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UNIVERSAL FAMILIES OF FOLIATIONS BY CURVES*

Xavier GOMEZ-MONT

For different reasons, the theories of Dynamical Systems and of Complex Analytic Geometry have been interested in parameter families of objects. A typical problem of dynamical systems is to know if for a given family of vector fields the orbit structure obtained by integrating the vector field remains topologically the same as the parameter moves, see Palis [14]. Complex analysts have been interested in constructing versal spaces of analytic objects, in such a way that one may reconstruct all parameter families of such objects by a simple method of pull-back, see Kuranishi [12] or Douady [4]. These theories have a non-trivial intersection in the theory of holomorphic foliations.

The main point of the present work is to exploit the fact that for a holomorphic foliation with singularities by one dimensional leaves, the line bundle tangent to the leaves on the non-singular points extends to an abstract holomorphic line bundle on all of M. Douady's parametrization of quotient sheaves in [4] applied to the tangent bundle of M plus the above fact produces a universal family of foliations by curves in M, where the parameter space is a complex analytic space. Douady has also shown that in order to solve parametrization problems of this sort one must impose on the families under consideration to have a flat variation with respect to the parameter. Again using the existence of the extended line bundle we show that this flatness condition is always satisfied if the parameter space is a reduced analytic space. We then proceed to describe the pattern of the universal families, where the Chern class of the line bundle

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springs as the discrete numerical invariant and then the continuous moduli appears.

The work is divided into three sections. In the first section we start from the geometric definition of a foliation, which is defined by local coordinate charts built on the model of the family of lines in $C^n = \bigcup w \times C$, $w \in C^{n-1}$. These local models are given in M-V, where V is a subvariety of codimension bigger than one. The main result of this section is in showing that the line bundle on M-V formed by vectors tangent to the leaves of the foliation has a natural extension as an abstract holomorphic line bundle L on all the manifold M and the existence of a holomorphic map $X:L \to TM$ from the line bundle L into the tangent bundle of M where the singularities of the foliation appear as the set of points where X vanishes. The proof relies on the Levi extension theorem. The fact that the foliation is non-singular in M-V is reflected in the fact that the map X is non-vanishing outside of a set of codimension bigger than one. This restriction is a natural one, since we want to avoid having an essential singularity along a codimension 1 subvariety.

Motivated by this geometric definition, we give an extended analytic definition of a foliation by curves as a non-identically equal to zero analytic map from the line bundle L into the tangent bundle of M, $X:L \to TM$. Two such maps define the same foliation if the line bundles are isomorphic as holomorphic bundles, and if after this identification the maps differ by multiplication by a never vanishing holomorphic function.

If X vanishes on a subvariety of codimension one, we could divide by an equation defining this subvariety and, modifying L accordingly, obtain a foliation $\frac{1}{h}X:L' \to TM$ as before. We consider these two foliations as distinct, since they have a different behaviour from the variational point of view that will interest us in section two.

Section two is devoted to proving the existence of a universal family of foliations. We begin by obtaining an equivalent definition of a foliation using subsheaves of the tangent sheaf, and then we use Douady's theorem of universal families of quotient sheaves to obtain the desired family. We then proceed to show that if the parameter space is reduced, then the flatness condition in Douady's theorem is always satisfied.

Section three is devoted to give a general outlook on what kind of analytic spaces one expects in this universal families. The first observation is that the Chern class remains constant in each connected component, which then brakes the problem into two: which Chern classes carry foliations, and what is the structure of those foliations that have constant Chern class.

We prove that for a compact Kähler manifold with vanishing first Betti number, the space of foliations with a given Chern class forms in a natural way a projective space . For projective manifolds we show in general that the space of foliations with a given Chern class is compact (and projective). Also for projective manifolds we show how for some Chern classes the variety of foliations having this Chern class has a structure of a holomorphic bundle of projective spaces over a complex torus of dimension one half the first Betti number of the manifold and that sometimes one can say exactly what is the dimension of this space. All these results use deep theorems of global analytic geometry, namely, the Kodaira-Nakano vanishing theorem, the existence of the Poincaré bundle on the Picard variety and the Riemann-Roch theorem.

Other applications of the line bundle technique may be found in [5], where we relate foliations to meromorphic vector fields, we sketch another construction of the universal family using the Poincaré bundle on the Picard variety of a projective variety and do explicit computations for projective spaces. In [6] we have analysed foliations by curves in singular spaces, where one only obtains a meromorphic line bundle, but interesting invariants appear as one measures the obstructions to make it a holomorphic line bundle. In [7] we persue the approach presented in this paper, where we let the complex structure of the base manifold to move and we give a method to construct tangent spaces and calculate universal derivatives. It turns out that the holomorphic cohomology of the line bundle carry information about the different complex structures that can be put on the leaves of the foliation. This fact is interesting since these groups depend only on the abstract analytic class of the foliation (i.e. the line bundle) and not on the topological foliation itself. In forthcoming works we will give a detailed picture for foliations in ruled surfaces and jointly with J. Seade and A. Verjovsky show that the Chern class is an invariant under homeomorphisms preserving the leaves of the foliation.

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1. Foliations by Holomorphic Curves

In this section we will define the objects that we will analyse throughout the paper: Foliations of complex manifolds by holomorphic curves. We will begin with a natural geometric definition, that motivates the extended analytic definition that we will use.

Let M be a complex manifold. A non-singular foliation of M by curves F can be given by an open covering of M by coordinate charts (U_i, ϕ_i) such that if

$$\phi_{ij} = \phi_i \circ \phi_j^{-1} = (\phi^1, \phi^2, \ldots, \phi^n) : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is the expression for a change of coordinates, then they satisfy

$$\frac{\partial \phi^1}{\partial z_n} = \ldots = \frac{\partial \phi^{n-1}}{\partial z_n} = 0 \qquad (1)$$

Such coordinate charts will be referred as foliated coordinate charts The differential condition (1) has the geometric meaning of leaving invariant the set of vertical lines given by $z_1 = K_1, \ldots, z_{n-1} = K_{n-1}$.

