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LOCAL AND GLOBAL RESULTS ON THE DESINGULARIZATION OF THREE-DIMENSIONAL VECTOR FIELDS

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INTRODUCTION

These notes are intended for presenting some results concerning the desingularization of vector fields by means of blowing-ups of the ambient space. The first part is devoted to the introduction of the general theory used in the sequel as well as to the presentation of some local results obtained in previous papers ([1],[4]). The second part is devoted to the proof of certain results about the global reduction of the singularities of three-dimensional vector fields.

I would like to thank Professor Aroca for his comments. (*).

PART I

1. ADAPTED VECTOR FIELDS

(1.1) Let X be a regular variety, i.e. a regular integral separated scheme of finite type over en algebraically closed field k. Let us denote by Ω_X, resp. Ξ_X, the cotangent, resp. tangent, sheaf of X relatively to k. Both Ω_X and Ξ_X are locally free $\mathcal{O}_X$-modules of rank n = dim X.

A closed subscheme $E \subset X$ is a "normal crossings divisor od X" iff for each closed point $P \in X$, the ideal $I(E)_P$ of $\mathcal{O}_{X,P}$ is generated by $\Pi_{i\in A}x_i$, where $A \subset \{1,\dots,n\}$ and $(x_1,\dots,x_n)$ is a regular system of parameters of the local ring $\mathcal{O}_{X,P}$. Let us denote by $\Xi_X[E]$ the sheaf of the "germs of vector fields tangent to E", which is given by

$$(1.1.1) \qquad \Xi_{X,P}[E] = \{D \in \Xi_{X,P}\ ;\ D(I(E)_P) \subset I(E)_P\}$$

for each closed point P of X. A base of $\Xi_{X,P}[E]$ is given by $\{x_i^{\phi(i)}\partial/\partial x_i\}_{i=1\dots n}$, where $\phi(i) = 1$ if $i \in A$ and $\phi(i) = 0$ otherwise.

(*) I would like also to thank the referee comments.

(1.2) Any invertible $\mathcal{O}_X$-submodule $\mathcal{D}$ of $\Xi_X[E]$ will be called an "unidimensional distribution over X adapted to E" (i.e., $\mathcal{D}$ is locally generated by a vector field $D \in \Xi_{X,P}[E]$).

Let E' be another normal crossings divisor. The "adaptation $(\mathcal{D},E')$ of $\mathcal{D}$ to E' "is defined to be $(\mathcal{D},E') = \mathcal{D} \cap \Xi_X[E']$. We know that $(\mathcal{D},E')$ is an unidimensional distribution over X adapted to E', (see [4]. I (1.3.3)).

$\mathcal{D}$ is said to be "multiplicatively irreducible and adapted to E" iff $\mathcal{D}$ coincides with its double orthogonal $\alpha_E(\mathcal{D})$ relatively to the natural pairing between $\Xi_X[E]$ and its dual sheaf. We know that $\alpha_E(\mathcal{D})$ is also an unidimensional distribution over X adapted to E which is multiplicatively irreducible. ([4].I. (1.2.3)). Moreover, if $E' \supset E$ one has

(1.2.1) $$\alpha_{E'}(\mathcal{D},E') = (\alpha_E(\mathcal{D}),E')$$

([4].I.(1.3.5.2)).

Let us assume that $\mathcal{D}_P$ is generated by $D = \sum_{i \in A} a_i x_i \partial/\partial x_i + \sum_{i \notin A} a_i \partial/\partial x_i$, then $\alpha_E(\mathcal{D})_P$ is generated by D/b, where b = g.c.d.(a_i). Note that in the case $E = \emptyset$, the singular saturated foliation given locally by D is the same as the one given by D/b ([4].Chap.I).

(1.3) Let $Y \subset X$ be a closed subscheme of X. Y has "normal crossings" with E at a closed point P iff there is a regular system of parameters $(x_1,\ldots,x_n)$ of $\mathcal{O}_{X,P}$ (called "suited for (E,Y) at P") and sets A, $B_j \subset 1,\ldots,n$ for each irreducible component Y_j of Y, such that

(1.3.1) $$I(Y_j)_P = \sum_{i \in B_j} x_i \mathcal{O}_{X,P} \ ; \ I(E)_P = (\prod_{i \in A} x_i)\mathcal{O}_{X,P}.$$

Y has "normal crossings with E" iff it is so at each closed point P.

Let $\pi: X' \to X$ be the blowing-up of X with a center Y having normal crossing with E. Then $E' = \pi^{-1}(E \cup Y)$ is a normal crossings divisor of X'. Moreover, there is a unique multiplicatively irreducible and adapted to E' unidimensional distribution $\mathcal{D}'$ over X' such that $\mathcal{D}'|_{\pi^{-1}(X-Y)} \simeq \mathcal{D}|_{X-Y}$ via the induced isomorphism between the tangent sheaves of $\pi^{-1}(X-Y)$ and X-Y (see [4].I.2). $\mathcal{D}'$ will be called the "strict transform of $\mathcal{D}$ adapted to E".

Assume that $(x_1,\ldots,x_n)$ is suited for (E,Y) at P and that P' is a closed point of X' such that $\pi(P') = P$. Then there is a regular system of parameters $(x'_1,\ldots,x'_n)$ of $\mathcal{O}_{X',P'}$ suited for (E',P'), an index $i_o \in B$ and scalars $\zeta_i \in k$, $i \in B-\{i_o\}$ such that $x_{i_o} = x'_{i_o}$, $x_i = (x'_i + \zeta_i)x'_{i_o}$, $i \in B-\{i_o\}$, $x_i = x'_i$, $i \notin B$, under the inclusion $\mathcal{O}_{X',P'} \subset \mathcal{O}_{X,P}$. Let us observe that $I(E')_{P'} = \prod_{i \in A'} x'_i$, where $A' = (A-B) \cup \{i_o\} \cup \{i \in A \cap B;\ \zeta_i = 0\}$. If $\mathcal{D}_P$ is generated by $D = \sum_{i \in A} a_i x_i \partial/\partial x_i +$

$+ \sum_{i\notin A} a_i \mu\, \partial/\partial x_i$, then there is a $\in \mathbb{Z}$ such that

(1.3.2) $$D' = [x'_{i_o}]^{\mu} a_{i_o} x'_{i_o} \partial/\partial x'_{i_o} +$$

$$+ \sum_{i\in A'\cap B-\{i_o\}} (a_i - a_{i_o}) x'_i \partial/\partial x'_i +$$

$$\sum_{i\in (A-A')\cap B-\{i_o\}} (a_i - a_{i_o})(x'_i + \zeta_i)\partial/\partial x'_i +$$

$$\sum_{i\in B-A} (a_i/x'_{i_o} - (x'_i+\zeta_i)a_{i_o})\partial/\partial x'_i +$$

$$+ \sum_{i\in A'-B} a_i x'_i \partial/\partial x'_i + \sum_{i\notin A'\cup B} a_i \partial/\partial x'_i].$$

if $i_o \in A$. If $i_o \notin A$, one has

(1.3.3) $$D' = [x'_{i_o}]^{\mu}[(a_{i_o}/x'_{i_o})x'_{i_o}\partial/\partial x'_{i_o} +$$

$$\sum_{i\in A'\cap B-\{i_o\}} (a_i - a_{i_o}/x'_{i_o}).x'_i\partial/\partial x'_i +$$

$$\sum_{i\in (A-A')\cap B} (a_i - a_{i_o}/x'_{i_o})(x'_i+\zeta_i)\partial/\partial x'_i +$$

$$\sum_{i\in B-A-\{i_o\}} (a_i/x'_{i_o} - (x'_i+\zeta_i)a_{i_o}/x'_{i_o})\partial/\partial x'_i +$$

$$+ \sum_{i\in A'-B} a_i x'_i\partial/\partial x'_i + \sum_{i\notin A'\cup B} a_i\partial/\partial x'_i].$$

(see [4].I.(2.2.5)). Let us define $\mu(\mathcal{D};E,P) = \mu$.

2. SINGULAR LOCUS

(2.1) Let $\mathcal{D}$ be a multiplicatively irreducible unidimensional distribution adapted to E and let Q be a point of X. The "adapted order $\nu(\mathcal{D},E,Q)$ of $\mathcal{D}$ at Q" is the maximum integer m such that $\mathcal{D}_Q \subset \eta^m.\Xi_{X,Q}[E]$, where η is the maximal ideal of $\mathcal{O}_{X,Q}$. The adapted order $\nu(\mathcal{D},E,Q)$ is then the minimum of the orders of the coefficients of a generator of $\mathcal{D}_Q$ along Q.

For each $r \geq 1$, let us denote $\mathrm{Sing}^r(\mathcal{D},E)$ the set of the points Q such that $\nu(\mathcal{D},E,Q) \geq r$. The "singular locus $\mathrm{Sing}(\mathcal{D},E)$" is defined to be $\mathrm{Sing}^1(\mathcal{D},E)$.

(2.2) Proposition.- Let π: X' $\to$ X be the blowing-up of X centered in a closed point P and let P' be a closed point of X' such that $\pi(P') = P$. Then

(2.2.1) $$\nu(\mathcal{D}',E',P') \leq \nu(\mathcal{D},E,P).$$

where $\mathcal{D}'$, E' are obtained as in (1.3). (The proof is easy, see ([4]. (3.1.4))).

The above proposition justifies the adapted viewpoint. In the non adapted case, we can assure that $\nu'+1 \leq \nu$. See, for example, Seidenberg's paper [13].

If the blowing-up π is not quadratic, (2.2.1) is not true in general. This motivates the following.

(2.3) Definition.- A closed subscheme $Y \subset X$ is said to be "weakly permissible for $\mathcal{D}$ adapted to E at the closed point P" if

a) $Y \subset \mathrm{Sing}(\mathcal{D},E)$, $\dim Y \leq \dim X-2$.

b) Y has normal crossings with E at P. (This implies that I is regular at P and that there is an open set $U \ni P$ such that Y and E have normal crossings in U).

c) Let $\pi: U' \to U$ be the blowing-up with center $Y \cap U$, then for each closed point P' such that $\pi(P') = P$, one has $\nu(\mathcal{D}',E',P') \leq \nu(\mathcal{D},E,P)$.

The closed subscheme Y is said to be "weakly permissible" if it is so for each closed point P.

The following definition is more directly related with the usual notion of permissible center, since it arises from the consideration of stationary sequences produced after quadratic blowing-ups (see [12] and [4] I.(3.3)).

(2.4) Definition.- Let Y be a closed curve in X. Y is said to be "permissible for $\mathcal{D}$ adapted to E at the closed point P" iff

a) $Y \subset \mathrm{Sing}(\mathcal{D},E)$, Y has normal crossings with E at P and $\dim Y=1 \leq \dim X-2$

b) Let D be a generator of $\mathcal{D}_P$, $r = \nu(\mathcal{D},E,P)$ and $\rho = \min(r+1, \nu_P(D(I(Y))))$. Then

$$\nu_Y(D(I(Y))) \geq \rho \tag{2.4.1}$$

$$\nu_Y(D(I(P))) \geq \rho-1$$

where ν_P, resp. ν_Y, is the order at P, resp. at Y, I(P) is the maximal ideal of $\mathcal{O}_{X,P}$ and I(Y) is the ideal of Y in $\mathcal{O}_{X,P}$.

The curve Y is said to be "permissible" if it is so for each closed point P. Permissibility is semicontinuous in the following sense: given a curve Y, the set of closed points P of Y such that Y is permissible at P is open in Y (See [4]. I.(3.4.5)).

(2.5) Proposition.- "Permissible" implies "Weakly permissible". (See [4].I.(3.4.6)).

Since permissible centers arise asymptotically in the stationary sequences, one can prove the following result of globalization ([4]. I. (3.4.9)).

(2.6) Theorem.- Assume that Y is a curve permissible for $\mathcal{D}$ adapted to E at the closed point P. Then by a finite sequence of quadratic blowing-ups centered at the non permissible points of Y (and of the succesive transforms), we can ensure that the strict transform Y' is globally permissible for $\mathcal{D}'$ adapted to E'.

3. DESINGULARIZATION STATEMENTS

(3.1) Fot the case dim X = 2 it is not possible in general to obtain $\nu(\mathcal{D},E,P) = 0$ for each closed point P of X. For instance, let $X = \mathbb{A}^2(k)$ and let $\mathcal{D}$ be globally generated by $D = x.x\partial/\partial x + (y-myx)\partial/\partial y$, $m \in \mathbb{N}$, where E is given by x=0. The, if one makes a quadratic blowing-up centered at the only singular point (the origin), one obtains only one singular point, which has the same local expression but with m' = m+1. (See [1] or [4].I.(4.1)). One has the following.

(3.2) <u>Theorem</u> ([1]) Assume dim X = 2. Then after a finite number of quadratic transformations centered at the points of Sing($\mathcal{D}$,E), one can obtain $\mathrm{Sing}^2(\mathcal{D},E)=\emptyset$.

(3.3) One can say something more for the case dim X = 2. Let P be a closed point of X and let R be the completion of the local ring $\mathcal{O}_{X,P}$. Let X^ = Spec(R), E^ = = induced normal crossings divisor in X^, $\mathcal{D}$^ = induced unidimensional distribution over X^ (for X^ one can develop a theory similar to the one for X in order to define unidimensional distribution, adapted order, etc.). Then one has the following.

<u>Theorem</u> ([1]) If dim X = 2, then after a finite number of quadratic blowing-ups centered at Sing($\mathcal{D}$,E) one can ensure that for each closed point P of X there is a regular hypersurface Y^ of X^ such that E^ ∪ Y^ is a normal crossings divisor of X^ and $\nu(\mathcal{D}\hat{}, E\hat{} \cup Y\hat{}, P\hat{}) = 0$.

The hypersurface Y^ is not necessarily convergent, as one can see by taking $\mathcal{D}$ globally generated by $D = x.x\,\partial/\partial x + (y-(m!)x-mxy)\partial/\partial y$ (see [4].I. (4.1.4)).

Let us remark that Seidenberg's result ([13] th. 12) is a direct corollary of the above theorem.

(3.4) <u>Reduction game</u>.- The "reduction game beginning at (X,E,$\mathcal{D}$,P)" is a game between two players A and B with the following rules. Let $r = \nu(\mathcal{D},E,P)$ and let status (0) = (X(0),E(0),$\mathcal{D}$(0),P(0)) = (X,E,$\mathcal{D}$,P). Assume that status (t) = = (X(t),E(t),$\mathcal{D}$(t),P(t)). First, the player A chooses an open set U(t) ∋ P(t) and a closed subscheme Y(t) of U(t) which is weakly permissible for $\mathcal{D}$(t) adapted to E(t) in the whole U(t) and makes the corresponding blowing up; let us denote $\pi(t+1)\colon X(t+1) \longrightarrow U(t)$ the blowing-up of U(t) with center Y(t); second, the player B chooses a closed point P(t+1) of X(t+1) such that $\pi(t+1)(P(t+1)) = P(t)$. Let $\mathcal{D}$(t+1),E(t+1) be the objects obtained form π(t+1) as in (1.3) from $\mathcal{D}$(t),E(t).

The play must stop if $\nu(\mathcal{D}(t+1),E(t+1),P(t+1)) < r$ and then the player A wins. Otherwise stat $(t+1) = (X(t+1),E(t+1),\mathcal{D}(t+1),P(t+1))$ and the game continues.

Let us denote by $mov(t+1) = (U(t),Y(t),\pi(t+1),P(t+1))$ the $(t+1)$-st move of the game. A "realization of the reduction game beginning at $(X,E,\mathcal{D},P)$" is any sequence $G = \{G(t) = (\text{status } (t), mov(t))\}_{t=0,1...}$ which respects the rules of the game. (Let us put status $(0) = \emptyset$ for completeness).

A "winning strategy for the reduction game beginning at $(X,E,\mathcal{D},P)$" is a sequence of functions $F(t)$, $t = 0,1,...$ such that:

a) $F(t)$ is defined over the set of realizations G of length equal to $t-1$ such that $\nu(\mathcal{D}(t-1),E(t-1),P(t-1)) = r$.

b) $F(t)(G) = (U_F(t-1)(G), Y_F(t-1)(G))$, where $U_F(t-1)(G)$ is an open subset of $X(t-1)$ and $Y_F(t-1)(G)$ is a closed subscheme of $U_F(t-1)(G)$ which is weakly permissible for $\mathcal{D}(t-1)$ adapted to $E(t-1)$ in all of $U_F(t-1)(G)$.

c) If G is any realization of the reduction game which satisfies

(3.4.1) $$mov(t) = (U_F(t-1)(G|_{t-1}), Y_F(t-1)(G|_{t-1}), \pi(t), P(t))$$

for each $t \leq$ length (G) (here $G|_{t-1}$ means $\{G(s)\}_{0 \leq s \leq t-1}$), then G is finite.

