Astérisque

F. ALMGREN

Basic techniques of geometric measure theory

Astérisque, tome 154-155 (1987), p. 267-306

<http://www.numdam.org/item?id=AST_1987__154-155__267_0>

EXPOSÉ n° XIV

BASIC TECHNIQUES OF GEOMETRIC MEASURE THEORY

F. ALMGREN

§1. EXAMPLES, CURRENTS AND FLAT CHAINS AND BASIC PROPERTIES, DEFORMATION THEOREMS AND THEIR CONSEQUENCES.

1.1 Examples.

(1) Suppose one wishes to find a two-dimensional surface of least area among all surfaces having a unit circle in $\mathbb{R}^3$ as boundary. It seems universally agreed that the unit disk having that circle as boundary is the unique solution surface. If we did not know the answer beforehand (as will be the case in general), then an appropriate way to find the surface of least area would be to choose a sequence S_1 , S_2 , S_3 ,... of surfaces having successively smaller areas approaching the least possible value of such areas and then "take the limit" of the sequence of surfaces in hopes that the limit would be the desired minimal surface. The "disk with spines" illustrated in Figure 1 is a good approximation in area to the unit disk if the spines are all of very small diameter and hence of very small area. If the S_i should be chosen to have a sufficiently large number of uniformly distribued slender spines, then there will be a great many points $p \in \mathbb{R}^3$ (possibly all of $\mathbb{R}^3$) for which there exist $p_i \in S_i$ for each i such that $\lim_i p_i = p$. If the surfaces S_i are regarded as measure theoretic surfaces (currents, flat chains mod ν , varifolds) then the limit does not "see" these extraneous points and is, as desired, the unit disk.

(2) Figure 2 illustrates a simple closed unknotted curve C of finite length for which any oriented least area surface spanning C must have infinite topological type. The oriented least area surface (integral current) is a regular minimal submanifold with boundary except at the one limit point.

(3) Figure 3 illustrates two special soap-film-like mathematical minimal surfaces S and T (which are, also realizable as physical soap films). As sets ∂S is a retract of S and ∂T is a deformation retract of T . Boundaries for such surfaces

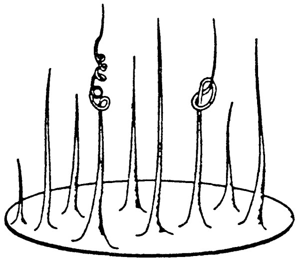

Figure 1. A disk with spines

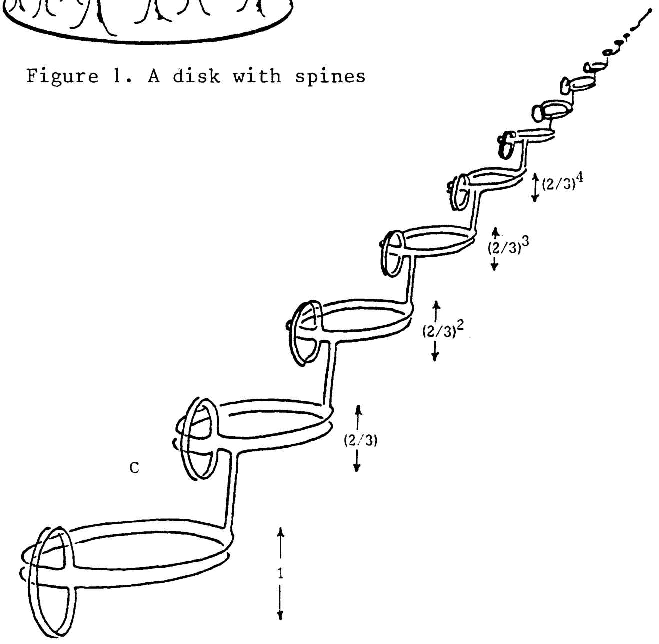

Figure 2. C is a simple closed unknotted curve of finite length. The distinct values of areas of regular minimal surfaces S with $\partial S = C$ contains a closed interval.

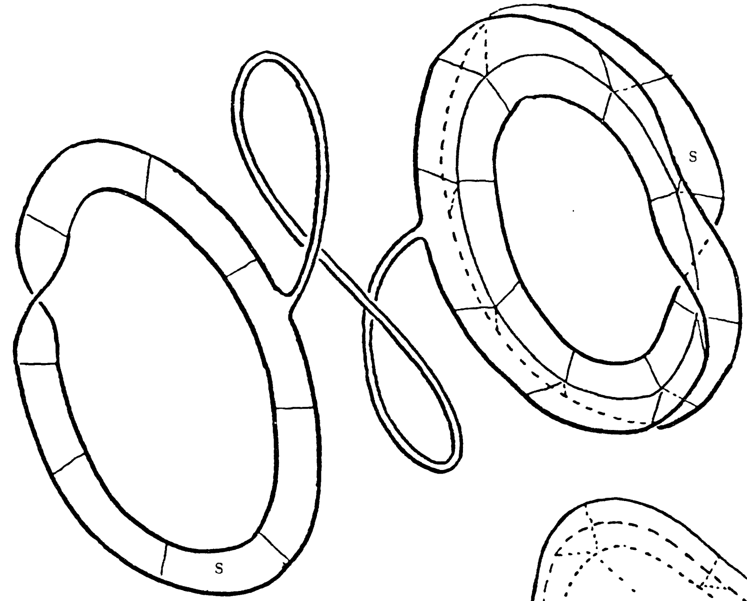

Figure 3A. ∂S is a retract of S (J.F. Adams).

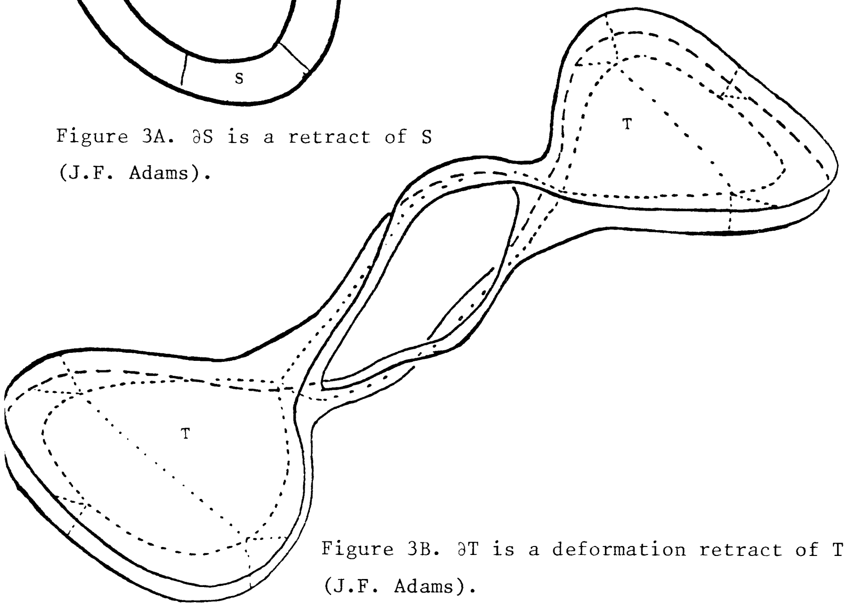

Figure 3B. ∂T is a deformation retract of T (J.F. Adams).

(integral varifolds) are defined as the support of the singular part of the associated first variation distributions.

(4) Holomorphic varieties in $\mathbb{C}^n$ are examples of oriented absolutely area minimizing (locally integral) currents. In particular, singularities in oriented area minimizing surfaces can be of real codimension two in the surfaces.

(5) Nontrivial cobordism is an obstruction to both regularity and real analytic geometric structure in area minimizing surfaces. If, for example, $\mathbb{C}P^2$ is smoothly embedded in $S^n \subset \mathbb{R}^{n+1}$, then the 5-dimensional least area integral current T having boundary such $\mathbb{C}P^2$ must be singular (since $\mathbb{C}P^2$ doesn't bound any manifold). Furthermore, sptT cannot even be a real analytic variety near its singularities ; otherwise, resolution of such singularities would give a manifold with $\mathbb{C}P^2$ boundary.

1.2 Roles of geometric measure theory. Modern geometric measure theory is useful in a number of different ways. It, for example, provides quite general techniques for studying measure and integration on singular k-dimensional surfaces in n-dimensional spaces. Additionally, it provides a framework in which mathematical questions involving complicated or unknown geometric configurations can be naturally formulated and studied.

1.3 Some terminology and assumptions in these notes.
(1) $B^n(p,r)$ [$U^n(p,r)$] denotes the closed [open] ball in $\mathbb{R}^n$ with center p and radius r . The integer $\beta(n)$ is the Besicovitch-Federer number in dimension n related to multiplicity of covering by closed balls in $\mathbb{R}^n$ in the Besicovitch-Federer covering theorems.

(2) L^n denotes n-dimensional Lebesgue measure in $\mathbb{R}^n$ and

$$\alpha(n) = L^n(B^n(0,1)).$$

(3) H^k denotes k-dimensional Hausdorff measure in $\mathbb{R}^n (0 \leq k \leq n)$. It is a regular Borel measure defined on all subsets of $\mathbb{R}^n$ with $H^n = L^n$. For reasonable k-dimensional subsets of $\mathbb{R}^n$, H^k gives the "right" value for k-dimensional subsets of R^n , H^k gives the "right" value for k-dimensional area ; it, of course, does assign a k-dimensional area to unreasonable sets as well.

(4) Two indispensable formulas in geometric measure theory are the Hausdorff area formula (a generalized unoriented change of variables formula) and the coarea

formula (a curvilinear form of Fubini's theorem). We here also call attention to the striking structure theorem for sets of finite Hausdorff measure which until recently was essential in rectifiable current compactness proofs (it is not used in the proof sketched in these lecture notes).

(5) $\mu(r), \tau(y) : \mathbb{R}^n \to \mathbb{R}^n$ for $r \in \mathbb{R}$, $y \in \mathbb{R}^n$ are specified by requiring

$$\mu(r)(x) = rx \text{ (homothety)}$$

and

$$\tau(y)(x) = x + y \text{ (translation)}$$

for $x \in \mathbb{R}^n$.

(6) As special terminology convenient for our purposes we let $\{\mathbb{R}^m \times \{0\}\}_\natural$: $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ and $\{\{0\} \times \mathbb{R}^n\}_\natural : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ denote the obvious orthogonal projections. Also we define $f \bowtie g : A \to B \times C$ by setting

$$(f \bowtie g)(a) = (f(a), g(a))$$

whenever $f : A \to B$, $g : A \to C$.

1.4 Measure theoretic surfaces. Mathematical surfaces have to be something ! In geometric measure theory surfaces are usually measures (or perhaps "k-vector valued measures").One of the biggest hurdles for new workers in the theory seems to be that of learning to think of surfaces as really being measures.

The basic ingredients of measure theoretic surfaces are :
S -- a k-dimensional rectifiable set in $\mathbb{R}^n$ (more carefully defined below) ;
$\theta : S \to \mathbb{R}^+$ -- an almost everywhere defined positive density function ;
$\xi : S \to$ (unit simple k-vectors in $\mathbb{R}^n$ "tangent" to S) -- an orientation function.

Out of such ingredients one assembles

$\mathcal{H}^k \llcorner S$ -- surface measure on S , i.e. for $A \subset \mathbb{R}^n$, $(\mathcal{H}^k \llcorner S)(A) = \mathcal{H}^k(S \cap A)$;
$t(S,\theta,\xi)$-- an oriented surface with density.

IMPORTANT. Any two surfaces which are $\mathcal{H}^k$ - almost equal are identified as the same surface.

An oriented surface with density is typically identified as a current, e.g. $t(S,\theta,\xi)$: (differential k - forms ϕ on $\mathbb{R}^n$ having compact support) $\to \mathbb{R}$

$$\phi \to \int_{x \in S} \langle \xi(x), \phi(x) \rangle \; \theta(x) \, d\mathcal{H}^k x \; .$$

Associated with such a current $T = t(S,\theta,\xi)$ are the following :

(i) the variation measure $\|T\|$ over $\mathbb{R}^n$ which assigns to $U \subset \mathbb{R}^n$ the number

$$\|T\|(U) = \int^*_{S\cap U} \theta \, dH^k .$$

(ii) the set of T ,
$set(T)$ = the set of points in $\mathbb{R}^n$ at which the k - dimensional density of $\|T\|$ at x ,

$$\Theta^k(\|T\|, x) = \lim_{r\downarrow 0} \|T\|B^n(x,r)/\alpha(k)r^k$$

is positive (which is H^k-almost equal to S).

(iii) the size of T ,

$$\mathbb{S}(T) = H^k(set(T)).$$

(iv) the mass of T ,

$$\mathbb{M}(T) = \|T\|(\mathbb{R}^n) .$$

(v) the restriction of T to W ,

$$T \llcorner W = t(S\cap W, \theta|S\cap W, \xi|S\cap W)$$

associated with any H^k-measurable subset W of $\mathbb{R}^n$.

The collection of such currents $t(S,\theta,\xi)$ forms a real vectorspace : the scalar multiple $r.t(S,\theta,\xi)$ equals $t(S,|r|\theta,sign(r)\xi)$ while the current sum $t(S,\theta,\xi) + t(S',\theta',\xi')$ equals $t(S'',\theta'',\xi'')$ for appropriate S'',θ'',ξ'' (due to cancellations of $\theta\xi$ and $\theta'\xi''$, $(S \cup S') \sim S''$ can have positive H^k-measure

1.5 Examples and spaces of measure theoretic surfaces. General measure theoretic surfaces are constructed from more elementary ones. Examples and general spaces include the following.

(1) $[\![p]\!] = t(\{p\},1,+1)$ is the 0-dimensional point mass current.