Since this local partition of C^n into a family of complex lines $C^n = C^{n-1} \times C \to C^{n-1}$ is being left invariant by the changes of coordinates, we may induce this partition into M , giving rise to a decomposition of M into leaves $M = \cup_\alpha L_\alpha$ obtained by gluing the local leaves. Each L_α will be a Riemann surface and the natural inclusion of L_α into M is an immersion, but it may wander around M and reaccumulate on itself, so it is not necessarily an embedding. The foliation actually may be defined by adjoining all coordinate charts compatible with the given cover in the sense of satisfying (1).

Geometric Definitions: Let M be a connected complex manifold of dimension n. A foliation (with singularities) F of M by curves is a non-singular foliation of M-V by curves where V is an analytic subvariety of M of codimension bigger than one. A point p in V is a removable singularity of the foliation F if we may find a coordinate chart on a neighborhood of p that is compatible with the atlas on M-V defining $\mathcal{F}$ in the sense of satisfying (1). The set of points that are non-removable singularities is called the <u>singular set of the foliation</u> and denoted by Sing($\mathcal{F}$).

Our aim now will be directed towards obtaining an analytic expression of the foliation $\mathcal{F}$. The main analytic concept that we will need in order to provide this expression is that of a <u>(holomorphic) line bundle</u> over M. A line bundle over M may be described by an open covering $\{U_i\}$ with local trivialisations $U_i \times \mathbb{C}$ and with transition functions for the intersections

$$\phi_i(U_i \cap U_j) \times \mathbb{C} \to \phi_j(U_i \cap U_j) \times \mathbb{C}$$

$$(z,t) \to (\phi_{ij}(z), \xi_{ij}(z)\cdot t)$$

where the ξ_{ij} are holomorphic non-vanishing functions satisfying the cocycle condition $\xi_{ik}=\xi_{ij}\cdot\xi_{jk}$ on $U_i \cap U_j \cap U_k$ (see Griffiths-Harris [9] p.66).

Let $\{(U_i,\phi_i)\}$ be a foliated coordinate cover of a foliation $\mathcal{F}$ on M-V. We obtain as a consequence of the compatibility conditions (1) that in the compositions of the Jacobians $D(\phi_{ik})=D(\phi_{ij})\cdot D(\phi_{jk})$ the term $\frac{\partial \phi^n_{ij}}{\partial z_n}$ at the lower right entry multiplies as a cocycle; so it defines a line bundle L' on M-V provided with the natural inclusion of L' into T(M-V) given by $t \to (0,0,\ldots,t)$. The image of L' in T(M-V) consists of those vectors tangent to the leaves of the foliation in M-V.

<u>Theorem</u> 1 ([5]): Let $\mathcal{F}$ be a singular foliation of M by curves, non-singular on M-V, with V an analytic subset of codimension bigger than one, and let L' be the holomorphic line bundle on M-V of tangent vectors to the foliation. Then:

1) L' has a canonical extension to an abstract holomorphic line bundle L on all of M and a bundle map $X:L \to TM$ whose image restricted to M-V are the tangent spaces to the leaves. X is unique up to multiplication by a nowhere zero holomorphic function on M.

2) A point p of V is a removable singularity of $\mathcal{F}$ if and only if X(p) is not the zero map $X(p):L_p \to T_pM$.

3) The singular set of the foliation is an analytic subvariety of M.

Proof: 1) The proof relies on the Levi extension theorem, which can be seen as a geometric version of Hartog's extension theorem (see Griffiths-Harris [9] p.396).

Let p be a point of V and choose a coordinate patch U of p which is biholomorphic to a polycilinder D in C^n. Let $V'=V\cap D$ so that L' restricts to a line bundle L' on D-V' and it comes with a map on D-V', $X:L'\to T(D-V')$, but since the tangent bundle on D is trivial, X has an expression $X=(X_1,\ldots,X_n)$ where each X_i is a bundle map

$$X_i:L'\to(D-V')\times C$$

Suppose X_i is not identically equal to zero, and let A the divisor of its zeroes $\{X_i=0\}$. By the Levi extension theorem $\bar{A}$ is a divisor on D. Now since D is a contractible Stein manifold we may find a holomorphic function g on D vanishing exactly on $\bar{A}$. Consider then the map

$$\frac{X_i}{g}:L'\to(D-V')\times C$$

it is holomorphic and nonvanishing on D-V', so it is an isomorphism on D-V'. But the right hand side extends as a line bundle to all of D, so we consider the extended line bundle L obtained by adjoining these extended coordinates.

That all this mappings glue correctly is caused by the following fact. Let L_1 and L_2 be two line bundles on the open set U, then any isomorphism f of L_1 to L_2 defined outside of a set of codimension two extends to an isomorphism on U. By Hartog's theorem we may extend f locally, but in local expression f is given by a function, so the set of points where it vanishes is of codimension 1. But by hypothesis it must be nonvanishing outside of one of codimension 2 so it is never vanishing. So the isomorphism extends to all of U.

Since at least one of the X_i is non-identically zero we obtain an extension at every point, and hence a unique global extension to all of M. Also the map $X:L\to TM$ extends to give an injection over M-V, and as a map from the abstract L is unique up to an automorphism of L, which is given by multiplication by a nonvanishing function on M.

2) If $X(p)\neq 0$ then the local triviality of the bundle and the local flow box theorem imply that p is a removable singularity.

Conversely, if p is a removable singularity then we may find an extension of L so that X is nonvanishing in a neighborhood. By the uniqueness of X we imply that $X(p)\neq 0$.

3) X can be thought of as a global section of the bundle Hom (L,TM) hence the set of points where it vanishes is an analytic subset.||

Motivated by this result we give:

Analytic Definitions: A holomorphic foliation (with singularities) on M by curves is a holomorphic map from a line bundle L on M into the tangent bundle of M, $X: L\to TM$. Two such (L,X) and (L',X') define the same foliation if and only if L is biholomorphic to L', and after identifying L with L' $X=fX'$ where f is a never-vanishing holomorphic function. The singular set Sing (X) of the foliation is the analytic subvariety defined by the zeroes of X, $X=0$. In M-Sing(X) we may define a non-singular foliation by curves obtained by integrating vector fields given by local expressions for X.