(3.5) The paper [4] is mainly devoted to the proof of the following.

<u>Theorem</u> ([4].I.(4.2.9)). If $\dim X = 3$ and $r \geq 2$, then there is a winning strategy for the reduction game beginning at $(X,E,\mathcal{D},P)$.

The existence of a winning strategy does not imply directly a result of global reduction: one needs to show that the strategies may be chosen "coherently enough" in order to "path" in each step of the process. Obviously, the following global statement is stronger than the requirement of the existence of a winning strategy.

(3.6) <u>Global reduction statement</u>.- Given $(X,E,\mathcal{D})$ as usual, let r = biggest integer a such that $\mathrm{Sing}^s(\mathcal{D},E) \neq \emptyset$, and assume that $r \geq 2$. Then there is a finite sequence of blowing-ups with weakly permissible centers

$$X = X(0) \xleftarrow{\pi(1)} X(1) \longleftarrow \cdots \xleftarrow{\pi(N)} X(N)$$

such that $\mathrm{Sing}^r(\mathcal{D}(N),E(N)) = \emptyset$

(3.7) By (3.2), the above statement is true if $\dim X = 2$. The rest of this paper is devoted to proving the statement (3.6) in the case $\dim X = 3$, for a certain kind of unidimensional distributions.

PART II

1. THE DIRECTRIX

(1.1) Let $\mathcal{D}$ be a multiplicatively irreducible unidimensional distribution over X adapted to E, let P be a closed point of X and let us fix a generator D of $\mathcal{D}_P$. Let us denote $R = \mathcal{O}_{X,P}$, $r = \nu(\mathcal{D},E,P) \geq 1$.

For each $f \in R$ such that $\nu_M(f) \geq r$ (M = maximal ideal of R), let $J^r(f)$ be the ideal of $Gr_M(R)$ given by $J^r(f) = 0$ if $\nu_M(f) > r$, $J^r(f)$ = the smallest ideal generated by linear forms such that

$$(1.1.1) \qquad in(f) \in J^r(f)$$

(i.e. $J^r(f)$ is the ideal of the strict tangent space of $f = 0$).

(1.2) Definition.- The ideal $J(\mathcal{D},E,P)$ is defined to be

$$(1.2.1) \qquad \sum_{f \in R} J^r(D(f)) = J^r(\mathcal{D},E,P)$$

if (1.2.1) is different from zero (i.e. $\nu_P(D(R)) = r$) or

$$(1.2.2) \qquad \sum_{\{f \in R;\ f.R \supset I(E)\}} J^r(D(f)/f)$$

otherwise (i.e. $\nu_P(D(R)) = r+1$). The "directrix of $\mathcal{D}$ at P adapted to E", denoted $Dir(\mathcal{D},E,P)$, is defined to be the linear subspace of the tangent space T_PX given by $V(J(\mathcal{D},E,P))$.

(1.3) Theorem.- Let $\pi: X' \to X$ be the blowing-up of X centered at the closed point P. Assume that $P' \in \pi^{-1}(P)$ is a closed point such that $\nu(\mathcal{D}',E,P') = r$. Then

$$(1.3.1) \qquad P' \in Proj(Dir(\mathcal{D},E,P)),$$

under the identification $\pi^{-1}(P) \simeq Proj(T_PX)$. (See [4].I.(3.2.7)).

(1.4) Corollary.- Let r = biggest s such that $Sing^s(\mathcal{D},E) \neq \emptyset$, $r \geq 1$, and assume that for each closed point $P \in Sing^r(\mathcal{D},E)$ one has $\dim Dir(\mathcal{D},E,P) = 0$. Then there is a sequence of quadratic blowing-ups

$$(1.4.1) \qquad X = X(0) \xleftarrow{\pi(1)} \ldots \xleftarrow{\pi(N)} X(N)$$

such that $Sing^r(\mathcal{D}(N),E(N)) = \emptyset$.

Proof.- It is enough to show that $Sing^r(\mathcal{D},E)$ is composed of isolated points. Let P be a closed point of $Sing^r(\mathcal{D},E)$ and let $\pi: X' \to X$ be the quadratic blowing-up of X centered at P. In view of (1.3.1) one has $\nu(\mathcal{D}',E'P') < r$ for each $P'\in \pi^{-1}(P)$. If there is a component Y of $\mathrm{Sing}^r(\mathcal{D},E)$ with dim $Y \geq 1$, $Y \ni P$, then the strict transform Y' satisfies $Y'\cap \pi^{-1}(P)\neq \emptyset$ and for each $P'\in Y'$, $\nu(\mathcal{D}'E',P')=r$. Contradiction. Thus the points of $\mathrm{Sing}^r(\mathcal{D},E)$ are isolated (hence there are finitely many of them) and the quadratic blowing-ups centered at these points give the result.

2. TYPE ZERO POINTS

(2.1) Let $(X,E,\mathcal{D},P)$ be as above. Let $R = \mathcal{O}_{X,P}$. For each $W\subset \mathrm{Gr}\ R$, let us denote

(2.1.1) $$H(W) = \{f\in R,\ \mathrm{In}(f)\in W\}.$$

Let us define $W_i(\mathcal{D},E,P)$ inductively as follows:

(2.1.2) $$W_o(\mathcal{D},E,P) = \mathrm{Gr}\ R$$ $$W_i(\mathcal{D},E,P) = J^r(D(H(W_{i-1}(\mathcal{D},E,P)))).$$

If $r = \nu(D(R))$, $W_1(\mathcal{D},E,P) = J(\mathcal{D},E,P)$. Moreover $W_{i+1}\subseteq W_i$ and since W_i is generated by linear forms ther is an N such that for $i\geq N, W_{i+1} = W_i$. Let us denote

(2.1.3) $$W(\mathcal{D},E,P) = \bigcap_i W_i(\mathcal{D},E,P)$$

(2.2) <u>Definition</u>.- The closed point P is said to be of "type zero" iff $W(\mathcal{D},E,P)\neq 0$. In this case, the "W-directrix $\mathrm{Dir}_W(\mathcal{D},E,P)$" is defined to be the linear subvariety $V(W(\mathcal{D},E,P))$ of T_PX given by $W(\mathcal{D},E,P)$.

(<u>Remark</u>: The points considered in [4].II(1.1.3) are exactly the points of type zero such that $e(E,P)\neq 0$, where $e(E,P)$ = number of components of E at P, and such that dim $\mathrm{Dir}(\mathcal{D},E,P)\neq 0$).

(2.3) <u>Proposition</u>.- Let $Y\subset X$ be a curve permissible for $\mathcal{D}$ adapted to E at the closed point P. Let π: $X'\longrightarrow X$ be the blowing-up of X centered at P and let $E',\mathcal{D}',Y'$ be the corresponding transforms. Let $P' = \pi^{-1}(P)\cap Y'$. Assume that P is a point of type zero. Then Y is tangent to $\mathrm{Dir}_W(\mathcal{D},E,P)$ iff $\nu(\mathcal{D}',E',P') = \nu(\mathcal{D},E,P) = r$.

<u>Proof</u>.- Let $D = \sum_{i\in A} a_i x_i \partial/\partial x_i + \sum_{i\notin A} a_i \partial/\partial x_i$, where $I(E) = (\Pi_{i\in A} x_i).R$ ($R = \mathcal{O}_{X,P}$). One can assume that $I(Y) = (x_2,\ldots,x_n)$. Assume first that Y is tangent to $\mathrm{Dir}_W(\mathcal{D},E,P)$. Then $W(\mathcal{D},E,P)\subset \sum_{i\geq 2}\bar{x}_i$. Gr R and hence

(2.3.1) $$J^r(D(I(Y)))\neq 0.$$

This implies that $r = \nu_P(D(I(Y)))$. If $1 \in A$, the equations I(1.3.2) (where $\mu = r-1$) together with I.(2.4.1) show that $\nu(\mathcal{D}',E',P') = r$. If $1 \notin A$, one has $\nu(\mathcal{D}',E',P') < r$ iff $In(a_1) = \Psi + \bar{x}_1 \cdot \phi$, where Ψ and ϕ are homogeneous in $\bar{x}_2,\ldots,\bar{x}_n$ of degree $r, r-1$ respectively, and $\phi \neq 0$. But this implies that $W_1(\mathcal{D},E,P) \not\subset (x_2,\ldots,x_n)$. Since for each $i \notin A$, $i \neq 1$ one has $In^r(a_i) = \Psi_i(\bar{x}_2,\ldots,\bar{x}_n)$, one deduces that $W_j(\mathcal{D},E,P) \not\subset (\bar{x}_2,\ldots,\bar{x}_n)$ for each $j \geq 1$, thus Y cannot be tangent to $Dir_W(\mathcal{D},E,P)$. Contradiction. The converse can be proved by reversing the arguments.

Corollary.- Let $r = \max(j;\ Sing^j(\mathcal{D},E) \neq \emptyset)$. Assume that each closed point in $Sing^r(\mathcal{D},E)$ is of type zero and let T be a curve in X which is globally permissible. Then the set of points P in $Y \cap Sing^r(\mathcal{D},E)$ such that Y is tangent to $Dir_W(\mathcal{D},E,P)$ is closed.

Proof.- If $P \in Y \cap Sing^r(\mathcal{D},E)$ and Y is not tangent to $Dir_W(\mathcal{D},E,P)$, then after a blowing-up one deduces that $\nu(\mathcal{D},E,Q) < r$ for $P \neq Q \in U$, $U \ni P$ open.

Remarks.- If $Y \subset Sing^r(\mathcal{D},E)$, then necessarily Y is tangent to $Dir_W(\mathcal{D},E,P)$ for all $P \in Y$.

The above proof shows also that the points P of $Y \cap Sing^r(\mathcal{D},E)$ such that Y is not tangent to $Dir_W(\mathcal{D},E,P)$ are finite in number. In particular, blowing-up these points one can obtain that Y is tangent to $Dir_W(\mathcal{D},E,P)$ for each $P \in Y \cap Sing^r(\mathcal{D},E)$.

(2.4) Theorem.- Let P be a type zero point of X and let Y be {P} or a permissible curve at P which is tangent to $Dir_W(\mathcal{D},E,P)$ at P. Let U be an open set, $U \ni P$, such that Y is globally permissible in U and let $\pi : X' \rightarrow U$ be the blowing-up of U with center Y. Then, for each closed point P' of X' such that $\pi(P') = P$ and $\nu(\mathcal{D}',E',P') = \nu(\mathcal{D},E,P) = r$, we have

a) $P' \in Proj(Dir_W(\mathcal{D},E,P)/T_PY)$

b) P' is of type zero and

$$\dim Dir_W(\mathcal{D}',E',P') \leq \dim Dir_W(\mathcal{D},E,P).$$

Proof.- First of all, let us reduce the problem to the case $e(E,P) < e(E',P')$ (actually, $e(E,P) = e(E',P')-1$): Let F be the union of all the components of E which contain the point P' (as an infinitely near point). One sees easily that $\nu(\mathcal{D},F,P) = r$, P is of type zero for $(\mathcal{D},F)$, $Dir_W(\mathcal{D},E,P)=Dir_W(\mathcal{D},F,P)$, Y is permissible for $(\mathcal{D},F)$ at P and tangent to $Dir_W(\mathcal{D},F,P)$. If $(\mathcal{D}'',F')$ denotes the strict transform of $(\mathcal{D},F)$ at the point P', one has

(2.4.1) $\mathcal{D}' = \mathcal{D}''$, $E' = F'$ (at P').

Then we can suppose that $e(E,P) < e(E',P')$. In this case, one can take a regular system of parameters $(x_1,\dots,x_n)$ at P such that $I(E) = (\Pi_{i\in A} x_i).R$, $I(Y) = \sum_{i\in B} x_i R$ where $R = \mathcal{O}_{X,P}$ and such that the transformation π at P' is given by

(2.4.2)
$$x'_i = x_i \quad i \notin B \ \text{ or } \ i = i_o$$
$$x'_i x'_{i_o} = x_i \quad i \in B - \{i_o\}$$

where $i_o \notin A$.

a) Since Y is tangent to $\mathrm{Dir}_W(\mathcal{D},E,P)$, one has $W(\mathcal{D},E,P) \subset \sum_{i\in B} \bar{x}_i \mathrm{Gr}R$ and thus

(2.4.3) $$0 \neq W(\mathcal{D},E,P) \subset J^r(D(I(Y))).$$

This implies that $\nu_P(D(I(Y))) = r$. In view of I.(2.4) ($\rho=r$) one deduces that Y is tangent to $V(J^r(D(I(Y))))$ and looking at the equations I.(1.3.3) (where $\mu = r-1$) one has that

(2.4.4) $$P' - \mathrm{Proj}(V(J^r(D(I(Y))))/T_P Y),$$

and a) follows immediatly.

Let us remark that from (2.4.4) and (2.4.2) one deduces that

(2.4.5) $$J^r(D(I(Y))) \subset \sum_{i\in B, i\neq i_o} \bar{x}_i.\mathrm{Gr}\ R$$

b) Let $Z = \sum_{i\in B'} \bar{x}_i.\mathrm{Gr}\ R$, where $B' = (\{1,\dots,n\}-B) \cup \{i_o\}$. Let us define $W_{Z,0} = Z$, $W_{Z,j} = J^r(D(H(W_{Z,j-1})))$, $W_Z = \bigcap_j W_{Z,J}$. By (2.4.5) one has

(2.4.6) $$W_{Z,1} \subset W_{Z,0} = Z$$

and hence $W_{Z,j} \subset W_{Z,j-1}$ for all j.

On the other hand one has $W(\mathcal{D},E,P) \subset Z$, hence $W(\mathcal{D},E,P) = W_Z$.

Let α: $\mathrm{Gr}\ R \longrightarrow \mathrm{Gr}\ R'$ ($R' = \mathcal{O}_{X',P'}$) be the isomorphism of graded k-algebras given by $\alpha(\bar{x}_i) = \bar{x}'_i$, $1 \leq i \leq n$. Let $Z' = \alpha(Z)$. As above, let us define $W_{Z',0} = Z'$, $W_{X',j} = J^r(D'(H(W_{Z',j-1})))$, $W_{Z'} = \bigcap_j W_{Z',j}$. One sees easily that

(2.4.7) $$W_{Z'} \subset W(\mathcal{D}',E',P')$$

Thus, in order to prove b) it is enough to show that $\dim_k W^*_{Z'} \geq \dim_k W(\mathcal{D},E,P)^* = \dim_k W^*_Z$, where * means the linear part.

Let $\bar{R}' = R'/\sum_{i\in B'} x_i R'$. Let β: $\mathrm{Gr}\ R' \rightarrow \mathrm{Gr}\ \bar{R}'$ be the induced morphism.

Since $W_Z \subset Z$, one sees easily that

(2.4.8) $$\dim_k W_Z^* = \dim_k \beta\alpha(W_Z)^*.$$

Then, it is enough to prove that

(2.4.9) $$\beta\alpha(W_Z)^* \subset \beta(W_{Z'})^*.$$

In order to prove (2.4.9), let us show inductively that

(2.4.10) $$\beta\alpha(W_{Z,j})^* \subset \beta(W_{Z',j})^*$$

If $j = 0$, then (2.4.10) follows by assumption. Now, remark that

(2.4.11) $$\beta\alpha(\mathrm{In}^r(D(x_i))) = \beta\,\mathrm{In}^r(D'(x_i')),\quad i \notin B'$$

recalling I.(1.3.3) and (2.4.2). Let $j > 0$ and let $\overline{\phi} \in \beta\alpha(W_{Z,j})^* = \beta\alpha(W_{Z,j^*})$. Let $\Psi = \sum_{i \notin B'} \lambda_i x_i$ be an element of $W_{Z,j-1}{}^*$ (recall that $W_{Z,j-1} \subset Z$) such that there exists a ϕ satisfying $\overline{\phi} = \beta\alpha(\phi)$ and

(2.4.12) $$\phi \in J^r(D(\textstyle\sum_{i \notin B'} \lambda_i x_i))$$

By the induction hypothesis, theres is $\Psi' = \sum_{i \notin B'} \lambda_i x_i' + \sum_{j \in B'} \mu_j x_j'$ such that $\Psi' \in W_{Z',j-1}$. One has

(2.4.13) $$\beta\,\mathrm{In}^r(D'(\sum_{i \notin B'} \lambda_i x_i' + \sum_{j \in B'} \mu_j x_j'))) = \beta(\mathrm{In}^r(D'(\sum_{i \notin B'} \lambda_i x_i')))$$

since looking at I(1.3.3) one deduces that

(2.4.14) $$\beta(\mathrm{In}^r(D'(x_i))) = 0 \text{ if } i \in B'.$$

Now, from (2.4.11) one deduces easily that there is $\phi' \in J^r(D'(\sum_{i \notin B'} \lambda_i x_i' + \sum_{j \in B'} \mu_j x_j))$, (hence $\phi' \in W_{Z',j}$), such that $\beta(\phi') = \overline{\Phi}$. This ends the proof of (2.4.10) and thus b) is proved.

