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EXPOSÉ n° XVI

SURFACES OF MEAN CURVATURE ONE IN HYPERBOLIC SPACE

Robert L. BRYANT

In this paper, we study the surfaces in hyperbolic space of constant mean curvature 1. These surfaces share many properties with minimal surfaces in Euclidean space, the most important being that these surfaces possess a "Weierstrass representation" in terms of holomorphic data (Theorem A). We concentrate on the complete surfaces of finite total curvature in H^3 and show that many of the "regularity" results of minimal surface theory carry over to hyperbolic space, though the hyperbolic analogue of the Gauss map fails to have all of the good features of the Euclidean Gauss map for minimal surfaces.

0. INTRODUCTION.

This paper grew out of the author's investigations into the theory of second order partial differential equations on surfaces whose solutions could be represented in terms of holomorphic functions on Riemann surfaces. The most famous example is, of course, Laplace's equation $\Delta u = 0$, but classically other equations were considered with this property, for example Liouville's equation $\Delta u = e^u$ can be solved in terms of holomorphic data. The famous example from geometry is the minimal surface equation in $\mathbb{E}^3$; the holomorphic representation being, of course, the famous "Weierstrass representation" (see[3]). It is natural to ask if the minimal surface equation in spaces of other constant sectional curvature admits a "holomorphic resolution" like the Weierstrass representation. This turns out not to be the case. However, it turns out that in a space form of sectional curvature c , the surfaces with mean curvature satisfying $H \equiv \pm\sqrt{-c}$ have a Weierstrass representation. When $c = 0$, this is just the Weierstrass representation of minimal surfaces. When $c > 0$, there are no such surfaces. When $c < 0$ we are reduced by dilation to the case $c = 1$, i.e. hyperbolic space.

In §1, we set up the basic geometry of hyperbolic space in terms of the moving frame. It turns out that the space of isometries of hyperbolic space, $PS\ell(2,\mathbb{C})$, has a natural complex structure and holomorphic metric and we show how this is reflected in the structure equations.

In §2, we study surface theory in hyperbolic space and introduce the hyperbolic Gauss map. We show that, in analogy with the Euclidean case, the hyperbolic Gauss map is conformal iff the surface is either totaly umbilic or has mean curvature 1. We then prove the Weierstrass representation result, Theorem A, that the surfaces of mean curvature 1 in H^3 are locally the projections from $S\ell(2,\mathbb{C})$ of holomorphic null curves where $S\ell(2,\mathbb{C})$ is given its Cartan-Killing metric.

In §3, we examine the immersions $f : M^2 \to H^3$ of mean curvature 1 which are complete and of finite total curvature. Just as in the minimal surface case in $\mathbb{E}^3$, such M^2 are conformally compact Riemann surfaces minus a finite number of points, and the (2,0) part of the second fundamental from is a holomorphic quadratic form on M^2 which extends meromorphically to the punctured points. However, in contrast to the case of Euclidean minimal surfaces, the Gauss map need not extend meromorphically across the punctures and the total curvature need not be an integral multiple of 4π. We present examples related to the Enneper surfaces and the catenoids in Euclidean space to illustrate these differences. We conclude with a criterion relating the degree of singularity of the (2,0) part of the second fundamental form with the meromorphicity of the Gauss map at the punctures. This allows us to recover some of the results of Gary Kerbaugh's thesis.

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1. HYPERBOLIC SPACE.

In this paper, we shall be concerned with surface theory in hyperbolic 3-space, i.e., the unique complete, simply connected Riemannian 3-manifold of constant sectionnal curvature -1. We describe the standard Minkowski model as follows : Let $\mathbb{L}^4$ denote $\mathbb{R}^4$ endowed with linear coordinates x^o, x^1, x^2, x^3 ; an inner product $\langle\ ,\ \rangle$ given by the quadratic form $-(x^o)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$; an orientation for which $dx^o \wedge dx^1 \wedge dx^2 \wedge dx^3 \neq 0$; and a time orientation given by $x^o > 0$. We set

$$H^3 = \{v \in \mathbb{L}^4 \mid \langle v,v \rangle = -1 \text{ and } x^o(v) > 0\}$$

and give H^3 the induced metric it inherits as a space-like hypersurface of $\mathbb{L}^4$. We also give H^3 the orientation for which $v_1, v_2, v_3 \in T_vH^3$ form an oriented basis iff $\{v, v_1, v_2, v_3\}$ forms an oriented basis of $\mathbb{L}^4$. It is well-known that H^3 is homeomorphic to a 3-ball and that the induced metric on H^3 is complete with constant sectional curvature -1. The linear transformations of $\mathbb{L}^4$ which preserve the quadratic form and the two orientations form a group which is the identity component of the group SO(3,1). These linear transformations preserve H^3 and restrict to H^3 to form the group of orientation preserving isometries of H^3. The geodesics of H^3 are formed by the intersections of H^3 with 2-planes in $\mathbb{L}^4$ which pass through the origin of $\mathbb{L}^4$.

The space H^3 is not compact, but can be compactified by adding on a two-sphere "at infinity", the ideal boundary, in such a way that the rigid motions of H^3 extend to continuous (in fact, smooth) homeomorphisms of $\overline{H}^3 = H^3 \cup S^2_\infty$. In the Minkowski model, this ideal boundary S^2_∞ may be most conveniently viewed as the space of null lines in $\mathbb{L}^4$, while H^3 is viewed as the space of lines on which the inner product is negative. Each geodesic in H^3 intersects S^2_∞ in two distinct points and conversely each pair of distinct points in S^2_∞ is joined by a unique geodesic in H^3. For more details, the reader should consult [6] or [5].

For doing differential geometric calculations we shall use moving frames. Let F be the (connected) six-manifold of bases (e_o, e_1, e_2, e_3) of $\mathbb{L}^4$ which satisfy the three conditions

$$e_o \wedge e_1 \wedge e_2 \wedge e_3 > 0$$

$$x^o(e_o) > 0$$

$$< e_\alpha , e_\beta > = \begin{cases} -1 \text{ if } \alpha, \beta = 0 \\ 0 \text{ if } \alpha \neq \beta \\ 1 \text{ if } \alpha = \beta = 1,2, \text{ or } 3 \end{cases}$$

The space F is a smooth submanifold of $\mathbb{L}^4 \times \mathbb{L}^4 \times \mathbb{L}^4 \times \mathbb{L}^4$. Regarding the projections on the factors $e_\alpha : F \to \mathbb{L}^4$ as vector-valued functions on F, we see that there exist unique 1-forms on F, $\{\omega^\alpha_\beta | \alpha, \beta = 0,1,2,3\}$ so that

$$de_\alpha = e_\beta \omega^\beta_\alpha . \tag{1.1}$$

Using the index range $1 \leqslant i,j,k \leqslant 3$, using the formulae for the inner products above, and writing ω^i for ω^i_o, we see that these equations can be written

$$\begin{aligned} de_o &= e_i \omega^i \\ de_i &= e_o \omega^i + e_j \omega^j_i \\ 0 &= \omega^i_j + \omega^j_i . \end{aligned} \tag{1.2}$$

Moreover, differentiating these equations, we get the remaining structure equations

$$
(1.3)\qquad
\begin{aligned}
d\omega^i &= -\omega^i_j \wedge \omega^j \\
d\omega^i_j &= -\omega^i_k \wedge \omega^k_j - \omega^i \wedge \omega^j \ .
\end{aligned}
$$

The map $e_o : F \to H$ is a smooth submersion. We clearly have $e_1, e_2, e_3 \in T_{e_o} H^3$ and, denoting the metric on H^3 by ds^2, we have

$$
(1.4)\qquad e_o^*(ds^2) = \langle de_o, de_o \rangle = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 \ .
$$

Thus, we may regard F as the oriented orthonormal frame bundle of H^3. Equations (1.3), (1.4) then show that ds^2 has sectional curvature -1 as claimed above.

Another map will be useful in our discussion. Let $N^3 \subseteq \mathbb{L}^4$ denote the positive null cone. Thus $N^3 = \{v \in \mathbb{L}^4 \mid \langle v,v \rangle = 0 \text{ and } x^o(v) > 0\}$. The map $e_o + e_3 : F \to N^3$ is a smooth submersion. We let $d\sigma^2$ denote the induced "metric" on N^3. We have

$$
(1.5)\qquad
\begin{aligned}
d(e_o+e_3) &= (e_o+e_3)\omega^3 + e_1(\omega^1+\omega^1_3) + e_2(\omega^2+\omega^2_3) \\
(e_o+e_3)^*(d\sigma^2) &= \langle d(e_o+e_3), d(e_o+e_3) \rangle = (\omega^1+\omega^1_3)^2 + (\omega^2+\omega^2_3)^2
\end{aligned}
$$

Let $[e_o+e_3]$ denote the line spanned by e_o+e_3. Then $[e_o+e_3] : F \to S^2_\infty$. We have exhibited S^2_∞ as the quotient $N^3/\mathbb{R}^+$. The "metric" $d\sigma^2$ is only well-defined up to a factor on S^2_∞. Thus S^2_∞ inherits a natural conformal structure as a quotient of N^3. We fix the orientation of S^2_∞ so that a positive 2-form on S^2_∞ pulls back under $N^3 \to S^2_\infty$ to a positive multiple of $(\omega^2+\omega^2_3) \wedge (\omega^1+\omega^1_3)$. We leave the verification that this is well defined to the reader.