2. The Universal Family

In this section we will show that the set of foliations by curves on a compact complex manifold has a natural structure of a complex analytic variety. This result is far from trivial when viewed from the geometric definition and we obtain it since the analytic definition allows one to use Douady's approach to the moduli problem [4]. In order to phrase our problem in Douady's language, we have to give a sheaf theoretic definition of a foliation:

Lemma 2: There is a one to one correspondence between foliations by curves on M and invertible subsheaves of the sheaf of holomorphic vector fields on M, $\mathcal{L}\subset\Theta_M$.

Proof: If $X: L\to TM$ is the map defining a foliation, let $\mathcal{L}$ be the invertible sheaf of local sections of L and clearly $X(\mathcal{L})\subset\Theta_M$ is an invertible subsheaf of Θ_M. Conversely, given an invertible subsheaf $\mathcal{L}\subset\Theta_M$, let L be the line bundle associated to $\mathcal{L}$. The inclusion $\mathcal{L}\subset\Theta_M$ induces a holomorphic non-identically equal to zero map $X: L\to TM$, defining a foliation. If we multiply X by a never vanishing holomorphic function then the associated sheaves in Θ_M coincide, hence the correspondence between foliations and invertible subsheaves is one to one.||

By the Lemma then, a foliation is completely determined by the subsheaf $L \subset \Theta_M$ that we will call the sheaf tangent to the foliation. The set of foliations by curves then is in one to one correspondence with the set of invertible subsheaves L of Θ_M. One would like to put a complex analytic structure in this set in such a way that one may construct all holomorphic families by a simple method of pull back. Such a task was done in a more general context by Douady [4], where he shows that in order to solve such a problem it is natural to restrict to families with flat quotients with respect to the parameter. We will henceford restrict to such families and show that in the space of all subsheaves of Θ_M the sheaves that are invertible form an open subset. Hence, restricting Douady's family to this open set provides a universal family of foliations by curves. We will begin with some definitions.

If E is a coherent sheaf on the complex space T, the corank of E at t is the dimension over C of $E_t \otimes_{\mathcal{O}_t} (\mathcal{O}_t/m_t)$, where m_t is the maximal ideal in the local ring $\mathcal{O}_t$. By Nakayama's lemma, the corank is the minimum number of generators that E_t needs over $\mathcal{O}_t$.

Let S be a complex analytic space, not necessarily reduced, M a compact smooth manifold, $\Pi: S\times M \to S$, $\Pi_2: S\times M \to M$ the two projection mappings and denote by $\Theta_\pi = \Pi_2^* \Theta_M$ the sheaf of tangent vectors to the fibers of Π.

Definition: A family of foliations parametrized by S will consist of an exact sequence of coherent sheaves on SxM

$$0 \to L \xrightarrow{X} \Theta_\pi \xrightarrow{\rho} N \to 0 \qquad (2)$$

where the kernel sheaf L is of corank one at every point of SxM and the quotient sheaf N is $\mathcal{O}_S$-flat (i.e. for every point $(s,p) \in S\times M$, $N_{s,p}$ is $\mathcal{O}_s$-flat).

Remarks: 1) The process of restricting to a fibre sxM is obtained by tensoring with $\mathcal{O}_s/m_s$. We use the notation $L(s)_p = L_{s,p} \otimes_{\mathcal{O}_s} \mathcal{O}_s/m_s$ for the stalk of the restriction to sxM at (s,p). Restricting (2) we obtain a family of sequences

$$0 \to L(s) \to \Theta_{s\times M} \to N(s) \to 0 \qquad (2)_s$$

where $\Theta_{sxM}=\Theta_\pi(s)$ is the tangent sheaf to the complex manifold sxM. It is easy to see that the hypothesis imply that $L(s)$ has also corank one at every point of sxM. For reduced complex spaces the notion of corank one and invertible sheaf coincide (see Grauert-Remmert [8] p. 91) and since M is a complex manifold it follows that $L(s)$ is an invertible sheaf on sxM. This fact means that we may interpret $(2)_s$ as a family of foliations on M parametrized by S.

2) If S is a reduced complex space, it is equivalent to recquire that L is of corank one at every point or that L is an invertible sheaf on SxM.

We will show the existence of a universal family of foliation by curves on M:

Theorem 3: Let M be a compact complex manifold. There is a complex analytic space $\mathcal{D}$ and a family of foliations parametrized by $\mathcal{D}$ with defining sheaf sequence on $\mathcal{D}$xM

$$0 \to \tilde{L} \overset{\tilde{X}}{\to} \Theta_\pi \overset{\rho}{\to} \tilde{N} \to 0 \qquad (3)$$

such that for any other family of foliations parametrized by S with sheaf sequence (2), there is a unique holomorphic map $f:S\to\mathcal{D}$ such that $f^*(\tilde{L})=L$ via the natural identification $\Theta_\pi = f^*\,\Theta_\pi$.

Proof: Let $\bar{\mathcal{D}}$ be the Douady space of quotient sheaves of Θ_M (see Douady [4]), with its universal sequence

$$0 \to \bar{L} \overset{\bar{X}}{\to} \Theta_\pi \overset{\rho}{\to} \bar{N} \to 0 \qquad (4)$$

By definition, there is a one to one correspondance between points of $\bar{\mathcal{D}}$ and quotient sheaves of Θ_M, and hence with subsheaves of Θ_M. Let $\mathcal{D}$ be the subset formed by those points of s in $\bar{\mathcal{D}}$ such that the corank of $L_{s,p}$ is one for every point p in M.

We claim that $\mathcal{D}$ is an open subset of $\bar{\mathcal{D}}$, and hence has a natural structure of a complex analytic space. Granting this claim for a moment, define sequence (3) as the restriction of (4) to this open subset. It follows that the map $f:S\to\bar{\mathcal{D}}$ obtained from the universality properties of the Douady space applied to (2) actually has values in $\mathcal{D}$, $f:S\to\mathcal{D}$ and $f^*(\tilde{L})=L$ as desired.

Now we show that $\mathcal{D}$ is an open subset of $\bar{\mathcal{D}}$ by means of the following lemma:

Lemma 4: Let M be a compact complex manifold and $\mathcal{E}$ a coherent sheaf on SxM that is $\mathcal{O}_S$-flat. If the corank of $\mathcal{E}$ is one at every point of s_oxM, then there is a neighgorhood U of s_o in S such that $\mathcal{E}$ has corank 1 at every point of UxM.