3. MAIN RESULTS

The rest of this paper is devoted to the proof of the following.

(3.1) Theorem.- Let r be the biggest s such that $Sing^s(\mathcal{D},E) \neq \emptyset$. Assume that $\dim X = 3$, $r \geq 2$ and that each closed point of $Sing^r(\mathcal{D},E)$ is of type zero. Then there is a sequence of permissible blowing-ups

$$X = X(0) \xleftarrow{\pi(1)} X(1) \longleftarrow \dots \xleftarrow{\pi(N)} X(N)$$

such that $Sing^r(\mathcal{D}(N),E(N)) = \emptyset$.

(3.2) With the hypothesis of (3.1), let us denote by $S_i(\mathcal{D},E)$, $i = 0, 1,2$ the set of all the closed points P of $Sing^r(\mathcal{D},E)$ such that $\dim Dir_W(\mathcal{D},E,P) = i$.

If $S_2(\mathcal{D},E) = S_1(\mathcal{D},E) = \emptyset$, then (3.1) is a corollary of (1.4) since $Dir_W(\mathcal{D},E,P) \supset Dir(\mathcal{D},E,P)$.

The stability Theorem (2.4) will allow us to proceed by eliminating first $S_2(\mathcal{D},E)$, then $S_1(\mathcal{D},E)$ and, finally, $S_o(\mathcal{D},E)$.

(3.3) For $j = 1,2$, let us denote by $S_j^*(\mathcal{D},E)$ the set of one dimensional irreducible components Y of $Sing^{r-1}(\mathcal{D},E)$ having the following property:

(3.3.1) "There is a closed point $P \in S_j(\mathcal{D},E)$ and a sequence of quadratic blowing-ups $\pi(i)$: $X(i) \longrightarrow X(i-1)$, $1 \leq i \leq N$, $X(0) = X$, $P(0) = P$, $Y(0) = Y$, such that $\pi(i)$ is centered at the closed point $P(i-1) \in \pi(i-1)^{-1}(P(i-2)) \cap \cap Y(i-1)$, where $Y(i)$ is the strict transform of $Y(i-1)$, in such a way that

a) $\nu(\mathcal{D}(i),E(i),P(i)) = r$, $P(i) \in S_j(\mathcal{D}(i),E(i))$, $0 \leq i \leq N$.

b) $Y(N)$ is permissible and tangent to $Dir_W(\mathcal{D}(N),E(N),P(N))$ at $P(N)$"

(3.4) Definition.- Assume that $S_2(\mathcal{D},E) = \emptyset \neq S_1(\mathcal{D},E)$ and let π: $X' \longrightarrow X$ be a blowing-up with center Y. Let S be the first one among the following statements which is true:

I. There is a closed point $Q \in X$ such that $S_1^*(\mathcal{D},E)$ does not have normal crossings with E at Q or such that Q is contained in at least two elements of $S_1^*(\mathcal{D},E)$.

II. There is a curve $Z \in S_1^*(\mathcal{D},E)$ such that Z is a component of $Sing^r(\mathcal{D},E)$.

III. There is a curve $Z \in S_1^*(\mathcal{D},E)$ which is permissible and tangent to $Dir_W(\mathcal{D},E,P)$ at each closed point $P \in Z$.

IV. There is a $Z \in S_1^*(\mathcal{D},E)$ and a closed point $Q \in Z$ such that Z is not permissible and tangent to $Dir_W(\mathcal{D},E,Q)$ at Q.

V. There is a closed point $Q \in S_1(\mathcal{D},E)$.

Then π "respects the 1-global procedure of reduction" iff:

(3.4.1) "If S = I, IV or V then Y = Q. If S = II or III, then Y = Z.

(3.5) Lemma.- Under the assumptions of (3.1), if Z is a curve contained in $Sing^r(\mathcal{D},E)$ and having normal crossings with E, then Z is globally permissible and for each closed point $P \in Z$ one has that Z is tangent to $Dir_W(\mathcal{D},E,P)$.

Proof.- Let $P \in Z$ be a closed point. Blowing-up P, one has that $Z' \cap \pi^{-1}(P) \subset Proj(Dir_W(\mathcal{D},E,P))$ by (2.4), thus Z is tangent to $Dir_W(\mathcal{D},E,P)$ (actually, Z is tangent to $Dir(\mathcal{D},E,P)$). Now, since dim $Dir_W(\mathcal{D},E,P) \geq 2$ and Z is tangent to it, one deduces that $\nu_P(D(I(Z))) = r$ and thus Z is permissible since $\nu(\mathcal{D},E,Z)=r$.

(3.6) Assume that $\pi: X' \longrightarrow X$ respects the 1-global procedure of reduction. By (2.4) and in view of (3.5) all the closed points in $Sing^r(\mathcal{D}',E')$ are of type zero and $S_2(\mathcal{D}',E') = \emptyset$. So, one can make the following

(3.7) Definition.- Assume that $S_2(\mathcal{D},E) = \emptyset \neq S_1(\mathcal{D},E)$. Let $I = \{1,2,..,N\}$ or $I=\mathbb{N}$. A sequence of permissible blowing-ups.

$$(3.7.1) \qquad \mathcal{S} = \{\pi(i): X(i) \longrightarrow X(i-1)\}_{i \in I}$$

with X(0) = X, is said to "respect the 1-global procedure of reduction" iff each $\pi(i)$, $i \in I$, respects the 1-global procedure of reduction (hence $S_1(\mathcal{D}(i-1),E(i-1)) \neq \emptyset$ and if it is finite one has $S_1(\mathcal{D}(N),E(N)) = \emptyset$.

(3.8) By (3.4) and (3.6), there is always a sequence $\mathcal{S}$ respecting the 1-global procedure of reduction (maybe infinite). Paragraph 4 is devoted to the proof of the following theorem.

Theorem.- Any sequence $\mathcal{S}$ which respects the 1-global procedure of reduction is finite.

One deduces (3.1) from (3.2) and (3.8) for the case $S_2(\mathcal{D},E) = \emptyset$.

(3.9) Let establish now a priority order as in (3.4) for the case $S_2(\mathcal{D},E) \neq \emptyset$. It is quite similar to the one that one can establish for the case of the points of a surface which have two-dimensional strict tangent space (see [9]). The essential diference is that once a normal crossing situation is obtained one has to consider the possibility of the existence of a curve which is locally permissible but not globally permissible.

(3.10) Definition.- Assume that $S_2(\mathcal{D},E) \neq \emptyset$ and let $\pi: X' \to X$ be a blowing-up with center Y. Let S be the first one among the following statements which is true.

I. There is a closed point $Q \in X$ such that $S_2^*(\mathcal{D},E)$ does not have normal crosings with E at Q.

II. There is a $Z \in S_2^*(\mathcal{D},E)$ with $Z \subset \mathrm{Sing}^r(\mathcal{D},E)$

III. There is a $Z \in S_2^*(\mathcal{D},E)$ which is permissible and tangent to $\mathrm{Dir}_W(\mathcal{D},E,P)$ at each closed point $P \in Z$.

IV. There is a $Z \in S_2^*(\mathcal{D},E)$ which is permissible and tangent to $\mathrm{Dir}_W(\mathcal{D},E,P)$ at a closed point $P \in S_2(\mathcal{D},E) \cap Z$ but it is not so at a closed point $Q \in Z$, where $Q \notin S_2(\mathcal{D},E)$.

V. The same as IV, but $Q \in S_2(\mathcal{D},E)$ and there is no curve permissible at Q and tangent to $\mathrm{Dir}_W(\mathcal{D},E,Q)$ passing through Q.

VI. The same as IV, but $Q \in S_2(\mathcal{D},E)$.

VII. There is a closed point $Q \in S_2(\mathcal{D},E)$.

Then, the blowing-up π is said to "respect the 2-global procedure of reduction" iff Y satisfies the following

(3.10.1) "If S = I,IV,V,VI or VII, then Y = such a Q.
If S = II or III, then Y = such a Z".

(3.11) Assume that $\pi: X' \to X$ respects the 2-global procedure of reduction. As in (3.6), all the closed points in $\mathrm{Sing}^r(\mathcal{D},E)$ are of type zero. Let us make the following:

Definition.- Assume that $S_2(\mathcal{D},E) \neq \emptyset$. Let $I = \{1,\ldots,N\}$ or $I=\mathbb{N}$. A sequence of permissible blowing-ups

(3.11.1) $$\mathcal{S} = \{\pi(i): X(i) \to X(i-1)\}_{i \quad I}$$

with $X(0) = X$, is said to "respect the 2-global procedure of reduction" iff each $\pi(i)$, $i \in I$, respects the 2-global procedure of reduction (hence $S_2(\mathcal{D}(i-1),E(i-1)) \neq \emptyset$) and if it is finite one has $S_2(\mathcal{D}(N),E(N)) = \emptyset$.

As in (3.7), there always exists a sequence $\mathcal{S}$ which respects the 2-globla procedure of reduction, maybe infinite. The paragraphs 5,6 and 7 are devoted to the proof of the following theorem.

(3.12) Theorem.- There is a finite sequence which respects the 2-global procedure of reduction.

Once this theorem is proved, one deduces the theorem (3.1) from (3.12), (3.8) and (3.2).

4. ONE DIMENSIONAL W-DIRECTRIX CASE

(4.1) This paragraph is devoted to the proof of Theorem (3.8). Let us fix a sequence $\mathcal{S}$ which respects the 1-global procedure of reduction.

One has $\mathrm{Sing}^r(\mathcal{D}(i),E(i)) = S_1(\mathcal{D}(i),E(i)) \cup S_o(\mathcal{D}(i),E(i))$. Assume that

(4.1.1) $$\mathrm{Sing}^r(\mathcal{D}(i),E(i)) = Z_1(i) \cup \dots\ Z_{m(i)}(i) \cup \{P_1(i),\dots,P_{q(i)}(i)\}$$

where $Z_j(i)$ are irreducible curves and $P_j(i)$ are isolated points. Let us reason by contradiction and assume that $\mathcal{S}$ is infinite.

In view of (2.4), if $\pi(i)$ is quadratic, one has $q(i+1) \leq q(i)$ and $m(i+1) = m(i)$, thus $m(i+1) + q(i+1) \leq m(i) + q(i)$. If $\pi(i)$ is monoidal, then one has $q(i+1) \leq q(i)-1$ and $m(i+1) = m(i)$ (if the statement S of (3.4) is S = III) or $q(i+1) = q(i)$, $m(i+1) = m(i)-1$ (if the statement S is S = II); then one has $m(i+1) + q(i+1) < m(i) - q(i)$. Thus one can assume without loss of generality that $\pi(i)$ is quadratic for all i (otherwise, if for each $N \geq 0$ there is $i \geq N$ such that $\pi(i)$ is monoidal, then $\mathcal{S}$ must be finite).

(4.2) Let H(i) be the set of the closed points P of X(i) such that there exists $Z \in S_1^*(\mathcal{D}(i),E(i))$, with $P \in Z$ and Z is not both permissible and tangent to $\mathrm{Dir}_W(\mathcal{D}(i),E(i),P)$ at the point P, or there exists $Z' \neq Z$, $Z' \in S_1^*(\mathcal{D}(i),E(i))$ with $P \in Z'$.

One has that $\pi(i+1)$ is centered at a point of H(i), since it is not monoidal (see (3.4)). Let us define a tree $\mathcal{A}$ in the following way. If $P \in H(i)$, $Q \in H(j)$, $j > i$, let us denote

(4.2.1)
$$P < Q \Leftrightarrow (\pi(i+1) \circ \dots \circ \pi(j))(Q) = P$$
$$P \sim Q \Leftrightarrow Q = (\pi(i+1) \circ \dots \circ \pi(j))^{-1}(P),$$

and let $\mathcal{A} = (\cup_i H(i))/\sim$ with the induced order. Recall that a branch of $\mathcal{A}$ is any maximal totally ordered subset of $\mathcal{A}$. We shall see later (see (4.6)) that the minimal points of $\mathcal{A}$ are precisely the points $P_\sim$, where $P \in H(0)$.

(4.3) Lemma.- Each H(i) is finite.

Proof.- Let us show first that for each $Z \in S_1^*(\mathcal{D}(i),E(i))$ the set of points where Z is not permissible is finite. In view of the semicontinuity of the permissibility, it is enough to show that there exists $Q \in Z$ such that Z is permissible at Q. Let us fix $P \in Z$. Since $Z \subset \mathrm{Sing}^{r-1}(\mathcal{D}(i),E(i))$, blowing-up this point and following a branch of Z, one generates a stationary sequence (see [4]. (3.3)), thus the strict transform of Z becomes permissible in a nonempty open set, hence also Z. The proof is completed by the remarks of (2.3).

(4.4) Proposition.- Let $\pi: X' \longrightarrow X$ be a quadratic blowing-up centered ata a closed point $P \in S_1(\mathcal{D},E)$ and assume that $P' \in \pi^{-1}(P) \cap S_1(\mathcal{D}',E')$. Then $\mathrm{Dir}_W(\mathcal{D}',E',P')$ and $\pi^{-1}(P)$ are not tangent to each other

Proof.- Let us keep the notation of (2.4). We know that $\dim \mathrm{Dir}_W(\mathcal{D},E,P) = \dim \mathrm{Dir}_W(\mathcal{D}',E',P')$ hence

(4.4.1) $$\dim_k(\mathcal{D},E,P)^* = \dim_k W(\mathcal{D}',E',P')^*$$

One has

(4.4.2) $$\begin{aligned}\dim_k W(\mathcal{D},E,P)^* &= \dim_k (W_Z)^* = \\ &= \dim_k \beta\alpha(W_Z)^* \leq \dim_k \beta(W_{Z'})^* \leq \\ &\leq \dim_k (W_{Z'})^* \leq \dim_k W(\mathcal{D}',E',P')^*\end{aligned}$$

(see the proof of (2.4)). Now, by (4.4.1) one has equalities everywhere in (4.4.2) and hence

(4.4.3) $$\dim_k W(\mathcal{D}',E',P')^* = \dim_k \beta(W_{Z'})^*$$

which is equivalent to the desired transversality.

(4.5) Corollary.- $S_1^*(\mathcal{D}(i),E(i))$ is the strict transform of $S_1^*(\mathcal{D}(i-1),E(i-1))$ by $\pi(i)$.

Proof.- It is enough to show that no curve is added to the strict transform of $S_1^*(\mathcal{D}(i-1),E(i-1))$ in order to obtain $S_1^*(\mathcal{D}(i),E(i))$. Let P be the center of $\pi(i)$. If $P \notin S_1(\mathcal{D}(i-1),E(i-1))$, then $\pi(i)^{-1}(P) \cap S_1(\mathcal{D}(i),E(i)) = \emptyset$ and thus there is no curve in $S_1^*(\mathcal{D}(i),E(i))$ contained in $\pi(i)^{-1}(P)$. Assume now that $P \in S_1(\mathcal{D}(i-1),E(i-1))$ and let Z be a curve contained in $\pi(i)^{-1}(P)$ and passing through $P' \in S_1(\mathcal{D}(i),E(i))$. Since $\pi(i)^{-1}(P)$ and $Dir_W(\mathcal{D}(i),E(i),P')$ are not tangent, no branch of Z is tangent to $Dir_W(\mathcal{D}(i),E(i),P')$ and after a quadratic blowing-up $\pi: X' \longrightarrow X(i)$, centered at P', the strict transform of Z does not touch $S_1(\mathcal{D}',E')$ hence $Z \notin S_1^*(\mathcal{D}(i),E(i))$.

(4.6) Corollary.- Assume that $\pi(i)$ is centered at $P \in H(i-1)$. Let $Y_1,\ldots,Y_s$ be the elements of $S_1^*(\mathcal{D}(i-1),E(i-1))$ which contain P. Then

(4.6.1)
$$\pi(i)^{-1}(H(i-1)-\{P\}) \subset H(i) \subset$$
$$\subset \pi(i)^{-1}(H(i-1)-\{P\}) \cup \{P'_s,\ldots,P'_s\}$$

where $P'_j = Y'_j \cap \pi(i)^{-1}(P)$, Y'_j being the strict transform of Y_j.