(2) $[\![p_0,p_1,\ldots,p_k]\!] = t(\Delta,1,\ (p_1-p_0)\wedge \ldots\wedge (p_k-p_{k-1})/|(p_1-p_0)\wedge \ldots \wedge(p_k-p_{k-1})|)$

is the oriented k -dimensional simplex with vertices $p_0,\ldots,p_k$; here $\Delta = spt[\![p_0,\ldots,p_k]\!]$ is the convex hull of $p_0,\ldots,p_k$ (spt denotes support). The boundary of $[\![p_0,\ldots,p_k]\!]$ is the(k-1) dimensional current

$$\partial[\![p_0,\ldots,p_k]\!] = \Sigma_j(-1)^j \, [\![p_0,\ldots,p_{j-1},p_{j+1},\ldots,p_k]\!] .$$

According to Stokes's theorem,

$$< \partial [\![p_0,\dots,p_k]\!] \, , \, \psi > \; = \; < [\![p_0,\dots,p_k]\!] \, , \, d\psi >$$

for each smooth differential (k-1) form ψ on $\mathbb{R}^n$. Also, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism then under the induced mapping $f_\#$ of currents,

$$f_\# [\![p_0,\dots,p_k]\!] = t(S,1,\xi)$$

where $S = f(\Delta)$ and ξ is the obvious orientation function. A short calculation shows that

$$f_\# \circ \partial = \partial \circ f_\# \quad .$$

(3) $\mathbb{P}_k(\mathbb{R}^n)$ denotes the k dimensional polyhedral chains in $\mathbb{R}^n$; it is the real vectorspace (of currents) generated by the oriented k-dimensional simplexes. In particular, for each $P \in \mathbb{P}_k(\mathbb{R}^n)$ one can write

$$P = \sum_i r(i) [\![p_0(i),\dots,p_k(i)]\!] \quad (\text{finite sum } r(j) \in \mathbb{R})$$

so that the associated simplexes $\{\Delta_i\}_i$ intersect pairwise in dimensions at most k-1.

One notes

$$\mathbb{S}(P) = \sum_i \mathcal{H}^k(\Delta_i),$$
$$\mathbb{M}(P) = \sum_i |r(i)| \mathcal{H}^k(\Delta_i)$$

and sets

$$\partial P = \sum_i r(i) \; \partial [\![p_0,\dots,p_k]\!] \qquad (\text{the } \underline{\text{boundary}} \text{ of } P)$$

and

$$f_\# P = \sum_i r(i) f_\# [\![p_0,\dots,p_k]\!]$$

for each diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$. If the $\{r(i)\}_i$ are restricted to be integers one obtains the abelian group $\mathbb{IP}_k(\mathbb{R}^n)$ of k-dimensional integral polyhedral chains. For $\nu \in \{2,3,4,\dots\}$ the quotient

$$\mathbb{IP}_k(\mathbb{R}^n) / \nu \, \mathbb{IP}_k(\mathbb{R}^n) = \mathbb{IP}_k^\nu(\mathbb{R}^n)$$

gives the abelian group of k-dimensional polyhedral chains mod ν ; a polyhedral chain mod ν written

$$\sum_i (s(i) \bmod \nu) [\![p_0,\dots,p_k]\!]$$

has size

$$\Sigma\{\mathcal{H}^k(\Delta_i) : z(i) \notin \nu\mathbb{Z}\}$$

and mass

$$\sum_i |z(i) \bmod \nu| \mathcal{H}^k(\Delta_i)$$

with

$$|z(i) \bmod \nu| = \inf\{|z| : z \in \mathbb{Z} \text{ with } z - z(i) \in \nu\mathbb{Z}\}.$$

(4) $S_k(\mathbb{R}^n)$ denotes the k-dimensional size bounded rectifiable currents in $\mathbb{R}^n$ defined by calling $t(S,\theta,\xi)$ a size bounded rectifiable current if and only if for each $\varepsilon > 0$ there is a polyhedral chain P and a (continuously differentiable) diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$(\mathbf{M} + \mathbf{S})(t(S,\theta,\xi) - f_\# P) < \varepsilon .$$

The sets S which appear in such expressions are called $(\mathcal{H}^k, k)$ rectifiable and $\mathcal{H}^k$-measurable subsets of $\mathbb{R}^n$.

(5) $I_k(\mathbb{R}^n)$ denotes the k-dimensional integer multiplicity rectifiable currents consisting of those size bounded rectifiable currents $t(S,\theta,\xi)$ for which θ is almost everywhere positive integer valued.

(6) $\mathbf{S}_k(\mathbb{R}^n)$, $\mathbf{I}_k(\mathbb{R}^n)$ denote those members of $S_k(\mathbb{R}^n)$, $I_k(\mathbb{R}^n)$ whose boundaries belong to $\mathbf{S}_{k-1}(\mathbb{R}^n)$, $I_{k-1}(\mathbb{R}^n)$ respectively. Such currents are called size bounded real currents and integral currents, respectively.

CAUTION. This terminology does not agree with that of Federer's treatise.

By definition we say that $T \in S_k(\mathbb{R}^n)$ has boundary $W \in S_{k-1}(\mathbb{R}^n)$ if and only if for each $\varepsilon > 0$ there is a k-dimensional polyhedral chain P and a continuously differentiable diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$(\mathbf{M} + \mathbf{S})(T - f_\# P) < \varepsilon \text{ and } (\mathbf{M} + \mathbf{S})(W - f_\# \partial P) < \varepsilon .$$

A corresponding definition holds for the integral currents. It follows that $\partial \circ \partial = 0$.

(8) We set

$$\mathbf{S}_{k,0}(\mathbb{R}^n) = \mathbf{S}_k(\mathbb{R}^n) \cap \{T : \operatorname{spt} T \text{ is compact}\} .$$

Also, for each $B \subset A \subset \mathbb{R}^n$ we set

$$\mathbf{S}_{k,0}(A,B) = \mathbf{S}_{k,0}(\mathbb{R}^n) \cap \{T : \operatorname{spt} T \subset A, \operatorname{spt} \partial T \in B\} .$$

Similar definitions apply for integral currents.

(8) Sometimes it is useful to consider more general "locally size bounded rectifiable" currents $T \in S_{k,loc}(\mathbb{R}^n)$ which are only locally of finite mass and size. Related terminology has the obvious meanings. An especially useful example is the current

$$\mathbb{E}^n = t(\mathbb{R}^n, 1, e_1 \wedge \ldots \wedge e_n)$$

which is a member of $\mathbb{I}_{n,loc}(\mathbb{R}^n)$.

1.6 Remark . Certainly, to date, rectifiable currents with integer coefficients have been of more interest and importance than those with general real number coefficients. In these lectures and notes we will explicity discuss primarily the real currents, however. Such currents have generally not been discussed elsewhere. Additionally, our constructions and theorems about real currents apply virtually whithout change to integral currents and to flat chains mod ν ; the main difference is that for these later two types of surfaces it is frequently possible to neglect the size of currents since it is always dominated by mass.

1.7 Basic properties and terminology of real chains.

(1) The mass function $\mathbb{M}$ is a norm which the size function $\mathbb{S}$ is not. Sometimes, for convenience, we abbreviate $\mathbb{MS} = \mathbb{M} + \mathbb{S}$ and $\mathbb{NS} = \mathbb{MS} + \mathbb{MS} \circ \partial$.
The associated functions

$$\mathbb{M}, \mathbb{S}, \mathbb{MS} : S_k(\mathbb{R}^n) \times S_k(\mathbb{R}^n) \to \mathbb{R}^+, \ \mathbb{NS} : \$_k(\mathbb{R}^n) \times \$_k(\mathbb{R}^n) \to \mathbb{R}^+, \mathbb{M}(T,T') = \mathbb{M}(T-T'), \text{ etc.}$$

are metrics. $S_k(\mathbb{R}^n)$ $[\$_k(\mathbb{R}^n)]$ is complete with respect to $\mathbb{MS}$ convergence [with respect to $\mathbb{NS}$-convergence] (obvious from the definitions).

(2) Two other basic norms are

$$F : \$_k(\mathbb{R}^n) \to \mathbb{R}^+ \qquad \text{(Whitney's flat norm)}$$

$\mathbb{F}(T) = \inf\{\mathbb{M}(X) + \mathbb{M}(Y) : X \in \$_k(\mathbb{R}^n) \text{ and } Y \in \$_{k+1}(\mathbb{R}^n) \text{ with } T = X + \partial Y\}$ for $T \in \$_k(\mathbb{R}^n)$, and

$$\mathbb{G} : \$_k(\mathbb{R}^n) \cap \{T : \partial T = 0\} \to \mathbb{R}^+$$

$$\mathbb{G}(S) = \inf\{\mathbb{M}(Y) : Y \in \$_{k+1}(\mathbb{R}^n) \text{ with } \partial Y = S\}$$

for each $S \in S_k(\mathbb{R}^n)$ with $\partial S = 0$; I do not know if either infimum is attained. Other indispensable functions are

$$\mathbb{GS} : \mathbb{S}_k(\mathbb{R}^n) \cap \{T : \partial T = 0\} \to \mathbb{R}^+ \qquad \text{(not a norm)},$$

$$\mathbb{GS}(T) = \inf\{\mathbb{MS}(Y) : Y \in \mathbb{S}_{k+1}(\mathbb{R}^n) \text{ with } \partial Y = T\}$$

for $T \in \mathbb{S}_k(\mathbb{R}^n)$ with $\partial T = 0$, and the associated metric

$$\mathbb{GS} : \mathbb{S}_k(\mathbb{R}^n) \times \mathbb{S}_k(\mathbb{R}^n) \cap \{(T,T') : \partial(T - T') = 0\} \to \mathbb{R}^+ ,$$

$$\mathbb{GS}(T,T') = \mathbb{GS}(T - T') ;$$

see the compactness Theorem 3.1 below.

CAUTION. For $0 \neq T \in \mathbb{S}_k(\mathbb{R}^n)$ with $\partial T = 0$ the function $\mathbb{R} \to \mathbb{S}_k(\mathbb{R}^n)$ seending $t \in \mathbb{R}$ to $tT \in S_k(\mathbb{R}^n)$ is not $\mathbb{GS}$ continuous ; however, see the homotopy Theorem 1.16 below.

(3) For each $B \subset A \cap \mathbb{R}^n$

$$\bigoplus_{k=0}^{n} \mathbb{S}_{k,0}(A,B)$$

is a chain complex with boundary operator ∂ (e.g. $\partial\circ\partial = 0$) and associated homology groups

$$H_*(A,B:\mathbb{R}) = \bigoplus_{k=0}^{n} H_k(A,B;\mathbb{F}) ;$$

see the homology Corollary 1.15 below.

(4) Each Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^m$ induces a natural chain mapping

$$f_\# : \oplus_k S_{k,0}(\mathbb{R}^n) \to \oplus_k S_{k,0}(\mathbb{R}^m)$$

of degree 0 (e.g. $f_\#\circ\partial = \partial\circ f_\#$). The construction of such $f_\#$ is largely a direct application of the Hausdorff area formula mentioned above ; commutation with ∂ seems easiest proved with the factorization

$$\{\{0\} \times \mathbb{R}^m\}_\natural \circ (1_{\mathbb{R}^n} \bowtie f) : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m .$$

As a direct consequence of the formula definition one estimates for $T \in \mathbb{S}_k(\mathbb{R}^n)$ that

$$\mathbb{M}(f_\# T) \leq \mathrm{Lip}(f)^k\, \mathbb{M}(T), \quad \mathbb{S}(f_\# T) \leq \mathrm{Lip}(f)^k\, \mathbb{S}(T),$$

and $f_\# T = g_\# T$ whenever $g : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz with $f|\mathrm{set}(T) = g|\mathrm{set}(T)$. More detailed estimates on $\mathbb{M}(f_\# V)$ and $\mathbb{S}(f_\# V)$ follow from the area formula.

(5) For $T = t(S,\theta,\xi) \in \mathcal{R}_k(\mathbb{R}^n)$ and Lipschitz $f : \mathbb{R}^n \to \mathbb{R}^p (p < n)$ one uses the coarea formula to construct for L^p almost every $y \in \mathbb{R}^p$ the slice of T by f at y,

$$< T,f,y > = t(S \cap f^{-1}\{y\},\theta',\xi') \in \mathcal{R}_{k-p}(\mathbb{R}^n) ;$$

here $\theta' = \theta | f^{-1}\{y\}$ and ξ' is the naturally defined orientation function. In particular, for L^p -almost every $y \in \mathbb{R}^p$,

$$\partial < T,f,y > = (-1)^p < \partial T,f,y > .$$

Detailed estimates on $\mathbb{M}(< T,f,y >)$ and $\mathbb{S}(< T,f,y >)$ follow from the coarea formula.

(6) Associated with each $T = t(S,\theta,\xi) \in \mathcal{R}_k(\mathbb{R}^n)$ and each $T' = t(S',\theta',\xi') \in \mathcal{R}_p(\mathbb{R}^m)$ is the Cartesian product

$$T \times T' = t(\{(x,y) : \Theta^{k+p}(\|T\| \times \|T'\|, (x,y)) > 0\},(\theta.\theta'),(\xi\times\xi')) \in \mathcal{R}_{k+p}(\mathbb{R}^{n+m})$$

where $(\theta . \theta')(x,y) = \theta(x).\theta'(y)$ and $(\xi \times \xi')(x,y) = \xi(x) \times \xi'(y)$.

CAUTION. The Cartesian product of some sets of measure zero can have infinite measure in the obvious dimensions.

We check

$$\partial(T \times T') = (\partial T) \times T' + (-1)^k T \times \partial T' .$$

(7) Whenever $f , g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : [0,1] \times \mathbb{R}^n \to \mathbb{R}^m$ are Lipschitz with $f = h(0,.)$ and $g = h(1,.)$ one has the homotopy formula for currents

$$g_\# T - f_\# T = \partial h_\#([[0,1]] \times T) + h_\#([[0,1]] \times \partial T)$$

for each $T \in \mathcal{R}_{k,0}(\mathbb{R}^n)$. In case $m = n$ and $h(t,x) = tx$ for each t, x then

$$[[p]] ⨯\!\!\!\!⨯ T = h_\#([[0,1]] \times T)$$

denotes the oriented cone over T with vertex p. If $k > 0$ then

$$\partial([[p]] ⨯\!\!\!\!⨯ T) = T - [[p]] ⨯\!\!\!\!⨯ \partial T .$$

In particular, $H_k(\mathbb{R}^n;\mathbb{R}) = 0$ for each $k = 1,2,\ldots,n$.