There is another way of describing H^3, F, N^3 and S^2_∞ which will be quite useful in our calculations. We identify $\mathbb{L}^4$ with the space of 2×2 Hermitian symmetric matrices by identifying (x^o, x^1, x^2, x^3) with the matrix

$$
(1.6)\qquad \begin{pmatrix} x^o + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^o - x^3 \end{pmatrix} \ .
$$

The complex Lie group, $S\ell(2,\mathbb{C})$ of 2×2 complex matrices with determinant 1, acts naturally on $\mathbb{L}^4$ by the representation

$$
(1.7)\qquad g.v = g\, v\, g^*
$$

where we regard v as a 2×2 Hermitian symmetric matrix as above and $g^* = {}^t\bar{g}$. Under this identification, we clearly have $\langle v,v \rangle = -\det(v)$. Thus $S\ell(2,\mathbb{C})$ preserves

$\langle\ ,\ \rangle$ and, since $S\ell(2,\mathbb{C})$ is connected, it must also preserve the two orientations. The kernel of this representation is $\{\pm I_2\} \subseteq S\ell(2,\mathbb{C})$, so $PS\ell(2,\mathbb{C}) = S\ell(2,\mathbb{C})/\{\pm I_2\}$ is seen to act faithfully on $\mathbb{L}^4$ as the identity component of $SO(3,1)$. We now recognize H^3 as the space of unimodular positive definite Hermitian 2×2 matrices. We can use $S\ell(2,\mathbb{C})$ to parametrize F as follows : Set

$$\text{(1.8)} \qquad \underline{e}_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \underline{e}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \underline{e}_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},\ \underline{e}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and let $e_\alpha(g) = g.\underline{e}_\alpha = g\, e_\alpha\, g^*$. Then the map $g \mapsto (e_\alpha(g))$ is a 2-1 map of $S\ell(2,\mathbb{C})$ onto F . The canonical forms ω^i, ω^i_j pull back under this map to be left-invariant 1-forms. In fact, one easily verifies that we have the equation on $S\ell(2,\mathbb{C})$

$$\text{(1.9)} \qquad g^{-1}dg = \frac{1}{2}\begin{bmatrix} \omega^3+i\omega^2_1 & (\omega^1-\omega^1_3) + i(\omega^2-\omega^2_3) \\ (\omega^1+\omega^1_3)-i(\omega^2+\omega^2_3) & -(\omega^3+i\omega^2_1) \end{bmatrix}$$

This equation will be important when we compare features of the complex geometry of $S\ell(2,\mathbb{C})$ with features of the Riemannian geometry of H^3 .

The space N^3 becomes the space of positive semi-definite 2×2 Hermitian matrices of determinant 0 . Such a matrix can always be written in the form $a\,{}^t\bar{a}$ where ${}^ta = (a^1,a^2)$ is a non-zero vector in $\mathbb{C}^2$ uniquely defined "up to phase", i.e. up to multiplication by $e^{i\theta}$. The map $a\,{}^t\bar{a} \mapsto [a^1,a^2] \in \mathbb{CP}^1$ represents the map $N^3 \to S^2_\infty$ and identifies S^2_∞ with $\mathbb{CP}^1$. In this way, the map $[e_o(g) + e_3(g)]$: $S\ell(2,\mathbb{C}) \to \mathbb{CP}^1$ is seen to be holomorphic. Indeed, the natural action of $S\ell(2,\mathbb{C})$ on S^2_∞ becomes simply the action of $S\ell(2,\mathbb{C})$ on $\mathbb{CP}^1$ by linear fractional transformations.

II. <u>SURFACE THEORY IN HYPERBOLIC SPACE AND THE CASE</u> H = 1.

Throughout this section, M^2 will denote a connected, smooth oriented surface. $f : M^2 \to H^3$ will denote a smooth immersion. Weaker differentiability hypotheses would do, but for simplicity we stay in the smooth category.

We let $F_f^{(1)} \subseteq M \times F$ denote the first order frame bundle of f . Thus $(m;e_o,e_1,e_2,e_3) \in F_f^{(1)}$ if $e_o = f(m)$ and $e_1 \wedge e_2 = f_*(T_mM)$ as oriented 2-planes. By projection on the first factor, $F_f^{(1)}$ is identified as the circle bundle of oriented orthonormal frames for the induced metric $ds^2_f = f^*(ds^2) = \langle df,df \rangle$ on M . We restrict all forms and maps to $F_f^{(1)}$. It follows that $e_3 \in T_{f(m)}H^3$ is the oriented unit normal to $f_*(T_mM)$ and hence, we may regard e_3 as well-defined as a map $e_3 : M \to \mathbb{L}^4$.

We have $e_3.df = \omega^3 = 0$ on $F_f^{(1)}$, so $ds^2_f = (\omega^1)^2 + (\omega^2)^2$, and the area form induced on M is $dA_f = \omega^1 \wedge \omega^2 > 0$. Since $d\omega^3 = -\omega_1^3 \wedge \omega^1 - \omega_2^3 \wedge \omega^2 = 0$, it follows that there exist smooth functions $h_{ij} = h_{ji}$ $(i,j = 1,2)$ on $F_f^{(1)}$ so that

$$\begin{pmatrix} \omega_1^3 \\ \omega_2^3 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} \tag{2.1}$$

One easily verifies that $II = h_{11}(\omega^1)^2 + 2h_{12}\omega^1 \circ \omega^2 + h_{22}(\omega^2)^2$ is a well-defined smooth quadratic form on M^2 , just as in Euclidean surface theory. The condition $II \equiv 0$ is, of course, the condition that $f(M^2)$ lies in a totally geodesic $H^2 \subseteq H^3$.

It is convenient to regard M^2 as a Riemann surface where we use the complex structure on M^2 which is compatible with the given orientation and the conformal structure determined by ds_f^2 . That this uniquely determines a complex structure on M^2 is well known, see [5] . We set $\omega = \omega^1 + i\omega^2$ and note that $ds_f^2 = \omega \circ \overline{\omega}$, $dA_f = (\frac{i}{2})\, \omega \wedge \overline{\omega}$ and that $d\omega = -i\omega_1^2 \wedge \omega$. Moreover

$$d\omega_1^2 = -\omega_3^2 \wedge \omega_1^3 - \omega^1 \wedge \omega^2 = (1 + h_{12}^2 - h_{11}h_{22})\omega^1 \wedge \omega^2 . \tag{2.2}$$

It follows that $(-K) = 1 + h_{12}^2 - h_{11}h_{22}$ where K is the Gauss curvature of ds_f^2 . The mean curvature, H, is defined by $H = \frac{1}{2}(h_{11}+h_{22})$.

We will now consider an analogue of the Euclidean Gauss map for surfaces in $\mathbb{E}^3$. It is easy to see that, at each $e_o = f(m)$, the oriented geodesic in H^3 passing through e_o with tangent vector $e_3(m)$ (in other words, the oriented normal geodesic) meets the ideal boundary in the points $[e_o \pm e_3] \in S_\infty^2$. Since the geodesic is oriented, we may speak of $[e_o - e_3]$ as the initial point and $[e_o + e_3]$ as the final point without fear of ambiguity. In particular, we have a well-defined map $[e_o+e_3] : M \to S_\infty^2$, roughly analogous to the Euclidean Gauss map. Recall ([Ossermann]) that for $M^2 \subseteq \mathbb{E}^3$, the Gauss map is conformal iff M^2 is either totally umbilic or minimal. We have the following analogue in hyperbolic space.

PROPOSITION 1. <u>The map</u> $[e_o+e_3]:M \to S_\infty^2$ <u>is conformal iff the immersion</u> f <u>is either totally umbilic</u> <u>(in which case</u> $[e_o+e_3]$ <u>reverses orientation)</u> <u>or</u> f <u>satisfies</u> $H \equiv 1$ <u>(in which case</u> $[e_o+e_3]$ <u>preserves orientation</u>).

<u>Proof</u> : The map $[e_o+e_3]$ is conformal iff $d\sigma_f^2 = \langle d(e_o+e_3), d(e_o+e_3) \rangle$ is a multiple of ds_f^2 . We compute

$$\begin{aligned} d\sigma_f^2 &= (\omega^1+\omega_3^1)^2 + (\omega^2+\omega_3^2)^2 \\ &= ((1-h_{11})\omega^1-h_{12}\omega^2)^2 + (-h_{12}\omega^1 + (1-h_{22})\omega^2)^2 \\ &= (2(H^2-H)-K)ds_f^2 + (H-1)((h_{11}-h_{22})(\omega^1)^2 + 4h_{12}\omega^1\circ\omega^2 - (h_{11}-h_{22})(\omega^2)^2) . \end{aligned}$$

It follows that $d\sigma_f^2$ is a multiple of ds_f^2 iff $(H-1)(h_{11}-h_{22}) = (H-1)h_{12} = 0$.

If $H \equiv 1$, then $d\sigma_f^2 = (-K)ds_f^2$. Thus $-K \geqslant 0$. Moreover,

$$(\omega^2+\omega_3^2) \wedge (\omega^1+\omega_3^1) = (-K)dA_f \geqslant 0$$

so $[e_o+e_3] : M^2 \to S_\infty^2$ is either orientation preserving or a constant map.

If $H \not\equiv 1$, then let $U = \{m \in M | H(m) \neq 1\}$. The set U is open and $f(U)$ is clearly totally umbilic. Let V be a connected component of U . Then, H must be constant on V by a standard argument ,and hence, constant on $\overline{V}$ since H is continuous on M . Thus $H \neq 1$ on $\overline{V}$ and so $\overline{V} \subseteq U$. Since V was a connected component of U it follows that $\overline{V} = V$, so by the connectedness of M , $V = M$. Thus f is totally umbilic. We compute

$$(\omega^2+\omega_3^2) \wedge (\omega^1+\omega_3^1) = -(H-1)^2\omega^1 \wedge \omega^2 < 0 \quad ,$$

so $[e_o+e_3]$ reverses orientation. ∎

<u>Remark</u> : The case where $H \equiv 1$ <u>and</u> f is totally umbilic is easily seen to be the case where $f(M) \subseteq H^3$ is an open subset of a horosphere in H^3 . In this case, the value of $[e_o+e_3]$ is simply the point of tangency of the horosphere with the ideal boundary S_∞^2 .