Proof.- The only points of H(i) in $\pi(i)^{-1}(P)$ are points in some Y'_j, $j = 1,\ldots,s$, in view of (4.5).

(Remark: We deduce that the minimal points of $\mathcal{A}$ are the points $p^\sim$ with $P \in H(0)$).

(4.7) As a consequence of (4.3) and (4.6), the tree $\mathcal{A}$ has an infinite branch which corresponds to the infinitely near points of a branch Z_o of some $Z \in S_1^*(\mathcal{D}(0),E(0))$ at a closed point Q(0). Let us denote by Q(i) the points of this infinite branch. Since $Z \subset Sing^{r-1}(\mathcal{D}(0),E(0))$, one obtains a stationary sequence and hence the strict transform of Z is permissible at Q(N) for some N. Moreover, taking N large enough, one can assume that the strict transform of Z has normal crossings with E(N), is the only element of $S_1^*(\mathcal{D}(N),E(N))$ passing through Q(N) (see the corollary (4.5)) and is tangent to $Dir_W(\mathcal{D}(N),E(N))$ (see the remarks of (2.3)). In this way one obtains a contradiction, since $Q(N) \in H(N)$. (Remark: In order to simplify the notation we have identified $Q(N) = Q(N)_\sim$).

This ends the proof of Theorem (3.8).

5. GENERAL CASE. REDUCTION TO THE LOCAL CONTROL

(5.1) Let us begin the proof of Theorem (3.12). Assume that $S_2(\mathcal{D},E) \neq \emptyset$ and let us fix a sequence $\mathcal{S}$ which respects the 2-global procedure of reduction. We shall reason by contradiction, assuming that $\mathcal{S}$ is infinite.

For a closed point $P \in S_2(\mathcal{D}(i),E(i))$ and a closed point $Q \in S_2(\mathcal{D}(j),E(j))$, $i < j$, let us define $P < Q$ and $P \sim Q$ exactly in the same way as in (4.2.1). Let

(5.1.1) $$\mathcal{A} = (\cup_i S_2(\mathcal{D}(i),E(i)))/\sim$$

with the induced order. Then $\mathcal{A}$ is a tree such that the minimal points of $\mathcal{A}$ are the points $P_\sim$ with $P \in S_2(\mathcal{D}(0),E(0))$. THis paragraph is mainly devoted to the proof of the following.

Theorem.- If every branch of $\mathcal{A}$ is finite, then one obtains a contradiction and hence $\mathcal{S}$ must be finite.

The paragraphs 6 and 7 are devoted to the "local control", i.e. to proving that each branch of $\mathcal{A}$ is finite, in order to finish the proof of Theorem (3.12).

(5.2) Proposition.- Assume that $P \in S_2(\mathcal{D},E)$ and that $\pi: X' \longrightarrow X$ is the blowing-up of X with center P. Then

(5.2.1) $$\mathrm{Sing}(\mathcal{D}',E') \subset \mathrm{Sing}(\mathcal{D},E)' \cup \mathrm{Proj}(\mathrm{Dir}_W(\mathcal{D},E,P)),$$

where $\mathrm{Sing}(\mathcal{D},E)'$ is the strict transform of $\mathrm{Sing}(\mathcal{D},E)$ and $\mathrm{Proj}(\mathrm{Dir}_W(\mathcal{D},E,P)) \subset \pi^{-1}(P)$.

Proof.- Let us take a regular system of parameters $(x_1,\dots,x_n)$ such that $I(E) = (\Pi_{i \leq e} x_i)\cdot\mathcal{O}_{X,P}$ and such that $W(\mathcal{D},E,P) = (\bar{x}_n)$ (this is always possible). Assume that $\mathcal{D}_P$ is generated by

(5.2.2) $$D = \textstyle\sum_{i \leq e} a_i x_i \,\partial/\partial x_i + \sum_{i > e} a_i \,\partial/\partial x_i$$

One deduces that $J^r(a_n) = (\bar{x}_n)$ and thus $\mathrm{In}(a_n) = \lambda \bar{x}_n^r$, $\lambda \in k$. Now, looking at the equations I.(1.3.2) and I.(1.3.3) one deduces that $\mathrm{Sing}(\mathcal{D}',E') \cap \pi^{-1}(P)$ is contained in $\mathrm{Proj}(\bar{x}_n = 0)$.

(5.3) Corollary.- With the hypotheses of (5.2), one of the following two statements is true:

a) $S_2^*(\mathcal{D}',E') = S_2^*(\mathcal{D},E)'$
b) $S_2^*(\mathcal{D}',E') = S_2^*(\mathcal{D},E)' \cup \mathrm{Proj}(\mathrm{Dir}_W(\mathcal{D},E,P))$.

Proof.- Since $r \geq 2$, $Z \in S_2^*(\mathcal{D},E) \Rightarrow Z \subset \mathrm{Sing}(\mathcal{D},E)$.

(5.4) Corollary.- For some N, $S_2^*(\mathcal{D}(N),E(N))$ has normal crossings with E(N).

Proof.- Let $\pi(i)$ be centered at the closed point P such taht $S_2^*(\mathcal{D}(i-1),E(i-1))$

does not have crossings with E(i-1) at P. If $P \notin S_2(\mathcal{D}(i-1),E(i-1))$, then

(5.4.1) $$S_2^*(\mathcal{D}(i),E(i)) = S_2^*(\mathcal{D}(i-1),E(i-1))'$$

since there are no points of $S_2(\mathcal{D}(i),E(i))$ in $\pi(i)^{-1}(P)$. Now, the result follows from (5.3) and from the standard facts abour desingularization for curves (see e.g. [9]).

Remark.- No assumption about the finiteness of the branches of $\mathcal{A}$ has been made.

(5.5) The following theorem will be proved in the Appendix.

Theorem.- Let $P \in S_2(\mathcal{D},E)$, let Y be a globally permissible curve with $P \in Y$ and let Z be another component of $\mathrm{Sing}^{r-1}(\mathcal{D},E)$ such that $P \in Z$. Let $\pi: X' \longrightarrow X$ be the blowing-up with center Y and let Z' be the strict transform of Z. Then

(5.5.1) $$Z' \in S_2^*(\mathcal{D}',E') \Rightarrow Z \in S_2^*(\mathcal{D},E).$$

The converse of the above theorem is not true, i.e. an element of $S_2^*(\mathcal{D},E)$ may be eliminated by means of a monoidal transformation with another center. For instance, let Y be given by (y,z) and let Z be given by (x,z). Assume that $\mathcal{D}$ is globally generated in $\mathbb{A}^3(k)$ by

(5.5.2) $$D = xz^{r-1}x\,\partial/\partial x + (x^{r-1}y^r)\partial/\partial y + z^r\partial/\partial z.$$

Then $Z - S_2^*(\mathcal{D},E)$ but $Z' \notin S_2^*(\mathcal{D}',E')$, since $\mathrm{Sing}^r(\mathcal{D}',E') = \emptyset$.

(5.6) Proposition.- Assume that $\pi(i)$ is given by a monoidal blowing-up centered at $Y \in S_2^*(\mathcal{D}(i-1),E(i-1))$. Then one of the following two statements is true:

a) $S^*(\mathcal{D}(i),E(i))$ is contained in the strict transform of $S_2^*(\mathcal{D}(i-1),E(i-1))-Y$ under $\pi(i)$.

b) There is exactly one curve $Y' \subset \pi(i)^{-1}(Y)$ such that $S_2^*(\ (i),E(i)) - \{Y'\}$ is contained in the strict transform of $S_2^*(\mathcal{D}(i-1),E(i-1))-\{Y\}$, and Y' has normal crossings with E(i) (hence $S_2^*(\mathcal{D}(i),E(i))$ has normal crossings with E(i)) and Y' is isomorphic to Y by means of $\pi(i)$.

Proof.- If there is no $Y' \in S_2^*(\mathcal{D}(i),E(i))$ contained in $\pi(i)^{-1}(Y)$, then a) is true by Theorem (5.5). Assume there is such a Y'. Let $P' \in Y'$ have the property of (3.3.1). Let $P = \pi(i)(P')$. Then $P \in S_2(\mathcal{D}(i-1),E(i-1))$. One can take a regular system of parameters (x,y,z) at P in such a way that $W(\mathcal{D}(i-1), E(i-1),P) = (\bar{z})$, $E(i-1) \subset (xy = 0)$ (locally at P), $I(Y) = (x,z)$. Then $\mathcal{D}(i-1)$ is generated at P by

$$(5.6.1)\qquad \begin{aligned} & D = a\,\partial/\partial x + b\,\partial/\partial y + c\,\partial/\partial z \\ \text{or}\quad & D = ax\,\partial/\partial x + b\,\partial/\partial y + c\,\partial/\partial z \\ \text{or}\quad & D = a\partial/\partial x + by\partial/\partial y + c\partial/\partial z \\ \text{or}\quad & D = ax\partial/\partial x + by\partial/\partial y + c\partial/\partial z. \end{aligned}$$

where $J^r(c) = (\bar{z})$. By (2.4), P' corresponds to the transformation $x = x'$, $y = y'$, $z = x'z'$. Then $\mathcal{D}(i)$ is generated at P' by

$$(5.6.2)\qquad \begin{aligned} & D' = a'x'\partial/\partial x' + b'\partial/\partial y' + c'\partial/\partial z' \\ \text{or}\quad & D' = a'x'\partial/\partial x' + b'y'\partial/\partial y' + c'\partial/\partial z' \end{aligned}$$

where $a' = a/x'^r$ (resp. $a' = a/x'^{r-1}$), $b' = b/x'^{r-1}$, $c' = c/x'^r - z'a'$ in the cases 1 and 3 (resp. 2 and 4) of (5.6.1). One has

$$(5.6.3)\qquad In^r(c') = \bar{z}'^r + \bar{y}'\psi(\bar{y}',\bar{z}') = \prod_{1\le i\le d} (\bar{z}' + \lambda_i\bar{y}')^{r_i} \qquad (\text{mod } \bar{x}').$$

Moreover, in the first case of (5.6.2), one has $In^r(b') = \bar{y}'\phi(\bar{y}',\bar{z}')$ (mod $\bar{x}'$). Since $P' \in S_2(\mathcal{D}(i),E(i))$ one deduces that $\phi = 0$ and $d = 1$. Now one can make the change $z' \mapsto z' + \lambda_1 y'$ and we can assume that $\lambda_1 = 0$. Thus, since $Y' \subset Sing^{r-1}(\mathcal{D}(i))$, the only possible tangent to Y' at P' is $x' = z' = 0$. Making the blowing-up centered at P' in the direction $x' = z' = 0$, the adapted order does not drop and repeating the above argument infinitely many times one can deduce that after a change $z' \mapsto z' + \sum \lambda_i y^i$, the equation of Y' must be $x' = z' = 0$.

Thus, locally at P', Y' satisfies the conditions of b). Since $\pi(i)$ is proper, $\pi(i)(Y') = Y$ and by Zariski's Main Theorem $\pi(i)\colon Y' \to Y$ is an isomorphism. Moreover Y' is transversal to the fibers of $\pi(i)$: and hence Y' has normal crossings with E(i).

Let Z' be another curve, $Z' \subset \pi(i)^{-1}(Y)$ and $Z' \in S_2^*(\mathcal{D}(i),E(i))$. Let Q' be a point in $S_2(\mathcal{D}(i),E(i)) \cap \pi(i)^{-1}(Y)$ which satisfies the property of (3.3.1) for Z'. We have $\pi(i)(Q') = Q \in S_2(\mathcal{D}(i-1),E(i-1))$ and thus Q' is the only point in $Sing(\mathcal{D}(i),E(i)) \cap \pi(i)^{-1}(Q)$. This implies that $Q' \in Y'$, since $Y' \subset Sing^{r-1}(\mathcal{D}(i),E(i))$ Now, reasonning as at the beginning, there is only one curve in $Sing(\mathcal{D}(i),E(i))$ passing through Q' and contained in $\pi(i)^{-1}(Y)$. Then Y' = Z' locally at Q' and hence Y' = Z' globally.

(5.7) Corollary 1.- There is an index N such that if $i \geq N$, then $S_2^*(\mathcal{D}(i),E(i))$ has normal crossings with E(i)).

Proof.- Follows from (5.3), (5.4) and (5.5).

(Note that no assumption about the finiteness of the branches of $\mathcal{A}$ has been made).

(5.8) Corollary 2.- Assume that each branch of $\mathcal{A}$ is finite. Then, given $N \geq 0$ there is $i \geq N$ such that $S_2(\mathcal{D}(i),E(i))$ is finite.

Proof.- In view of the above Corollary, one can assume that $S_2^*(\mathcal{D}(N),E(N))$ has normal crossings with $E(N)$. Let $Z_j(N)$, $j = 1,\dots,m(N)$ be the elements of $S_2^*(\mathcal{D}(N),E(N)$ which are contained in $\mathrm{Sing}^r(\mathcal{D}(N),E(N))$. If $m(N) = 0$, then the corollary is true. Assume that $m(N) \geq 1$. It is enough to prove that for some $N' \geq N$ one has $m(N') < m(N)$.

Since $S_2^*(\mathcal{D}(N),E(N))$ has normal crossings with $E(N)$, there is $j(N)$ such that $Z_{j(N)}(N)$ is the center of $\pi(N+1)$. In view of the above proposition, one has two possibilities:

a) The elements of $S_2^*(\mathcal{D}(N+1),E(N+1))$ which are contained in $\mathrm{Sing}^r(\mathcal{D}(N+1),E(N+1))$ are exactly the strict transforms of $Z_j(N)$, $j \neq j(N)$. In this case $m(N+1) = m(N)-1$ and it is enough to make $N' = N+1$.

b) The elements of $S_2^*(\mathcal{D}(N+1),(N+1))$ which are contained in $\mathrm{Sing}^r(\mathcal{D}(N+1),E(N+1))$ are the strict transforms $Z_j(N+1)$ of $Z_j(N)$, $j \neq j(N)$, and a curve $Z_{j(N+1)}(N+1)$ contained in the exceptional divisor of $\pi(N+1)$.

If b) holds, let us repeat the procedure. If b) occurs infinitely many times, there is an index j such that $Z_j(i)$ has been blown-up infinitely many times. Taking a point $P \in Z_j(N) \cap S_2(\mathcal{D}(N),E(N))$, it is possible to construct an infinite branch of $\mathcal{A}$ from P. Contradiction.

(5.9) Let $i_1 < i_2 < \dots$ be such that $S_2(\mathcal{D}(i_j),E(i_j))$ is finite. Let us consider the set

(5.9.1) $$\cup_{j \geq 1} S_2(\mathcal{D}(i_j),E(i_j)).$$

and let us define the same order and equivalence relations as in (4.2.1). We obtain a tree $\mathcal{A}'$. (under the assumption that each branch of $\mathcal{A}$ is finite).

Since each branch of $\mathcal{A}$ is finite, each branch of $\mathcal{A}'$ is finite. Now, since $\mathcal{A}'$ has only finitely many points at each level, $\mathcal{A}'$ itself is finite. Now, in view of (5.8), there is an N such that for each $i \geq N$ the center of $\pi(i+1)$ does not contain any point of $S_2(\mathcal{D}(i),E(i))$, hence it is a closed point not in $S_2(\mathcal{D}(i),E(i))$, such that $S_2^*(\mathcal{D}(i),E(i))$ does not have normal crossings at this point with $E(i)$, or such that some curve of $S_2^*(\mathcal{D}(i),E(i))$ is not both permissible and tangent to $\mathrm{Dir}_W(\mathcal{D}(i),E(i),P)$ at this point P. By (5.7) and by the remarks of

(2.3), the above situation cannot be repeated infinitely many times. This ends the proof of the theorem of (5.1).

6. LOCAL CONTROL. FIRST REDUCTIONS

(6.1) This paragraph is devoted to proving that (if $\mathcal{S}$ satisfies, certain conditions) the finiteness of the branches of $\mathcal{A}$ follows from the finiteness of a special type of branches, called "good branches". The next paragraph is devoted to proving that a good branch is always finite.

In order to do that, we shall restrict the sequence $\mathcal{S}$ by requiring it to respect the "strong 2-global procedure of reduction". This will consist in assigning priorities to the centers which are possible for each value of the statement S of (3.10) and, in this way, such a sequence always exists. This priority will be given according to a certain invariant called "date of birth" attached to each element of $S_2^*(\mathcal{D}(i),E(i))$. This kind of invariant has been used before (see [6]) and it will allow us to read the "trees" generated by $\mathcal{S}$ "horizontally" rather than "vertically".