(8) A basic tool in controlling the geometry of currents is the following

CONSTANCY THEOREM. Suppose P is a k-dimensional affine subspace of $\mathbb{R}^n$ with constant orientation function ξ_0, $U \subset P$ is relatively open and pathwise connected, and $T \in \mathcal{F}_k(\mathbb{R}^n)$ with $\operatorname{spt}(\partial T) \cap U = 0$. Then

$$T \llcorner U = r.t(P \cap U, 1, \xi_0)$$

for some $r \in \mathbb{R}$.

Special applications of this constancy theorem occur in association with central projections within cubes.

1.8 Central projections within cubes. Suppose $K = [0,1]^m \subset \mathbb{R}^m$, $p \in [1/4,3/4]^m \subset K$ and $0 < \varepsilon < 1/2$. We define

$$\begin{aligned} \sigma(p) &: K \sim \{p\} \to \partial K, \\ \sigma(p,\varepsilon) &: K \to K, \\ \tau(p,\varepsilon) &: [0,1] \times K \to K \end{aligned}$$

by setting

$$\sigma(p)(x) = q \in \partial K \text{ if and only if } q = p + t(x-p) \text{ for some } t \geq 1,$$

$$\sigma(p,\varepsilon)(x) = \begin{cases} \sigma(p)(x) \text{ if } |x-p| \geq \varepsilon \\ p + \varepsilon^{-1}|x-p|(\sigma(p)(x) - p) \text{ if } |x-p| \leq \varepsilon, \end{cases}$$

$$\tau(p,\varepsilon)(t,x) = (1-t)p + t\sigma(p,\varepsilon)(x) \text{ for each } t, x.$$

If $k \leq m$ and $T = t(S,\theta,\xi) \in \mathcal{F}_k(K)$, then one confirms the existence of

$$\begin{aligned} \sigma(p)_\# T &= \lim_{\varepsilon \downarrow 0} \sigma(p,\varepsilon)_\# T \in \mathcal{F}_k(K) \quad [\in \mathcal{F}_k(\partial K) \text{ if } k < m], \\ \sigma(p)_\# \partial T &= \lim_{\varepsilon \downarrow 0} \sigma(p,\varepsilon)_\# \partial T \in \mathcal{F}_{k-1}(\partial K), \\ \tau(p)_\#([\![0,1]\!] \times T) &= \lim_{\varepsilon \downarrow 0} \tau(p,\varepsilon)_\#([\![0,1]\!] \times T) \in \mathcal{F}_{k+1}(K), \\ \tau(p)_\#([\![0,1]\!] \times \partial T) &= \lim_{\varepsilon \downarrow 0} \tau(p,\varepsilon)_\#([\![0,1]\!] \times \partial T) \in \mathcal{F}_k(K) \end{aligned}$$

(with all limits in the $\mathbf{M}\mathcal{F}$ metric) with

$$\partial \sigma(p)_\# T = \sigma(p)_\# \partial T$$

and

$$\sigma(p)_\# T - T = \partial \tau(p)_\#([\![0,1]\!] \times T) + \tau(p)_\#([\![0,1]\!] \times \partial T)$$

provided

(a) $\displaystyle\int_{x \in \operatorname{set}(T) \cap \operatorname{Int}(K)} |x-p|^{-k} \, d\mathcal{H}^k x + \int_{x \in \operatorname{Int}(K)} |x-p|^{-k} \, d\|T\| x < \infty$

(if $k < m$)

or

$\Theta^k(\mathcal{H}^k \llcorner \mathrm{set}(T),p) \in \{0,1\}$ and θ is approximately continuous at p (if $k = m$), and

(b) $\int_{x \in \mathrm{set}(\partial T) \cap \mathrm{Int}(K)} |x-p|^{1-k} d\mathcal{H}^{k-1}x + \int_{x \in \mathrm{Int}(K)} |x-p|^{1-k} d\|\partial T\| x < \infty$.

For a general Radon measure μ over K , one checks

$$\int_{p \in [1/4,3/4]^m} \int_{x \in \mathrm{Int}(K)} |x-p|^{-k} d\mu x \, d\mathcal{L}^m p$$

$$= \int_{x \in \mathrm{Int}(K)} \int_{p \in [1/4,3/4]^m} |x-p|^{-k} d\mathcal{L}^m p \, d\mu x \leq \Gamma \mu(\mathrm{Int}(K))$$

in case $k < m$; here Γ is a readily estimatable constant depending only on m . For $k = m$, one has

$$\int_{p \in \mathrm{Int}(K)} \Theta^k (\mathcal{H}^k \llcorner \mathrm{set}(T),p) d\mathcal{L}^k p = \mathcal{H}^k[\mathrm{set}(T) \cap \mathrm{Int}(K)],$$

$$\int_{p \in \mathrm{set}(T) \cap \mathrm{Int}(K)} \theta(p) d\mathcal{L}^k p = \|T\| (\mathrm{Int}(K)).$$

One uses the area formula to check that the integrals (or densities) in (a) and (b) dominate $\mathbf{N}\mathbf{S}(\sigma(p)_\# T)$, $\mathbf{M}\mathbf{S}[\tau(p)_\#([\![0,1)]\!] \times T)]$, and $\mathbf{M}\mathbf{S}[\tau(p)_\#([\![0,1]\!] \times \partial T)]$. One concludes finally the existence of a constant $\Gamma_{1,9} < \infty$ for which the following holds.

1.9 Basic central projection estimates. Corresponding to each $T \in \mathbf{S}_k(K)$, there is $p \in \mathrm{Int}(K)$ such that $\sigma(p)_\# T$, $\sigma_\# \partial T$, $\tau(p)_\#([\![0,1]\!] \times T)$, $\tau(p)_\#([\![0,1]\!] \times \partial T)$ all exist with

$$\sigma(p)_\# T \in \mathbf{S}_k(\partial K) \text{ if } k < m ,$$

$$\sigma(p)_\# T \in \mathbf{S}_k(K) \text{ and } \sigma(p)_\# \partial T \in \mathbf{S}_{k-1}(\partial K) \text{ if } k = m,$$

$$\partial \sigma(p)_\# T = \sigma(p)_\# \partial T$$

(so that, for $k = m$, the Constancy Theorem is applicable to $\sigma(p)_\# T$),

$\mathbf{M}(\sigma(p)_\# T) \leq \Gamma_{1.9} \mathbf{M}(T)$, $\mathbf{S}(\sigma(p)_\# T) \leq \Gamma_{1.9} \mathbf{S}(T)$,

$\mathbf{M}(\partial\sigma(p)_\# T) \leq \mathbf{M}(\partial T)$, $\mathbf{S}(\partial\sigma(p)_\# T) \leq \Gamma_{1.9} \mathbf{S}(\partial T)$,

$\sigma(p)_\#([\![0,1]\!] \times T) \in \mathbf{S}_{k+1}(K)$ if $k < m$,

$\tau(p)_\#([\![0,1]\!] \times T) = 0$ if $k = m$,

$\tau(p)_\#([\![0,1]\!] \times \partial T) \in \mathbf{S}_k(K)$,

$\mathbf{M}(\tau(p)_\#([\![0,1]\!] \times T)) \leq \Gamma_{1.9} \mathbf{M}(T)$, $\mathbf{S}(\tau(p)_\#([\![0,1]\!] \times T) \leq \Gamma_{1.9} \mathbf{S}(T)$,

$\mathbf{M}(\tau(p)_\#([\![0,1]\!] \times \partial T)) \leq \Gamma_{1.9} \mathbf{M}(\partial T)$,

$\mathbf{S}(\tau(p)_\#([\![0,1]\!] \times \partial T)) \leq \Gamma_{1.9} \mathbf{S}(\partial T)$.

If $0 < r < 1$ and

$$L = \mu(r)K \quad ,$$

then maps of the form

$$\mu(r) \circ \sigma(p,\varepsilon) \circ \mu(1/r) \quad \text{and} \quad \mu(r) \circ \tau(p,\varepsilon) \circ (1_{\mathbb{R}} \times \mu(1/r))$$

are used to define central projections in L in the obvious way. Corresponding estimates result, e.g. for $T \in S_k(L)$,

$$\mathbf{M}(\mu(r)_{\#} \circ \sigma(p)_{\#} \circ \mu(1/r)_{\#} T) \leq \Gamma_{1.9} \mathbf{M}(T) \quad ,$$

$$\vdots$$

$$\mathbf{S}(\mu(r)_{\#} \circ \tau(p)_{\#} \circ (1_R \times \mu(1/r))_{\#}([\![0,1]\!] \times T)) \leq \Gamma_{1.9}\, r\, \mathbf{S}(T) \quad ;$$

note the factor r .

1.10 Standard cubes in $\mathbb{R}^n$ and admissible families of cubes and their associated complexes.

(1) We denote by $K(0)$ the collection of closed unit n-dimensional cubes in $\mathbb{R}^n$ associated with the integer lattice. In particular, $K \in K(0)$ if and only if there are $z_1, \ldots, z_n \in \mathbb{Z}$ with

$$K = \tau(z_1, \ldots, z_n)\, [0,1]^n \quad .$$

For $N \in \mathbb{Z}$ we set

$$K(N) = \{\mu(2^{-N})K : K \in K(0)\} \quad .$$

The members of $K(N)$ are said to be standard cubes of level N .

(2) By an admissible family of cubes, one means a subset F of $\cup_N K(N)$ for which the following three conditions hold.

(2.1) $K, L \in F$ with $K \neq L$ implies $\operatorname{Int}(K) \cap \operatorname{Int}(L) = 0$.

(2.2) $K, L \in F$ with $K \cap L \neq 0$ implies $|\operatorname{level}(K) - \operatorname{level}(L)| \leq 1$.

(2.3) $K \in F$ implies $\partial K \subset \cup\{L : K \neq L \in F\}$.

The empty family is admissible. Any nonempty admissible family is infinite. The union of the members of an admissible family is open.

Whenever F is an admissible family of cubes and $K \in F$, one sets

$$\operatorname{nbs}(K) = F \cap \{L : L \cap K \neq \emptyset\} \quad .$$

(3) Corresponding to each open subset U of $\mathbb{R}^n$ one defines

$$WF(U) \quad \underline{\text{(Whitney family)}}$$

to consist of those cubes $K \in \cup_N K(N)$ for which the following two conditions hold.

(3.1) $\operatorname{dist}(K, \mathbb{R}^n \sim U) > n\, 2^{1-\operatorname{level}(K)}$.

(3.2) $\operatorname{dist}(L, \mathbb{R}^n \sim U) \leq n\, 2^{1-\operatorname{level}(L)}$ for that unique $L \in K(\operatorname{level}(K) - 1)$ with $K \subset L$.

$WF(U)$ is an admissible family of cubes with $\cup WF(U) = U$.

(4) Associated with each admissible family F of cubes is the <u>cubical complex</u> of F

$$CX(F)$$

consisting of all cubes K for which the following two conditions hold.

(4.1) K is a cubical face (of some dimension) of some n-dimensional cube in F .

(4.2) In case $\dim(K) > 0$, then $\operatorname{level}(K) \geq \operatorname{level}(L)$ whenever L is a face of some n-dimensional cube in F with $\dim(K) = \dim(L)$ and $\operatorname{Int}(K) \cap \operatorname{Int}(L) \neq \emptyset$.

Additionally, for each $k \in \{0,1,\ldots,n\}$ we set

$$CX_k(F) = CX(F) \cap \{K : \dim(K) = k\}.$$

1.11 <u>Iterated central projections of currents associated with admissible families of cubes</u>. Suppose F is an admissible family of cubes and $T \in \mathbf{S}_k(\mathbb{R}^n)$ ($\cup F$ need not contain $\operatorname{spt}T$ and, in many important applications, will not). Suppose also that $K_1, K_2, K_3, \ldots$ is a listing of the members of F . The central projection technique described in 1.8, 1.9 is a local construction which is readily adapted to produce a sequence of currents $T = T(n-1,0), T(n-1,1), T(n-1,2), T(n-1,3), \ldots$ such that for each $k = 1, 2, 3, \ldots$

$$T(n-1,k) \llcorner (\mathbb{R}^n \sim K_k) = T(n-1,k-1) \llcorner (\mathbb{R}^n \sim K_k)$$

and $T(n-1,k) \llcorner K$ is the centrally projected image of $T(n-1,k-1) \llcorner K$ onto ∂K with center of projection in $\operatorname{Int}(K)$ chosen to provide appropriate mass and size estimates in accordance with 1.9 . The existence of $T(n-1) = \lim_{k\to\infty} T(n-1,k)$ (with limit in the $\mathbb{N}\mathbf{S}$ metric) is easily checked (the "admissibility" of F gives the necessary local finiteness conditions). Within $\cup F$ $\operatorname{spt}T(n-1)$ lies on $\cup CX_{n-1}(F)$. A similar sequential application of the central projection technique to the $(n-1)$-dimensional cubes in $CX_{n-1}(F)$ produces a current $T(n-2)$ which, within $\cup F$, lies on $\cup CX_{n-2}(F)$. Continuing in this manner one ultimately obtains a current $T(k-1)$ which, within $\cup F$, lies on $\cup CX_k(F)$ and whose boundary $\partial T(k-1)$, within $\cup F$, lies on $\cup CX_{k-1}(F)$.

The results of this procedure combined with the various estimates of 1.8, 1.9, and 1.7(8) (the Constancy Theorem) are summarized in the following theorem.