Proof: By the compactness of M, it is enough to show that every point (s_o,p_o) has a neighborhood where $\mathcal{E}$ has corank one at each point. Recall that by Nakayama's lemma, the corank is also the minimal number of generators of $\mathcal{E}_{s,p}$ as an $\mathcal{O}_{s,p}$-module. Let F be a local section of $\mathcal{E}$ that generates $\mathcal{E}_{s_o,p_o}$, and consider the sheaf map $\mathcal{O} \to \mathcal{E}$ obtained by multiplication with F. By coherence of $\mathcal{E}$ it follows that this map is surjective perhaps on a smaller neighborhood of (s_o,p_o). Define $\mathcal{J}$ by the exact sheaf sequence

$$0 \to \mathcal{J} \to \mathcal{O} \overset{xF}{\to} \mathcal{E} \to 0 \qquad (5)$$

Let S' and U be neighborhoods of s_o and p_o such that (5) is defined on S'xU and U is a coordinate neighborhood. We now assume acquaintance with Douady [4]. Let K be a privileged polycilinder for $\mathcal{J}(s_o)$. By [4], we have that there exists a neighborhood S" of s_o in S' such K is $\mathcal{J}(s)$ privileged for $s \in S"$ and that it is possible to define a Banach vector bundle over S" whose fibre at any $s \in S"$ are the sections of $\mathcal{J}(s)$ over sxK, $B(K,\mathcal{J}(s))$. But on s_oxM we have by flatness of S"xU over S"

$$0 \to \mathcal{J}(s_o) \to \mathcal{O}_{s_o xM} \overset{xF}{\to} \mathcal{E}(s_o) \to 0$$

As mentioned before, since U is reduced $\mathcal{E}(s_o)$ is an invertible sheaf and hence $\mathcal{J}(s_o)$ is identically zero over s_oxU. But then $B(K,\mathcal{J}(s))=0$ for all s in S", since they are fibers of a Banach vector bundle with fiber $B(K,\mathcal{J}(s_o))=0$.

By Cartan's theorem A, it follows that $B(K,\mathcal{J}(s))$ generate each stalk $\mathcal{J}(s)_p$, hence $\mathcal{J}(s)=0$ for all s. Interpreting $\mathcal{J}$ as the ideal defining the support of $\mathcal{E}$, $\mathcal{J}(s)=0$ means that sxM is completely contained in the support of $\mathcal{J}(s)$ for every $s \in S$; hence the variety defined by $\mathcal{J}$ is all of SxM, which means that $\mathcal{J}$ is contained in the nilradical ideal $\mathcal{N}$ of $\mathcal{O}_{S"xU}$. This will produce an exact sequence of sheaves on S"xU

$$0 \to \mathcal{N}/\mathcal{J} \to \mathcal{E} \to \mathcal{O}^{red}_{S"xU} \to 0 \qquad (6)$$

where $\mathcal{O}^{red}=\mathcal{O}/_{N}$ is the reduction of S"xU. The corank of $\mathcal{O}^{red}_{S"xU}$ is always one, and since (6) is surjective the corank of $\mathcal{E}$ is bigger or equal to one at every point. Since F generates each stalk, the corank is less than or equal to one. So it is exactly one, and this proves the Lemma and hence the theorem.‖

The following result tells us that the condition that the quotient sheaf is flat with respect to the parameter is automatically satisfied in a very general situation.

Theorem 5: Let L be a line bundle on SxM and $X:L \to \Pi_2^* TM$ a holomorphic bundle map on SxM such that for every s in S the restriction of X to sxM does not vanish identically on an open set. Then the action of X on the invertible sheaf associated to L gives rise to a family of foliations parametrized by S

$$0 \to \mathcal{L} \xrightarrow{X} \Theta_\pi \xrightarrow{\rho} \mathcal{N} \to 0 \tag{7}$$

Proof: What we have to prove is that the quotient sheaf $\mathcal{N}$ is S-flat. Since this is a local assertion, and assuming momentarily that S is smooth, we are reduced to proving the following: Let $X=(X_1,.,X_m)$ be an m-tuple of holomorphic functions in UxV, where U and V are polydiscs in $\mathbb{C}^l$ and $\mathbb{C}^k$ with $t=(t_1,.,t_l)$ and $z=(z_1,.,z_k)$ as variables, X induces by multiplication a holomorphic map $X:\mathcal{O}_{UxV}\to\mathcal{O}^{\oplus m}_{UxV}$ such that its restriction to txV is injective for every t in U, then we have to show that the quotient sheaf is U-flat at (0,0). Let W=Ux0 be the subvariety of UxV defined by z=0. The idea of the proof is to show first that the formal completion of the quotient sheaf along W is U-flat, and then use that the process of formal completion is faithfully flat.

Let $\mathcal{B}$ be the sheaf $\mathcal{O}_{UxV}$ restricted to W, so that its stalks at every point are the convergent power series in t and z, and let

$$0 \to \mathcal{B} \xrightarrow{X} \mathcal{B}^{\oplus m} \to \mathcal{Q} \to 0 \tag{8}$$

be the exact sequence in W obtained by restricting (7) to W. The objective is to prove that $\mathcal{Q}$ is W-flat at (0,0). Let $\mathcal{I}$ be the sheaf of ideals in $\mathcal{B}$ generated by $(z_1,...,z_k)$ and consider the inverse system formed by $\mathcal{B}/\mathcal{I}^n$. Let $\hat{\mathcal{B}}$ be the sheaf of rings obtained as limit of the inverse limit. $(W,\hat{\mathcal{B}})$ is what is known as the formal completion of

UxV along W, see Hartshorne [10] p.194. The stalk of $\hat{\mathcal{B}}$ at a point consists of power series in z with coefficients convergent power series in t.