(6.2) In this paragraph, let us assume without loss of generality that $S_2^*(\mathcal{D},E)$ has normal crossings with E and that each closed point $P \in S_2(\mathcal{D},E)$ is contained in at most two elements of $S_2^*(\mathcal{D},E)$. This may be obtained after finitely many steps in view of (5.3), (5.6) and (5.7).

(6.3) Definition.- Let $\mathcal{S}$ be a sequence which respects the 2-global procedure of reduction. Given an element Z of $S_2^*(\mathcal{D}(i),E(i))$ the "date of birth" dat(Z) of Z is defined by

$$\text{(6.3.1)} \qquad \operatorname{dat}(Z) = \begin{cases} \operatorname{dat}(\pi(i)(Z)). \text{ If } Z \text{ is not contained in} \\ \text{the exceptional divisor of } \pi(i). \\ i. \text{ Otherwise.} \end{cases}$$

If i=0, then dat(Z) = 0 for all Z. For each closed point $P \in S_2(\mathcal{D}(i),E(i))$, the invariant "date", dat(P) is defined to be dat(P) = 0 if no element of $S_2^*(\mathcal{D}(i),E(i))$ contains P and $\operatorname{dat}(P) = \sum_j \operatorname{dat}(Z_j)$ where $Z_j \in S_2^*(\mathcal{D}(i),E(i))$ are the elements containing P (at most two of them).

(6.4) Definition.- The sequence $\mathcal{S}$ is said to "respect the strong 2-global procedure of reduction" if at each step i, i = 0,1,..., the center Y of $\pi(i+1)$ has the following property: "dat(Y) is minimal among all dat(Z) where Z runs over all the centers given by the statement S of (3.10) applied to $\mathcal{D}(i),E(i)$".

Obviously, a sequence $\mathcal{S}$ which respects the strong 2-global procedure of reduction always exists. In order to prove the theorem (3.12), let us prove that any sequence which respects the strong 2-global procedure of reduction is always finite.

(6.5) Definition.- Let Γ be a branch of the tree $\mathcal{A}$. Γ is said to be a "good branch" iff the following property holds. Let P be an element of Γ which is not the last one. Write

$$P = \{P_i, P_{i+1}, \ldots, P_j\} \qquad (6.5.1)$$

where $P_{i+s} \in X(i+s)$. Then if $\pi(j+1)$ is quadratic (necessarily centered at P_j) there is no curve $Z \in S_2^*(\mathcal{D}(j), E(j))$ which is permissible and tangent to $\mathrm{Dir}_W(\mathcal{D}(j), E(j), P_j)$ at the point P_j.

(6.6) This paragraph is mainly devoted to proving the following.

Theorem.- Assume that each good branch of any sequence which respects the 2-global procedure of reduction is finite. Let us consider a sequence $\mathcal{S}$ which respects the strong 2-global procedure of reduction and let $\mathcal{A}$ be the associated tree. Then each branch of $\mathcal{A}$ is finite and hence $\mathcal{S}$ is finite.

(6.7) In order to prove that from a certain step onwards, we can assume that each branch is a good branch, let us introduce the concepts of "cycle" and "bad cycle".

Definition.- A "cycle γ of length s" for $\mathcal{D}$,E, is a pair $\gamma = (\mathcal{P}(\gamma), \mathcal{C}(\gamma))$ where $\mathcal{P}(\gamma) \subset S_2(\mathcal{D},E)$ has exactly s elements, $\mathcal{C}(\gamma) \subset S_2^*(\mathcal{D},E)$ has also exactly s elements and the following properties hold:

a) Each point of $\mathcal{P}(\gamma)$ is contained in exactly two curves of $\mathcal{C}(\gamma)$.

b) Each curve of $\mathcal{C}(\gamma)$ contains exactly two points of $\mathcal{P}(\gamma)$.

The cycle γ is said to be a "bad cycle" iff it has, in addittion, the following properties.

c) Given a point $P \in \mathcal{P}(\gamma)$ there is exactly one curve $Z \in \mathcal{C}(\gamma)$ which is permissible and tangent to the W-directrix at P.

d) Given a curve $Z \in \mathcal{C}(\gamma)$ there is exactly one point $P \in \mathcal{P}(\gamma)$ such that Z is permissible and tangent to the W-directrix at P.

Let us denote by $\mathrm{cycl}(\mathcal{D},E)$, resp. $\mathrm{bcycl}(\mathcal{D},E)$, the set of cycles, resp. bad cycles.

The set $\mathrm{cycl}(\mathcal{D},E)$ is always finite, since $S_2^*(\mathcal{D},E)$ is finite and hence the intersection points of the elements of $S_2^*(\mathcal{D},E)$ are a finite set.

Given a cycle γ of length s, one can represent it by a sequence

(6.7.1) $$(P_1, c_1, \ldots, P_s, c_s)$$

where $P_i \in C_{i-1} \cap C_i$, $C_o = C_s$. (Think of the index i as lying in $\mathbb{Z}/(s)$).

Given a subset $A \subset \mathrm{cycl}(\mathcal{D},E)$, let us denote $\bigcup_{\gamma \in A} \mathcal{P}(\gamma)$ by $\mathcal{P}(A)$. Given a subset $Z \subset X$, let us denote by

(6.7.2) $$\mathrm{cycl}(\mathcal{D},E,Z)$$
$$\text{resp. } \mathrm{bcycl}(\mathcal{D},E,Z)$$

the set of the cycles, resp. bad cycles, γ such that $\mathcal{P}(\gamma) \cap Z \neq \emptyset$. Similarly, given a subset $B \subset S_2^*(\mathcal{D},E)$, let us denote by

(6.7.3) $$\mathrm{cycl}(\mathcal{D},E,B)$$
$$\text{resp. } \mathrm{bcycl}(\mathcal{D},E,B)$$

the set of cycles, resp. bad cycles, γ such that $\mathcal{C}(\gamma) \cap B \neq \emptyset$.

(6.8) Proposition.- Let Γ be a branch of the tree $\mathcal{A}$ associated to a sequence $\mathcal{S}$ which respects the 2-global procedure of reduction. Then if Γ is not a good branch then there is an element P of Γ, not maximal in Γ.

(6.8.1) $$P = \{P_i, P_{i+1}, \ldots, P_j\},\ P_{i+s} \in X(i+s)$$

such that $\pi(j+1)$ is centered at P_j, there is a curve which is permissible and tangent to the W-directrix at P_j and there is another curve in $S_2^*(\mathcal{D}(j),E(j))$ passing through P_i which does not have the above property. Moreover, $\mathrm{bcycl}(\mathcal{D}(j),E(j)) \neq \emptyset$.

Proof.- Assume that Γ is not good. Then one can choose $P \in \Gamma$ such that $\pi(j+1)$ is quadratic centered at P_j, but there is a curve $Z_1 \in S_2^*(\mathcal{D}(j),E(j))$ which is permissible and tangent to $\mathrm{Dir}_W(\mathcal{D}(j),E(j),P_j)$ at P_j. Let us denote $Q(1) = P_j$. Since $\pi(j+1)$ is quadratic and $S_2^*(\mathcal{D}(j),E(j))$ has normal crossings with E(j), there is a curve $Y \in S_2^*(\mathcal{D}(j),E(j))$ such that $Q(1) \in Y$ but Y is not both permissible and tangent to $\mathrm{Dir}_W(\mathcal{D}(j),E(j),Q(1))$ at Q(1). Now, since $\pi(j+1)$ is not monoidal, then Z_1 is not globally permissible and tangent to the W-directrix, hence there is a point $Q(2) \in Z_1$ such that is not both permissible and tangent to the W-directrix at Q(2). Now, since the center of $\pi(j+1)$ is Q(1) and not Q(2), one deduces from the priorities of (3.10) that $Q(2) \in S_2(\mathcal{D}(j),E(j))$ and that there exists $Z_2 \in S_2^*(\mathcal{D}(j),E(j))$ which is permissible and tangent to the W-directrix at Q(2). But Z_2 cannot be globally permissible and tangent to the W-directrix, hence there is a point Q(3), etc. In this way one obtains a sequence

(6.8.1) $$Z_o = Y,\ Q(1),\ Z_1,\ Q(2),\ Z_2,\dots$$

such that for $i \geq 1$, Z_i is both permissible and tangent to the W-directrix at Q(i) while Z_{i-1} is not. Since $S_2^*(\mathcal{D}(j),E(j))$ is finite, there exists a minimal $s \geq 2$ such that ther is h, $0 \leq h \leq s-2$ with $Z_s = Z_h$. Then $(Q(h+1),Z_{h+1},\dots,Q(s),Z_s)$ is a bad cycle and thus $\text{bcycl}(\mathcal{D}(j),E(j)) \neq \emptyset$.

(6.9) In view of the above proposition, in order to prove (6.6) it is enough to show that after finitely many steps one has $\text{bcycl}(\mathcal{D}(i),E(i)) = \emptyset$ for all $i \geq N$. But instead of controling the number of bad cycles at each step, let us control the number of cycles. This is enough for our purposes since $\text{bcycl}(\mathcal{D},E) \subset \text{cycl}(\mathcal{D},E)$

(6.10) Proposition.- Let $\mathcal{S}$ be a sequence which respects the 2-global procedure of reduction. The one has, for each $i \geq 0$:

a) $\pi(i+1)^{-1}(\mathcal{P}(\text{cycl}(\mathcal{D}(i),E(i))) \supset \mathcal{P}(\text{cycl}(\mathcal{D}(i+1),E(i+1)))$.

b) $\#\ \text{cycl}(\mathcal{D}(i),E(i)) \geq \#\ \text{cycl}(\mathcal{D}(i+1),E(i+1))$.

(# = number of elements).

Proof.- Assume first that $\pi(i+1)$ is quadratic centered at P. By (5.3), $\pi(i+1)$ induces a bijection

(6.10.1) $$\text{cycl}(\mathcal{D}(i+1),E(i+1)) - \text{cycl}(\mathcal{D}(i+1),E(i+1),\pi(i+1)^{-1}(P)) \longrightarrow$$
$$\longrightarrow \text{cycl}(\mathcal{D}(i),E(i)) - \text{cycl}(\mathcal{D}(i),E(i),P)$$

given by

(6.10.2) $$(P_1,c_1,\dots,P_s,c_s) \mapsto (\pi(i+1)(P_1),\dots,\pi(i+1)(c_s)).$$

Let $\gamma' \in \text{cycl}(\mathcal{D}(i+1),E(i+1),\pi(i+1)^{-1}(P))$, with $\gamma' = (P'_1,c'_1,\dots,P'_s,c'_s)$. Then there is an index l such that $P'_l \in \pi(i+1)^{-1}(P)$. Now, in view of (5.3) and (6.2), neither c'_{l-1} nor c'_l is contained in $\pi(i+1)^{-1}(P)$. Without loss of generality, one can assume that $c'_l \subset \pi(i+1)^{-1}(P)$ (hence it its the only element of $S_2^*(\mathcal{D}(i+1),E(i+1))$ which is contained in the exceptional divisor $\pi(i+1)^{-1}(P)$). Let us denote $Q = P_l$, $Z = c'_l$, $T = P'_{l+1}$, $Y = c'_{l-1}$, $V = c'_{l+1}$. In view of (5.3) and (6.2), Q,Z,T,V do not depend on γ', since Y and V are necessarily the strict transforms of the two elements of $S_2^*(\mathcal{D}(i),E(i))$ passing through P. One deduces that if $\text{cycl}(\mathcal{D}(i+1),E(i+1),\pi(i+1)^{-1}(P)) \neq \emptyset$, then the map

(6.10.3) $$\mathrm{cycl}(\mathcal{D}(i+1),E(i+1),\pi(i+1)^{-1}(P)) \longrightarrow \longrightarrow \mathrm{cycl}(\mathcal{D}(i),E(i),P)$$

given at γ' by

(6.10.4) $$(P_1,c_1,\ldots,Y,Q,Z,T,V,P_{l+2},c_{l+2},\ldots,P_s,c_s) \longmapsto \longmapsto (\pi(i+1)(P_1),\pi(i+1)(c_1),\ldots,\pi(i+1)(Y),P,\pi(i+1)(V),\ldots \ldots,\pi(i+1)(P_s),\pi(i+1)(c_s))$$

is bijective. The proposition in this case follows immediatly.

Assume now that $\pi(i+1)$ is monoidal centered at Y.

Let $B \subset S_2^*(\mathcal{D}(i+1),E(i+1))$ be $B = \emptyset$ if one has a) in (5.6) and $B = \{Y'\}$ if one has b) in (5.6). By (5.6), $\pi(i+1)$ induces an injective map

(6.10.5) $$\mathrm{cycl}(\mathcal{D}(i+1),E(i+1)) - \mathrm{cycl}(\mathcal{D}(i+1),E(i+1),B) \longrightarrow \mathrm{cycl}(\mathcal{D}(i),E(i)) - \mathrm{cycl}(\mathcal{D}(i),E(i),Y)$$

given as in (6.10.2). Thus, the proposition is true if $B = \emptyset$. Let $\gamma = (P_1,Y',P_2,c_2,\ldots,P_s,c_s) \in \mathrm{cycl}(\mathcal{D}(i+1),E(i+1),\{Y'\})$, then the map

((6.10.6) $$\gamma \longmapsto (\pi(i+1)(P_1),Y,\pi(i+1)(P_2),\ldots,\pi(i+1)(c_s))$$

is clearly injective. This ends the proof.

(6.11) Lemma.- Let $(X,\mathcal{D},E)$ satisfy the hypotheses of (6.2). Assume that $\pi\colon X' \longrightarrow X$ is a blowing-up centered at Y, where $Y = \{P\}$ (a closed point) or $Y \in S_2^*(\mathcal{D},E)$ and Y is permissible and tangent to the W-directrix at each closed point. Let us fix a closed point $Q \in Y$ and let $Z \in S_2^*(\mathcal{D},E)$ with $Q \in Z$. Let Z' be the strict transform of Z and let $Q' = Z' \cap \pi^{-1}(Y)$. Assume that $Q' \in S_2(\mathcal{D}',E')$ and that Z' is permissible and tangent to $\mathrm{Dir}_W(\mathcal{D}',E',Q')$ at Q'. Then $Q \in S_2(\mathcal{D},E)$ and Z is permissible and tangent to $\mathrm{Dir}_W(\mathcal{D},E,Q)$ at Q.

Proof.- Assume first that π is quadratic (then $Q = P$). Obviously, $Q \in S_2(\mathcal{D},E)$ and in view of (2.3) it is enough to show that Z is permissible. One can take a regular system of parameters (x,y,z) at Q suited for (E,Z) such that $I(Z) = (y,z)$. Now from I.(1.3.2) and I.(1.3.3) one deduces that if Z' is permissible and tangent to the W-directrix, then Z is permissible, too.

Assume now that π is monoidal. Obviously, $Q \in S_2(\mathcal{D},E)$. One can take a regular system of parameters (x,y,z) suited for (E,Y) and suited for (E,Z) such that $W(\mathcal{D},E,Q) = (\bar{z})$, $I(Y) = (x,z)$. Then $I(Z) = (x,y)$ or $I(Z) = (y,z)$, but since $Q' \in S_2(\mathcal{D}',E')$, necessarily $I(Z) = (y,z)$, (see (2.4)). If Z' is permissible, then Z is permissible by (5.6.2).

(6.12) Let Y and Z be two regular curves in X such that $Y \cup Z$ has normal crossings with E. Assume that P is a closed point in $Y \cap Z$. Let π_Y: $X' \longrightarrow X$ be the blowing-up with center Y and let P' be the closed point of Z'(= strict transform of Z by π_Y) over P. Let $\pi'(1)$: $X'(1) \longrightarrow X'$ be the quadratic blowing-up centered at P' and let P'(1) be the closed point of Z'(1) (=strict transform of Z' by $\pi'(1)$) over P'. Let $\pi(1)$: $X(1) \longrightarrow X$ be the quadtatic blowing-up centered at P. Let P(1) and R(1) be the closed points of Z(1) and Y(1) (=strict transforms of Z and Y by $\pi(1)$) over P and let T be the projective line of $\pi(1)^{-1}(P)$ joinning P(1) and R(1). Finally, let π_T: $X(1)' \longrightarrow X(1)$ be the blowing-up with center T and let P(1)' be the closed point of Z(1)' (= strict transform of Z(1) by π_T) over P(1). The following results will be proved in the Appendix:

<u>Theorem</u>.- (a). (Denote $\mathcal{D}'$, $\mathcal{D}'(1)$,etc. the transforms by $\pi, \pi'(1)$,etc). Assume that $P \in S_2(\mathcal{D},E)$ and Y is both globally permissible and tangent to the W-directrix. Assume that $P'(1) \in S_2(\mathcal{D}'(1),E'(1))$ (hence $P' \in S_2(\mathcal{D}',E')$). Then $T = \mathrm{Proj}(\mathrm{Dir}_W(\mathcal{D},E,P))$ and T is permissible at P(1).