As another instance of the analogy between minimal surfaces in $\mathbb{E}^3$ and surfaces of mean curvature 1 in H^3 , we recall that, if $M^2 \subseteq \mathbb{E}^3$ is a smooth minimal surface, ds^2 is the induced metric, and $K \leqslant 0$ is the Gauss curvature of this metric, then $(-K)ds^2$ is the pull-back via the Gauss map of the metric of Gauss curvature 1 on S^2. Hence $(-K)ds^2$ has Gauss curvature +1 when $K \neq 0$. Conversely, if ds^2 is a metric on M^2 with $(-K)ds^2$ a metric of Gauss curvature +1, then there exists (locally) an isometric immersion of M^2 into $\mathbb{E}^3$ as a minimal surface (in fact, there is a 1-parameter family of such immersions). Now, consider the case of $f : M^2 \to H^3$ with $H \equiv 1$. If we set $\eta = (\omega^1+\omega_3^1) - i(\omega^2+\omega_3^2)$, then we have

$$\eta = ((1-h_{11}) + ih_{12})\omega$$

$$(i/2)\eta \wedge \bar{\eta} = -K((i/2)\omega \wedge \bar{\omega}).$$

The structure equations give

$$d\omega = -i\omega_1^2 \wedge \omega$$
$$d\eta = i\omega_1^2 \wedge \eta$$
$$d\omega_1^2 = (-K)(i/2)\omega \wedge \bar{\omega} = (i/2)\eta \wedge \bar{\eta} .$$

It follows immediately that $d\sigma_f^2 = \eta \wedge \bar{\eta} = -K\omega \wedge \bar{\omega} = (-K)ds_f^2$ has Gauss curvature +1. Conversely, by applying the Frobenius theorem, see [3], one can prove that if ds^2 is a metric on M^2 so that $(-K)ds^2$ has Gauss curvature +1, then M^2 can be locally isometrically immersed into H^3 with $H \equiv 1$.

It is interesting to note that, whereas dilation in $\mathbb{E}^3$ carries minimal surfaces to minimal surfaces, implying that ds^2 satisfies our integrability condition iff c^2ds^2 does where $c \neq 0$ is a constant, there does not seem to be any corresponding motion in H^3 preserving the surfaces of mean curvature 1.

We conclude this section by proving an analogue for surfaces satisfying $H \equiv 1$ in H^3 of the Weierstrass formula for minimal surfaces in $\mathbb{E}^3$. Recall that $S\ell(2,\mathbb{C})$ is a complex Lie group and that $g^{-1}dg$ is a holomorphic $S\ell(2,\mathbb{C})$-valued 1-form on $S\ell(2,\mathbb{C})$. The Cartan-Killing metric on $S\ell(2,\mathbb{C})$ is given by the holomorphic quadratic form

$$\phi = -4 \det(g^{-1}dg). \tag{2.3}$$

If M^2 is a Riemann surface, a holomorphic map $F : M^2 \to S\ell(2,\mathbb{C})$ is said to be <u>null</u> if $F^*(\phi) = 0$.

THEOREM A : <u>Let</u> M^2 <u>be a Riemann surface and let</u> $F : M^2 \to S\ell(2,\mathbb{C})$ <u>be a null immersion. Then</u> $e_o \circ F = f:M^2 \to H^3$ <u>is a smooth conformal immersion with</u> $H \equiv 1$. <u>Conversely, if</u> M^2 <u>is simply connected and</u> $f:M^2 \to H^3$ <u>is an immersion with</u> $H \equiv 1$, <u>then there exists an</u> $F:M^2 \to S\ell(2,\mathbb{C})$ <u>which is holomorphic with respect to the induced complex structure on</u> M^2 <u>and so that</u> $f = e_o \circ F$. <u>Moreover</u>, F <u>is unique up to right multiplication by a constant</u> $g \in SU(2) \subseteq S\ell(2,\mathbb{C})$.

<u>Remarks</u> : Note that this theorem is a close analogue of the Weierstrass theorem : In the standard Weierstrass theorem, we replace H^3 by $\mathbb{E}^3$, $S\ell(2,\mathbb{C})$ by $\mathbb{C}^3$, $e_o:SL(2,\mathbb{C}) \to H^3$ by $\mathbb{R}\mathrm{e} :\mathbb{C}^3 \to \mathbb{E}^3$, ϕ by the natural complex inner product on $\mathbb{C}^3$,

and of course $H \equiv 1$ by $H \equiv 0$. Theorem A then holds verbatim for minimal surfaces in $\mathbb{E}^3$.

In the second part of the theorem, if we do not assume M to be simply connected, then we still get a holomorphic map $F:\widetilde{M}^2 \to S\ell(2,\mathbb{C})$ where $\widetilde{M}^2$ is the simply connected cover of M^2 and we get a representation $\chi : \pi_1(M,m_o) \to SU(2)$ in the obvious way.

Proof of Theorem A : To simplify our calculations, we will use the notation $\omega = \omega^1 + i\omega^2$ and $\pi = \omega_1^3 - i\omega_2^3$. Assume that $F : M^2 \to S\ell(2,\mathbb{C})$ is a holomorphic null immersion of a Riemann surface M . Then we have, by (1.9)

$$\begin{aligned} F^*(\omega^3+i\omega_1^2) &= 2\alpha \\ F^*(\overline{\omega}-\pi) &= 2\beta \\ F^*(\omega+\overline{\pi}) &= 2\gamma \end{aligned} \tag{2.4}$$

where α , β , and γ are holomorphic 1-forms on M^2. Since F is null, we have

$$F^*(\phi) = 4(\alpha^2 + \beta\gamma) = 0$$

and

$$\begin{aligned} F^*(\omega^3) &= \alpha + \overline{\alpha} \\ F^*(\omega) &= \overline{\beta} + \gamma \ . \end{aligned}$$

Now let $f = e_o \circ F$, then we have

$$\begin{aligned} f^*(ds^2) &= F^*((\omega^3)^2 + \omega\circ\overline{\omega}) = (\alpha+\overline{\alpha})^2 + (\overline{\beta}+\gamma)(\beta+\overline{\gamma}) \\ &= (\alpha^2+\beta\gamma) + (2\overline{\alpha}\circ\alpha + \beta\circ\overline{\beta} + \gamma\circ\overline{\gamma}) + (\overline{\alpha}^2 + \overline{\beta}\circ\overline{\gamma}) \\ &= 2\alpha\circ\overline{\alpha} + \beta\circ\overline{\beta} + \gamma\circ\overline{\gamma} \ . \end{aligned}$$

Since F is an immersion, this last expression is positive definite. Since it clearly determines the same conformal structure as the given complex structure, it follows that $f : M^2 \to H^3$ is a conformal immersion with induced metric $ds_f^2 = f^*(ds^2)$. We will now show that for this immersion $H \equiv 1$ by computing H in a first order adapted frame. Let $U \subseteq M$ be a simply connected open set on which there exists a smooth 1-form ϕ of type (1,0) so that $ds_f^2 = \phi\circ\overline{\phi}$ on U . Clearly M is covered by such open sets. Restricting to U , we see that there exist functions A, B, C on U satisfying

$$\begin{aligned} F^*(\omega^3+i\omega_1^2) &= 2A\phi \\ F^*(\overline{\omega} - \pi) &= 2B\phi \\ F^*(\omega + \overline{\pi}) &= 2C\phi \end{aligned} \tag{2.5}$$

$$(2.6) \qquad A^2 + BC = 0 \ , \ 2A\bar{A} + B\bar{B} + C\bar{C} = 1 \ .$$

By elementary algebra (and the simple connectivity of U), we see that there exist smooth functions p,q on U (unique up to replacement by (-p,-q)) so that

$$\begin{aligned} A &= pq \\ B &= p^2 \\ C &= -q^2 \\ p\bar{p} + q\bar{q} &= 1 \ . \end{aligned}$$

Note that because $A\phi$, $B\phi$, and $C\phi$ are holomorphic on U, it follows that p/q is meromorphic on U (unless $q \equiv 0$). In particular, the 1-form pdq - qdp must be of type (1,0) since we have the representations

$$pdq - qdp = \begin{cases} p^2 d(q/p) & \text{where } p \neq 0 \\ -q^2 d(p/q) & \text{where } q \neq 0 \ . \end{cases}$$

Let $h : U \to SU(2)$ be defined by

$$h = \begin{pmatrix} q & -\bar{p} \\ p & \bar{q} \end{pmatrix} .$$

Then $e_o \circ (Fh) = e_o \circ F$ since h has values in SU(2). Moreover, we compute

$$\begin{aligned} (Fh)^{-1}d(Fh) &= h^{-1}(F^{-1}dF)h + h^{-1}dh \\ &= h^{-1}\begin{bmatrix} pq & -q^2 \\ p^2 & -pq \end{bmatrix}\phi\, h + h^{-1}dh \\ &= \begin{bmatrix} \bar{q}dp + \bar{p}dp & \bar{p}d\bar{q} - \bar{q}d\bar{p} - \phi \\ qdp - pdq & pd\bar{p} + qd\bar{q} \end{bmatrix} \\ &= \frac{1}{2}(Fh)^*\begin{bmatrix} \omega^3 + i\omega_1^2 & \omega + \bar{\pi} \\ \bar{\omega} - \pi & -(\omega^3 + i\omega_1^2) \end{bmatrix} . \end{aligned}$$

It follows that $(Fh)^*(\omega) = -\phi$

$(Fh)^*(\omega^3) = 0$.

Thus $Fh : U \to S\ell(2,\mathbb{C})$ is an oriented adapted frame field on U for the immersion $f = e_o \circ F = e_o(Fh)$. Moreover, since

$$(Fh)^*(\pi) = \bar{\omega} - 2(qdp - pdq)$$

and since qdp - pdq has type (1,0) it follows that the mean curvature of the immersion f is identically 1 as promised.

