^+} \{\frac{\partial\rho}{\partial\bar{z}} u'(z)^2\} dxdy = 0 ,$$

where $\mathcal{R}e$ stands for real part, and here $\frac{\partial}{\partial z} = \frac{\partial}{\partial y} - i \frac{\partial}{\partial x}$, $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial y} + i \frac{\partial}{\partial x}$.

Thus, (7) is our local variational condition, a condition which is expressed in a coordinate neighborhood. In order to conclude that $u'(z)^2$ is C^∞ "up to the boundary", we need a more general version of the Schwarz reflection principle, and this author is indebted to F. Tomi for helpful suggestions on this point.

LEMMA (General Schwarz Reflection Principle). Let $\mathcal{D}_r^+$ be the upper half of the disc of radius r in the complex plane. Suppose that f is an L^1, complex valued holomorphic function on the interior of $\mathcal{D}_r^+$ satisfying

$$\operatorname{Re} \int_{\mathcal{D}_r^+} \frac{\partial \rho}{\partial \bar{z}} f \, dxdy = 0 \tag{8}$$

for all $C^2 \rho$. Then, f reflects to a holomorphic function in the interior of the whole disc $\mathcal{D}_r$ and thus is C^∞ up to the boundary section $(-r,r) \times \{0\}$ of $\mathcal{D}_r^+$.

Proof. Since f is holomorphic, (4) can be rewritten as

$$\operatorname{Re} \int_{\mathcal{D}_r^+} \frac{\partial}{\partial \bar{z}} (\rho f) \, dxdy = 0$$

for all ρ. Thus

$$\operatorname{Re} \{\int_{\mathcal{D}_r^+} \frac{\partial}{\partial x} (\rho f) dxdy + i \int_{\mathcal{D}_r^+} \frac{\partial}{\partial y} (\rho f) \, dxdy\} = 0 .$$

If we take ρ to have compact support disjoint from the boundary $x^2 + y^2 = r^2$ in $\mathcal{D}_r^+$, the first term must vanish. Thus,

$$\operatorname{Re} i \int_{\mathcal{D}_r^+} \frac{\partial}{\partial y}(\rho f) dxdy = 0 = \lim_{\epsilon \to 0} \operatorname{Re} i \int_{y \geqslant 0} \frac{\partial}{\partial y} (\rho f) dxdy . \tag{9}$$

Moreover, (9) implies that

$$\lim_{\epsilon \to 0} \operatorname{Im} \int_{y=\varepsilon} (\rho f) dx = 0 \tag{10}$$

for all ρ with support chosen as above. Define a reflected function F on the entire disc $\mathcal{D}_r$ by

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in \mathcal{D}_r^+ \\ \overline{f(x,-y)} & \text{if } (x,y) \in \mathcal{D}_r^- \end{cases}$$

the bar over f denoting complex conjugation, and $\mathcal{D}_r^-$ is the bottom half of $\mathcal{D}_r$. Thus, F is well defined a.e., and is in L^1 on $\mathcal{D}_r$. We claim that F is a.e. equal to a holomorphic function on $\mathcal{D}_r$. In order to establish this by Weyl's lemma, we, need only show that F is weakly holomorphic, that is

$$\int_{\mathcal{D}_r} F \frac{\partial \rho}{\partial \bar{z}} = 0$$

for all real valued C^∞ functions ρ with compact support in $\mathcal{D}_r$.

$$\int_{\mathcal{D}_r} F \frac{\partial \rho}{\partial \bar{z}} \, dxdy = \int_{\mathcal{D}_r} \frac{\partial}{\partial \bar{z}}(F\rho) \, dxdy$$

$$= \int_{\mathcal{D}_r} \frac{\partial}{\partial x}(\rho F) dxdy + i \int_{\mathcal{D}_r} \frac{\partial}{\partial y}(\rho F) dxdy .$$