<u>Theorem</u>.- (b). There are isomorphic neighborhoods of P'(1) and P(1)' (where P'(1), E'(1), Z'(1) correspond, respectively, to P(1)', E(1)', Z(1)' under this isomorphism), in such a way that

(6.12.1) $$\pi_Y \circ \pi'(1) = \pi(1) \circ \pi_T$$

over these neighborhoods.

<u>Corollary</u>.- (Under the hypothesis that each good branch is finite and the hypotheses of (6.2)). Let $\mathcal{S}$ be any sequence respecting the 2-global procedure of reduction. Let us fix $Z \in S_2^*(\mathcal{D},E)$ and assume that there is a closed point $P \in Z$ with $P \in S_2(\mathcal{D},E)$, but that Z is not both permissible and tangent to the W-directrix at P. Let Γ be the branch of the tree $\mathcal{A}$ associated to $\mathcal{S}$ which is defined by the infinitely near points of Z over P. Then Γ is finite.

<u>Proof</u>.- Assume that Γ is not finite. This implies that Γ_N is not a good branch for each N, where Γ_N denotes the branch beginning at the step N. Then one can find a quadratic blowing-up at a step $i \geq N$ for each N. This allows us to construct, by means of the above theorem, a stationary sequence of quadratic blowing-ups for Z. This leads to a contradiction since then Z must be permissible and tangent to the directrix at some step and hence at the first step by the following easy result

(6.12.2) "Assume that Z has normal crossings with E and $P \in Z$, $P \in S_2(\mathcal{D},E)$. Let Z' be the strict transform of Z by the quadratic transformation centered at P and let P' be the closed point of Z' over P. Assume that $P' \in S_2(\mathcal{D}',E')$ and Z' is permissible and tangent to the W-directrix at P'. Then Z is permissible and tangent to the W-directrix at P."

Remark.- For the problem of the reduction of singularities of a surface, one can find a result similar to the theorem above in ([14|.[17]).

(6.13) Theorem.- Let $\mathcal{S}$ be a sequence respecting the strong 2-global procedure of reduction. Assume that each branch of the corresponging tree $\mathcal{A}$ is finite. Then there is a step N with $\mathrm{cycl}(\mathcal{D}(N),E(N)) = \emptyset$.

Proof.- By (6.10) it is enough to show that ther is a step N such that $\#\mathrm{cycl}(\mathcal{D}(N),E(N)) < \#\mathrm{cycl}(\mathcal{D},E)$. Let us first asume the following property.

(6.13.1)

"Let us fix $P \in \mathcal{P}(\mathrm{cycl}(\mathcal{D},E))$. THen $P(\mathrm{mod}\sim)$ (where $\sim$ is the equivalence of (4.2.1)) has a finite number of elements

$$P(\mathrm{mod}\sim) = \{P_o = P, P_1, \ldots, P_s\}$$

where $P_i \in S_2(\mathcal{D}(i),E(i))$".

and let us prove the theorem under this assumption. For a step l there is a $P \in \mathcal{P}(\mathrm{bcycl}(\mathcal{D}(l),E(l))$ since otherwise all the branches are good branches (see 6.8)) and $\mathcal{S}$ is finite, hence in the last step there are no cycles. Let us assume without loss of generality that l = 0. Let Z be an element of $S_2^*(\mathcal{D},E)$, $P \in Z$, such that Z is not both permissible and tangent to $\mathrm{Dir}_W(\mathcal{D},E,P)$ at P (such a Z always exists). Let us use the notation of (6.13.1). If $\#\mathrm{cycl}(\mathcal{D}(s),E(s)) = \#\mathrm{cycl}(\mathcal{D},E)$ (otherwise one has won), by the proof of (6.10), necessarily $P_s \in \mathcal{P}(\mathrm{cycl}(\mathcal{D}(s),E(s))$. Let Z(s) be the strict transform of Z. We have $Z(s) \in S^*(\mathcal{D}(s),E(s))$ since otherwise the number of cycles decreases. (Note that $\mathcal{O}_{X,P} \simeq \mathcal{O}_{X(s),P}$). Now, let Z(s+1) be the strict transform of Z(s) by $\pi(s+1)$ and let $P_{s+1} = Z(s+1) \cap (\pi(s+1)^{-1}(\mathrm{center}))$. If $P_{s+1} \notin \mathcal{P}(\mathrm{cycl}(\mathcal{D}(s+1),E(s+1)))$, then by the proof of (6.10) one deduces that the number of cycles must decrease. Thus, one may assume that $P_{s+1} \in \mathcal{P}(\mathrm{cycl}(\mathcal{D}(s+1),E(s+1))$ and that $Z(s+1) \in S^*(\mathcal{D}(s+1),E(s+1))$. Moreover, by (6.11), Z(s+1) is not both permissible and tangent to the W-directrix at P(s+1). Now, let us repeat the procedure. Corollary (6.1.2) says that one cannot repeat this procedure infinitely many times, hence at some step the number of cycles decreases.

Now, let us prove (6.13.1). First of all, one can assume that for each N there is $i \geq N$ such that $\mathrm{bcycl}(\mathcal{D}(i),E(i) \neq \emptyset$ and the blowing-up $\pi(i+1)$ is quadratic centered at a closed point Q(i) such that there are exactly two curves C(i), $C'(i) \in S^*(\mathcal{D}(i),E(i))$ passing through Q(i). Moreover, exactly one of them is both permissible and tangent to the W-directrix at Q(i) (see (6.8)). Hence the blowing-up $\pi(i+1)$ corresponds to the statement S = IV of (3.10). Let us fix $P \in \mathcal{P}(\text{cycl}(\mathcal{D},E))$, $P(\text{mod} \sim) = \{P_o = P, P_1, \ldots\}$. Assume that $P_i \neq$ center of $\pi(i+1)$ for all i (i.e., $\pi(i+1)$ is either monoidal or quadratic with a center different from P_i). With this assumption let us prove that there is s such that $\pi(s+1)$ is monoidal centered at a curve containing P_s (hence $P(\text{mod}\sim)$ is finite).

Assume that Y,Z are the only curves of $S_2^*(\mathcal{D},E)$ passing through P. Assume first that neither Y nor Z is both permissible and tangent to the W-directrix at P. Since there is an index N such that $\pi(N+1)$ corresponds to S = VI, from (3.10) one deduces that for some $i \geq N$, the point P_i has been blown-up. Indeed, at $\mathcal{D}(N),E(N)$ each closed point contained in a curve of $S_2^*(\mathcal{D}(N),E(N))$ which is not both permissible and tangent to the W-directrix must also be contained in a curve permissible and tangent to the W-directrix. Assume now that Y is permissible and tangent to the W-directrix at P and that $\pi(i+1)$ is never monoidal centered at Z_i (strict transform of Z). Now let us prove that there is a step l such that Y_l is globally permissible and $\pi(l+1)$ is monoidal centered at Y_l.

Let us denote by Q(j), j = 1,...,m, the closed points of Y such that Y is not permissible and tangent to the W-directrix at Q(j). One may assume without loss of generality that through each Q(j) passes another curve $T(j) \in S_2^*(\mathcal{D},E)$ such that T(j) is permissible and tangent to the W-directrix at Q(j) (this occurs always at the index N such that $\pi(N+1)$ is given by S = VI). Let us denote $Q_o(j) = Q(j)$, $Q_{i+1}(j)$ = only closed point in Y_{i+1} over $Q_i(j)$. In order to prove that Y_l is permissible at a certain step, it is enough to prove that if Y_i is not both permissible and tangent to the W-directrix at $Q_i(j)$, then there exists $i' \geq i$ such that the center of $\pi(i'+1)$ contains $Q_{i'}(j)$. See Corollary (6.12). This is easy if there is no curve through $Q_i(j) \in S_2(\mathcal{D}(i),E(i))$ permissible and tangent to the W-directrix (see the reasoning just before). Assume that it is not so. In view of the priorities of (3.10) and since $\mathcal{S}$ follows the strong 2-global procedure of reduction, each time when $\pi(N+1)$ corresponds to S = VI one increases strictly the date of birth of the new points $Q \in S_2(\mathcal{D}(N+1),E(N+1))$ such that there is a curve permissible and tangent to the W-directrix through Q and another curve in $S_2^*(\mathcal{D}(N+1),E(N+1))$ which is not so. Then, at last $Q_N(j)$ is the only point with minimal date of birth and $\pi(N+1)$ must be centered at it, (of course, one wins if $\pi(i+1)$ contains $Q_i(j)$ in its center before step N).

Thus, one can assume without loss of generality that Y is permissible. Since each good branch is finite, necessarily there is a step N such that no curve of $S_2^*(\mathcal{D}(N),E(N))$ is contained in $\text{Sing}^r(\mathcal{D}(N),E(N))$ (otherwise, we always have S = II, and necessarily all the branches are good branches). Thus, for the blowing-up $\pi(N+1)$, one has S = III. In this way, the date of birth of the possible centers when S = III is strictly increased and finally Y_l is the only one with minimal date of birth (if is has not blown-up before), hence the center of $\pi(l+1)$ must necessarily be Y_l.

(6.14) Corollary.- The theorem (6.6) is true

Proof.- It is enough to consider $\mathcal{S}$ as a sequence beginning at $(X(N),\mathcal{D}(N), E(N)$ where $\text{cycl}(\mathcal{D}(N),E(N)) = \emptyset$.

7. LOCAL CONTROL. POLYGONS

(7.1) In order to prove the theorem (3.12) it is now enough to show that any good branch of the tree associated to a sequence which respects the strong 2-global procedure of reduction is finite. This will be done by means of controlling the characteristic polygon $\Delta(\mathcal{D},E,P,p)$ associated to $\mathcal{D}$ at the point P, relative to a regular system of parameters $p = (x,y,z)$. This polygon plays a role very similar to the one for surfaces (see [9]).

In order to simplify the treatement of ([4].I), let us construct an invariant which does not increase and which cannot be stationary instead of constructing an invariant which decreases strictly at each step.

Like in paragraph 6, let us assume without loss of generality that $S_2^*(\mathcal{D},E)$ has normal crossings with E and that for each closed point $P \in S_2(\mathcal{D},E)$ there are at most two elements of $S_2^*(\mathcal{D},E)$ through P.

Lex us fix a sequence $\mathcal{S}$ which respects the strong 2-global strategy and let Γ be a good branch of the corresponding tree $\mathcal{A}$. Since we are only interested in the local control of Γ, one can assume without loss of generality that if $\bar{P} \in \Gamma$, then $\bar{P} = \{P\}$ (only one point) and thus

(7.1.1) $$\Gamma = \{P(0),P(1),\dots\}$$

where $P(i) \in X(i)$ and the center of $\pi(i+1)$ contains $P(i)$.

Let us denote $R(i) = \mathcal{O}_{X(i),P(i)}$ and let $\hat{R}(i)$ be the completion of R(i). The system of parameters used for this local control are in general system of parameters in $\hat{R}(i)$. This will be possible since

(7.1.2) $$\text{Der}_k(R(i)) \subset \text{Der}_k(\hat{R}(i)).$$

Finally, in order to simplify notation, let us denote $R = R(0)$, $\hat{R} = \hat{R}(0)$.

Obviously, for $i \geq 1$ one has $e(E(i),P(i)) \geq 1$, where $e(E(i),P(i))$ denotes the number of components of E(i) passing through P(i). Thus, let us assume without loss of generality that

(7.1.3) $e(E(i),P(i)) \geq 1$ for all $i \geq 0$.

(7.2) Definition.- Let $p = (x,y,z)$ be a regular system of parameters in $R^\wedge$ and let us identify $R^\wedge = k[[x,y,z]]$. Assume that D generates $\mathcal{D}_P$. The "cloud of points Exp(D,p)" is defined by

(7.2.1) $\mathrm{Exp}(D;p) = \mathrm{Exp}(zD(x)/x;p) \cup \mathrm{Exp}(zD(y)/y;p) \cup \mathrm{Exp}(D(z);p)$

Here if $f = \sum f_{hij} x^h y^i z^j \in R^\wedge[x^{-1},y^{-1}]$ one denotes

(7.2.2) $\mathrm{Exp}(f;p) = \{(h,i,j); f_{hij} \neq 0\} \subset \mathbb{R}^3.$

Let us denote

(7.2.3) $m(\mathcal{D},E,P;p) = \min\{h; (h,-1,r) \in \mathrm{Exp}(D;p)\}.$

For each $n \in \mathbb{N} \cup \{\infty\}$, let us denote

(7.2.4) $\mathbb{H}(n) = \{(u,v)\ ;\ u \geq 0,\ v+u/n \geq 0\} \subset \mathbb{R}^2$

Now, the polygon $\Delta(\mathcal{D},E,P;p)$ is defined to be the convex hull of

(7.2.5) $(\psi(\mathrm{Exp}(D,p)) + \mathbb{H}(m(\mathcal{D},E,P;p))) \cap \{(u,v);\ v \geq -1\} \subset \mathbb{R}^2$

where $\psi(\mathrm{Exp}(D,p))$ denotes the projection from $(0,0,r)$ of $\mathrm{Exp}(D,p) \cap \{(h,i,j); j \leq r-1\}$ onto the hyperplane $j = r-1$.

One knows that $\Delta(\mathcal{D},E,P,p)$ has its vertices in $(1/(r+1)!).\mathbb{Z}^2$. Let us denote by $(\alpha(\mathcal{D},E,P;p)), \beta(\mathcal{D},E,P;p))$ the vertex of lowest abscissa. This vertex will be called "main vertex". The number $\beta(\mathcal{D},E,P;p)$ will be the main invariant for the control of the branch Γ.

Remark.- Although the above definitions work in general, they will be useful only for a special kind of regular systems of parameters and for points in $S_2(\mathcal{D},E)$.

(7.3) Definition.- Assume that $P \in S_2(\mathcal{D},E)$. A regular system of parameters $p = (x,y,z)$ of $R^\wedge$ is a "good system of parameters" iff the following properties hold:

a) $(x=0) \subset E \subset (xy = 0)$ (locally at P).

b) $W(\mathcal{D},E,P) \pmod{\bar{x}} \neq (\bar{y}).Gr(R) \pmod{\bar{x}}$

(Note that GrR = GrR^).

(7.4) Lemma.- With the above hypotheses, let $p = (x,y,z)$ be a regular system of parameters, let $\Delta = \Delta(\mathcal{D},E,P;p)$, then:

a) $\Delta \subset \{(u,v);\ u \geq 0,\ v \geq -1\}$.

b) $\Delta \subset \{(u,v);\ u+v \geq 1\}$.

c) (x,z) is permissible at the point P iff $\Delta \subset \{(u,v);\ u \geq 1\}$.

d) (y,z) is permissible at the point P iff $\Delta \subset \{(u,v);\ v \geq 1\}$.

Proof.- $\mathcal{D}_P$ is generated by

(7.4.1) $$D = ax\partial/\partial x + b\partial_y + c\partial/\partial z$$

where $\partial_y = \partial/\partial y$ or $y\partial/\partial y$ depending on e(E,P). Now the result follows easily from the definition of Δ and the definition of permissibility (I.(2.4)). (See also [9] for the analogous facts on surfaces).

(7.5) Proposition.- Assume that $P \in S_2(\mathcal{D},E)$, let $p = (x,y,z)$ be a good system of parameters, let $\pi: X' \longrightarrow X$ be a blowing-up centered at P or centered at (x,z) or (y,z). Assume that the center of π is permissible and tangent to the W-directrix.