Let $\hat{\mathcal{Q}}$ be the inductive limit of $\mathcal{Q}/I^n\mathcal{Q}$ in the category of sheaves of abelian groups on W. The sequence (8) gives rise to an exact sequence of inverse systems of coherent sheaves on W.

$$0 \to (\mathcal{B}/I^n\mathcal{B}) \overset{X}{\to} (\mathcal{B}/I^n\mathcal{B})^{\oplus m} \to (\mathcal{Q}/I^n\mathcal{Q}) \to 0$$

If W' is any subpolydisc in W, Cartan's theorem B guarantees that we obtain a similar inverse system of groups of global sections of W'. The first term $H^o(W',\mathcal{B}/I^n\mathcal{B})$ consists of polynomials in z of degree n-1 with coefficients holomorphic functions on t, so it forms a surjective system (see Atiyah-MacDonald [1] p.104) hence the inductive limit of the sequences is exact on W'. Since polydiscs from a fundamental system of neighborhoods of W we obtain an exact sequence of sheaves in W

$$0 \to \hat{\mathcal{B}} \overset{X}{\to} \hat{\mathcal{B}}^{\oplus m} \to \hat{\mathcal{Q}} \to 0 \tag{9}$$

For every point t in W, let N(t) be the degree of the first nonzero term of X evaluated at t in its power series expansion in z, that exists by hypothesis. This number is upper semicontinuous as a function of t, so after perhaps shrinking the domain W we may find an upper bound N for N(t) along W. It is then easy to see that the vector bundle maps induced from the $\mathcal{O}_W$-sheaf mappings

$$\mathcal{B}/_{I^n} \overset{X}{\to} \mathcal{B}^{\oplus m}/_{I^{n+N}\mathcal{B}^{\oplus m}}$$

are always injective on W. This implies that the quotient is an $\mathcal{O}_W$-locally free sheaf, hence $\mathcal{O}_W$-flat. The inductive limit of all this sequences converge to (9), and since flatness is preserved under inductive limits (see Matsumura [13]) we obtain that $\hat{\mathcal{Q}}$ is $\mathcal{O}_W$-flat. In particular the stalk at t=0, $\hat{\mathcal{Q}}_o$, is $\mathcal{O}_{W,0}$-flat.

For any $\mathcal{B}_o$-module M, we have the maps $\mathcal{B}_o \to \hat{\mathcal{B}}_o$ and $M \to \hat{M}$, where we are completing with the ideal $(z_1,\ldots,z_k)$. These maps induce the $\hat{\mathcal{B}}_o$-module homomorphism

$$\hat{\mathcal{B}}_o \otimes_{\mathcal{B}_o} M \to \hat{\mathcal{B}}_o \otimes_{\mathcal{B}_o} \hat{M} \to \hat{\mathcal{B}}_o \otimes_{\hat{\mathcal{B}}_o} \hat{M} = \hat{M}$$

Tensoring the stalk over 0 of the sequence (8) with $\hat{\mathcal{B}}_o$ over $\mathcal{B}_o$, using right exactness of the tensor product and the above maps to the stalk 0 of (9) we obtain the commutative exact diagram

$$\begin{array}{ccccccc} & & \mathcal{B}_o \otimes_{\mathcal{B}_o} \hat{\mathcal{B}}_o & \longrightarrow & \mathcal{B}_o^{\oplus m} \otimes_{\mathcal{B}_o} \hat{\mathcal{B}}_o & \longrightarrow & \mathcal{Q}_o \otimes_{\mathcal{B}_o} \hat{\mathcal{B}}_o & \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & \hat{\mathcal{B}}_o & \longrightarrow & \hat{\mathcal{B}}_o^{\oplus m} & \longrightarrow & \hat{\mathcal{Q}}_o & \longrightarrow \end{array}$$

the left two vertical arrows are isomorphism. A diagram chase shows then that $\hat{\mathcal{Q}}_o = \mathcal{Q}_o \otimes_{\mathcal{B}_o} \hat{\mathcal{B}}_o$.

Now we will use that the ring extension $\mathcal{B}_o \to \hat{\mathcal{B}}_o$ is faithfully flat (Atiyah-MacDonald [2] p.114). Let

$$0 \to L \to M \to N \to 0$$

be a sequence of $\mathcal{O}_{W,0}$-modules. Tensoring with $\hat{\mathcal{Q}}_o$, using that it is $\mathcal{O}_{W,0}$-flat and the associativity of tensor products gives the exact sequence

$$0 \to (L \otimes_{\mathcal{O}_{W,0}} \mathcal{Q}_o) \otimes_{\mathcal{B}_o} \hat{\mathcal{B}}_o \to (M \otimes_{\mathcal{O}_{W,0}} \mathcal{Q}_o) \otimes_{\mathcal{B}_o} \hat{\mathcal{B}}_o \to (N \otimes_{\mathcal{O}_{W,0}} \mathcal{Q}_o) \otimes_{\mathcal{B}_o} \hat{\mathcal{B}}_o \to 0$$

Using faithful flatness we can cancel $\hat{\mathcal{B}}_o$, proving that $\mathcal{Q}_o$ is $\mathcal{O}_{W,0}$-flat.

In case S is not smooth, embed S in an open subset of $\mathbb{C}^n$ and extend the local functions X to a neighbourhood. The hypothesis that X does not vanish for a fiber identically is valid in some neighbourhood. Then one uses that flatness is preserved under base extensions (see Hartshorne [10] p.254) to reduce the result from $\mathbb{C}^n$ back to S. ‖

Corollary 6.- Let $\mathcal{L}$ be a corank one sheaf at every point of S x M and let X: $\mathcal{L} \to \Theta_{\Pi}$ be an injective sheaf mapping. Suppose that S is reduced and that, for each $s \in S$, the map $X(s): \mathcal{L}(s) = \mathcal{L}/m_s \cdot \mathcal{L} \to \Theta_M$ is injective. Then X induces a family of foliation parametrized by S.

Proof: As mentioned, for reduced spaces corank one is equivalent to invertible, so the theorem applies. ‖

3. Structure of the Universal Parameters

This section is devoted to give a more detailed description of the structure of the parameter spaces of the universal family of foliations by curves in a fixed compact complex manifold. We begin by pointing out that the Chern class of the line bundle tangent to the foliation is a discrete numerical invariant.

Theorem 7: Let $\mathcal{D}$ be the universal parameter space of foliations by curves of M, then to each connected component of $\mathcal{D}$ we may associate a cohomology class in $H^2(M,\mathbb{Z})$ such that the line bundle of any foliation represented in this component has this cohomology class as Chern class.