Since ρ has compact support in $\mathcal{D}_r$ the first integral vanishes, and our expression is equal to

$$i \int_{\mathcal{D}_r} \frac{\partial}{\partial y}(\rho F) dxdy = \lim_{\varepsilon \to 0} i \int_{|y| \geqslant \varepsilon} \frac{\partial}{\partial y} (\rho F) dxdy$$

$$= - \lim_{\varepsilon \to 0} i \int_{y=\varepsilon} \rho(x,\varepsilon) f(x,\varepsilon) dx + \lim_{\varepsilon \to 0} i \int_{y=-\epsilon} \rho(x,-\varepsilon) \bar{f}(x,\varepsilon) dx$$

$$= - \lim_{\varepsilon \to 0} i \int_{y=\epsilon} \rho(x,\varepsilon)[f-\bar{f}] dx + \lim_{\varepsilon \to 0} i \int [\rho(x,-\varepsilon) - \rho(x,\varepsilon)] \bar{f}(x,\varepsilon) dx$$

$$= - 2 \lim_{\varepsilon \to 0} \operatorname{Im} \int_{y=\varepsilon} \rho f + \lim_{\varepsilon \to 0} i \int [\rho(x,-\varepsilon) - \rho(x,\varepsilon)] \bar{f}(x,\varepsilon) dx .$$

By (7) the first limit is zero. Since f is L^1, it follows that there is a sequence $\varepsilon_n \to 0$ such that

$$\lim_{\varepsilon_n \to 0} \varepsilon_n \int_{y=\varepsilon_n} |f| dx = 0 .$$

This implies that the second limit is zero. Thus

$$\int_{\mathcal{D}_r} \frac{\partial \rho}{\partial \bar{z}} F = 0$$

and F is holomorphic on the interior of the disc $\mathcal{D}_r$. This concludes the proof of the reflection principle.■

We may now continue with our proof. Our assumption that the minimum $u \in H^1(\mathcal{D}, \mathbb{R}^n)$ clearly implies that $u'(z)^2$ is in L^1 of the disc. Therefore, an immediate application of the previous lemma shows that the holomorphic quadratic differential $u'(z)^2 dz^2$ may be reflected across the boundary of $\mathcal{D}$ to a holomorphic quadratic differential on the sphere. Since there are none (except 0), $u'(z)^2 \equiv 0$ which finishes theorem A . ■

It still remains to establish the existence of a minimum for E . In this case we need the famous

LEMMA (Courant-Lebesgue).- In a domain G of the plane, we consider a sequence of mappings $u_m : G \to \mathbb{R}^n$ with piecewise continuous first derivatives so that their Dirichlet integrals are bounded by a constant A ,

$$E(u_m) \leqslant A .$$

About a fixed point Q , we draw a circle of radius β . Denote by C_β an arc contained in G , and let s denote arc length on C_β . Then, there exists for each positive $\beta < 1$ a value ρ with

$$\beta \leqslant \rho \leqslant \sqrt{\beta}$$

so that

$$\int_{C_\rho} \left(\frac{du_m}{ds}\right)^2 ds \leqslant \frac{\varepsilon(\beta)}{\rho}$$

with

$$\varepsilon(\beta) = \frac{2A}{\log(1/\beta)} \to 0$$

as $\beta \to 0$.

Consequently, for the length L_ρ of the image C^1_ρ of C_ρ in $\mathbb{R}^n$ we have

$$L^2_\rho \leqslant 2\pi\varepsilon(\beta) .$$

The proof of this lemma can be found in Courant's book "Dirichlet's Principle."

THEOREM B.- Dirichlet's functional $E : N_\Gamma \to \mathbb{R}$ has an absolute minimum.

Proof. Let (u_m) be a minimizing sequence for E. Thus $E(u_m) \to \inf(E)$, the infimum of E on N_Γ we may assume that each u_m is harmonic. Let A_1, A_2, A_3 be three distinct points on Γ and Q_1 , Q_2, Q_3 three distinct points on S^1 . Since Dirichlet's functional is invariant under the three dimensional[1] conformal group of the disc we may, without loss of generality farther assume that $u_m(Q_i) = A_i$ for all m and i .

For the next step we will show that the sequence $\{u_m\}$ is equicontinuous in $C(D,R^n)$, and it is here that we use the Courant-Lebesgue lemma. We start by showing that the u_m are equicontinuous on S^1 .