Let $P' \in X'$ be a closed point such that $\pi(P') = P$ and assume that $P' \in S_2(\mathcal{D}',E')$. Assume that π is given at P' by one of the following expressions

(7.5.1)
$$\begin{aligned} (T-1,0)&: x = x';\ y = x'y';\ z = x'z'. \\ T-2&: x = x'y';\ y = y';\ z = y'z'. \\ T-3&: x = x';\ y = y';\ z = x'z'. \\ T-4&: x = x';\ y = y';\ z = y'z'. \end{aligned}$$

Then $p' = (x',y',z')$ is a good system of parameters at P' and if $\Delta = \Delta(\mathcal{D},E,P;p)$, $\Delta' = \Delta(\mathcal{D}',E',P';p')$ one has

(7.5.2) $$\Delta' = \text{convex hull of } \sigma(\Delta) + \mathbb{R}^2_0$$

where $\sigma(u,v) = (u+v-1,v)$ if (T-1,0), $\sigma(u,v) = (u,u+v-1)$ if T-2, $\sigma(u,v) = (u-1,v)$ if T-3 and $\sigma(u,v) = (u,v-1)$ if T-4.

Proof.- The asertion (7.5.2) follows easily from the equations I(1.3.2) and I.(1.3.3), from the usual behaviour of the characteristic polygon under the transformations (7.5.1) (see [9]) and from the fact that

(7.5.3)
$$\begin{aligned} m(\mathcal{D}',E',P';p') &= m(\mathcal{D},E,P;p)-1 \quad (T-1,0) \\ &= m(\mathcal{D},E,P;p) \quad T-3 \\ &= \infty \quad T-2,\ T-4. \end{aligned}$$

Next, we prove that p' is good. Clearly, we have part a) of (7.3). Since $W(\mathcal{D},E,P) \neq (\bar{y})$, we have $W(\mathcal{D},E,P) = (\bar{z}+\lambda\bar{x}+\mu\bar{y})$. Since

$P' - S_2(\mathcal{D}',E')$, one has $\lambda = 0$ if (T-1,0), $\mu = 0$ if (T-2) and $\lambda = \mu = 0$ if T-3, T-4. Now the result is easy: essentially, one has to consider the last coefficient in (7.4.1) and its directrix cannot be $y' = 0$ by known arguments (see [9]). (For more details, see [4] II.3 and II.4).

(7.6) <u>Lemma</u>.- Assume that $P \in S_2(\mathcal{D},E)$, let $p = (x,y,z)$ be a good system of parameters and let $\Delta = \Delta(\mathcal{D},E,P;p)$. Let (α,β) be the main vertex of Δ.

a) Let us consider a coordinate change

(7.6.1) $$z_1 = z + \sum \lambda_{pq} x^p y^q$$

where $(p,q) \geq (\alpha,\beta)$ in the lexicographical order. Then $p_1 = (x,y,z_1)$ is a good system of parameters and iff (α_1,β_1) is the main vertex of $\Delta_1 = \Delta(\mathcal{D},E,P;p_1)$, one has

(7.6.2) $$(\alpha_1,\beta_1) \geq (\alpha,\beta)$$

in the lexicographical order. Moreover, if $\lambda_{\alpha\beta} = 0$ one has equality in (7.6.2).

b) Let us consider a coordinate change

(7.6.3) $$y_1 = y + \sum_{i \geq 1} \mu_i x^i$$

Let $p_1 = (x,y_1,z)$. Then $(\alpha,\beta) = (\alpha_1,\beta_1)$. Moreover, if $e(E,P) = 1$, p_1 is a good system of parameters.

c) Let us consider a coordinate change

(7.6.4) $$z_1 = z + \lambda x^p y^q, \quad (0,0) \neq (p,q) \in \mathbb{Z}^2$$

where $(p,q) < (\alpha,\beta)$ in the lexicographical order. Then $p_1 = (x,y,z_1)$ is a good system of parameters and (p,q) is the main vertex of Δ_1.

<u>Proof</u>.- a) Follows from the general computations for surfaces ([9]) if one remarks that

(7.6.5) $$D(z_1) = D(z) + \sum \lambda_{pq} x^p y^q (p\, D(x)/x + qD(y)/y)$$

b) As in a), noting that

(7.6.6) $$zD(y_1)/y_1 = z(D(y) + \sum \mu_i . i x^i D(x)/x)/y_1$$

c) Since $(0,1) \leq (p,q) < (\alpha,\beta)$, we have $(0,1) < (\alpha,\beta)$; one has $W(\mathcal{D},E,P) = (\bar{z} + \lambda\bar{x})$. This implies that z^r appears as a monomial in $D(z)$ and hence the desired result follows easily from the general computations ([9]) and from

(7.6.5). (For more details, see: [4] II.3.2 and II.3.3).

(7.7) Definition.- Let $P \in S_2(\mathcal{D},E)$ and let $p = (x,y,z)$ be a good system of parameters. Let $\Delta = \Delta(\mathcal{D},E,P,p)$ and let (α,β) be the main vertex of Δ. One says that p is "main vertex-prepared" iff one of the following holds:

a) $(\alpha,\beta) \notin \mathbb{Z}_0^2$.

b) $(\alpha,\beta) \in \mathbb{Z}_0^2$ but there is no change of coordinate $z_1 = z + \lambda x^\alpha y^\beta$ such that $(\alpha_1,\beta_1) > (\alpha,\beta)$ in the lexicographical order (notations of (7.6) a)).

If $p = (x,y,z)$ is not main vertex-prepared, one can obtain by a sequence of changes $z_1 \mapsto z + \lambda x^\alpha y^\beta$ a system of parameters which is main vertex-prepared. The resulting change

(7.7.1) $$z_1 = z + \sum \lambda_{pq} x^p y^q$$

is called a "main vertex preparation" of p. (This result is an easy corollary of (7.6)).

Remark.- No change of the type

(7.7.2) $$z_1 = z + \sum_{(p,q) > (\alpha,\beta)} \lambda_{pq} x^p y^q$$

may affect the fact that the main vertex (α,β) is or is not solvable (may be eliminated after a change $z \mapsto z + \lambda x^\alpha y^\beta$) since it does not affect the monomials which may contribute to (α,β) in (7.2.5).

(7.8) Proposition.- Let $P \in S_2(\mathcal{D},E)$ and let $p = (x,y,z)$ be a good system of parameters which is main-vertex-preparated. Assume that there is a curve $Y \in S_2^*(\mathcal{D},E)$ which is permissible tangent to the W-directrix at P and contained in $x = 0$. Then $I(Y) = (x,z)$ at P.

Proof.- One has $I(Y) = (x, z + \sum_{i \geq s} \lambda_i y^i)$. Assume that $\lambda_s \neq 0$. Let (α,β) be the main vertex of $\Delta(\mathcal{D},E,P;p) = \Delta$. Assume first that $(\alpha,\beta) > (0,s)$ in the lexicographical order. Then, if $z_1 = z + \lambda_s y^s$ one deduces that $(0,s)$ is the main vertex of $\Delta_1 = \Delta(\mathcal{D},E,P;(x,y,z_1))$. Now, after the change $z_2 = z_1 + \sum_{i>s} \lambda_i y^i$, $(0,s)$ remains the main vertex of Δ_2. But this contradicts (7.4) c), since (x,z_2) is permissible. If $(\alpha,\beta) < (0,s)$ one can reason as above, since the change $z_1 = z + \sum_{i \geq s} \lambda_i$ does not modify the main vertex. If $(\alpha,\beta) = (0,s)$, the change $z_1 = z + \lambda_s y^s$ does not modify the main vertex since P is main-vertex-prepared and one can reason as above. Hence the only possibility is $\lambda_s = 0$ and thus $I(Y)=(x,z)$.

(7.9) Lemma.- Let $P \in S_2(\mathcal{D},E)$ and let $p = (x,y,z)$ be a good system of parameters which is main-vertex prepared. Let us consider the coordinate change

(7.9.1) $$y_1 = y + \sum_{i \geq 1} \mu_i x^i$$

Then:

a) If $e(E,P) = 1$, then $p = (x,y_1z)$ is also a good system of parameters which is main-vertex-prepared and the main vertices of $\Delta = \Delta(\mathcal{D},E,P,p)$ and $\Delta_1 = \Delta(\mathcal{D},E,P;p_1)$ agree.

b) If $e(E,P) = 2$, let E' be $(x=0)$. Then $r = \nu(\mathcal{D},E',P)$, $P \in S_2(\mathcal{D},E')$, p is a good system of parameters which is main-vertex prepared and $\Delta(\mathcal{D},E,P;p) = \Delta(\mathcal{D},E',P;p)$ (now one can apply a)).

Proof.- a) The change (7.9.1) does not affect the monomials which may contribute to the fact that after a change of main vertex preparation the main vertex would be dissolved.

b) It is enough to observe that

(7.9.2) $$J^r(\mathcal{D},E,P) = J^r(\mathcal{D},E',P) = J^r(D(z)).$$

(7.10) Lemma.- Let $P \in S_2(\mathcal{D},E)$ and let $p = (x,y,z)$ be a good system of parameters which is main-vertex-prepared. Then $W(\mathcal{D},E,P) = (\bar{z} + \lambda\bar{x})$ and if $\lambda \neq 0$, then $(1,0) > (\alpha,\beta)$ in the lexicographical order where (α,β) is the main vertex of $\Delta = \Delta(\mathcal{D},E,P;p)$.

Proof.- Assume first that $W(\mathcal{D},E,P) = (\bar{z} + \lambda\bar{x} + \mu\bar{y})$, $\mu \neq 0$, let us make the coordinate change $z_1 = z + \mu y$, then $W(\mathcal{D},E,P) = (z_1 + \lambda x)$. This implies easily that $(0,1) \notin \Delta_1 = \Delta(\mathcal{D},E,P;p_1)$. Then necessarily $(0,1) \in \Delta$ and thus $(0,1) = (\alpha,\beta)$ (see (7.6)).

The change $z \longmapsto z_1$ dissolves (α,β), contradiction, hence $\mu = 0$. Assume now that $(1,0) \leq (\alpha,\beta)$, then after $z_1 = z + \lambda x$, one obtains that $W(\mathcal{D},E,P) = (\bar{z}_1)$ and hence $(1,0) \notin \Delta_1$, contradiction.

Remark.- The above lemma implies that after a coordinate change $z_1 = z + \lambda x$ one obtains a good system of parameters $p_1 = (x,y,z_1)$ which is main-vertex-prepared, such that $W(\mathcal{D},E,P) = (z_1)$ and such that the main vertex of Δ_1 coincides with the main vertex of Δ. Such a system p_1 will be called "dir-prepared" and the coordinate change $z \longmapsto z_1$ will be called a "directrix preparation".

(7.11) Proposition.- Let $P \in S_2(\mathcal{D},E)$ and let $P = (x,y,z)$ be a good system of parameters which is dir-prepared. Assume that there is a curve $Y \in S_2^*(\mathcal{D},E)$ which is permissible and tangent to the W-directrix at P and that Y is not contained in (x=0). The there is a coordinate change

(7.11.1)
$$y_1 = y + \sum_{i\geq 1}\mu_i x^i$$
$$z_1 = z + \sum_{j\geq 2}\lambda_j x^j$$

having the following properties:

a) If e(E,P) = 2, then $\mu_i = 0$ for all i.

b) $p_1 = (x,y_1,z_1)$ is dir-prepared and the main vertex of $\Delta_1 = \Delta(\mathcal{D},E,P;p_1)$ coincides with the main vertex of $\Delta = \Delta(\mathcal{D},E,P;p)$.

c) $I(Y) = (z_1,y_1)$ at P.

Proof.- Since Y is tangent to $Dir_W(\mathcal{D},E,P)$ and $W(\mathcal{D},E,P) = (\bar{z})$, then one can always to make a change like (7.11.1) in order to obtain $I(Y) = (y_1,z_1)$. Obviously, if e(E,P) = 2, then $Y \subset E$ and a) follows immediatly. Let s be the first index such that $\lambda_s \neq 0$. Reasoning as in (7.8), one deduces that $(s,0) \geq (\alpha,\beta)$ in the lexicographical order and thus one obtains b).

Remark.- Under the hypothesis of (7.1), there is at most one curve Y which is permissible and tangent to the W-directrix at P.

(7.12) Definition.- Let $P \in S_2(\mathcal{D},E)$. A good system of parameters p = (x,y,z) is said to be "prepared" iff it is dir-prepared and has the addtitional property that if Y is curve which is permissible at P and tangent to $Dir_W(\mathcal{D},E,P)$, then I(Y) = (x,z) or I(Y) = (y,z).

If p is not prepared, one can obtain a system which is prepared by making first a preparation of the main vertex, then a directrix-preparation and finally the change (7.11.1).

(7.13) Theorem.- Let us consider a step $\pi(i+1): X(i+1) \longrightarrow X(i)$ of the sequence $\mathcal{S}$. Assume that p(i) = (x(i),y(i),z(i)) is a good system of parameters which is prepared at P(i). Then there is a good system of parameters p(i+1) = (x(i+1),y(i+1),z(i+1)) which is prepared at P(i+1) such that

(7.13.1)
$$\beta(\mathcal{D}(i+1),E(i+1),P(i+1);p(i+1)) \leq$$
$$\leq \beta(\mathcal{D}(i),E(i),P(i);p(i))$$

Moreover, if one has equality in (7.13.1) then $P(i+1) \notin$ (strict transform of x(i) = 0) and x(i+1) = 0 is the exceptional divisor of $\pi(i+1)$ at P(i+1).

Proof.- In order to simplify the notation, let us denote X = X(i), X' = X(i+1), $\pi = \pi(i+1)$, etc. Assume first that π is monoidal. Since p is prepared, π must be centered at (x,z) or at (y,z). If π is centered at (y,z), then it must be given at P' by T-4, in view of the fact $W(\mathcal{D},E,P) = (\bar{z})$ (for notation, see (7.5)). Let $p' = (x',y',z')$ be the resulting system of parameters, which is a good system of parameters in view of (7.5). By (7.5.2).

(7.13.2) $$(\alpha',\beta') = (\alpha,\beta-1)$$

where (α,β) is the main vertex of $\Delta = \Delta(\mathcal{D},E,P;p)$ and (α',β') is the main vertex of $\Delta' = \Delta(\mathcal{D}',E',P';p')$. Now, assume that (α',β') may be eliminated by a change $z_1' = z' + \lambda x'^{\alpha'} y'^{\beta'}$ this implies easily that (α,β) could be eliminated by means of the change $z_1 = z + \lambda x^{\alpha'} y^{\beta'+1}$, contradiction. Hence p' is main-vertex-prepared and after a preparation one obtains the desired system of parameters p(i+1). In this case one has the strict inequality in (7.13.1) since $\beta' = \beta-1$.

Assume now that π is centered at (x,z). Then π must be given at P' by T-3 and one has

(7.13.3) $$(\alpha',\beta') = (\alpha-1,\beta)$$

As above, p' is main-vertex prepared. Moreover $x' = 0$ is the exceptional divisor of and $P' \notin$ (strict transform of $x=0$). Making a preparation, which does not change $x'=0$, one obtains the result.

Assume now that π is quadratic. Now, since $W(\mathcal{D},E,P) = (z)$, π is given at P' by one of the following equations

(7.13.4) $$(T\text{-}1,\zeta):\ x = x';\ y = x'(y'+\zeta);\ z = x'z'$$
$$T\text{-}2:\ x = x'y';\ y = y';\ z = y'z'.$$

Assume first that π is given by T-2. Now, in view of (7.5.2) one has

(7.13.5) $$(\alpha',\beta') = (\alpha,\alpha+\beta-1).$$

Now, since π is quadratic and since the branch Γ is a good branch, (x,z) cannot be permissible and hence $\alpha < 1$, thus $\beta' < \beta$. If a coordinate change $z_1' = z'+x'^{\alpha'}y'^{\beta'}$ can dissolve the main vertex (α',β'), then the change, $z_1 = z + x^{\alpha'}y^{\beta'-\alpha'+1}$ dissolves (α,β), contradiction. Now, it is enough to make a preparation of p'.

Assume now that π is given by $(T\text{-}1,\zeta)$. Let us first reduce the problem to the case $\zeta = 0$. Since there is no permissible curves through P, in order to obtain a prepared system of parameters it is enough to obtain a dir-prepared system. If $e(E,P) = 1$, after the change $y_1 = y - \zeta x$, one obtains from p a system of parameters p_1 which is dir-prepared (hence prepared) and such that π is given at P' by (T-1,0) from p_1. Assume now that $e(E,P) = 2$, let E^* be given by x=0. One knows that p is a good system of parameters which is prepared for $(\mathcal{D},E^*)$ and moreover, the strict transform of $(\mathcal{D},E^*)$ at P' coincides with the corresponding strict transform of $(\mathcal{D},E)$ (recall that $\zeta \neq 0$). Since $S_2^*(\mathcal{D},E)$ has normal crossings with E, there is no permissible curve tangent to the W-directrix through P relatively to $(\mathcal{D},E^*)$ (if such a curve Y is not both permissible and tangent to the W-directrix relative to $(\mathcal{D},E)$, then necessarily $Y \not\subset E$, and hence Y has not normal crossings with E. On the other hand clearly $Y \in S_2^*(\mathcal{D},E)$). Now, one can reason exactly as above.