To establish the converse proposition, we suppose that M^2 is simply connected and we let $f : M^2 \to H^3$ be an immersion with $H \equiv 1$. By simple connectivity, there exists a (1,0) form ϕ globally defined on M^2 for which $ds_f^2 = \phi \circ \bar{\phi}$. We may therefore choose a lifting $g : M^2 \to S\ell(2,\mathbb{C})$ for which the associated frame field $\{e_o(g)\}$ is adapted with $\omega = -\phi$. Thus, we have

$$g^{-1}dg = \frac{1}{2}g^*\begin{bmatrix} \omega^3 + i\omega_1^2 & \omega + \bar{\pi} \\ \bar{\omega} - \pi & -(\omega^3 + i\omega_1^2) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} i\rho & -2\phi - \bar{\eta} \\ \eta & -i\rho \end{bmatrix}$$

where by the hypothesis, ρ is a real form and η is of type (1,0) (since the mean curvature of f is $\equiv 1$). Consider the SU(2)-valued 1-form on M

$$\mu = \frac{1}{2}\begin{bmatrix} i\rho & -\bar{\eta} \\ \eta & -i\rho \end{bmatrix} .$$

It is easy to see that μ satisfies $d\mu = -\mu \wedge \mu$ (since both η and ϕ are of type (1,0)). It follows by the Frobenius theorem that there exists a smooth map $h : M^2 \to SU(2)$ (unique up to left translation by a constant) so that $\mu = h^{-1}dh$. Let us write

$$h = \begin{pmatrix} q & -\bar{p} \\ p & \bar{q} \end{pmatrix}$$

for smooth functions p and q on M^2 . Then, if we set $F = gh^{-1}$, we easily compute

$$F^{-1}dF = \begin{bmatrix} pq & -q^2 \\ p^2 & -pq \end{bmatrix}\phi .$$

It follows that, since dF is of type (1,0) we must have that $F : M^2 \to S\ell(2,\mathbb{C})$ is holomorphic. Clearly , F is a null immersion and satisfies

$$e_o \circ F = e_o \circ (gh^{-1}) = e_o \circ g = f .$$

Now we show uniqueness up to constants as follows : if $F_1, F_2 : M^2 \to S\ell(2,\mathbb{C})$ are two holomorphic liftings of f , we must have

$$F_1 = F_2 h$$

where $h : M^2 \to SU(2)$ is a holomorphic map of M^2 into $S\ell(2,\mathbb{C})$. However, SU(2) is a totally real submanifold of $S\ell(2,\mathbb{C})$ and hence h must be constant. ∎

III. FINITE TOTAL CURVATURE.

In the theory of minimal immersions $f : M^2 \to \mathbb{E}^3$, the complete surfaces of finite total curvature play an important role. The main results, due to Osserman , are as follows. If $f : M^2 \to \mathbb{E}^3$ is a minimal immersion for which the induced metric is complete and of finite total curvature, then there exists a unique compact Riemann surface $\overline{M}^2$ and a finite set of points $E \subseteq \overline{M}^2$ so that $M^2 = \overline{M}^2 - E$ as Riemann surfaces. The (1,0)-part of df, namely ∂f, is holomorphic on M^2 and extends meromorphically to $\overline{M}^2$. Finally, the Gauss map $\gamma_f : M^2 \to S^2$ is holomorphic and extends to a holomorphic mapping $\overline{\gamma}_f : \overline{M}^2 \to S^2$ whose degree is the total curvature divided by -4π. These results go a long way toward describing the complete minimal surfaces of finite total curvature in purely algebraic terms, via Riemann surface theory.

In this section, we pursue the analogy between $H \equiv 0$ in $\mathbb{E}^3$ and $H \equiv 1$ in H^3 by studying the immersions $f : M^2 \to H^3$ satisfying the conditions $H \equiv 1$, ds_f^2 is complete, and ds_f^2 has finite total curvature. These assumptions on f will hold throughout this section. We continue to denote the first order adapted frame bundle by $F_f^{(1)}$ and we assume all forms and functions are restricted to it.

PROPOSITION 2 : <u>The quadratic form</u> $Q = (1 - h_{11} + ih_{12})\,(\omega)^2$ <u>is a well-defined holomorphic quadratic form on</u> M .

<u>Proof</u> : If we set $\eta = (\omega^1 + \omega_3^1) - i(\omega^2 + \omega_3^2)$, then the structure equations read

$$d\omega = -i\omega_1^2 \wedge \omega \qquad d\eta = i\omega_1^2 \wedge \eta$$
$$\eta = (1 - h_{11} + ih_{12})\omega\ .$$

It immediately follows that $Q = \eta \circ \omega$ is well-defined, holomorphic, and of type (2,0). ∎

If $Q \equiv 0$, then it follows that $\eta \equiv 0$ so $d\sigma_f^2 = \eta \circ \overline{\eta} \equiv 0$, so the mapping $f : M^2 \to H^3$ is totally umbilic. We have already seen that this implies that $f(M^2)$ is a horosphere in H^3. We set this case aside now and assume henceforth that $Q \not\equiv 0$. Let B be the zero divisor of Q. Thus, as a formal sum

$$B = \sum_{p \in M} \nu_p(Q).p \geqslant 0$$

where $\nu_p(Q)$ is the order of vanishing of Q at $p \in M$. We let $|B|$ denote the <u>support</u> of B, i.e., the set of $p \in M$ for which $\nu_p(Q) > 0$. Then $|B|$ is a discrete set in M since $Q \not\equiv 0$.

PROPOSITION 3 . <u>The quadratic form</u> $d\sigma_f^2$ <u>is a pseudo-metric on</u> M <u>of constant Gauss curvature</u> +1 .

Proof : Let $p \in M$ be fixed and let $z : U \to \mathbb{C}$ be a local holomorphic coordinate with $p \in U$ and $z(p) = 0$. Then we may write

$$\mathcal{Q}\big|_U = (z)^{\nu_p(\mathcal{Q})} h(z)(dz)^2$$

where $h(z)$ is holomorphic and $h(0) \neq 0$. We have $ds_f^2\big|_U = e^{2\lambda} dz \circ d\bar{z}$ for some smooth λ on U . It follows from

$$\mathcal{Q} \circ \bar{\mathcal{Q}} = (\eta \circ \bar{\eta}) \circ (\omega \circ \bar{\omega}) = d\sigma_f^2 \circ ds_f^2$$

that

$$d\sigma_f^2\big|_U = |z|^{2\nu_p(\mathcal{Q})} |h(z)|^2 e^{-2\lambda} dz \circ d\bar{z} \ .$$

Thus $d\sigma_f^2$ is indeed a pseudometric on M in the usual sense. Moreover, the structure equations $d\eta = i\omega_1^2 \wedge \eta$, $d\omega_1^2 = (i/2)\eta \wedge \bar{\eta}$ show that $\eta \circ \bar{\eta} = d\sigma_f^2$ has constant Gauss curvature $+1$. ∎

We now need some elementary facts about pseudo-metrics of curvature $+1$. First, if we regard $\mathbb{CP}^1$ as $\mathbb{C} \cup \{\infty\}$ and let $\zeta : \mathbb{CP}^1 - \{\infty\} \to \mathbb{C}$ be the standard meromorphic coordinate, then the standard metric of Gauss curvature $+1$ on $\mathbb{P}^1$ can be written in the form

$$\mu = \frac{4\, d\zeta \circ d\bar{\zeta}}{(1+\zeta\bar{\zeta})^2} \ .$$

Second, if D is a simply connected Riemann surface and $d\sigma^2$ is a pseudo-metric on D of constant Gauss curvature $+1$, then there exists a meromorphic function ξ on D so that, as a holomorphic map $\xi : D \to \mathbb{P}^1$ satisfies $\xi^*(\mu) = d\sigma^2$, i.e.

$$d\sigma^2 = \frac{4\, d\xi \circ d\bar{\xi}}{(1+\xi\bar{\xi})^2} \ .$$

(This second fact follows from the Frobenius Theorem.) Moreover, ξ is unique up to rotations of $\mathbb{P}^1$. Thus, if $\xi' : D \to \mathbb{P}^1$ also satisfies $\xi'^*(\mu) = d\sigma^2$, then

$$\xi' = \frac{a\xi - \bar{b}}{b\xi + \bar{a}}$$

where a,b are complex constants satisfying $a\bar{a} + b\bar{b} = 1$. Third, we will need the following identity. If $z : D \to \mathbb{C}$ is a holomorphic coordinate imbedding D in $\mathbb{C}$ (we are assuming $D \neq S^2$), then we have

$$d\sigma^2 = \frac{4|\xi'(z)|^2}{(1+\xi(z)\overline{\xi(z)})^2}\, dz\circ d\bar{z} = e^{2\lambda} dz\circ d\bar{z}$$

where the smooth function λ satisfies the identity

$$\frac{\partial^2\lambda}{\partial z^2} - \left(\frac{\partial\lambda}{\partial z}\right)^2 = \frac{2\xi'(z)\xi'''(z) - 3(\xi''(z))^2}{4(\xi'(z))^2}$$

$$= \frac{1}{2} S_z(\xi)$$

where $S_z(\xi)$ is, by definition, the <u>Schwarzian derivative</u> of ξ with respect to z. Finally, we shall need the following extension of the above results to the case of non-simply connected domains.

PROPOSITION 4. *Let $d\sigma^2$ be a pseudo-metric of Gauss curvature $+1$ on the punctured disk $\Delta^* = \{w \in \mathbb{C} | 0 < |w| < 1\}$. Suppose moreover that the $d\sigma^2$ - area of Δ^* is finite. Then, there exists a local holomorphic coordinate z on $\Delta_\varepsilon = \{w \in \mathbb{C} | |w| < \varepsilon\}$ for some $\varepsilon > 0$ with $z(0) = 0$, and a real number $\beta > -1$ so that, on Δ_ε, we have*

$$d\sigma^2\Big|_{\Delta_\varepsilon} = \frac{4(\beta+1)^2(z\bar{z})^\beta}{(1+(z\bar{z})^{\beta+1})^2}\, dz\circ d\bar{z} .$$

Moreover, β is unique, and z is unique up to replacement by λz where $|\lambda| = 1$.