1.12 <u>General deformation theorem.</u> There is a constant $\Gamma_{1.12}$ with the following properties. Corresponding to each admissible family F of n-dimensional cubes in $\mathbb{R}^n$ and each $T \in \mathbb{S}_k(\mathbb{R}^n)$ there exist $P, W \in \mathbb{S}_k(\mathbb{R}^n)$ and $Q \in \mathbb{S}_{k+1}(\mathbb{R}^n)$ with the following properties :

(1) $P - T = \partial Q + W$;

(2) $P \llcorner (\mathbb{R}^n \sim \cup F) = T \llcorner (\mathbb{R}^n \sim \cup F)$;

(3) $\operatorname{spt}(P) \cap \cup F \subset \cup CX_k(F)$ and $\operatorname{spt}(\partial P) \cap \cup F \subset \cup CX_{k-1}(F)$;

(4) for each $K \in CX_k(F)$,

$$P \llcorner K = t(K, \gamma_0, \xi_0)$$

for some constant density θ_0 and some constant orientation ξ_0 ;

(5) for each $L \in CX_{k-1}(F)$,

$$\partial P \llcorner L = t(L, \omega_0, \eta_0)$$

for some constant density ω_0 and some constant orientation η_0 ;

(6) $Q = Q \llcorner \cup F$ and $W = W \llcorner \cup F$;

(7) whenever $K_1, K_2, K_3, \ldots \in F$ then

$$\mathbf{M}(P \llcorner \cup_i K_i) \leq \Gamma_{1.12}\, \mathbf{M}(T \llcorner \cup_i \operatorname{nbs}(K_i))$$
$$\mathbb{S}(P \llcorner \cup_i K_i) \leq \Gamma_{1.12}\, \mathbb{S}(T \llcorner \cup_i \operatorname{nbs}(K_i)),$$
$$\mathbb{S}(\partial P \llcorner \cup_i K_i) \leq \Gamma_{1.12}\, \mathbf{M}(\partial T \llcorner \cup_i \operatorname{nbs}(K_i)),$$
$$\mathbb{S}(\partial P \llcorner \cup_i K_i) \leq \Gamma_{1.12}\, \mathbb{S}(\partial T \llcorner \cup_i \operatorname{nbs}(K_i)) ;$$

one can, of course, have $\cup F = \cup_i K_i$. ;

(8) $\mathbf{M}(Q) \leq 2^{-N}\Gamma_{1.12}\mathbf{M}(T)$, $\mathbb{S}(Q) \leq 2^{-N}\Gamma_{1.12}\mathbb{S}(T)$,
$\mathbf{M}(W) \leq 2^{-N}\Gamma_{1.12}\mathbf{M}(\partial T)$, $\mathbb{S}(W) \leq 2^{-N}\Gamma_{1.12}\mathbb{S}(\partial T)$;

here $N = \inf\{\operatorname{level}(K) : K \in F\}$.

1.13 COROLLARY (Isoperimetric inequality). <u>There is a constant</u> $\Gamma_{1.13} < \infty$ <u>with the following property</u>. <u>Corresponding to each</u> $T \in S_k(\mathbb{R}^n)$ <u>with</u> $\partial T = 0$ <u>there is</u> $Q \in S_{k+1}(\mathbb{R}^n)$ <u>with</u>

(1) $\partial Q = T$,

(2) $\mathbb{S}(Q) \leqslant \Gamma_{1.13} \mathbb{S}(T)^{(k+1)/k}$,

(3) $\mathbb{M}(Q) \leqslant \Gamma_{1.13} \mathbb{M}(T) \, \mathbb{S}(T)^{(k+1)/k}$.

Proof. We let $N \in \mathbb{Z}$ so that

$$2^{-k(N+1)} \leqslant \Gamma_{1.12} \, \mathbb{S}(T) < 2^{-kN}$$

and apply 1.12 with $F = K(N)$. One notes $H^k(K) = 2^{-kN}$ for each $K \in CX_k(F)$, uses conclusions (3), (4), (8) of 1.12 to conclude $P = 0$, $W = 0$ there, and sets Q above equal to $-Q$ there and $\Gamma_{1.13} = 2\Gamma_{1.12}^{(k+1)/k}$. ∎

1.14 COROLLARY (Cubical approximation theorem). Corresponding to each $T \in \mathbb{S}_k(\mathbb{R}^n)$ and each $0 < \varepsilon < 1$ there exist (a cubical polyhedral chain) $P \in P_k(\mathbb{R}^n)$, $W \in \mathbb{S}_k(\mathbb{R}^n)$, and $Q \in \mathbb{S}_{k+1}(\mathbb{R}^n)$ such that

(1) $P - V = \partial Q + W$.

(2) $\mathbb{M}(P) \leqslant \Gamma_{1.12}\mathbb{M}(T)$, $\mathbb{S}(P) \leqslant \Gamma_{1.12}\mathbb{S}(T)$,

(3) $\mathbb{M}(\partial P) \leqslant \Gamma_{1.12}\mathbb{M}(\partial T)$, $\mathbb{S}(\partial P) \leqslant \Gamma_{1.12}\mathbb{S}(\partial T)$,

(4) $\mathbb{MS}(Q) < \varepsilon$, $\mathbb{MS}(W) < \varepsilon$.

Proof. Take $F = K(N)$ for large N in 1.12. ∎

REMARK. Note in 1.14(2) that the estimate on $\mathbb{M}(P)$ is independent of $\mathbb{M}(\partial T)$ in contrast with 4.2.9(2) of Federer's treatise.

1.15 COROLLARY (Homology groups). Suppose A and B are compact Lipschitz neighborhood retracts in $\mathbb{R}^n$ (i.e., there are open neighborhoods U of A and V of B and Lipschitz retractions $U \to A$ and $V \to B$) with $B \subset A$. Then the homology groups $H_*(A,B:\mathbb{R})$ defined in 1.7(3) are naturally isomorphic with the usual singular homology groups of the pair (A,B) with coefficients in $\mathbb{R}$.

Proof. Each singular chain is homologous to a Lipschitz chain in (A,B) and each member of $S_{k,0}(A,B)$ is homologous to a cubical chain in (U,V). ∎

1.16 THEOREM (Homotopy groups). Suppose A is a compact Lipschitz neighborhood retract in $\mathbb{R}^n$ and j , $k \in \{0,1,\dots,n\}$ with $j + k \leqslant n$. If $\mathbb{SZ}_{k,0}(A) = \mathbb{S}_{k,0}(A) \cap \{T : \partial T = 0\}$ is topologized as the inductive limit of its $\mathbb{MS}$ bounded subsets with the $\mathbb{GS}$ metric topology, then

$$\pi_j(\mathbf{SZ}_k(A);0) \cong H_{j+k}(A;\mathbb{R})\ ;$$

here π_j refers to homotopy in dimension j .

(2) If $\mathbf{SZ}_k(A)$ is topologized as the inductive limit of its $\mathbf{MS}$ bounded subsets with the $\mathbf{G}$-metric topology, then $\mathbf{SZ}_k(A)$ is homotopically trivial.

Proof. In establishing (1), one argues as in the integer coefficient case using the isoperimetric inequalities of 1.13. In (2) scalar multiplication by $t \in [0,1]$ generates a homotopy between various mappings and the 0-map. ∎

1.17 THEOREM (Isoperimetric inequality for cycles with small projections). There is a constant $\Gamma_{1.17} < \infty$ with the following property. Suppose $T \in \mathbf{S}_k(R^n)$ with $\partial T = 0, 0 < \lambda < \infty$, and

$$L^k[\pi\ (\mathrm{set}T)] \leqslant \lambda$$

for each orthogonal projection $\pi : \mathbb{R}^n \to \mathbb{R}^k$ having as kernel an $(n-k)$-dimensional coordinate subspace of $\mathbb{R}^n$. Then there is $Q \in \mathbf{S}_{k+1}(\mathbb{R}^n)$ with

(1) $\partial Q = T$,

(2) $\mathbf{M}(Q) \leqslant \Gamma_{1.17}\mathbf{M}(T)\lambda^{1/k}$,

(3) $\mathbf{S}(Q) \leqslant \Gamma_{1.17}\mathbf{S}(T)\lambda^{1/k}$.

Proof. One initially applies 1.12 or 1.14 with $F = K(M)$ for sufficiently large M to obtain a cubical polyhedral cycle P with $\mathbf{GS}(T,P)$ very small and $L^k[\pi(\mathrm{set}P)] \leqslant 2\lambda$ for each coordinate plane orthogonal projection $\pi : \mathbb{R}^n \to \mathbb{R}^k$ as above. One then chooses $N \in \mathbb{Z}$ so that 2^{kN} is considerably larger than 2λ (but still within a preassigned bounded factor of 2λ) and applies the following adaptation of the deformation construction used in 1.12. We fix $F = K(N)$ and require that the centers of projection (in 1.8, 1.9, 1.11, 1.12) be the exact centers of the cubes in question. One applies the total deformation σ to $\tau(x)_{\#}P$; here $x \in \mathbb{R}^n$, in particular, is carefully chosen with respect to the various $\pi(\mathrm{spt}P)$ projected sets so that at the stage of the deformation construction at which one projects within k-dimensional cubes K , one can guarantee that the support of the current being projected there does not meet the center of K . One concludes $\sigma_{\#}(\tau(x)_{\#}P) = 0$ and hence $\tau(x)_{\#}P = \partial Q$ with $\mathbf{MS}(Q)$ comparable to $\lambda^{1/k}\mathbf{MS}(P)$. The theorem follows readily. ∎

§2. LIPSCHITZ MULTIPLE VALUED FUNCTIONS AND APPROXIMATIONS OF CURRENTS.

2.1 Multiple valued functions. We frequently wish to regard the "graph" or the image of a function $\mathbb{R}^m \to \mathbb{P}_0(\mathbb{R}^n)$ as a multiply sheeted m-dimensional surface with possible branching, folds, or other singular behavior. It is with this geometric image in mind that we refer to $\mathbb{P}_0(\mathbb{R}^n)$-valued functions as "multiple valued functions".

Central to the usefulness of such multiple valued functions is the fact that general m-dimensional surfaces in $\mathbb{R}^{m+n}$ with complicated topological or singularity structure can indeed be represented by (or strongly approximated by) the graphs or images of such functions with $\mathbb{R}^m$ as fixed simple domain. Functional analytic techniques then become applicable in ways novel to geometric problems for obtaining estimates, generating comparison surfaces, etc.

2.2 Examples.

(A) Suppose $0 \leqslant \varepsilon < \infty$ and $0 < \theta < \pi/2$ is defined by requiring $1 + 2\varepsilon = 2(1+\varepsilon)\cos\theta$. We set

$$B = [\![(\cos\theta,\ \sin\theta)]\!] + [\![(\cos\theta,\ -\sin\theta)]\!] - 2\,[\![(-1,0)]\!] \in \mathbb{IP}_0(\mathbb{R}^2).$$

$$P = [\![(0,0),\ (\cos\theta,\ \sin\theta)]\!] + [\![(0,0),\ (\cos\theta,\ -\sin\theta)]\!] - 2\,[\![(-1,0),\ (0,0)]\!] \in \mathbb{IP}_1(\mathbb{R}^2).$$

Then $\partial P = B$, $\mathbb{S}(P) = 3$, $\mathbb{M}(P) = 4$, and

$$\mathbb{S}(P) + \varepsilon\mathbb{M}(P) = 3 + 4\varepsilon = \inf\{\mathbb{S}(Q) + \varepsilon\mathbb{M}(Q) : Q \in I_1(\mathbb{R}^2) \text{ with } \partial Q = B\}.$$

For $\varepsilon = 1$, $\mathbb{GS}(B) = \mathbb{MS}(P) = 7$.

(B) Suppose $N \in \{1,2,3,\dots\}$, $p_1, p_2, \dots, p_N$ are distinct points in $\mathbb{R}^n$, and $q_1, q_2, \dots, q_N$ are distinct points in $\mathbb{R}^n$ with

$$a = \inf_{i \neq j} |p_i - p_j| \quad \text{and} \quad b = \sup_i |p_i - q_i|.$$

We note that any path in $\mathbb{R}^n$ connecting distinct points among the collection $\{p_1,\dots,p_N,q_1,\dots,q_N\}$ other than a path connecting p_i to q_i for some i must have length (hence size) at least $a-2b$.

Suppose also $\sigma_1,\dots,\sigma_N$, $\tau_1,\dots,\tau_N \in \mathbb{R}$ with $\sum_i \sigma_i = \sum_i \tau_i$ and

$$P = \sum_i \sigma_i [\![p_i]\!], \quad Q = \sum_i \tau_i [\![q_i]\!] \in P_0(\mathbb{R}^n).$$

(B.1) If $\mathbb{GS}(P,Q) < a - 2b$ then $\sigma_i = \tau_i$ for each i and $W = \sum_i \sigma_i [\![p_i, q_i]\!]$ is the unique member of $\mathbf{S}_1(\mathbb{R}^n)$ with $\partial W = Q - P$ and $\mathbf{MS}(W) = \mathcal{GS}(P,Q)$.

(B.2) Corresponding to each $T \in \mathbb{P}_0(\mathbb{R}^n)$ with $\mathbf{S}(T) \leq N$, $\langle P-T, 1 \rangle = 0$, and $\mathcal{GS}(P,T) < a/2$ there exist $v_1,\ldots,v_N \in \mathbb{R}^n$ with $|p_i - v_i| < a/2$ for each i so that $T = \Sigma\ \sigma_i [\![v_i]\!]$.