Proof: Let $\tilde{\mathcal{L}}$ be the kernel sheaf of the universal family (3), and let $\mathcal{O}_{\mathcal{D}}^{red}$ be the reduction sheaf of $\mathcal{O}_{\mathcal{D}}$ (i.e. obtained by modding out all nilpotent elements), and $\mathcal{D}^{red}$ the analytic space with the same topological space as $\mathcal{D}$ but with the structure sheaf $\mathcal{O}_{\mathcal{D}}^{red}$. Let $\tilde{\mathcal{L}}^{red}$ be the sheaf $\tilde{\mathcal{L}}\otimes_{\mathcal{O}_{\mathcal{D}xM}}\mathcal{O}_{\mathcal{D}^{red}xM}$, where $\tilde{\mathcal{L}}$ is the universal kernel of (3). $\tilde{\mathcal{L}}^{red}$ is a corank one sheaf over $\mathcal{D}^{red}xM$, but since this space is reduced, it is actually an invertible sheaf. Now, there is a one to one correspondence between invertible sheaves and line bundles (see Griffiths-Harris [9] p.698), so we obtain that there is a line bundle L on $\mathcal{D}^{red}xM$ such that its restriction to each fibre sxM is the line bundle associated to L(s) in $(2)_s$. Topologically, L may be obtained as the pull back bundle to a classifying space and any curve in the parameter space provides a homotopy between the classifying maps at the two extremes. Since the Chern class is just the pull back of a cohomology class in the classifying space, we obtain the result in the statement of the theorem since the induced maps on cohomology is a homotopy invariant.‖

The next result gives us a procedure to construct a family consisting of all the foliations by curves with given line bundle as tangent bundle.

Proposition 8: Let L be a holomorphic line bundle on the compact manifold M, and let P_L be the complex projective space formed by lines through 0 in the finite dimensional vector space of holomorphic maps from L to the tangent bundle of M, $P_L=\text{Proj}H^o(M,\text{Hom}(L,TM))$. There is a family of foliations on M parametrized by P_L such that at Mx {X} we have the foliation specified by $X\varepsilon P_L$.

Proof: Consider the family of foliations on $Mx(H^o(M,\text{Hom}(L,TM))-\{0\})$ given by a map from $\Pi_1^*L\to\Pi_1^*TM$ given by associating at (p,X) the map

$X(p): L_p \to T_pM$. If we consider the map associated at the point $(p, \lambda X)$, where λ is a scalar, we obtain the map $\lambda X(p)$, so we can incorporate this linear variation of the scalar into the hyperplane line bundle H of the projective space P_L to obtain a tautological map on $M \times P_L$

$$\Pi_1^* L \otimes \Pi_2^* H^{-1} \to \Pi_1^* TM$$

By construction it is not identically equal to zero on any fiber, so theorem 5 applies to show that it is parametrizing a family of foliations by curves, and by construction there is a one to one correspondence with the foliations it is parametrizing and foliations having L as a tangent bundle.||

The next result tells us that the above two results are already enough to give a clear picture for a class of complex manifolds.

Theorem 9: Let M be a compact Kähler manifold with vanishing first Betti number. Then the space of foliations by curves has a structure of a disjoint union of complex projective spaces.

Proof: By the Hodge decomposition theorem of cohomology (see Griffiths-Harris [9]), the hypothesis imply that $H^1(M, \mathcal{O}_M) = 0$. The exponential sequence of sheaves on M

$$0 \to \mathbb{Z} \to \mathcal{O}_M \overset{e}{\to} \mathcal{O}_M^* \to 0 \tag{10}$$

gives at the level of cohomology the injective map

$$0 = H^1(M, \mathcal{O}_M) \to H^1(M, \mathcal{O}_M^*) \to H^2(M, \mathbb{Z})$$

which may be interpreted as saying that holomorphic line bundles in M are completely determined by its topological class, (i.e. its Chern class in $H^2(M, \mathbb{Z})$).

For every holomorphic line bundle L in $H^1(M, \mathcal{O}^*)$ construct the projective space $P_L = \mathrm{Proj}H^0(M, \mathrm{Hom}(L, TM))$ as in Proposition 8, and let $f_L: P_L \to \mathcal{D}$ be the holomorphic map obtained from the universal properties of $\mathcal{D}$. To see that the universal space $\mathcal{D}$ is equal to the disjoint union of P_L, it suffices to show that at every point X in P_L the derivative of f_L is an isomorphism.

Douady in [4] p.77 shows that the tangent space to $\mathcal{D}$ at X may be canonically identified with $H^0(M, \mathrm{Hom}(\mathcal{L}, \Theta_M / X(\mathcal{L})))$. Applying the functor Hom $(\mathcal{L},\ )$ to the sequence

$$0 \to L \overset{X}{\to} \Theta_M \to \Theta_M/X(L) \to 0$$

and then considering the long exact sequence of cohomology gives the short exact sequence

$$0 \to H^0(M,\mathcal{O}_M) \overset{X}{\to} H^0(M,\mathrm{Hom}(L,\Theta_M) \to H^0(M,\mathrm{Hom}(L,\Theta_M/X(L))) \to 0$$

This exact sequence tells us that the dimension of every tangent space of $f_L(P_L)$ is the same as the dimension of P_L. By construction $f_L:P_L \to \mathcal{D}$ is injective and by Proposition 8 and the above observation on the holomorphic structures in the same topological bundle, $f_L(P_L)$ is a connected component of $\mathcal{D}$. Its topological dimension is the same as the one of P_L, hence it is actually a smooth complex manifold. Any bijective map between smooth complex manifolds is a biholomorphism (see Griffiths-Harris [9] p.19), so the theorem is proved.‖

Corollary 10: The universal family of foliations of $\mathbb{C}P^n$ form a countable union of projective spaces indexed by the Chern class of the tangent bundle to the foliation.