<u>Proof</u> : Let $L = \{y \in \mathbb{C} | \mathrm{Re}(y) < 0\}$. Then $\exp : L \to \Delta^*$ given by $\exp(y) = e^y = w$ is a universal covering map of Δ^*. Let $d\tilde{\sigma}^2 = \exp^*(d\sigma^2)$. Then $d\tilde{\sigma}^2$ is a pseudometric on L of constant Gauss curvature $+1$. Moreover, $d\tilde{\sigma}^2$ is invariant under the deck transformation $y \mapsto y + 2\pi i$. Since L is simply connected, it follows that there exists a meromorphic $\xi : L \to \mathbb{P}^1$ so that $\xi^*(\mu) = d\tilde{\sigma}^2$. The invariance of $d\tilde{\sigma}^2$ implies that there exists a point $(a,b) \in \mathbb{C}^2$ with $a\bar{a} + b\bar{b} = 1$ so that ξ satisfies

$$\xi(y + 2\pi i) = \frac{a\xi(y) - \bar{b}}{b\xi(y) + \bar{a}} .$$

Since each element of the rotation group of $\mathbb{P}^1$ is conjugate to a rotation fixing 0 and ∞, we see that, by composing ξ with a suitable rotation of $\mathbb{P}^1$, we may assume that ξ actually satisfies

$$\xi(y + 2\pi i) = e^{2\pi i\alpha}\xi(y)$$

where α is some real number, $0 \leqslant \alpha < 1$. It follows that there is a well-defined meromorphic function $\psi(w)$ on Δ^* so that $\psi(e^y) = e^{-\alpha y}\xi(y)$ for all $y \in L$. It follows that, on Δ^*, we have the formula

$$d\sigma^2 = \frac{4d(w^\alpha\psi)\circ d(\overline{w^\alpha\psi})}{(1 + (w\bar{w})^\alpha\psi\bar{\psi})^2}$$

$$= \frac{4|w|^{2(\alpha-1)}|\alpha\psi+w\psi'|^2 dw\circ d\bar{w}}{(1 + |w^\alpha\psi|^2)^2}$$

where the last expression shows that the middle expression is well defined even though $w^\alpha\psi(w)$ is a multivalued meromorphic function on Δ^*. We are going to use the hypothesis that Δ^* has finite $d\sigma^2$-area to show that ψ is meromorphic at $w = 0$, i.e., that ψ does not have an essential singularity at $w = 0$. Let us assume this for the moment. Then, there exists an integer n so that $w^{-n}\psi(w)$ is a non-vanishing holomorphic function on a neighborhood, say $U_\rho = \{w \in \mathbb{C} \mid |w| < \rho\}$ of 0. Then, on U_ρ there exists a holomorphic function $g(w)$ so that $\psi(w) = w^n e^{g(w)}$ for $w \in U_\rho$. There are now three cases :

(i) $\alpha+n > 0$. In this case, we set $\beta = \alpha+n-1$ and $z = we^{g/(\beta+1)}$. Then we easily compute that

$$d\sigma^2\Big|_{U_\rho} = \frac{4(\beta+1)^2(z\bar{z})^\beta}{(1 + (z\bar{z})^{\beta+1})^2}\, dz\circ d\bar{z} \;;$$

(ii) $\alpha+n < 0$. In this case, we set $\beta = -(\alpha+n+1)$ and $z = we^{-g/(\beta+1)}$ and again the above formula holds;

(iii) $\alpha+n = 0$. Since n is an integer and $0 \leqslant \alpha < 1$, it follows that $\alpha = n = 0$. Thus $\psi = e^g$ and $d\sigma^2 = 4d\psi\circ d\bar{\psi}/(1+\psi\bar{\psi})^2$ where ψ is holomorphic and non-vanishing on U_ρ. By a rotation of $\mathbb{P}^1$ we may replace ψ by ϕ where $\phi(0) = 0$. It follows that there exists a unique integer $\beta \geqslant 0$ so that ϕ can be written in the form $\phi = z^{\beta+1}$ for some local holomorphic coordinate z on a (possibly smaller) neighborhood of $0 \in U_\rho$. We then have

$$d\sigma^2 = \frac{4d\phi\circ d\bar{\phi}}{(1+\phi\bar{\phi})^2} = \frac{4(\beta+1)^2(z\bar{z})^\beta}{(1+(z\bar{z})^{\beta+1})^2}\, dz\circ d\bar{z} \quad .$$

The stated uniqueness of β and z now follows easily.

It remains to show that the finiteness of the integral

$$A = \int_{\Delta^*} \frac{2i\; d(w^\alpha\psi)\wedge d(\overline{w^\alpha\psi})}{(1 + |w^\alpha\psi|^2)^2} < \infty$$

implies that ψ has, at worst, a pole at $w = 0$. If α were known to be rational, say $\alpha = p/q$ with $p,q \in \mathbb{Z}$, $q \neq 0$, then $qA < \infty$ would be the area covered by the map

$h(x) = x^p\psi(x^q)$ $h : \Delta^* \to \mathbb{P}^1$ (counting multiplicities). If h had an essential singularity we would then have a contradiction since by Picard's theorem, h would cover any point of $\mathbb{P}^1$ infinitely often with at most 2 exceptions. Since h has no essential singularity, it follows that ψ does not. Unfortunately, this simple argument does not generalize to the case of irrational α's .

Our argument is based on a Nevanlinna-type inequality. Our basic reference is [1 §1b] and we will use their notation. Thus, our finiteness condition is

$$\int_{\Delta^*} dd^c \log(1 + |w^\alpha \psi(w)|^2) < \infty$$

where $\psi(w)$ is meromorphic in the deleted disk $\Delta^* = \{w|0 < w < 1\}$, and α is some real number. By dilating, we may actually assume that $\psi(w)$ is meromorphic on the disk $\{w|0 < |w| < 1 + \varepsilon\}$ for some $\varepsilon > 0$ and that ψ has no poles or zeroes on $|w| = 1$. Then, we may apply Corollary (1.5) of [1, § 1b] to $h(w) = 1 + |w^\alpha\psi(w)|^2$ and conclude that $N(D,r) \leqslant C_1 \log r + C'$ and hence that $\psi(w)$ has only a <u>finite</u> number of poles in Δ^* . It follows that by dilating and restricting to Δ^* , we may assume that $\psi(w)$ has no poles in Δ^* and is smooth on the boundary $|w| = 1$.

Let $\Delta^- = \{z \in \Delta^* | z$ is not negative and real$\}$. For every $\beta \in \mathbb{R}$, we let z^β denote the unique branch well-defined on Δ^- so that $1^\beta = 1$. Then, $\zeta(y) = y^{2\alpha}\psi(y^2) : \Delta^- \to \mathbb{C}$ is well-defined and holomorphic on Δ^- . Now, the integrand $4\pi dd^c \log(1 + |w^\alpha\psi(w)|^2)$ is the pull back via $w^\alpha\psi(w) : \Delta^- \to \mathbb{C}$ of the spherical area form on $\mathbb{P}^1 \supseteq \mathbb{C}$. Since $y \mapsto y^2$ is a double cover of Δ^* , it follows that

$$\int_{\Delta^-} 4\pi dd^c \log(1 + |\zeta|^2) < \infty .$$

Thus the area of the spherical image $\zeta(\Delta^-) \subseteq \mathbb{P}^1$, counting multiplicities, is finite. It follows that there is a set $K \subseteq \mathbb{C}$ of positive measure so that $\zeta(y) = k$ has only a finite number of solutions for all $k \in K$. By shrinking Δ^* and dilating if necessary, we may assume that $\zeta(y) \notin K$ for all $y \in \Delta^-$. For $m \geqslant 0$, set

$$B_m = \{z \in \Delta^- | 2^{-m-2} < |z| < 3.2^{-m-2} \text{ and } -3/4\,\pi < \arg z < 3/4\,\pi\}.$$

Consider the functions $\zeta_m(y) = \zeta(2^{-m}y)$ for $y \in B_o$. This sequence $\{\zeta_m(y)\}$ does not assume values in K for any $y \in B_o$. It follows by a theorem of Montel [2 , vol.II, p .248, Thm.15.2.8] that the sequence $\{\zeta_m(y)\}$ constitutes a normal family. Hence, there exists a subsequence $\{\zeta_{m_k}(y)\}$ so that either $\{\zeta_{m_k}(y)\}$ converges uniformly on compact sets to a holomorphic function or else converges uniformly in the spherical sense to $\infty \in \mathbb{P}^1$.

If the limit function is holomorphic, it is bounded on the arc $|z| = \frac{1}{2}$, $\mathbb{R}e(z) \geqslant 0$. It follows that there exists an $M < \infty$ so that $|\zeta_{m_k}(y)| \leqslant M$ for $|y|=\frac{1}{2}$ $\mathbb{R}e(y) \geqslant 0$, and all $k \geqslant 0$. Let $a \geqslant \alpha$ be an integer. Then we have

$$|4^{-m_k a} y^{2a} \psi(4^{-m_k} y^2)| \leqslant | 4^{-m_k \alpha} y^{2\alpha} \psi(4^{-m_k} y^2)| \leqslant M$$

for all $k \geqslant 0$ and $y = \frac{1}{2}$, $\mathbb{R}e\,(y) \geqslant 0$. It follows that the holomorphic function $w^a \psi(w)$ is uniformly bounded by a constant M on a sequence of concentric circles in Δ^* whose radii go to zero. It follows by the maximum modulus principle that $|w^a \psi(w)|$ is bounded on a neighborhood of zero and hence by the removable singularities theorem $w^a \psi$ has a holomorphic extension to $w = 0$.