Let us assume that $\zeta = 0$. Let us denote by $-1/\varepsilon$ the slope of the first segment of Δ, i.e. the segment joining the first and the second vertices (if there is no second vertex, set $\varepsilon = \infty$). Let us distinguish two cases: $\varepsilon \leq 1$ and $\varepsilon > 1$. If $\varepsilon > 1$, after (T-1,0) one has

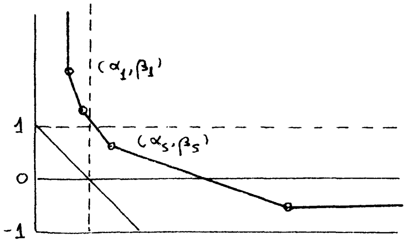
(7.13.6) $$(\alpha',\beta') = (\alpha+\beta-1,\beta)$$

and reasonning as above, p' is main vertex prepared. Moreover, $P' \notin$ (strict transform of $x = 0$), and $x' = 0$ is the exceptional divisor of π at P' (this is not changed by the previous change $y_1 = y-\zeta x$). Now, after a preparation of p', one obtains the result.

Assume now that $\varepsilon \leq 1$. Let $(\alpha_1,\beta_1) = (\alpha,\beta)$, $(\alpha_2,\beta_2),\dots,(\alpha_s,\beta_s)$ be the first vertices of Δ in the sense of increasing abscissas, let us denote

(7.13.7) $$\varepsilon_i = (\alpha_{i+1}-\alpha_1)/(\beta_i-\beta_{i+1})$$

and assume that $\varepsilon_i \leq 1$ for $i < s$ and that $\varepsilon_s > 1$.



After (T-1,0), one has that the main vertex of Δ' is

(7.13.8) $$(\alpha',\beta') = (\alpha_s + \beta_s-1,\beta_s)$$

Assume now that p' is not main vertex prepared and that a coordinate change $z_1 = z' + \lambda x^{\alpha'} y^{\beta'}$ dissolves (α',β'). Let us make a coordinate change $z_1=z+x^{\alpha_s} y^{\beta_s}$ in order to obtain p(1). One has that p(1) is prepared and the main vertex $(\alpha(1),\beta(1))$ of $\Delta(1)$ coincides with (α,β). There are two possibilities: $\varepsilon(1) > 1$ or $\varepsilon(1) \leq 1$. If $\varepsilon(1) > 1$ one has the above case. If $\varepsilon(1) \leq 1$, let us repeat the procedure. Since one has always

$$(7.13.9)\qquad \begin{array}{c} \alpha_s > \alpha \\ \alpha_s + \beta_s \leq \alpha + \beta \end{array}$$

one can repeat only finitely many times until the corresponding p' is main-vertex prepared (thus one obtains the strict inequality in (7.13.1)) or one obtains the case $\varepsilon > 1$.

(7.14) Corollary.- The branch Γ must be finite.

Proof.- Let us fix p(0) and let us obtain p(i) like in (7.13). If Γ is not finite, then there is N such that for $i \geq N$ one has that

$$(7.14.1)\qquad \beta(\mathcal{D}(i), E(i), P(i); p(i)) = \beta(\mathcal{D}(N), E(N); p(N))$$

Assume without loss of generality that N = 0. Now, looking at the polygons, for each N there is $i \geq N$ such that $\pi(i+1)$ is quadratic. By (7.13), the points P(i) correspond to the infinitely near points of a regular curve Y under quadratic or monoidal transformations with center contained in the exceptional divisor of the preceding transformation. In this situation, Y has normal crossings with E at P at a certain step, which may be supposed to be the first one.

Now, let us apply Theorem (6.12). It implies that Y generates a stationary sequence under quadratic blowing-ups and, since Y has normal crossings with E (see (6.12.2)) then Y must be permissible and tangent to the W-directrix at P(i) for each i. Since Γ is good, this contradicts the fact that for some i, $\pi(i+1)$ is quadratic.

(7.15) Thus (3.12) is proved and hence the main result (3.1).

APPENDIX

1. Proof of II.(5.5)

(1.1) Let π_Y: $X' \to X$ be the blowing-up of X centered at Y, let us denote by Z' the strict transform of Z by π_Y, let P' be a closed point of $\pi_Y^{-1}(Y) \cap Z'$ having the property II.(3.3.1) for a certain branch of Z' and let $P = \pi_Y(P')$. Without loss of generality, let us assume that Z and Z' have only one branch. Let us denote by $P' = P'(0), P'(1), \ldots$ the infinitely near points of Z' and let $Z' = Z'(0), Z'(1), \ldots$ be the strict transforms of Z' passing through these points. Analogously, let us denote by $P = P(0), P(1), \ldots$ the infinitely near points of Z and let $Z = Z(0), Z(1), \ldots$ be the corresponding transforms. One has the following diagram

$$\begin{array}{ccccccc} X'(0) & \xleftarrow{\pi'(1)} & X'(1) & \xleftarrow{\pi'(2)} & X'(2) & \longleftarrow & \ldots \\ \downarrow{\scriptstyle \pi_Y} & & & & & & \\ X(0) & \xleftarrow{\pi(1)} & X(1) & \xleftarrow{\pi(2)} & X(2) & \longleftarrow & \ldots \end{array}$$

where $\pi'(i)$ (resp. $\pi(i)$ is centered at $P'(i-1)$ (resp. $P(i+1)$), $i \geq 1$.

Let us denote by t the minimum number such that $Z'(t)$ is permissible and tangent to the W-directrix at $P'(t)$. Let us proceed by induction on t. Instead of proving directly II.(5.5), let us prove the following stronger assertion

(1.1.2) "$Z'(t)$ permissible and tangent to the W-directrix at $P'(t) \Rightarrow P(t) \in S_2(\mathcal{D}(t), E(t))$ and $Z(t)$ is permissible and tangent to the W-directrix at $P(t)$".

(1.2) Case $t = 0$. Let us choose a system of parameters $p = (x,y,z)$ suited for (E,Y) such that $I(Y) = (y,z)$ and such that $W(\mathcal{D},E,P) = (\bar{z})$. In view of II. (2.4), since $P' \in S_2(\mathcal{D}',E')$, then π_Y is given at P' by

(1.2.1) $$x = x'; \; y = y'; \; z = y'z'$$

In particular, this implies that Z is tangent to $\mathrm{Dir}_W(\mathcal{D},E,P)$, since $P' \in Z'$.

Since $P' \in S_2(\mathcal{D}',E')$, in view of (1.2.1), one sees that $W(\mathcal{D}',E',P') = (\bar{z} + \lambda \bar{x}' + \mu \bar{y}')$. Let us make the coordinate change $z_1 = z + \lambda xy + \mu y^2$, which does not change $I(Y)$, nor $W(\mathcal{D},E,P)$ nor (1.2.1). This allows us to assume that $\lambda = \mu = 0$, hence $W(\mathcal{D}',E',P') = (\bar{z}')$.

One has that Z' is tangent to $\mathrm{Dir}_W(\mathcal{D}'E',P') = (\bar{z}' = 0)$. Moreover, since Z' is not contained in $y' = 0$ (exceptional divisor of π_Y) and since Z' has normal crossings with E' at P', one deduces that Z' is transversal to y' = 0. This implies that after a coordinate change of the type

$$(1.2.2)\qquad \begin{aligned} z_1' &= z' + \textstyle\sum_{i\geq 2}\mu_i y'^{\,i} \\ x_1' &= x' + \textstyle\sum_{i\geq 1}\lambda_i y'^{\,i} \end{aligned}$$

one can assume that I(Z') = (x',z'). (Note that if $(x = 0)\subset E$, then $(x' = 0)\subset E'$ and hence $\lambda_i = 0$ for all i). Now, making a coordinate change

$$(1.2.3)\qquad \begin{aligned} z_1 &= z + \textstyle\sum_{i\geq 2}\mu_i y^{i+1} \\ x_1 &= x + \textstyle\sum_{i\geq 1}\lambda_i y^{i} \end{aligned}$$

which does not affect the conditions above, one can assume that $\mu_i = \lambda_i = 0$ for all i in (1.2.2).

In this situation, I(Z) must be (x,z) and looking at the equations I.(1.3.2) and I.(1.3.3) one deduces easily that Z is permissible at P (obviously, Z is tangent to $\mathrm{Dir}_W(\mathcal{D},E,P) = (\bar{z} = 0)$). Note that $P\in S_2(\mathcal{D},E)$ by II.(2.4), since $P'\in S_2(\mathcal{D}',E')$.

(1.3) Lemma. Without loss of generality, one can assume that one of the following statement is true:

a) $Z\subset E$ and the component of E which contains Z does not contain Y.

b) $e(E,P)\leq 1$ and if $e(E,P) = 1$, then $Z\not\subset E$ and $Y\subset E$.

Proof. Assume that E_1 is a component of E which contains Z, Y. Taking p = (x,y,z) as in the first part of (1.2), necessarily $I(E_1) = (y)$. This implies that $Z'\subset E_1$ = strict transform of E_1, contradiction, since $P'\notin E_1$ in view of (1.2.1). Lemma is proved.

Let $E^* = \bigcup$ components of E which contain {Y}; we have $e(E^*,P)\leq 1$. If $Z\not\subset E$, then for some s, $P'(s)\notin E^*(s)$ = strict transform of E^* by $\pi_Y\circ\pi'(1)\circ\cdots\circ\pi'(s)$ and $P(s)\notin E^*(s)$ = strict transform of E^* by $\pi(1)\circ\cdots\circ\pi(s)$

This implies that one can consider $(\mathcal{D},E^*)$ instead of $(\mathcal{D},E)$ in order to prove II.(5.5). (See also the proof of II. (7.13), case T-1,ζ).

(1.4) First case: Z and Y transversal. Let p = (x,y,z') be suited for (E,Y), I(Y), I(Y) = (y,z) and $W(\mathcal{D},E,P) = (\bar{z})$. Then π_Y is given by (1.2.1) and Z is tangent to $\mathrm{Dir}_W(\mathcal{D},E,P) = (\bar{z}=0)$. Since Z and Y are transversal, by making a coordinate change $x_1 = x + \lambda y$, if necessary (see the above lemma), we can assume Z is is tangent to (x,z). Let us assume as in (1.2) that $W(\mathcal{D}',E',P') = (\bar{z}')$.

Since $Z'_2 \in S^*(\mathcal{D}',E'), Z'$ is tangent to $Dir_W(\mathcal{D}',E',P') = (\bar{z}' = 0)$. Then, by (1.2.1), Z' is tangent to (x',z'). Now, the equations of $\pi'(1)$ and $\pi(1)$ at $P'(1)$, respectively $P(1)$, are respectively:

$$\pi'(1):\ x' = x'(1)\ y'(1),\quad y' = y'(1),\quad z' = y'(1)\ z'(1)$$
$$\pi(1):\ x = x(1)\ y(1),\quad y = y(1),\quad z = y(1)\ z(1)$$

Lek K be the field of the rational fuctions of X. O e has the following equalities in K:

(1.4.2) $$x'(1) = x(1),\ y'(1) = y(1),\ z'(1) = z(1)/y(1)$$

Now, let T be the curve given locally at P(1) by $(y(1),z(1))$. (Note that T is globally defined by $T = Proj(Dir_W(\mathcal{D},E,P)) \subset \pi(1)^{-1}(P)$).

Assume that T is permissible at P(1) (see lemma (1.5)) and let U(1) be an open neighborhood of P(1) such that T is globally permissible at U(1). Let

(1.4.3) $$\pi_T : V'(1) \longrightarrow U(1)$$

be the blowing-up with center T and let Q'(1) be the point over P(1) which corresponds to the strict transform of $z(1) = 0$. By (1.4.2) W'(1) of P'(1) and U'(1) of Q'(1) which are isomorphic and such that $\pi_Y \circ \pi'(1) = \pi(1) \circ \pi_T$ over them. In particular, Z'(1) goes into the strict transform H'(1) of Z(1) under π_T. This implies that

(1.4.3) $$t(H'(1)) = t(Z'(1)) = t(Z')-1$$

and by induction hypothesis $Z(1) \in S_2^*(\mathcal{D},E)$ and hence $Z \in S_2^*(\mathcal{D},E)$.

(1.5) Lemma. T is permissible at P(1).

Proof. Let $p = (x,y,z)$, $p' = (x',y',z')$, $p(1) = (x(1),\ y(1),\ z(1))$ and $p'(1) = (x'(1),\ y'(1),\ z'(1))$. Since p(1) is a good system of parameters, T is permissible at P(1) iff

(1.5.1) $$\Delta(\mathcal{D}(1),E(1),P(1);p(1)) \subset \{(u,v);\ \ v > 1\}.$$

Let us adopt the obvious notation Δ, Δ', $\Delta(1)$, $\Delta'(1)$ for the corresponding polygons. One has

(1.5.2)
$$\Delta' = \text{convex hull of } (\sigma_2(\Delta) + \mathbb{R}^2_o)$$
$$\Delta'(1) = \text{convex hull of } (\sigma_1(\Delta') + \mathbb{R}^2_o)$$
$$\Delta(1) = \text{convex hull of } (\sigma_1(\Delta) + \mathbb{R}^2_o)$$

where $\sigma_1(u,v) = (u,u+v-1)$, $\sigma_2(u,v) = (u,v-1)$. One obtains (1.5.1) from (1.5.2) by noting that

(1.5.3)
$$\Delta' \cap \{(u,v)\ ;\ u+v < 1\} = \emptyset.$$

(1.6) Second case: Z and Y tangent. Let $p = (x,y,z)$ be suited for (E,Y), $I(Y) = (y,z)$ and $W(\mathcal{D},E,P) = (\bar{z})$. Moreover, assume that $W(\mathcal{D}',E',P') = (\bar{z}')$. Z is tangent to (x,y) since by assumption Z' is tangent to (z'). One deduces from (1.2.1) that Z' is tangent to (y',z'). Now, the equations of $\pi'(1)$, $\pi(1)$ at $P(1)$, $P(1)$ are respectively

(1.6.1)
$$\pi'(1):\ x' = x'(1),\ y' = y'(1)x'(1),\ z' = z'(1)x'(1)$$
$$\pi(1):\ x = x(1),\quad y = y(1)x(1),\quad z = z(1)x(1)$$

One has

(1.6.2)
$$x'(1) = x(1),\ y'(1) = y(1),\ z'(1) = z(1)/y(1)x(1)$$

Now, let T be the curve given at $P(1)$ by $(x(1),z(1))$ (as above $T = \mathrm{Proj}(\mathrm{Dir}_W(\mathcal{D},E,P)) \subset \pi(1)^{-1}(P))$. By an argument like (1.5), one can prove that T is permissible at $P(1)$. Let

(1.6.3)
$$\pi_T:\ W''(1) \longrightarrow U(1)$$

be the blowing-up as in (1.4.3) and let $Q''(1)$ be the point over $P(1)$ which corresponds to the strict transform of $z(1) = 0$. Let $Y(1)$ be the strict transform of Y under $\pi(1)$. One has $P(1) \in Y(1)$. Now let $Y'(1)$ be the strict transform of $Y(1)$ under π_T. Then, we have $Q''(1) \in Y'(1)$. Moreover $Y'(1)$ is permissible since $Y(1)$ is permissible. Let

(1.6.4)
$$\pi_{Y'(1)}:\ V'(1) \longrightarrow V''(1)$$

be the blowing-up centered at $Y'(1)$ and let $Q'(1)$ be the point over $Q''(1)$ corresponding to the strict transform of $z(1) = 0$. From (1.6.2), one deduces that there are neighborhoods $W'(1)$ and $U'(1)$ of $P'(1)$, $Q'(1)$ respectively, which are isomorphic. In particular, $Z'(1)$ maps into the strict transform of $Z(1)$ under $\pi_T \circ \pi_{Y'(1)}$. Moreover $Q''(1) \in S_2(\mathcal{D}''(1),E(1))$. Let $Z''(1)$ be the strict transform of $Z(1)$ by π_T. By induction hypothesis, $t(Z''(1)) \leq t(Z'(1)) = t(Z'(1)) = t(Z')-1$, and thus $Z(1) \in S^*_2(\mathcal{D}(1),E(1))$, hence $Z \in S^*_2(\mathcal{D},E)$.

2. Proof of II. (6.12). (Theorems (a) and (b)). Use the arguments of (1.4) and (1.5) above.

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