(C) Suppose $f : \mathbb{R} \to \mathbb{P}_0(\mathbb{R}^2)$ is defined by setting

$$\begin{aligned} f(x) &= [\![(x,x^{1/2})]\!] - [\![(x,-x^{1/2})]\!] \qquad \text{if } x > 0 \\ &= 0 \quad \text{if } x \leq 0 . \end{aligned}$$

Then

$$\mathcal{GS}(f(x),f(y)) \leq 4\ |x-y|^{1/2}(1 + |x-y|)^{1/2}$$

for each x, $y \in \mathbb{R}$, e.g. f is locally $\mathcal{GS}$-Hölder continuous with exponent 1/2. We set

$$f_\# E^1 = t(\{(x,y) : x = y^2\}, 1, (2y,1)/(4y^2+1)^{1/2}) \in \mathbf{I}_{1,\mathrm{loc}}(\mathbb{R}^2).$$

For each $x \in \mathbb{R}$,

$$f(x) = \langle f_\# E^1, \{\mathbb{R}\times\{0\}\}_\natural, x \rangle$$

(the slice of $f_\# E^1$ by $\{\mathbb{R}\times\{0\}\}_\natural$ at x).

(D) We identify $\mathbb{R}^2 = \mathbb{C}$, $\mathbb{R}^4 = \mathbb{C}^2$ and define $f : C \to \mathbb{P}_0(\mathbb{C}^2)$ by setting

$$f(z) = \Sigma\{[\![(z,w)]\!] : w^2 = z^3\}$$

for each $z \in C$. Then f is locally $\mathcal{GS}$-Lipschitz and $f_\# E^2 \in \mathbf{I}_{2,\mathrm{loc}}(\mathbb{R}^4)$ is the algebraic variety $w^2 = z^3$ regarded as a locally integral current. If $\Gamma = \partial(E^2 \llcorner U^2(0,1)) \in \mathbf{I}_1(\mathbb{R}^2)$ (an oriented circle), then $f_\# \Gamma$ will be a single oriented curve in $\mathbb{R}^4$ lying in the variety $w^2 = z^3$ which curve projects to cover Γ twice.

A basic combinatorial estimate related to the $\mathcal{GS}$-topology on $\mathbb{P}_0(\mathbb{R}^n)$ is the following.

2.3 PROPOSITION. <u>Corresponding to each</u>

$$B = \sum_{j=1}^{N} s(j)\ [\![q(j)]\!] - \sum_{i=1}^{M} r(i)\ [\![p(i)]\!] \in \mathbb{P}_0(\mathbb{R}^n)$$

<u>with</u> $\sum_j s(j) = \sum_i r(i)$ <u>there exist</u>

(a) $K \in \{1,2,\dots,3M + 3N - 3\}$ (note the a priori bound here),

(b) $a(1),\dots,a(K),b(1),\dots,b(K) \in \mathbb{R}^n$,

(c) positive numbers $t(1),\dots,t(K) \leqslant r(1) + \dots + r(N)$ (note the a priori bound here),

(d) $P = \sum_k t(k)\,[[a(k), b(k)]] \in \mathbb{P}_1(\mathbb{R}^n)$

such that

(1) $\mathbb{M}(P) + \mathbb{S}(P) = \mathbb{GS}(B)$,

(2) spt(P) lies within the convex hull of spt(B) and contains no closed loops.

2.4 Extension of multiple valued mappings in [0,1] . Suppose one has P , $Q \in \mathbb{P}_0(\mathbb{R}^n)$ with $< P - Q, 1 > = 0$ and (only as a convenient simplifying assumption for the exposition here) $\operatorname{spt}P \cap \operatorname{spt}Q = \emptyset$. Suppose also that

$$W = \sum_{i=1}^{K} t(i)\,[[a(i), b(i)]] \in \mathbb{P}_1(\mathbb{R}^n)$$

with $\partial W = Q - P$ and $\mathbb{MS}(W) = \mathbb{GS}(P,Q)$. We will indicate how to construct a $\mathbb{GS}$ Lipschitz function

$$f = f(1) + \dots + f(K) : [0,1] \to \mathbb{P}_0(\mathbb{R}^n)$$

with $f(0) = P$ and $f(1) = Q$. The definition of $f(k)$ depends on which of several possibilities occurs. We set $w(k) = [a(k) + b(k)]/2$.

(i) if $a(k), b(k) \in \operatorname{spt}P$ then

$$\begin{aligned} f(k)(s) &= t(k)\,[[(1-2s)a(k) + 2sw(k)]] - t(k)\,[[(1 - 2s)b(k) + 2sw(k)]] \text{ for } 0 \leqslant s \leqslant 1/2 \\ &= 0 \quad \text{for } 1/2 \leqslant s \leqslant 1 . \end{aligned}$$

(ii) if $a(k) \in \operatorname{spt}P$, $b(k) \in \operatorname{spt}Q$, then

$$f(k)(s) = t(k)\,[[(1 - s)a(k) + sb(k)]] \quad \text{for } 0 \leqslant s \leqslant 1 .$$

(iii) if $a(k) \in \operatorname{spt}P$, $b(k) \notin \operatorname{spt}P \cup \operatorname{spt}Q$,then

$$\begin{aligned} f(k)(s) &= t(k)\,[[(1 - 2s)a(k) + 2sb(k)]] \quad \text{for } 0 \leqslant s \leqslant 1/2 \\ &= 0 \quad \text{for } 1/2 \leqslant s \leqslant 1 . \end{aligned}$$

(iv) if $a(k), b(k) \notin \operatorname{spt}P \cup \operatorname{spt}Q$, then

$$\begin{aligned} f(k)(s) &= 0 \quad \text{if } 0 \leqslant s \leqslant 1/2 \\ &= t(k)\,[[(2-2s)b(k) + (2s - 1)w(k)]] - t(k)\,[[(2 - 2s)a(k) + (2s - 1)w(k)]] \text{ if } 1/2 \leqslant s \leqslant 1 \end{aligned}$$

with corresponding definitions holding for other possibilities. One checks

$$\mathbb{S}(f(s)) \leq 2K, \ \mathbb{M}(f(s)) \leq 2\sum_k t(k) ,$$

$$\mathbb{GS}(f(s), f(t)) \leq 4|s - t| \ \mathbb{GS}(P,Q)$$

for each $s, t \in [0,1]$.

2.5 Extension of multiple valued mappings in admissible families of cubes. Suppose F is an admissible family of cubes in $\mathbb{R}^m$ and

$$f_0 : \cup CX_0(F) \to \mathbb{P}_0(\mathbb{R}^n) \cap \{P: \mathbb{S}(P) \leq N, \ \mathbb{M}(P) \leq M, \ < P,1 > = L, \ \operatorname{spt}P \subset B^n(0,R)\}$$

(L, M, N, R fixed) is given and is $\mathbb{GS}$-Lipschitz. The extension procedure in 2.4 adapts readily to give a $\mathbb{GS}$-Lipschitz function

$$f_1 : \cup CX_1(F) \to \mathbb{P}_0(\mathbb{R}^n) .$$

Corresponding to each $k \in \{2,3,\ldots,m\}$ and each $K \in CX_k(F)$, we let

$$\Gamma(K,1),\ldots,\Gamma(K,\gamma(K)) \subset \mathbb{R}^n$$

denote the pathwise connected components of

$$\operatorname{Clos}(R^n \cap \cup\{\operatorname{spt}(f(x)) : x \in L \in CX_1(F) \text{ with } L \subset \partial K\})$$

and choose $q(K,i) \in \Gamma(K,i)$ for $i = 1,\ldots,\gamma(K)$.

Assuming inductively that $f_{k-1} : \cup CX_{k-1}(F) \to \mathbb{P}_0(\mathbb{R}^n)$ has been defined we set

$$f_k((1-t)\operatorname{center}(K) + tx) = \sum_{i=1}^{\gamma(K)} \tau(-q(K,i))_{\#} \circ \mu(t) \circ \tau(q(K,i))_{\#}[f_{k-1}(x) \llcorner \Gamma(K,i)]$$

for each $K \in CX_k(F)$, each $t \in [0,1]$, and each $x \in \partial K$. One checks that $f = f_m$ is $\mathbb{GS}$ Lipschitz with $\mathbb{S}(f(x)) \leq 12N - 6$, $\mathbb{M}(f(x)) \leq \mathbb{M}(12N - 6)$, and

$$\mathbb{GS}(f(x), f(y)) \leq \Gamma_{2.5} \ |x - y| \ \operatorname{Lip}_{\mathbb{GS}}(f_0) \ (1 + M)N$$

for each $x, y \in \cup F$; here $\Gamma_{2.5}$ is a constant depending only on m and n .

2.6 THEOREM (Extension of Lipschitz multiple valued functions). Corresponding to each subset A of $\mathbb{R}^m$ and each $\mathbb{G}\mathbb{S}$-Lipschitz function

$$f : A \to \mathbb{P}_0(\mathbb{R}^n) \cap \{P : \mathbb{S}(P) \leq N, \mathbb{M}(P) \leq M, \langle P,1 \rangle = L, \operatorname{spt} P \subset B^n(0,R)\}$$

(L, M, N, R fixed) there is a $\mathbb{G}\mathbb{S}$-Lipschitz function

$$g : \mathbb{R}^m \to \mathbb{P}_0(\mathbb{R}^n)$$

such that $g|A = f$, $\mathbb{S}(g(x)) \leq 12N - 6$, $\mathbb{M}(g(x)) \leq M(12N - 6)$, $\langle g(x),1 \rangle = L$, $\operatorname{spt}(g(x)) \subset B^n(0,R)$, and

$$\mathbb{G}\mathbb{S}(g(x),g(y)) \leq \Gamma_{2.6}|x - y| \operatorname{Lip}_{\mathbb{G}\mathbb{S}}(f)\ (1 + M)N$$

for each $x, y \in \mathbb{R}^m$; here $\Gamma_{2.6}$ is a constant depending only on m and n .

Proof. Assuming A is closed we set $F = WF(\mathbb{R}^m \sim A)$ (1.10(3)) and choose g_0 : $\cup CX_0(F) \to \mathbb{P}_0(\mathbb{R}^n)$ subject to the requirement that for each $x \in \cup CX_0(F)$, $g_0(x) = f(z)$ for some $z \in A$ with $|x - z| = \operatorname{dist}(x,A)$.

2.7 REMARK. In case $N = 1$ the procedure outlined in 2.4, 2.5, 2.6 will give $\mathbb{S}(g(x)) = 1$ for each x so that 2.6 is a weak form of Kirszbraun's theorem for single-valued functions, i.e. without the best constant.

2.8 How to map rectifiable currents by Lipschitz multiple valued functions. Suppose

$$f : \mathbb{R}^m \to \mathbb{P}_0(\mathbb{R}^n) \cap \{P: \mathbb{S}(P) \leq N, \mathbb{M}(P) \leq M, \langle P,1 \rangle = L, \operatorname{spt} P \subset B^n(0,R)\}$$

(L, M, N, R fixed) is $\mathbb{G}\mathbb{S}$-Lipschitz. We will define

$$([\![1_{R^m}]\!] \bowtie f)_\#: S_k(\mathbb{R}^m) \to S_k(\mathbb{R}^{m+n})$$

and

$$f_\# = \{\{0\} \times \mathbb{R}^n\}_\# \circ ([\![1_{\mathbb{R}^m}]\!] \bowtie f)_\#: S_k(\mathbb{R}^m) \to S_k(\mathbb{R}^n) .$$

To do this we fix Borel sets $A_\nu = \mathbb{R}^m \cap \{x : \mathbb{S}(f(x)) = \nu\}$ for $\nu = 0,1,\ldots,N$ (each A_ν is relatively open in $\mathbb{R}^m \sim (A_{\nu+1} \cup \ldots \cup A_N)$). For $\nu \in \{1,\ldots,N\}$ and $p \in A_\nu$ we use statement (B.2) in example (B) in 2.2 together with Kirszbraun's theorem to conclude the existence of $\sigma_1,\ldots,\sigma_\nu \in \mathbb{R}$ (with $L = \sigma_1 + \ldots + \sigma_\nu$ $|\sigma_1| + \ldots + |\sigma_\nu| \leq M$) together with an open neighborhood U of p in $\mathbb{R}^m$ and Lipschitz functions $g_1,\ldots,g_\nu : U \to \mathbb{R}^n$ such that

$$0 < \inf\{|g_i(x) - g_j(x)| : i \neq j,\ x \in U\}$$

and

$$f(x) = \sum_i \sigma_i [\![g_i(x)]\!] \qquad \text{for each } x \in U \cap A_\nu .$$

For $T \in S_k(\mathbb{R}^m)$ we set $T_\nu = T \llcorner A_\nu$ and define

$$([\![1_{\mathbb{R}^m}]\!] \bowtie f)_\# T = \sum_\nu \; ([\![1_{\mathbb{R}^m}]\!] \bowtie f)_\# T_\nu$$

where each $([\![1_{\mathbb{R}^m}]\!] \bowtie f)_\# T_\nu$ is characterized by the requirement

$$[([\![1_{\mathbb{R}^m}]\!] \bowtie f)_\# T_\nu] \llcorner U \times \mathbb{R}^n = \sum_i \sigma_i (1_U \bowtie g_i)_\# (T_\nu)$$

whenever p, U, $\sigma_1, \ldots, \sigma_\nu$, $g_1, \ldots, g_\nu$ are as above.

2.9 THEOREM (Properties of multiple valued mappings). Suppose

$$f : \mathbb{R}^m \to \mathbb{P}_0(\mathbb{R}^n) \cap \{P : \mathbb{S}(P) \leq N, \mathbb{M}(P) \leq M, < P, 1 > = L, \operatorname{spt} P \subset B^n(0,R)\}$$

(L, M, N, R fixed) is $\mathbb{GS}$-Lipschitz with constant λ . Then

(1)
$$([\![1_{\mathbb{R}^m}]\!] \bowtie f)_\# : \bigoplus_{k=0}^{m} \mathbf{S}_k(\mathbb{R}^m) \to \bigoplus_{k=0}^{m} \mathbf{S}_k(\mathbb{R}^{m+n})$$

and

$$f_\# : \bigoplus_{k=0}^{m} \mathbf{S}_k(\mathbb{R}^m) \to \bigoplus_{k=0}^{m} \mathbf{S}_k(\mathbb{R}^n)$$

are chain mappings of degree 0 ; in particular, they are linear and commute with the boundary operator ∂ .