Proof: Projective spaces satisfy the conditions of Theorem 9.‖

One can be very explicit in constructing the universal families in projective spaces $\mathbb{C}P^n$. The following facts are proved in [5]:

1) For d>0 there is a one to one correspondance between:

a) Holomorphic maps from the line bundle L_{1-d} with Chern class 1-d into the tangent bundle of $\mathbb{C}P^n$.

b) Homogeneous vector fields $\sum_{i=0}^{n} P_i \frac{\partial}{\partial z_i}$ in $\mathbb{C}^{n+1}$ of degree d, modulo additions $h\Sigma z_i \frac{\partial}{\partial z_i}$, where h is a homogeneous polynomial of degree d-1.

c) Polynomial vector fields in affine coordinates $(w_1,.,w_n)$

$$g\sum_{i=1}^{n} w_i \frac{\partial}{\partial w_i} + \sum_{k=0}^{d} X_k \qquad (11)$$

where the X_k are homogeneous vector fields of degree k and g is a homogeneous polynomial of degree d; where we recquire that if g=0 then X_d is not a multiple of the radial vector field $\Sigma w_i \frac{\partial}{\partial w_i}$ with a polynomial of degree d-1.

2) In the expression (11), if $(w_1:..:w_n)$ are homogeneous coordinates for the hyperplane at infinity $\mathbb{C}P^{n-1}=\mathbb{C}P^n-\mathbb{C}^n$ then the set of points where g vanishes describes the set of tangencies of the folia-

tion with $\mathbb{C}P^{n-1}$. In particular g is identically equal to zero if and only if the hyperplane at infinity is saturated by the leaves of the foliation.

3) The set of foliations of $\mathbb{C}P^n$ with Chern class 1-d may be parametrized in a natural way by a projective space of dimension $\binom{d-1+n}{d}(d+n+1)-1$. For n=2 it is d^2+4d+2. The set of such foliations that contain a hyperplane saturated by the foliation is of codimension $\binom{d+n-1}{d}-n$; for n=2 it is of codimension d-1.

It is interesting to point out how in affine coordinates the left term in (11) appears. Point 2) clarifies which family of foliations by curves of projective space one obtains by "fixing the degree".

In order to see the pattern for complex manifolds with non-vanishing first Betti number, one is faced to understand the different holomorphic structures that can be put in the same topological line bundle. This is a very well understood problem if M is a projective manifold (i.e. embeddable in some projective space), and hence we restrict to these.

The long exact sequence of cohomology of the exponential sequence (10) gives the exact sequence

$$0 \to \frac{H^1(M,\mathcal{O})}{H^1(M,\mathbb{Z})} \to H^1(M,\mathcal{O}^*) \overset{c}{\to} H^2(M,\mathbb{Z})$$

$H^1(M,\mathcal{O}^*)$ represents the holomorphic classes of line bundles in M, it is an abelian group under tensor product and for projective varieties the kernel of c denoted by $Pic_o(M)$, is formed of those line bundles with trivial Chern class. Artin proved in [1] that $Pic_o(M)$ is a compact group variety embeddable in some projective space, what is called an Abelian variety. Hence $H^1(M,\mathcal{O}^*)$ has a structure of a countable number of projective complex torus, $Pic_\alpha(M)$, indexed by the admissible Chern classes α in $H^2(M,Z)$, each having the same dimension as $Pic_o(M)$. Mumford then proved the existence of a Poincaré bundle (see Griffiths-Harris [9]p.328), that is the existence of a line bundle P_α on $M\times Pic_\alpha(M)$ such that its restriction to MxL is biholomorphic to the line bundle represented by L, satisfying a universal property.

Consider in $M\times Pic_\alpha(M)$ the vector bundle $Hom(P_\alpha, \Pi_1^* TM)$ and let E_α be the coherent sheaf on $Pic_\alpha(M)$ obtained by direct image of its sheaf of sections, $E_\alpha = \Pi_{2*}(Hom(P_\alpha, \Pi_1^* TM))$. If for every line boundle L with Chern class α, the vector space $H^0(M, Hom(L,TM))$ has a fixed dimension, say r, then it follows from a theorem of Grauert (see Hartshorne [10], p.288) that E_α is a locally free sheaf of rank r, and if $\rho: E_\alpha \to Pic_\alpha(M)$ is the associated vector bundle the fibers of ρ are the groups $H^0(M, Hom(L,TM))$. Let $\rho: Proj(E_\alpha) \to Pic_\alpha(M)$ be the CP^{r-1} blundle obtained by considering the lines in each fiber of E_α. $Proj(E_\alpha)$ carries a line bundle H that restricts to each fiber as the hyperplane bundle of the fiber (see Hartshorne [10] p. 162 or [5]). The generalization of theorem 9 is then

Theorem 11: Let M be a projective manifold and let $\alpha \in H^2(M,Z)$ be such that for any holomorphic line bundle L on M with Chern class α the vector space $H^0(M, Hom(L,TM))$ has constant dimension $r > 0$, then $Proj(E_\alpha)$ is biholomorphic to the subvariety of $\mathcal{D}$ obtained by fixing the Chern class α. In particular this component of $\mathcal{D}$ is smooth, connected, compact and has dimension $r-1+\frac{1}{2}b_1$, (b_1=first Betti number of M).

Proof; Similar considerations as in Proposition 8 gives a bundle map over $M\times Proj(E_\alpha)$

$$X: \rho^* P_\alpha \otimes \Pi_2^* H^{-1} \to \Pi_1^* TM$$

where Π_1 and Π_2 are the projection to the factors and ρ is the map $M\times Proj(E_\alpha) \to M\times Pic_\alpha(M)$. X is tautological in the sense that if we restrict to $M\times e$, we obtain the foliation in M that e represents. By Theorem 5, X parametrizes a family of foliations by curves, and by the universality properties of $\mathcal{D}$ we obtain a map $Proj(E_\alpha) \to \mathcal{D}$. By Theorem 7 and by construction the image is a connected component of $\mathcal{D}$ and the mapping is bijective with its image. As in Theorem 9, to show that it is a biholomorphism it is enough to check that the dimension of each tangent space of the image is less that or equal than the dimension of $Proj(E_\alpha)$ (for its topological dimension is already the one of $Proj(E_\alpha)$, so this implies that it is smooth, and hence the map, being bijective, is a biholomorphism). Fix $e \in Proj(E_\alpha)$ representing $X: L \to TM$ and apply the functor $Hom(L, \ )$ to the sequence

$$0 \to L \xrightarrow{X} \theta_M \to \theta_M / X(L) \to 0$$

and then consider the long exact sequence of cohomology to obtain

$$0 \to H^0(M,\mathcal{O}_M) \to H^0(M,\mathrm{Hom}(L,\theta_M)) \to H^0(M,\mathrm{Hom}(L,\theta_M/X(L,))) \to H^1(M,\mathcal{O}_M)$$

The third term by Douady is the tangent space to $\mathcal{D}$ at the image of e, and since $\dim H^1(M,\mathcal{O}_M)=\dim \mathrm{Pic}_0(M)=\frac{1}{2}$(Betti number of M) we obtain that the tangent space of $\mathcal{D}$ at the image of e is less than or equal to the dimension of $\mathrm{Proj}(E_\alpha)$, and hence we are done.‖

Now the objective will be to give criteriums for the hypothesis of the above lemma and to estimate r in terms of topological data.