If the limit function of the sequence $|\zeta_{m_k}(y)|$ is ∞ , then we apply the above reasoning to $\{1/\zeta_{m_k}(y)\}$ and conclude that $w^{-a} \psi(w)^{-1}$ is holomorphic at zero. Thus,in either case $\psi(w)$ has at worst a pole at $w = 0$. ∎

Remarks : The Nevanlinna part of the above argument was suggested to the author by Phillip Griffiths. The normal families part was modeled on a similar proof by [2 , vol. II, pg.258, Thm.15.4.2]. The source of the complication in the proof is, of course, the branching term w^α . Note that Proposition 4 would definitely be false without the hypothesis of finite $d\sigma^2$-area. Also, note that $d\sigma^2$ extends to be a pseudo-metric on a neighborhood of $0 \in \Delta$ iff β is an integer. For further properties of pseudo-metrics, the reader is referred to the paper by Cowen and Griffiths cited above.

We now turn to the study of $f : M^2 \to H^2$ satisfying our hypotheses. Since ds_f^2 is complete with non-positive Gauss curvature, the finiteness of the total curvature implies that M^2 is conformally a compact Riemann surface $\overline{M}^2$ minus a finite number of points $E \subseteq \overline{M}^2$. Henceforth we will simply identify M^2 with $\overline{M}^2 - E$. The above is a theorem of Huber and a proof may be found in [3] . The set E cannot be empty : it is easy to see that, if $v \in H^3$ is fixed, then every point $p \in M$ for which $\langle v,f(p)\rangle$ is a local minimum of $\langle f,v\rangle : M \to \mathbb{R}$ satisfies $|H(p)| > 1$ where $H(p)$ is the mean curvature of f at p .

PROPOSITION 5 : The support of the divisor B is finite in M and Q extends to $\overline{M}$ as a meromorphic quadratic differential.

Proof : The support of B is discrete in M so it suffices to show that $|B|$ cannot accumulate at any $e \in E$. However, $d\sigma_f^2$ is a pseudo-metric on M of constant Gauss curvature +1 and finite total area since the area form of $d\sigma_f^2$ is (-K) times the area form of ds_f^2 . By Proposition 4, it follows that, if $e \in E$ then there exists a local holomorphic coordinate $z : U \to \mathbb{C}$ on $U \subseteq \overline{M}$ where $e \in U$, $z(e) = 0$ and a real number $\beta = \beta(e)$ so that

$$d\sigma_f^2 = \frac{4(\beta+1)^2(z\bar{z})^\beta}{(1+(z\bar{z})^{\beta+1})^2}\, dz\circ d\bar{z} \;.$$

In particular, $d\sigma_f^2$ does not vanish on a deleted neighborhood of e in U . Since $d\sigma_f^2$ vanishes on $|B|$, it follows that $|B| \cap U = \emptyset$. Thus B is finite. Now, there also exists a function $h(z)$ holomorphic on $z(U - \{e\}) \subseteq \mathbb{C} - \{0\}$ so that

$$\mathcal{Q}\big|_{U-\{e\}} = h(z)(dz)^2$$

since $\mathcal{Q}$ is holomorphic on M . Since $\mathcal{Q}\circ\overline{\mathcal{Q}} = d\sigma_f^2\circ ds_f^2$, we see that

$$ds_f^2\Big|_{U-\{e\}} = \frac{(1 + (z\bar{z})^{\beta+1})^2|h(z)|^2}{4(\beta+1)^2(z\bar{z})^\beta}\, dz\circ d\bar{z} \;.$$

If $b \geqslant \beta$ is an integer, it follows that

$$ds_f^2\Big|_V \leqslant c\left|\frac{h(z)}{z^b}\right|^2 dz\circ d\bar{z}$$

for some constant $c > 0$ and some deleted neighborhood V of e in U .

Since ds_f^2 is complete at the end e , it follows that the flat metric $|h(z)/z^b|^2 dz\circ d\bar{z}$ is also complete. By a lemma of Osserman [4 ,Lemma 9.6] , it follows that $h(z)z^{-b}$ has a pole of finite order at $z = 0$. Thus $h(z)$ is meromorphic at $z = 0$, so $\mathcal{Q}$ is meromorphic at e . Since $e \in E$ was arbitrary, this establishes our claim. ∎

It follows that we may now define $\nu_p(\mathcal{Q})$ for all $p \in \overline{M}$ .By Riemann's relation, we have

$$\chi(\overline{M}) = -\frac{1}{2}\sum_{p\in M}\nu_p(\mathcal{Q}) = -\frac{1}{2}\deg \overline{B}$$

where $\overline{B}$ is the divisor of $\mathcal{Q}$ as a meromorphic form on $\overline{M}$.

COROLLARY : <u>For all</u> $e \in E$, $\beta(e) \geqslant \nu_e(\mathcal{Q}) + 1$.

<u>Proof</u> : Near the end e , the metrics ds_f^2 and $|h(z)z^{-\beta}|^2 dz\circ d\bar{z}$ are obviously equivalent in that one is complete iff the other is. But the completeness of $|h(z)z^{-\beta}|^2 dz\circ d\bar{z}$ is clearly equivalent to the condition $\nu_e(\mathcal{Q}) - \beta(e) \leqslant -1$. ∎

Remark : Proposition 5 is quite analogous to a similar result in the theory of minimal surfaces $f : M^2 \to \mathbb{E}^3$ which are complete and of finite total curvature : in that case, the Gauss map $\gamma_f : M^2 \to S^2$ is holomorphic and extends uniquely to a holomorphic mapping $\gamma_f : \overline{M}^2 \to S^2$ (we use the opposite orientation on S^2 from the standard one since, in the standard orientation, the Gauss map of a minimal surface is anti-holomorphic), see [4]. Moreover, the analogue of $d\sigma_f^2$, namely $-K\,ds_f^2$, is $\gamma_f^*(\mu)$ where μ is the standard metric of Gauss curvature +1 on $S^2 \subseteq \mathbb{E}^3$. Thus, in the Euclidean case $d\sigma_f^2$ extends to be a pseudo-metric on all of $\overline{M}$ and has the normal form above at each $e \in E$ with $\beta(e)$ an integer. Moreover, the analogue of Q is the (2,0)-part of the second fundamental form II_f of $f : M^2 \to \mathbb{E}^3$. The fact that Q in the Euclidean case extends meromorphically to $\overline{M}^2$ follows from Lemma 9.6 in [4] coupled with the equation $Q\circ\overline{Q} = ds_f^2 \circ d\sigma_f^2$ and the assumed completeness of ds_f^2.

We are now going to examine some of the differences between the two theories by computing some examples. The following formulae will be useful. Suppose that M^2 is a Riemann surface and that $F : M^2 \to S\ell(2,\mathbb{C})$ is a holomorphic null immersion. Referring to the proof of Theorem A , we see that we may write

$$F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}, \quad F_1F_4 - F_2F_3 \equiv 1$$

$$F^{-1}dF = \begin{pmatrix} pq & -q^2 \\ p & -pq \end{pmatrix}\phi, \quad p\overline{p} + q\overline{q} = 1$$

$$h = \begin{pmatrix} q & -\overline{p} \\ p & \overline{q} \end{pmatrix}$$

where the F_i are holomorphic functions on M , ϕ is of type (1,0) on $U \subseteq M$ and p,q are smooth functions on U with p/q meromorphic. The mapping $Fh : U \to S\ell(2,\mathbb{C})$ is an adapted framing on U so we can compute

$$f = e_o = F\,\underline{e}_o\,{}^t\overline{F} = F\,{}^t\overline{F}$$

$$e_o + e_3 = Fh(\underline{e}_o + \underline{e}_3)\,{}^t\overline{h}\,{}^t\overline{F} = 2F\binom{q}{p}\,(\overline{q}\ \overline{p})^t\,\overline{F}\ .$$

Thus, using our identification of S^2_∞ with $\mathbb{CP}^2$, we see that

$$\begin{aligned}[e_o+e_3] &= [F_1q + F_2p,\ F_3q + F_3p] \in \mathbb{CP}^1 \\ &= [dF_1,\ dF_3]\ .\end{aligned}$$

Moreover, we have

$$ds_f^2 = \phi \circ \overline{\phi}$$

$$d\sigma_f^2 = 4(p\,dq - q\,dp) \circ (\overline{p\,dq} - \overline{q\,dp})$$

$$\mathcal{Q} = 2(p\,dq - q\,dp) \circ \phi \ .$$

Note, in particular, that, because of the relation $p\overline{p} + q\overline{q} = 1$, we have

$$d\sigma_f^2 = \frac{4d(p/q) \circ d(\overline{p/q})}{(1 + |p/q|^2)^2}$$

Thus ,the total curvature of ds_f^2 on the open set U is just the area (counted with multiplicities) of the spherical image of the meromorphic map $p/q : U \to \mathbb{P}^1$.

Example 1 (Enneper's Cousin) : Let $M = \mathbb{C}$ and let $\lambda \neq 0$ be a complex number. We define $F : \mathbb{C} \to S\ell(2,\mathbb{C})$ by

$$F(z) = \begin{bmatrix} \cosh \lambda z - \lambda z \sinh \lambda z & \lambda \sinh \lambda z \\ \lambda^{-1} \sinh \lambda z - z \cosh \lambda z & \cosh \lambda z \end{bmatrix} .$$

We compute

$$F^{-1}dF = -\lambda^2 \begin{pmatrix} z & -1 \\ z^2 & -z \end{pmatrix} dz \ ,$$

so F is a holomorphic null immersion. In this case, we may choose ϕ globally on $\mathbb{C}$ to be $\phi = -\lambda^2(1 + z\overline{z})dz$ and subsequently compute that for $f = F\,{}^t\overline{F}$,

$$ds_f^2 = (\lambda\overline{\lambda})^2 \ (1 + z\overline{z})^2 dz \circ d\overline{z}$$

$$d\sigma_f^2 = 4 \ dz \circ d\overline{z}/(1 + z\overline{z})^2$$

$$\mathcal{Q} = 2\lambda^2 (dz)^2$$

$$[e_o + e_3](z) = [\lambda \cosh \lambda z, \sinh \lambda z] \in \mathbb{CP}^1 \ .$$

Clearly ds_f^2 is complete on $\mathbb{C}$. In fact, the alert reader will recognize this as the metric induced on $\mathbb{C}$ by immersing it minimally in $\mathbb{E}^3$ as Enneper's surface (see [3]). Note that $d\sigma_f^2$ is just the standard metric on $\mathbb{P}^1$ restricted to $\mathbb{C} = \mathbb{P}^1 - \{\infty\}$. Thus, the total curvature is -4π . It is not difficult to see that two values of λ give rise to equivalent immersions f iff they have the same modulus. Thus, we may as well take λ to be real and positive.