(2) For each $T \in S_k(\mathbb{R}^m)$,

$$\mathbb{S}[([\![1_{\mathbb{R}^m}]\!] \bowtie f)_\# T] \leq N(1+\lambda^2)^{k/2} \mathbb{S}(T) ,$$

$$\mathbb{M}[([\![1_{\mathbb{R}^m}]\!] \bowtie f)_\# T] \leq M(1+\lambda^2)^{k/2} \mathbb{M}(T) ,$$

$$\mathbb{S}(f_\# T) \leq N \lambda^k \mathbb{S}(T), \qquad \mathbb{M}(f_\# T) \leq M \lambda^k \mathbb{M}(T).$$

Proof. The main thing to establish here is the relationship

$$([\![1_{\mathbb{R}^m}]\!] \times f)_\# \circ \partial T = \partial \circ ([\![1_{\mathbb{R}^m}]\!] \bowtie f)_\# T .$$

One verifies the obvious behavior of $([\![1_{\mathbb{R}^m}]\!] \times f)_\# \circ g_\#$ for g either a diffeomorphism of $\mathbb{R}^m$ or an isometric injection of $\mathbb{R}^k$ into $\mathbb{R}^m$. One recalls 1.5(4) and is able to reduce the problem to the case in which $T = E^m \llcorner K$ for a convex polyhedron $K \subset \mathbb{R}^m$. One recalls the terminology of 2.8 and approximates $A_\nu \cap K$ and $A_\nu \cap \partial K$ strongly (with respect to $\mathcal{L}^m$ and $\mathcal{H}^{n-1}$) on the inside by compact sets C_ν and D_ν respectively and applies the construction used in the Lipschitz extension theorem to extend $f| \bigcup_\nu (C_\nu \cup D_\nu)$ to $g : \mathbb{R}^m \to \mathbb{P}_0(\mathbb{R}^n)$. For points near $p \in C_\nu \cup D_\nu$ it is a property of our construction that $g = \sum_i \sigma_i [\![g_i(\cdot)]\!]$, in which case local commutation with ∂ if immediate. For $p \notin \bigcup_\nu (C_\nu \cup D_\nu)$, near p, g is a sum of affine mappings of polyhedra (with an obvious meaning) in which case local commutation with ∂ is readily checked. The asserted commutivity follows since ∂ is continuous with respect to $\mathbb{NS}$ convergence.

2.10 <u>How to construct a $\mathbb{CS}$-Lipschitz multiple valued function approximation to a rectifiable cycle with error of small projected area</u>. We suppose $0 < \mu < \infty$ and $\theta : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ is orthogonal such that for each increasing function $\lambda : \{1,2,\ldots,n\} \to \{1,2,\ldots,m+n\}$ there is a linear function $g(\lambda) : \mathbb{R}^m \to \mathbb{R}^n$ with $\|g(\lambda)\| \leq \mu$ such that

$$\operatorname{graph}[g(\lambda)] = \theta^{-1}(\mathbb{R}^{m+n} \cap \{x : x_{\lambda(1)} = \ldots = x_{\lambda(n)} = 0\}).\ .$$

We also suppose $T \in \mathbf{S}_m(\mathbb{R}^{m+n})$ with $\partial T = 0$ and $0 < N < \infty$.

<u>The construction</u>.

(A) We set $\Pi = \{\mathbb{R}^m \times \{0\}\}_\natural \circ \theta : \mathbb{R}^{m+n} \to \mathbb{R}^m$.

(B) We let A denote the set of points $a \in \mathbb{R}^m$ for which

$$\mathbf{MS}[T \llcorner \pi^{-1} B^m(0,r)] \leq N\,\alpha(m) r^m$$

for each $0 < r < \infty$.

(C) We set $B = \mathbb{R}^m \sim A$ so that, in particular, for each $b \in B$ there is $0 < r(b) < \infty$ with

$$\mathbf{MS}[T \llcorner \pi^{-1} B^m(0,r(b))] > N\,\alpha(m) r(b)^m .$$

B is clearly open, and one uses the Besicovitch-Federer covering theorem to infer the existence of $b_1, b_2, b_3, \ldots \in B$ with $B \subset \bigcup_i B^m(b_i, r(b_i))$ and

$$\beta(m) \geq \operatorname{card}\{i : z \in B^m(b_i, r(b_i))\} \qquad \text{for each } z \in \mathbb{R}^m ;$$

hence

$$\begin{aligned} \mathcal{L}^m(B) &\leqslant \sum_i \mathcal{L}^m(\mathbf{B}^m(b_i, r(b_i)) \\ &\leqslant \sum_i \alpha(m)\ r(b_i)^m \\ &< N^{-1} \sum_i \mathbf{M}\mathfrak{S}[T \llcorner \pi^{-1}\mathbf{B}^m(b_i, r(b_i))] \\ &\leqslant N^{-1}\ \beta(m)\mathbf{M}\mathfrak{S}(T). \end{aligned}$$

For each $b \in B$ we fix $a(b) \in A$ with $|b - a(b)| = \operatorname{dist}(b,A)$. Hence, for each $b \in B$ and each $0 < r < \infty$.

$$\mathbf{B}^m(b,r) \subset \mathbf{B}^m(a(b), r + |b - a(b)|) ,$$
$$\mathbf{M}\mathfrak{S}[T \llcorner \pi^{-1}\mathbf{B}^m(b,r)] \leqslant N\ \alpha(m)\ (r + |b - a(b)|)^m .$$

(D) We set $\mathcal{F} = \mathrm{WF}(\pi^{-1}B)$ (the admissible Whitney family of cubes as in 1.10(3)) and also, for each $k \in \{1,2,3,\ldots\}$, let

$$\mathcal{F}(k) = \mathcal{F} \cap \{K: \operatorname{level}(K) \leqslant k\} \cup \mathcal{K}(k) \cap \{K: \operatorname{Int}(K) \cap \operatorname{Int}(L) = \emptyset \text{ for each } L \in \mathcal{F} \text{ with } \operatorname{level}(L) < k\}$$

so that each $\mathcal{F}(k)$ is an admissible family of cubes with minimum cube level equal to k .

For each $k \in \{1,2,3,\ldots\}$ and each $z \in \mathbb{R}^m$ we set

$$\operatorname{level}(k,z) = \sup\{\operatorname{level}(K): K \in \mathcal{F}(k) \text{ with } z \in \pi(K)\} .$$

A straightforward computation shows the existence of a constant $\Gamma < \infty$ depending only on m and n with the following properties :

(D.1) for each $k \in \{1,2,3,\ldots\}$ and each $a \in A$,
$\cup\{\operatorname{nbs}(K): K \in \mathcal{F}(k) \text{ with } a \in \pi(K)\} \subset \pi^{-1}\mathbf{B}^m(a, \Gamma\ 2^{-\operatorname{level}(k,a)})$ (1.10(2));

(D.2) for each $k \in \{1,2,3,\ldots\}$ and each $b \in B$,
$\cup\{\operatorname{nbs}(K): K \in \mathcal{F}(k) \text{ with } b \in \pi(K)\} \subset \pi^{-1}\mathbf{B}^m(a(b), \Gamma\ 2^{-\operatorname{level}(k,b)})$.

These estimates are the geometric foundation of our construction.

(E) We let

$$W, W(1), W(2), W(3), \ldots, \in \mathfrak{S}_m(\mathbb{R}^{m+n})$$

be the currents which result from applying the deformations of 1.12 to T and the admissible families $\mathcal{F}$, $\mathcal{F}(1)$, $\mathcal{F}(2)$, $\mathcal{F}(3)$,... respectively.

One checks the existence of a constant $\Gamma_{2.10} < \infty$ depending only on m

and n with the following properties :

(E.1) for each $z, w \in \mathbb{R}^m$ and each k, the slice

$$f(k)(z) = \langle W(k), \pi, z \rangle \in \mathbb{P}_0(\mathbb{R}^{m+n})$$

exists with $\mathbf{S}(f(z)) \leq \Gamma_{2.10} N$, $\mathbf{M}(f(z)) \leq \Gamma_{2.10} N$, $\langle f(z), 1 \rangle = 0$ (since $W(k)$ is a cycle), and

$$\mathbf{GS}(f(k)(z), f(k)(w)) \leq \Gamma_{2.10} N(1 + \mu^2)^{m/2} |z - w| ;$$

(E.2) $W(k) = f(k)_{\#} E^m$ for each k ;

(E.3) for each $z, w \in \mathbb{R}^m$, the $\mathbf{GS}$-limit

$$f(z) = \lim_{k \to \infty} f(k)(z)$$

exists and equals the slice $\langle W, \pi, z \rangle$ with $\mathbf{S}(f(z)) \leq \Gamma_{2.10} N$, $\mathbf{M}(f(z)) \leq \Gamma_{2.10} N$, $\langle f(z), 1 \rangle = 0$,

$$\mathbf{GS}(f(z), f(w)) \leq \Gamma_{2.10} N(1 + \mu^2)^{m/2} |z - w| ;$$

(E.4) $W = f_{\#} E^m$;

(E.5) $\pi[\mathrm{set}(T - f_{\#} E^m)] \subset B$.

As a particular case in (E.1) one uses 1.12 and estimates for $b \in B$ that
$2^{-m[\mathrm{level}(k,b)]} \mathrm{card}\{J : J \in CX_m(F(k))$ with $b \in \pi(J)$ and $W(k) \llcorner \mathrm{Int}(J) \neq \emptyset\}$

$$\begin{aligned}
&\leq \mathbf{S}(W(k) \llcorner \cup\{K : K \in F(k) \text{ with } b \in \pi(K) \text{ and } W(k) \llcorner K \neq \emptyset\}) \\
&\leq \Gamma_{1.12} \mathbf{S}(T \llcorner \cup\{\mathrm{nbs}(K) : K \in F(k) \text{ with } b \in \pi(K) \text{ and } W(k) \llcorner K \neq 0\}) \\
&\leq \Gamma_{1.12} \mathbf{S}(T \llcorner \pi^{-1} B^m(a(b), \Gamma\, 2^{-\mathrm{level}(k,b)})) \\
&\leq \Gamma_{1.12} \mathbf{N}\, \alpha(m)\, (\Gamma\, 2^{-\mathrm{level}(k,b)})^m
\end{aligned}$$

so that

$$\mathrm{card}\{J : J \in CX_m(F(k) \text{ with } b \in \pi(J) \text{ and } W(k) \llcorner \mathrm{Int}(J) \neq \emptyset\}$$

$$\leq N(\Gamma_{1.12} \alpha(m) \Gamma^m) .$$

2.11 THEOREM (Lipschitz multiple valued approximations of rectifiable cycles with error of small projected areas). Suppose $1 < \mu < \infty$ and θ, $\theta(1)$, $\theta(2), \ldots, \theta(\nu) : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ are orthogonal mappings with the property that for each $k \in \{1, \ldots, \nu\}$ and for each increasing function $\lambda : \{1, \ldots, n\} \to \{1, \ldots, m+n\}$ there is a linear function $g(k,\lambda) : \mathbb{R}^m \to \mathbb{R}^n$ with $\|g(k,\lambda)\| \leq \mu$ such that

$$\operatorname{graph}[g(k,\lambda)] = [\theta(k)\circ\theta]^{-1}[\mathbb{R}^{m+n} \cap \{x: x_{\lambda(1)} = \dots = x_{\lambda(n)} = 0\}]$$

(for any given $\theta(1),\dots,\theta(\nu)$ this will hold for suitable μ and θ).

Suppose also $T \in \mathbf{\$}_m(\mathbb{R}^{m+n})$ with $\partial T = 0$ and $0 < N < \infty$.

Then there is a function

$$f : \mathbb{R}^m \to \mathbb{P}_0(\mathbb{R}^{m+n})$$

together with an (error) current $W \in \mathbf{\$}_m(\mathbb{R}^{m+n})$ with $\partial W = 0$ and open subset U of $\mathbb{R}^{m+n}$ with the following properties:

(1) $T = f_{\#}E^m + W$;

(2) $\mathbf{M\$}(f_{\#}E^m) \leqslant (\Gamma_{2.11})^{\nu}\mathbf{M\$}(T)$ and $\mathbf{M\$}(W) \leqslant (\Gamma_{2.11})^{\nu}\mathbf{M\$}(T)$;

(3) $\operatorname{set}(W) \subset U$ and

$$L^m[\{\mathbb{R}^m \times\{0\}\}_{\natural}\circ\theta(k)U] \leqslant N^{-1}\Gamma_{2.11}\mathbf{M\$}(T) ;$$

(4) for each x , $y \in \mathbb{R}^m$.

$$\mathbf{\$}(f(x)) \leqslant \Gamma_{2.11}\,\nu\,\mu^m N \ , \ \mathbf{M}(f(x)) \leqslant \Gamma_{2.11}\nu\ \mu^m\ N \ , \ \langle f(x),1\rangle = 0 \ .$$

$$\mathbf{G\$}\,(f(x),f(y)) \quad \leqslant \Gamma_{2.11}\nu\ \mu^m\ N|x-y| \ .$$

In the above $\Gamma_{2.11}$ is a constant depending only on m and n .

Proof. It is clearly sufficient to consider the case $\theta = 1_{\mathbb{R}^m}$. We will construct $f = f(1) + f(2) + \dots + f(\nu) : \mathbb{R}^m \to \mathbb{P}_0(\mathbb{R}^{m+n})$ where each $f(k)$ is obtained by suitably adapting the approximation construction of 2.10. Representative steps are the following.