[1] Each element of this group is given by $z \mapsto c\,\frac{z-a}{1-\bar{a}z}$ where $|c| = 1$ and $|a| < 1$.

It is well known that any Jordan curve Γ has the following property.

"There exists for each $T > 0$, a $\sigma(T)$ with $\sigma(T) \to 0$ as $T \to 0$ so that
(11) for any points Q, R on Γ whose distance is not yet greater than T
<u>one</u> of the <u>two</u> curves QR on Γ has a diameter not exceeding $\sigma(T)$."

We know that $u_m : S^1 \to \Gamma$ monotonically.

Pick two close points P and on S^1. Let R be a point between P and Q so that a circular arc of small radius β joins P and Q in $\mathcal{D}$. By the Courant-Lebesgue lemma we can draw a circulas arc C_ρ of radius $\rho > \beta$ around R joining two points P_1' and P_2' on S^1 so that the shorter arc joining P_1 and P_2 (i) is contained in the shorter arc joing P_1' and P_2', (ii) contains only one of the distinguished points Q_j and (iii) the length of the image of the circular arc C_ρ under u_m is small. This implies that the distance between $u_m(P_1')$ and $u_m(P_2')$ is small. By (11) this implies that the shorter arc joining $u_m(P_1')$ and $u_m(P_2')$ on Γ is small. If β is small enough the shorter arc on Γ must be the one which contains <u>only</u> one of the three points A_i. Therefore by monotonicity the image of the smaller $\widehat{P_1'P_2'}$ arc must be the small arc on Γ. This implies that the distance between $u_m(P_1)$ and $u_m(P_2)$ is also small. Thus $u_m|S'$ is an equicontinuous family and since u_m is harmonic it is an equicontinuous family in $(CD, \mathbb{R}^n)$.

We may therefore extract a convergent subsequence (u_{m_i}) which converges to $u \in C_o(D, \mathbb{R}^n) \cap H^1(D, \mathbb{R}^n)$, i.e., u is continuous and has finite Dirichlet integral. It is clear that u will be harmonic.

Our last step is to argue that Dirichlet's functional is lower semi-continuous with respect to uniform convergence as long as Dirichlet's integral remains bounded. For this, we use the following basic lemma.

LEMMA. <u>Let</u> Ω' <u>and</u> Ω <u>be open sets in the plane with</u> $\overline{\Omega}' \subset \Omega$, <u>and let</u> ω <u>be a real valued harmonic function on</u> Ω. <u>Then if</u> $\mu = \operatorname{dist}(\Omega', \partial\Omega)$

$$\sup_{x \in \Omega'} |\nabla\omega(x)| \leqslant \frac{\text{const}}{\mu} \sup_{x \in \Omega} |\omega(x)| . \tag{12}$$

Lower semi-continuity is now a consequence of (12) as follows :

Let $\mathcal{D}_r \subset \mathcal{D}$ be the disc of radius $r < 1$. Then

$$\sum_{i=1}^{n} \int_{\mathcal{D}_r} \nabla u_m^i \circ \nabla u_m^i \leqslant \sum_{i=1}^{n} \int_{\mathcal{D}} \nabla u_m^i . \nabla u_m^i$$

Thus

$$\lim_{m\to\infty} \inf \sum_{i=1}^{n} \int_{\mathcal{D}_r} \nabla u_m^i . \nabla u_m^i \leqslant \lim_{m\to\infty} \inf \sum_{i=1}^{n} \int_{\mathcal{D}} \nabla u_m^i . \nabla u_m^i$$

and by the lemma the left-hand side of this inequality is equal to $\sum_{i=1}^{n} \int_{\mathcal{D}_r} \nabla u^i . \nabla u^i$.

Since this is true for each $r < 1$,we obtain

$$\sum_{i=1}^{n} \int_{\mathcal{D}} \nabla u^i . \nabla u^i \leqslant \lim_{m\to\infty} \inf \sum_{i=1}^{n} \int_{\mathcal{D}} \nabla u_m^i . \nabla u_m^i$$

which establishes lower semi-continuity and also concludes the proof of theorem B and thus also the main theorem. ■

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