In contrast to the Euclidean case, note that the hyperbolic Gauss map, $[e_o + e_3] : \mathbb{C} \to S_\infty^2$ does not extend across $z = \infty$. In fact, this map omits exactly two values ($[\lambda, \pm 1]$) and covers the remainder of S^2_∞ infinitely many times. Writing c and s for $\cosh \lambda z$ and $\sinh \lambda z$, we easily compute

$$f(z) = (1 + z\bar{z})(c\bar{c} + s\bar{s}) \left[\frac{1}{(c\bar{c} + s\bar{s})} \begin{pmatrix} \lambda s \\ c \end{pmatrix} (\lambda \bar{s}, \bar{c}) + M(z) \right]$$

where the matrix $M(z)$ consists of terms of the order of $|z|^{-1}$ and smaller. If we regard S^2_∞ as the closure of H^3 as the hyperbolic ball, this formula implies that for $|z| >> 0$, $f(z)$ is close to $[\lambda \sinh \lambda z , \cosh \lambda z]$ in S^2_∞. It follows that the closure of $f(\mathbb{C})$ in $H^3 \cup S^2_\infty$ contains <u>all</u> of S^2_∞. Thus, the "asymptotic boundary" of $f(\mathbb{C})$ is the entire S^2_∞. This formula also implies that f is not an imbedding.

<u>Example 2</u> (Catenoid Cousins) : Let $M = \mathbb{C}^*$ and let μ be a real number satisfying $\mu > -\frac{1}{2}$, $\mu \neq 0$. We consider the <u>multi-valued</u> holomorphic map

$$F(z) = \frac{1}{\sqrt{2\mu+1}} \begin{bmatrix} (\mu+1)z^\mu & \mu z^{-(\mu+1)} \\ \mu z^{\mu+1} & (\mu+1)z^{-\mu} \end{bmatrix} .$$

We compute

$$F^{-1}dF = \frac{\mu(\mu+1)}{2\mu+1} \begin{bmatrix} z^{-1} & -z^{-2\mu-2} \\ z^{2\mu} & -z^{-1} \end{bmatrix} dz .$$

It follows that $\tilde{F} : \tilde{\mathbb{C}}^* \to S\ell(2,\mathbb{C})$ is a holomorphic null immersion where $\tilde{\mathbb{C}}^*$ is the universal cover of $\mathbb{C}^*$ and $\tilde{F}$ is the single-valued lift of F. Computing $f = F\,{}^t\bar{F}$, we find

$$f(z) = \frac{1}{2\mu+1} \begin{bmatrix} \mu+1 & \mu/z \\ \mu z & \mu+1 \end{bmatrix} \begin{bmatrix} (z\bar{z})^\mu & 0 \\ 0 & (z\bar{z})^{-\mu} \end{bmatrix} \begin{bmatrix} \mu+1 & \mu\bar{z} \\ \mu/\bar{z} & \mu+1 \end{bmatrix} .$$

Thus, $f : \mathbb{C}^* \to H^3$ is a well-defined immersion with mean curvature 1. We also compute

$$ds^2_f = \frac{\mu^2(\mu+1)^2(1+(z\bar{z})^{2\mu+1})^2}{(2\mu+1)^2(z\bar{z})^{2\mu+1}} \, dz \circ d\bar{z}$$

$$d\sigma^2_f = \frac{4(2\mu+1)^2(z\bar{z})^{2\mu}}{(1+(z\bar{z})^{2\mu+1})^2} \, dz \circ d\bar{z}$$

$$Q = -2\mu(\mu+1)(dz/z)^2$$

$$[e_o+e_3](z) = [1,z] \in \mathbb{CP}^1 = S^2_\infty .$$

Thus ds^2_f is complete. Its total curvature is easily seen to be $-4\pi(2\mu+1)$. The hyperbolic Gauss map is actually 1-1 and completes across the ends as a biholomorphism $\bar{M} \xrightarrow{\sim} S^2_\infty$.

It is easy to see that $f(\mathbb{C})^*$ is a surface of revolution whose axis is the geodesic joining the two points $\{[1,0],[0,1]\}$ in S^2_∞ .

Moreover, the profile curve is imbedded for $-\frac{1}{2} < \mu < 0$, but has a single self-intersection when $\mu > 0$.

Note that the value of β at the ends $z = 0,\infty$ is 2μ . Coupled with Example 1, this shows that, in contrast to the Euclidean minimal surface case, β can be any real number greater than -1 . Also, in contrast to the Euclidean minimal surface case, the total curvature can be any non-positive number, i.e. there is no "quantization" of the curvatures. Finally, this example shows that even when the hyperbolic Gauss map completes across the ends, there is, in general, no relation between the total curvature and the degree of this map.

In the simply-connected case, we have a complete classification.

THEOREM B : Suppose that M^2 is simply connected and that $f : M^2 \to H^3$ is a complete, mean curvature 1, finite total curvature immersion. Then M is conformally equivalent to $\mathbb{C}$. Any holomorphic lifting $F : M^2 \to S\ell(2,\mathbb{C})$ furnished by Theorem A satisfies

$$F^{-1}dF = \begin{pmatrix} r_1 r_2 & -r_2^2 \\ r_1^2 & -r_1 r_2 \end{pmatrix} dz$$

where z is a standard coordinate $z : M \to \mathbb{C}$ and r_1, r_2 are polynomials in z with no common zeroes. Conversely, given a pair of polynomials in z with no common zeroes, say (r_1,r_2), then there exists an $F : \mathbb{C} \to S\ell(2,\mathbb{C})$ unique up to left translation satisfying the above equation. The map $f = F\,{}^t\bar{F} : \mathbb{C} \to H^3$ is then a conformal, mean curvature 1, complete immersion whose total Gauss curvature is -4π times a non-negative integer. Two pairs $r = (r_1,r_2)$ and $s = (s_1,s_2)$ give rise to immersions congruent under rigid motions of H^3 iff there exists an $h \in SU(2)$ so that $r = sh$.

Proof : We already know that M cannot be the sphere or the Poincaré disk, so we may as well take $M = \mathbb{C}$. By Theorem A , there exists a holomorphic null lifting $F : \mathbb{C} \to S\ell(2,\mathbb{C})$ unique up to right translation by an element of SU(2). Since F is holomorphic and an immersion, the components of $F^{-1}dF$ must be holomorphic 1-forms on $\mathbb{C}$ with no common zeroes. Let us write

$$F^{-1}dF = \begin{pmatrix} F_1 & -F_3 \\ F_2 & -F_1 \end{pmatrix} dz$$

where the $F_i(z)$ are entire functions on $\mathbb{C}$ with no common zeroes and satisfying $F_1^2 = F_2F_3$. The induced metric on $f : \mathbb{C} \to H^3$ is given by

$ds_f^2 = (2|F_1|^2 + |F_2|^2 + |F_3|^2)dz\circ d\bar{z}$. If $F_2 \equiv 0$, then $F_1 \equiv 0$ and F_3 is an entire function with no zeroes. Since ds_f^2 is complete, Osserman's Lemma 9.6 implies that F_3 does not have an essential singularity at $z = \infty$. Thus F_3 is a polynomial in z with no zeroes and hence is constant. In this case, we may simply take $(r_1,r_2) = (0,(F_3(0))^{1/2})$. Now suppose $F_2 \not\equiv 0$. By our previous remarks, the total curvature of the metric ds_f^2 is the spherical area of the image (counting multiplicities) of the meromorphic map $F_2/F_1 = p/q : \mathbb{C} \to \mathbb{P}^1$. By hypothesis this is finite. By Picard's theorem, this can only happen if F_2/F_1 extends meromorphically to $z = \infty$. Thus $F_2/F_1 = r_1/r_2$ where r_1 and r_2 are polynomials in z with no common factors. Thus, we may write $F_2 = r_1G$, $F_1 = r_2G$ where G is an entire function (G is clearly meromorphic on $\mathbb{C}$ and cannot have any poles on $\mathbb{C}$ since r_1 and r_2 have no common zeroes while F_1 and F_2 have no poles on $\mathbb{C}$). It follows, since $G \not\equiv 0$, that $F_3 = r_1^2(G/r_2)$. Again, since F_3 has no poles and r_1 and r_2 have no common zeroes, it follows that $G/r_1 = H$ where H is an entire function on $\mathbb{C}$. We now have

$$F^{-1}dF = \begin{pmatrix} r_1r_2 & -r_2^2 \\ r_1^2 & -r_1r_2 \end{pmatrix} Hdz \ .$$

We can now compute $\mathcal{Q} = 2(r_1dr_2 - r_2dr_1)\circ Hdz$. By Proposition 5 , $\mathcal{Q}$ is meromorphic at $z = \infty$, so H extends meromorphically to $z = \infty$. However, H is entire on $\mathbb{C}$ and has no zeroes since $F^{-1}dF$ has no zeroes. Thus, H is constant and by scaling (r_1,r_2) we may as well assume $H \equiv 1$. This proves the first part of the theorem. The converse is as follows : Given a pair of relatively prime polynomials (r_1,r_2), we consider the linear differential equation

$$\frac{dF}{dz} = F(z) \begin{pmatrix} r_1r_2 & -r_2^2 \\ r_1^2 & -r_1r_2 \end{pmatrix}$$

with initial condition $F(0) = I_2$ (say). By the standard theory, this has a unique solution (globally defined) $F : \mathbb{C} \to S\ell(2,\mathbb{C})$. The corresponding metric on $f = F^t\bar{F}$ is given by

$$ds_f^2 = (|r_1|^2 + |r_2|^2)^2 dz\circ d\bar{z} \ .$$

Since $|r_1|^2 + |r_2|^2 > C_o > 0$ for some $C_o \in \mathbb{R}^+$, we see that ds_f^2 is complete. The total curvature is -4π times the degree of the rational mapping $r_1/r_2 : \mathbb{P}^1 \to \mathbb{P}^1$ and is therefore $4\pi \max(d_1,d_2)$ where d_i is the degree of r_i . The uniqueness statement is now elementary. ∎

Remark : Theorem B also follows from Lawson's correspondence between minimal surfaces in $\mathbb{E}^3$ and surfaces of mean curvature 1 in H^3 plus the classification of the finite total curvature simply connected minimal surfaces in $\mathbb{E}^3$. A big difference, first proved by Gary Kerbaugh (private communication), is that although the corresponding minimal surface in $\mathbb{E}^3$ is algebraic, the mean curvature 1 surface in H^3 is never algebraic unless (r_1,r_2) consists of a pair of constants (in this case, the resulting surface is a horocycle). Kerbaugh proves even more : if the resulting surface $f : \mathbb{C} \to H^3$ constructed above has only one point in its asymptotic boundary, then (r_1,r_2) is a pair of constants. This involves a study of the above differential equation for $F(z)$.