(a) $B(1) \subset \mathbb{R}^m$, $f(1)$ are obtained by the procedure of 2.10 with θ , B, T, f, F there corresponding to $\theta(1)$, $B(1)$, T, $f(1)$, $F(1)$ here. We then write

$$T = f(1)_{\#}E^m + [T - f(1)_{\#}E^m] \;= f(1)_{\#}E^m + W(1).$$

(b) $B(2) \subset \mathbb{R}^m$, $f(2)$ are obtained essentially by the procedure of 2.10 with θ , B , T, f, F there corresponding to $\theta(2)$, $b(2)$, $W(1)$, $f(2)$,

$$F(2) = WF([\{\mathbb{R}^m \times\{0\}\}_{\natural}\circ\theta(1)]^{-1}B(1) \cap [\{\mathbb{R}^m \times\{0\}\}_{\natural}\circ\theta(2)]^{-1}B(2))$$

here (see the comments below about relevant cube sizes in the Whitney family). We then write

$$T = f(1)_\# E^m + f(2)_\# E^m + [W(1) - f(2)_\# E^m] = f(1)_\# E^m + f(2)_\# E^m + W(2).$$

(c) In general, $B(k) \subset \mathbb{R}^m$, $f(k)$ are obtained essentially by the procedure of 2.10 with θ, B, T, f, F there corresponding to $\theta(k)$, $B(k)$, $W(k-1)$, $f(k)$,

$$F(k) = WF([\{\mathbb{R}^m \times \{0\}\}_\natural \circ \theta(1)]^{-1} B(1) \cap \ldots \cap [\{\mathbb{R}^m \times \{0\}\}_\natural \circ \theta(k)]^{-1} B(k))$$

here, and one writes

$$T = f(1)_\# E^m + \ldots + f(k)_\# E^m + [W(k-1) - f(k)_\# E^m] = f(1)_\# E^m + \ldots + f(k)_\# E^m + W(k).$$

Finally, of course, $f = f(1) + \ldots + f(\nu)$, $W = T - f_\# E^m$, and

$$U = \cap_k [\{\mathbb{R}^m \times \{0\}\}_\natural \circ \theta]^{-1} B(k) .$$

The geometric foundation of the construction described lies in inclusions analogous to those of 2.10(D.2) which do not hold in the present context without modification. To see what is true we write (in a symbolic way) $\sigma(k)_\# W(k-1)$ as the deformed image of $W(k-1)$ by a singular map $\sigma(k) : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ consisting of an iteration of central projections within cubes in $CX(F)$ (in particular, $W(0) = T$ and $f(k)_\# E^m = \sigma(k)_\# W(k-1)$). We note

$$W(1) = \sigma(1)_\# T \qquad \text{(with centers of projection determined only by T)},$$

$$W(2) = \sigma(2)_\# W(1) = \sigma(2)_\# [T - \sigma(1)_\# T] \; .$$

One checks that $\cup CX_m(F(1)) \cap \cup CX_m F(2)) \subset \cup CX_m(F(2))$ and one concludes by inspection of our construction of $\sigma(2)$ that $\sigma(2)_\# \circ \sigma(1)_\# T = \sigma(1)_\# T$ and that the choices of centers of projection determining $\sigma(2)$ depend solely on T (not on $\sigma(1)_\# T$). We require the same choices of centers in constructing $\sigma(2)$ as in constructing $\sigma(1)$ where this is possible. Hence

$$W(2) = \sigma(2)_\# T - \sigma(1)_\# T$$

and $W(2)$ will be nonzero on $J \in CX_m(F(2))$ only when $\sigma(2)_\# T$ and $\sigma(1)_\# T$ do not agree on J . An inspection of the combinatorial structure of the Whitney families $F(1)$ and $F(2)$ shows that the obvious reformulation of (D.2) holds provided we replace " $\Gamma\, 2^{-\mathrm{level}(k,b)}$" there by "$2\Gamma\, 2^{-\mathrm{LEVEL}(k,b)}$" where

$$\text{LEVEL}(k,b) = \sup\{\text{level}(J) : J \in CX_m(F(2)) \text{ with } b \in \{\mathbb{R}^m \times \{0\}\} \llcorner \circ \theta(2)J \text{ and}$$
$$\sigma(2)_{\#}T \llcorner J \neq \sigma(1)_{\#}T \llcorner J\}.$$

Such a reformulated estimate is readily checked to be adequate for the remainder of the construction provided the constants are adjusted appropriately.

Similar estimates hold for $\sigma(3)_{\#}W(2)$, $\sigma(4)_{\#}W(3), \ldots$ and the construction proceeds as described above.

2.12 A computational scheme for minimal surfaces based on multiple valued functions. B. Super has used multiple-valued function representations in his computations of surfaces of least area. For a given boundary he randomly generates an initial oriented surface (with orientation cancellations generating higher topological types) and then runs an area minimization routine vertex by vertex (generating and eliminating triangles as necessary). This and other important applications of multiple-valued function theory are indicated in the references.

§ 3. COMPACTNESS THEOREMS FOR RECTIFIABLE CURRENTS.

The main purpose of this third section is to sketch a proof of the following theorem.

3.1 THEOREM (Compactness theorem for size bounded real currents). Suppose $T_1, T_2, T_3, \ldots \in \mathcal{S}_{m,0}(\mathbb{R}^{m+n})$ with $\sup_i \mathbb{N}\mathcal{S}(T_i) < \infty$ and $\bigcup_i \operatorname{spt} T_i$ bounded.

Then there exists a subsequence $i(1), i(2), i(3), \ldots$ of $1, 2, 3, \ldots$ and $T \in \mathcal{S}_{m,0}(\mathbb{R}^{m+n})$ such that $0 = \lim_{j\to\infty} \mathbb{F}(T, T_{i(j)})$.

Furthermore,

$$\mathbf{M}(T) \leq \liminf_j \mathbf{M}(T_{i(j)}), \quad \mathcal{S}(T) \leq \liminf_j \mathcal{S}(T_{i(j)}),$$

$$\mathbf{M}(\partial T) \leq \liminf_j \mathbf{M}(\partial T_{i(j)}), \quad \mathcal{S}(\partial T) \leq \liminf_j \mathcal{S}(T_{i(j)}).$$

REMARK. This theorem and its integral current and flat chain mod ν analogues (which can be proved by virtually the same method) give basic existence theorems for solutions to parametric variational problems in the context of geometric measure theory.

The proof of the theorem requires several preliminary results.

3.2. Families of orthogonal projections.

(1) We say that orthogonal projections $\pi_1, \pi_2, \ldots, \pi_N : \mathbb{R}^{m+n} \to \mathbb{R}^m$ are m-spanning if and only if there is $\varepsilon > 0$ such that

$$\varepsilon < \sum_i L^m[\pi_i \circ \Theta([0,1]^m \times 0)]$$

for each orthogonal map $\Theta : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$.

(2) We denote by $\Lambda(m+n,m)$ the set of all increasing maps $\{1,2,\dots,m\} \to \{1,2,\dots,m+n\}$ and to each $\lambda \in \Lambda(m+n,m)$ associate the orthogonal projection $\pi(\lambda) : \mathbb{R}^{m+n} \to \mathbb{R}^m$ sending $x \in \mathbb{R}^{m+n}$ to $(x_{\lambda(1)}, \dots, x_{\lambda(m)}) \in \mathbb{R}^m$. The projections $\{\pi(\lambda)\}_\lambda$ are m-spanning. Clearly, $\operatorname{card}\Lambda(m+n,m) = \binom{m+n}{m} = BC$.

(3) We set

$$N_0 = (1 + 2^{m+n})^2\, 16^{m+n} - 2$$

and

$$N_{3.2} = 1 + N_0[BC - 1].$$

One checks that whenever F is an admissible family of cubes and $K \in F$ then

$$N_0 \geq \operatorname{card}\{J : J \in F \text{ and } J \cap L \neq \emptyset \text{ for some } L \in F \text{ s.t. } L \cap K \neq \emptyset\}$$

(4) We now choose and fix orthogonal mappings

$$\Theta_0,\ \omega(1) = 1_{\mathbb{R}^{m+n}}, \dots, \omega(N_{3.2}) : \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$$

and a number $1 < \mu_0 < \infty$ which satisfy the following conditions.

(4.1) Whenever $\lambda(1), \dots, \lambda(BC) \in \Lambda(m+n,m)$ with $(\lambda(i))_i = \Lambda(m+n,m)$ the family $\{\pi(\lambda(i)) \circ \omega(i)\}_i$ is m-spanning ; such a condition is realized by requiring $\omega(1), \dots, \omega(BC)$ all to be very close to $1_{\mathbb{R}^{m+n}}$.

(4.2.) For each sequence $\lambda(1), \dots, \lambda(BC) \in \Lambda(m+n,m)$ and each increasing sequence $i(1), \dots, i(BC) \in \{1, \dots, N_{3.2}\}$ the orthogonal projections $\{\pi(\lambda(j)) \circ \omega(i(j))\}_j$ are m-spanning ; to realize such a sequence one selects the $\{\omega(i)\}_i$ sequentially inferring from the real analyticity of degeneracy conditions that almost all "next choices" are suitable.

(4.3) Corresponding to each $\mu, \lambda \in \Lambda(m+n,m)$ and each $i \in \{1, \dots, N_{3,2}\}$ there is a linear function $g : \mathbb{R}^m \to \mathbb{R}^{m+n}$ with $\|g\| \leq (1 + \mu^2)^{m/2}$ such that

$$1_{R^m} = \pi(\mu) \circ \omega(i) \circ \Theta_0 \circ g$$

and

$$\operatorname{Im}(g) = \mathbb{R}^{m+n} \cap \{x : x_j = 0 \text{ for each } j \notin \lambda\{1,\dots,m\}\}$$

3.3. THEOREM (Deforming out parts of currents having small projected areas). *Suppose* $0 < \varepsilon < 1$, $N \in \{1,2,3,\dots\}$, *and* $V, W \in S_m(\mathbb{R}^{m+n})$ *with* $\partial(V + W) = 0$. *Suppose also*

$$H^m(\text{set}V \cup \text{set}W) \leq \Gamma_1\ \varepsilon^{-1}\ 2^{-mN}$$

and

$$L^m[\pi(\lambda)\circ\omega(i)\text{set}W] \leq \Gamma_2\ \varepsilon^m\ 2^{-mN}$$

for each $\lambda \in \Lambda(m+n,m)$ *and each* $i \in (1,\dots,N_{3.2})$. *Then there is an admissible family* F *of* $(m+n)$*-dimensional cubes and a suitable choice of centers of projections among members of* $CX_m(F) \cup \dots \cup CX_{m+n}(F)$ *such that the following estimates hold with* $V_* \in \mathbf{S}_m(\mathbb{R}^{m+n})$ *denoting the deformed image of* $V + W$ *in accordance with* 1.12.

(1) $V_* - (V + W) = \partial Q$ *for some* $Q \in \mathbf{S}_{m+1}(\mathbb{R}^{m+n})$ *with* $\text{set}Q \subset \cup F$ *and*

$$\mathbf{MS}(Q) \leq 2^{-N}\ \Gamma_{3.3}\ \mathbf{MS}(V + W).$$

(2) $V_* \llcorner (\mathbb{R}^{m+n} \sim \cup F) = V \llcorner (\mathbb{R}^{m+n} \sim \cup F)$.

(3) $\mathbf{MS}(V_* \llcorner \cup F) \leq \Gamma_{3.3}\ \mathbf{MS}(V \llcorner \cup F)$.

(4) $\cup F$ *is the union of Borel subsets* $A[\lambda(1), \dots,\lambda(BC);i(1),\dots,i(BC)]$ *corresponding to each sequence* $\lambda(1), \dots, \lambda(BC) \in \Lambda(m+n,m)$ *and each increasing sequence* $i(1), \dots, i(BC) \in \{1, \dots,N_{3,2}\}$ *such that*

$$L^m[\pi(\lambda(j))\circ\omega(i(j))A[\lambda(1),\dots,\lambda(BC);i(1),\dots,i(BC)]]$$

$$\leq \varepsilon\cdot H^m(\text{set}V \cup \text{set}W)$$

$$+\ \varepsilon^{-m}\ \Gamma_{3.3}\ \sup\{L^m[\pi(\lambda)\circ\omega(i)\text{set}W]:\ \lambda \in \Lambda(m+n,m)$$

$$\text{and}\quad i \in \{1, \dots,N_{3.2}\}\}$$

for each $j = 1, \dots, BC$.

In the above, $\Gamma_1, \Gamma_2, \Gamma_{3.3}$ *are constants depending only on* m *and* n.

Proof. The main device of the proof is careful construction of F. To do this we say that $K \in k(M)$ is manageable if the following two criteria are met :

(a) $\mathcal{H}^m[(\text{setV} \cup \text{setW}) \cap K] \leqslant \mu_1 2^{-mM}$ (suitable fixed μ_1) and also

(b) there do not exist $\lambda(1), \ldots, \lambda(BC) \in \Lambda(m+n,m)$ and increasing $i(1), \ldots, i(BC) \in \{1, \ldots, N_{3.2}\}$ such that

$$\mathcal{L}^m[\pi(\lambda(j))\circ\omega(i(j))(\text{setW} \cap K)] > \varepsilon_1 2^{-mM}$$

for each $j \in \{1, \ldots, BC\}$ (suitable fixed ε_1).

$F \subset \bigcup_\nu k(\nu)$ is that admissible family of cubes K characterized by the requirements

(i) $\text{Int}(K) \cap \text{Int}(L) = \emptyset$ for each $L \in F \cap \bigcup_{k<\text{level}(K)} k(k)$.

(ii) one or both of the following conditions hold : either there is $L \in F \cap k(\text{level}(K)-1)$ with $K \cap L \neq \emptyset$ or there is $L \in k(\text{level}(k)+1)$ with $L \subset K$ such that L is not manageable.

The main point of these conditions is the following. If $K \in F$ is manageable and $L \in k(\text{level}(K)+1)$ with $L \subset K$ is not manageable by failure only of criterion (b) above then the following will hold.