By hypothesis, we may embed our manifold M in some projective space CP^n, and by pulling back to M the hyperplane bundle of CP^n we obtain special line bundles that are called very ample. One says that a line bundle is ample if some power of it is very ample. One sees that the notion of ampleness depends only on the Chern class of the bundle, and not on the bundle itself.

<u>Theorem</u> 12: Let M be a projective manifold, L any line bundle on M and β the Chern class of an ample bundle on M. There exists an N, such that for all $n > N$ the hypothesis of Theorem 11 are satisfied for $c(L)-n\beta$ and r is equal to the Euler-Poincaré characteristic $\chi(TM \otimes L^* \otimes \beta^n)$.

<u>Proof</u>: Let α be the Chern class of L, and let $\mathcal{P}_\alpha$ be the Poincaré bundle on $M\mathrm{x}\mathrm{Pic}_\alpha(M)$, and H a line bundle with Chern class β. On $M\mathrm{x}\mathrm{Pic}_\alpha(M)$ consider the vector bundle $\mathrm{Hom}(\mathcal{P}_\alpha,\Pi_1^*TM)$ and the line bundle Π_1^*H, that is relatively ample with respect to the projection Π_2. Applying a theorem of Serre (see Hartshorne [10] p.228) we find that there exists an N such that for all $q > 0$ and all $n > N$ the higher direct image sheaves $R^q\Pi_{2*}\mathrm{Hom}(\mathcal{P}_\alpha,\Pi_1^*TM \otimes H^n)$ vanish identically and for $q = 0$ we obtain a vector bundle of rank $\chi(TM \otimes L^* \otimes \beta^n) =$ $= \deg(\mathrm{ch}(TM \otimes L^* \otimes \beta^n)\cdot \mathrm{td}(TM))_n$ by the Hirzebruch-Riemann-Roch theorem (see Hartshorne [10] p.432 or Hirzebruch [11]).‖

We will exploit now the above theorem by mapping certain subvarieties of $\mathrm{Proj}(E_\alpha)$ onto components of $\mathcal{D}$, obtaining in this form the compactness of the components of $\mathcal{D}$ with fixed Chern class.

<u>Theorem</u> 13: Let M be a projective manifold, α any class in $H^2(M,Z)$ and denote by $\mathcal{D}_\alpha$ the components of $\mathcal{D}$ that have α as a Chern class. Then $\mathcal{D}_\alpha$ is a compact analytic space.

Proof: Let β be an ample class in M and N as in Theorem 12 applied to α. Let D be an irreducible divisor in M having Poincaré dual $n\beta$ with $n>N$ (we can even assume D smooth since β is ample). Let f be the holomorphic section on M of the line bundle L_D vanishing exactly on D, and denote by $\mathcal{D}_\alpha$ and $\mathcal{D}_{\alpha+n\beta}=\mathrm{Proj}(E)$ the components of $\mathcal{D}$ having Chern class α and $\alpha+n\beta$, as well as

$$0 \to \mathcal{L}_\alpha \overset{X_\alpha}{\to} \Theta_\pi \to \mathcal{N}_\alpha \to 0$$

the defining sequence of the foliation in $M\times\mathcal{D}_\alpha$ and similarly for $\alpha+n\beta$.

We have shown that $\mathcal{D}_{\alpha+n\beta}$ is Proj(E) and that $\mathcal{L}_{\alpha+n\beta}$ is an invertible sheaf on $\mathcal{D}_{\alpha+n\beta}$. Denote by $\mathcal{Z}$ the set of points where $X_{\alpha+n\beta}$ vanishes (i.e. take local expressions of $X_{\alpha+n\beta}$ and equate them to zero). Consider the analytic subset $\psi:\mathcal{Z}\subset(D\times\mathrm{Proj}E) \to \mathrm{Proj}(E)$ as a subvariety of $D\times\mathrm{Proj}(E)$. If m is the dimension of M, the upper semicontinuity behavior of dimension guarantees that there is a closed subvariety W of Proj(E) where the fibers of ψ have dimension m-1, since that is the dimension of D, W represents those points of Proj(E) whose associated maps vanish on D. Put in W its reduced analytic structure and consider the restriction of the family $X_{\alpha+n\beta}$ to W. Consider the locally free sheaf mappings on $M\times W$

$$\frac{1}{f}X_{\alpha+n\beta}:\mathcal{L}_{\alpha+n\beta}/_W \otimes \Pi_1^*(L_D^{-1}) \to \Theta_\pi$$

It is holomorphic by construction, and since the domain is an invertible sheaf we conclude by Theorem 5 that it is parametrizing a family of foliations by curves. By the universality properties of $\mathcal{D}$ we obtain a holomorphic map $W \to \mathcal{D}_\alpha$. This mapping is surjective, since pointwise we may go in the opposite direction multiplying by f. Since W is compact the image is also compact, hence $\mathcal{D}_\alpha$ is a compact analytic space.‖

Remark: 1) If G denotes the group of biholomorphisms of M, one obtains from the universal properties of the space $\mathcal{D}$ a holomorphic action $G\times\mathcal{D}\to\mathcal{D}$. Small transversals to this G-action will produce versal spaces for the new equivalence relation: $X_i:L_i \to TM$ are equivalent if we can find a biholomorphism g of M such that $Dg(X_o(\mathcal{L}_o))=X_1(\mathcal{L}_1)$.

2) One may use Chern's theorems [3] to obtain a precise idea of

how many singular points one expects for a foliation.

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