On the other hand, using methods of algebraic geometry, it is possible to construct many holomorphic maps $F : M \to S\ell(2,\mathbb{C})$ which are null immersions where M is a compact Riemann surface minus a finite number of points and the components of F extend meromorphically to $\overline{M}$. It follows, in this case, that $F^{-1}dF$ is an $S\ell(2,\mathbb{C})$-valued meromorphic 1-form on $\overline{M}$ and hence by our formulae, the total curvature of $f = F\,{}^t\overline{F}$ will be an integral multiple of 4π and the hyperbolic Gauss map will complete holomorphically across the poles of F. Thus, the space of such surfaces is quite large.

We now turn to a condition relating the hyperbolic Gauss map and the holomorphic form Q. Let $\Delta^* = \{w \in \mathbb{C} | 0 < |w| < 1\}$ and let $f : \Delta^* \to H^3$ be a mean curvature 1, conformal immersion. We say that ds_f^2 is complete at $w = 0$ if any smooth curve $\gamma(t)$ in Δ^* which tends toward $w = 0$ as $t \to \infty$ has infinite ds_f-length. Clearly, the results of our earlier propositions apply, and we may conclude that if ds_f^2 has finite total curvature then Q completes meromorphically across $w = 0$.

PROPOSITION 6 : Let $f : \Delta^* \to H^3$ be a conformal, mean curvature 1 immersion which is complete at $w = 0$ and of finite total curvature, then the hyperbolic Gauss map $[e_0+e_3] : \Delta^* \to S^2_\infty$ completes holomorphically across $w = 0$ iff the order of Q at $w = 0$ is at least -2.

Proof : If $Q \equiv 0$ on Δ^*, then f is totally umbilic and $[e_0+e_3] : \Delta^* \to S^2_\infty$ is constant so of course it completes across $w = 0$. We set this case aside, and assume that $Q \not\equiv 0$ from now on. Then $d\sigma_f^2$ is a pseudo-metric of Gauss curvature +1 to which Proposition 4 applies. We replace Δ^* by a smaller disk $\Delta^*_\varepsilon = \{z \in \mathbb{C} | 0 < |z| < \varepsilon\} \subseteq \Delta^*$ where z is a holomorphic coordinate on Δ^*_ε so that for some $\beta > -1$,

$$d\sigma_f^2 = \frac{4(\beta+1)^2 (z\bar{z})^\beta dz \circ d\bar{z}}{(1+(z\bar{z})^{\beta+1})^2} .$$

Let us write $Q = h(z)(dz)^2$ where $h(z)$ is a non-zero holomorphic function on Δ^*_ε and

$h(z)$ is meromorphic at $z = 0$. Let $\tilde{\Delta}^*_\varepsilon = \tilde{\Delta}^*_\varepsilon = \{y \in \mathbb{C} \mid \mathrm{Re}(y) < \log \varepsilon\}$ and we denote the covering map by $y \mapsto e^y = z$. Note that $z^\beta = e^{\beta y}$ is well-defined on $\tilde{\Delta}^*_\varepsilon$. Tracing through the proof of Theorem A, we find that there exists a holomorphic map $F : \tilde{\Delta}^*_\varepsilon \mapsto S\ell(2,\mathbb{C})$ with $f(e^y) = F(y)\,{}^t\overline{F}(y)$ for $y \in \tilde{\Delta}^*_\varepsilon$ satisfying

$$F^{-1}dF = \frac{-zh(z)}{2(\beta+1)} \begin{bmatrix} 1 & -z^{-(\beta+1)} \\ z^{\beta+1} & -1 \end{bmatrix} dz .$$

Since $F(y)\,{}^t\overline{F}(y) = F(y+2\pi i)\,{}^t\overline{F}(y+2\pi i)$ for all $y \in \tilde{\Delta}^*_\varepsilon$, it follows that the holomorphic map $y \mapsto F^{-1}(y)F(y+2\pi i) \in S\ell(2,\mathbb{C})$ actually has values in the totally real submanifold $SU(2) \subseteq S\ell(2,\mathbb{C})$, and hence is constant. Let us write $F(y+2\pi i) = F(y)H$ where $H \in SU(2)$. Since this implies that $F^{-1}(y+2\pi i)dF(y+2\pi i) = H^{-1}F^{-1}(y)dF(y)H$, we conclude

$$H \begin{bmatrix} 1 & -e^{-(\beta+1)(y+2\pi i)} \\ e^{(\beta+1)(y+2\pi i)} & -1 \end{bmatrix} = \begin{bmatrix} 1 & -e^{-(\beta+1)y} \\ e^{(\beta+1)y} & -1 \end{bmatrix} H$$

for all $y \in \tilde{\Delta}^*_\varepsilon$. It immediately follows that

$$H = \sigma \begin{bmatrix} e^{\pi i(\beta+1)} & 0 \\ 0 & e^{-\pi i(\beta+1)} \end{bmatrix}$$

where $\sigma = \pm 1$. We now set

$$G(y) = F(y) \begin{bmatrix} e^{-(\beta+1)y/2} & 0 \\ 0 & e^{(\beta+1)y/2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and we see that $G(y+2\pi i) = \sigma G(y)$ for $y \in \tilde{\Delta}^*_\varepsilon$. It follows that if we write

$$G = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix},$$

then the ratios of the components u_i, v_i are well-defined on Δ^*_ε. In particular, we note that

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = G(y) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = F(y) \begin{pmatrix} e^{-(\beta+1)y/2} \\ e^{(\beta+1)y/2} \end{pmatrix} = e^{-(\beta+1)y/2} F(y) \begin{pmatrix} 1 \\ e^{(\beta+1)y} \end{pmatrix}.$$

By our previous formulae, $[e_0+e_3](e^y) = [u_1(y),u_2(y)]$. Thus, our problem reduces to showing that $u_2(y)/u_1(y)$ is a meromorphic function of z at $z = 0$ iff $h(z)$ has a pole of order no worse than 2 at $z = 0$. We regard u_i, v_i now as (possibly double-valued) functions of z and we compute that

$$\frac{d}{dz}\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} = \frac{1}{z}\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}\begin{bmatrix} -(\beta+1)/2 & z^2h(z)/2(\beta+1) \\ (\beta+1) & (\beta+1)/2 \end{bmatrix}.$$

Because $\det G \equiv 1$, it easily follows that u_1 and u_2 are linearly independent solutions of the equation (on Δ^*_ε)

$$z^2u'' + zu' - ((\beta+1)^2/4 + z^2h(z)/2)u = 0 .$$

Now $r = u_2/u_1$ is meromorphic on Δ^*_ε and is non-constant. Suppose that r extends meromorphically to $z = 0$. Then, in particular, r''/r' is meromorphic at $z = 0$ and has a simple pole at $z = 0$. It follows that the Schwarzian of r, $S(r) = (r''/r')' - \frac{1}{2}(r''/r')^2$ has a double pole at $z = 0$. On the other hand, we compute from $r = u_2/u_1$ that $S(r) = (1 - (\beta+1)^2)/2z^2 - h(z)$. Thus, $h(z)$ cannot have worse than a double pole at $z = 0$.

Now suppose that $h(z)z^2$ is holomorphic at $z = 0$. Then the above equation for u_1 and u_2 has a regular singular point at $z = 0$. If $\sigma = +1$, so that u_1 and u_2 are single valued on Δ^*, the theory of regular singular points implies that u_1 and u_2 are meromorphic at $z = 0$, and hence $r = u_2/u_1$ is meromorphic. If $\sigma = -1$, then $z^{1/2}u_1$ and $z^{1/2}u_2$ are known to be meromorphic at $z = 0$, so again $r = u_2/u_1$ is meromorphic at $z = 0$. ∎

Remark : It is not difficult to see that, if $h(z) = h_{-2}z^{-2} + h_{-1}z^{-1} + \dots$, then we must have $(\beta+1)^2 + 2h_{-2} = (n+1)^2$ where n is the branching order of the Gauss map at $z = 0$. In particular, note that h_{-2} must be real.

COROLLARY : *If $F : \mathbb{C} \to S\ell(2,\mathbb{C})$ is a holomorphic null immersion for which $f = F^t\overline{F}$ is complete and of finite total curvature, and for which the Gauss map extends holomorphically across $z = 0$, then $f(\mathbb{C}) \subseteq H^3$ is a horosphere.*

Proof : If f were not totally umbilic, then $\mathcal{Q} \not\equiv 0$ and is a meromorphic quadratic form on $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ with a pole of some order at $z = \infty$ and holomorphic elsewhere. However, the total degree of the divisor of $\mathcal{Q}$ is -4 by Riemann's relation, so the Gauss map cannot extend across $z = \infty$ by Proposition 6. ∎

Remark : This corollary is due to Gary Kerbaugh (private communication).

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