(A) there will exist $\lambda(1), \ldots, \lambda(BC) \in \Lambda(m+n,m)$ and increasing $i(1), \ldots, i(BC) \in \{1, \ldots, N_{3,2}\}$ such that

$$\mathcal{L}^m[\pi(\lambda(j))\circ\omega(i(j))(\text{setW} \cap L)]$$

is relatively large compared to $2^{-m\,\text{level}(K)}$. We then place K and relevant nearby smaller cubes in $A[\lambda(1), \ldots, \lambda(BC); i(1), \ldots, i(BC)]$ and use our estimate to guarantee that the $\pi(\lambda(j))\circ\omega(i(j))$ projections of $A[\lambda(1), \ldots, \lambda(BC); i(1), \ldots, i(BC)]$ are dominated by the corresponding projections of W.

(B) Since K is manageable we estimate

$$BC - 1 \geqslant \text{card}\{i: \mathcal{L}^m[\pi(\lambda)\circ\omega(i)(\text{setW} \cap K)] > \varepsilon_1 2^{-m\,\text{level}(K)} \text{ for some } \lambda \in \Lambda(m+n,m)\}.$$

by virtue of our choice of $N_{3.2}$ we conclude the existence of at least one

$i_0 \in \{1, \ldots, N_{3,2}\}$ such that

$$L^m[\pi(\lambda)\circ\omega(i_0)(\text{set}W \cap J)] \leq \varepsilon_1\, 2^{-m\,\text{level}(J)}$$

for each $\lambda \in \Lambda(m+n,m)$ whenever $J \in F$ and there is $H \in F$ with $J \cap H \neq \emptyset$ and $H \cap K \neq \emptyset$. This condition together with criterion (a) above are sufficient to guarantee that the size and mass of V_* near K can be dominated by the size and mass of V only near K; the details of the argument are a straightforward extension of the methods discussed in the proof of 1.17.

Proof of theorem 3.1. For $m = 0$ the theorem is left to the reader. For $m \geq 1$ we reduce the theorem to the case $T_i \in \mathbf{\$}_m(\mathbb{R}^{m+n})$ with $\partial T_i = 0$ for each i by an inductive argument. In particular, if

$$0 = \lim_i \mathbb{G}(W, \partial T_i) \quad \text{and} \quad 0 = \lim_i \mathbb{G}(T_*, T_i - [[0]] \times \partial T_i)$$

then

$$T = T_* - [[0]] \times W.$$

We henceforth assume $T_i \in \mathbf{\$}_m(\mathbb{R}^{m+n})$ with $\partial T_i = 0$ for each i and will show the existence of a subsequence $i(1), i(2), i(3), \ldots$ of $1, 2, 3, \ldots$ and $T \in \mathbf{\$}_m(\mathbb{R}^{m+n})$ with

$$0 = \lim_j \mathbb{G}(T, T_{i(j)}) \qquad (\text{not } \mathbb{G}\mathbb{S}).$$

The remainder of the proof is in four parts.

Part 1. We note 3.2.(4.3) and will apply Theorem 2.11 in a straightforway way with the orthogonal projections $\Theta, \Theta(1), \ldots \Theta(\nu)$ there replaced by the orthogonal projections Θ_0, $\{\pi(\lambda)\circ\omega(i): \lambda \in \Lambda(m+n,m)\ \ i \in \{1, \ldots, N_{3.2}\}\}$. In so doing we check the existence of a fixed constant Γ such that for each $i, \nu \in \{1, 2, 3, \ldots\}$ we can find

$$g_i(\nu): \mathbb{R}^m \to \mathbb{P}_0(\mathbb{R}^{m+n}),\ W_i(\nu) \in \mathbf{\$}_m(\mathbb{R}^{m+n}) \text{ with } \partial W_i = 0,\ U_i(\nu) \subset \mathbb{R}^{m+n}$$

for which, in particular, for each i and ν,

(a) $T_i = g_i(\nu)_\# E^m + W_i(\nu)$;

(b) $W_i(\nu) = W_i(\nu) \llcorner U_i(\nu)$ and

$$L^m[\pi(\lambda)\circ\omega(j)U_i(\nu)] \leq \Gamma\, 2^{-\nu} \quad \text{for each } \lambda \text{ and } j;$$

(c) for each $x, y \in \mathbb{R}^m$, $\mathbb{S}(g_i(\lambda)(x) \leq \Gamma\, 2^{\nu}$, $\mathbb{M}(g_i(\lambda)(x)) \leq \Gamma\, 2^{\nu}$, $\langle g_i(\nu)(x), 1 \rangle = 0$, and

$$\mathbb{GS}(g_i(\nu)(x),\ g_i(\nu)(y)) \leq \Gamma\, 2^{\nu}\, |x-y|\ ;$$

(d) $\mathbb{MS}(g_i(\nu)_{\#} E^m) + \mathbb{MS}(W_i(\nu)) \leq \Gamma$.

One uses Cantor's diagonal process in inferring for each ν the existence of a subsequence $i(1), i(2), i(3), \ldots$ and $g(\nu): \mathbb{R}^m \to \mathbb{P}_0(\mathbb{R}^{m+n})$ with

(e) $0 = \lim_{j\to\infty} \sup\{\mathbb{G}(g(\nu)(x), g_{i(j)}(\nu)(x)) : x \in \mathbb{R}^m\}$ (not $\mathbb{GS}$).

One uses 2.3 in verifying for each $x,y \in \mathbb{R}^m$ that

(f) $\mathbb{GS}(g(\nu)(x), g(\nu)(y)) \leq \Gamma\, 2^{\nu}\, |x - y|$.

One then uses (e), (f) above and an adaptation of 1.7(7) and 2.6 to construct a $\mathbb{GS}$-Lipschitz homotopy between $g(\nu)$ and $g_{i(j)}(\nu)$ for large j (possibly with large Lipschitz constant locally being balanced by small densities-in the integer coefficient or integer mod ν case this situation will not occur) in order to conclude

(g) $0 = \lim_{j\to\infty} \mathbb{G}(g(\nu)_{\#}E^m,\ g_{i(j)}(\nu)_{\#} E^m)$.

<u>Part 2</u>. For each i and ν, (a) and (b) of part 1 imply

$$g_i(\nu)_{\#} E^m - g_i(\nu+1)_{\#} E^m = -W_i(\nu) + W_i(\nu+1)$$

with

(a) $L^m[\pi(\lambda)\circ\omega(k)\operatorname{set}(g_i(\nu)_{\#} E^m - g_i(\nu+1)_{\#}\mathbb{R}^m)] \leq (3/2)\ \Gamma\, 2^{-\nu}$

for each λ and k ; for each ν we set

$$C(\nu) = \operatorname{set}(g(\nu)_{\#} E^m - g(\nu+1)_{\#} E^m).$$

Since

$$g(\nu)_{\#}E^m - g(\nu+1)_{\#}E^m \in \mathbf{S}_m(\mathbb{R}^{m+n})$$

(essential to know at this point) and

$$0 = \lim_{j\to\infty} G(g(\nu)_{\#}E^m - g(\nu+1)_{\#}E^m,\ g_{i(j)}(\nu)_{\#}E^m - g_{i(j)}(\nu+1)E^m)$$

one is able to infer from (a) that

(b) $\mathcal{L}^m[\pi(\lambda)\circ\omega(k)C(\nu)] \leq (3/2)\,\Gamma\, 2^{-\nu}$ for each λ and k.

We then set

$$V(\nu) = g(\nu)_{\#}E^m \llcorner \cap\{[\mathbb{R}^{m+n} \sim C(\mu)]: \mu = \nu, \nu+1, \ldots\}$$

and check that

(c) $V(\nu) = V(\nu+1) \llcorner [\mathbb{R}^{m+n} \sim C(\nu)]$.

In view of part 1(d) and 1.7(1) we conclude the existence of $V \in \mathcal{S}_m(\mathbb{R}^{m+n})$ (not necessarily in $S_m(\mathbb{R}^{m+n})$ or having boundary 0 at this point) with

(d) $0 = \lim_{\nu\to\infty} \mathbb{MS}(V - V(\nu))$;

this will be the T of the conclusion of our theorem once we show $V \in \mathbf{S}_m(\mathbb{R}^{m+n})$ with $\partial T = 0$ and with $\lim_{j\to\infty} \mathbb{G}(V,V_i) = 0$. We note initially that

(e) $V(\nu) = V \llcorner \cap\{[\mathbb{R}^{m+n} \sim C(\mu)]: \mu = \nu, \nu+1, \ldots\}$

Part 3. We choose suitable $\varepsilon = \varepsilon(\nu)$ and $N = N(\nu)$ and apply Theorem 3.3. with V, W there replaced by $V(\nu)$ and $g(\nu)_{\#}E^m \llcorner V\{C(\mu): \mu = \nu, \nu+1, \ldots\}$ respectively to obtain $V_*(\nu) \in \mathbf{S}_m(\mathbb{R}^{m+n})$ with $\partial V_*(\nu) = 0$ and $A_\nu[\lambda(1),\ldots,\lambda(BC);i(1),\ldots,i(BC)]\}_{\lambda,i}$, etc. so that, in particular,

(a) $0 = \lim_{\nu\to\infty} \mathbb{GS}(V_*(\nu),\ g(\nu)_{\#}E^m)$.

(b) $0 = \lim_{\nu\to\infty} \mathcal{L}^m[\pi(\lambda(j))\circ\omega(i(j))A_\nu[\lambda(1),\ldots;i(1),\ldots]]$.

For each j, λ, i.

(c) $\mathbf{MS}(V_*(\nu) \llcorner \bigcup_{\lambda,i} A_\nu[\lambda(1),\ldots;i(1),\ldots])$

$\leq \Gamma_{3.3}\mathbf{MS}(V(\nu) \llcorner \bigcup_{\lambda,i} A_\nu[\lambda(1),\ldots;i(1),\ldots])$

$\leq \Gamma_{3.3}\mathbf{MS}(V \llcorner \bigcup_{\lambda,i} A_\nu[\lambda(1),\ldots;i(1),\ldots])$

(recall part 2(e)).

In view of condition 3.2.(4.1) we conclude from (c) (since V is fixed) that

(d) $0 = \lim_{\nu\to\infty} \mathbf{MS}(V_*(\nu) \llcorner \bigcup_{\lambda,i} A_\nu[\lambda(1),\ldots;i(1),\ldots])$,

(e) $0 = \lim_{\nu\to\infty} \mathbf{MS}(V - V_*(\nu))$,

(f) $V \in S_m(\mathbb{R}^{m+n})$ with $\partial V = 0$.

Part 4. In order to estimate $\mathbb{G}(V,V_{i(j)})$ we write

$$\mathbb{G}(V, V_{i(j)}) \leq \mathbb{G}(V, V_*(\nu)) + \mathbb{G}(V_*(\nu), g(\nu)_\# E^m) + \mathbb{G}(g(\nu)_\# E^m, g_{i(j)}(\nu)_\# E^m) + \mathbb{G}(g_{i(j)}(\nu)_\# E^m, V_{i(j)}).$$

The first summand is estimated with part 3(e) and 1.13. The second summand is estimated by part 3(a). The third summand is estimated by part 1(g). The final summand is estimated by part 1(a)(b)(d) and 1.17 ; note here that the estimate is independent of $i(j)$. In showing $G(V,V_{i(j)})$ is small for large j one first fixes large ν to make the first, second, and fourth summands small and then chooses large j to make the third summand small. We have, as noted above, $T = V$ for the conclusion of our theorem. ■

REFERENCES

W.K. ALLARD, On the first variation of a varifold, Ann. of Math. 95 (1972), 417-491.

(This is the basic reference for the theory of varifolds mentionned in the initial examples. This paper also contains an existence theorem for mass minimizing integral currents based on varifold theory.)

F. ALMGREN , Deformations and multiple valued functions, Geometric Measure Theory and the Calculus of Variations, Proc. Symposia in Pure Math., 1985,29-130.

(This is the basic reference for the extended deformation theory and multiple valued function theory of these notes. A number of the oonstructions are illustrated with figures).

F. ALMGREN, The homotopy groups of the integral cycle groups, Topology 1 (1962), 257-299.

This is the reference needed in the proof of Theorem 1.16 for homotopy groups in the integer coefficient case).

F. ALMGREN and B. SUPER, Multiple valued functions in the geometric calculus of variations, Astérisque 118 (1984), 13 - 22.

(This paper contains a discussion of important applications of multiple valued function theory within geometric measure theory by F. Almgren, P. Mattila, D. Nance B. Solomon, B. Super and B. White. Additional important applications of multiple valued function theory appear in the volume, Geometric Measure Theory and the Calculus of Variations, mentioned above).

H. FEDERER, Geometric Measure Theory, Springer-Verlag, New-York, 1969.

(This is the basic treatise for the fundamentals of geometric measure theory. It should be consulted, in particular, for basic material on Hausdorff measure (2.10.2), the Besicovitch-Federer covering theory (2.8.14 ; our number $\beta(n)$ is called $2\zeta+1$ there in the obvious context), the Hausdorff area formula (3.2.19, 3.1.20), the coarea formula (3.2.22), currents (4.1), and slices of rectifiable currents (4.3.8, 4.3.10) discussed in these notes).

H. FEDERER, Flat chains with positive densities, Indiana Univ. Math.J.35 (1986), 413-424.

(This paper contains an alternative proof to the compactness theorem for size bounded rectifiable currents based on the structure theorem for sets of finite Hausdorff measure.)

B. SOLOMON, A new proof of the closure theorem for integral currents, Indiana Univ. Math.J.33 (1984), 393-419.

(This was the first of the compactness theorems based on multiple-valued function theory.)

B. WHITE, A new proof of the compactness theorem for integral currents, preprint.

(This is the newest, and perhaps the shortest, of the compactness proofs.)

F. ALMGREN
Department of Mathematics
Princeton University
PRINCETON N.J. 08544
(U.S.A.)