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by

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1. Introduction

Mertens conjectured [5] 1897 that

(1) $|M(x)| = |\sum_{n\leq x} \mu(n)| < \sqrt{x}$ for $x>1$,

where $\mu(n)$ is the Möbius function. It was known for a long time that (1) implies that all non-trivial zeros $\rho_\nu=\beta_\nu+i\gamma_\nu$ of the Riemann zeta-function $(0<\gamma_1\leq\gamma_2\leq\dots)$

(2) lie on the critical line $\sigma=1/2$,
(3) are simple,
(4) satisfy $|a_{\rho_\nu}| := |\rho_\nu\zeta'(\rho_\nu)|^{-1} \leq 1$.

Supposing (2)-(4) it is possible to show [2] that certain mean value of $M(x)/\sqrt{x}$ equals

(5) $K_1(v,T) = 2 \sum_{0<\gamma_\nu<T} \kappa_1\left(\frac{\gamma_\nu}{T}\right) \frac{\cos(\gamma_\nu v - \pi\psi_\nu)}{|\rho_\nu\zeta'(\rho_\nu)|}$, where

(6) $\kappa_1(\tau) = (1-\tau)\cos(\pi\tau) + \pi^{-1}\sin(\pi\tau)$, $\pi\psi_\nu = \arg(\rho_\nu\zeta'(\rho_\nu))$.

Ingham [1] showed this with the simpler choice $\kappa_0(\tau)=(1-\tau)$.

Using the lattice basis reduction algorithm of A.K.Lenstra, H.W. Lenstra and Lovász [3], A.M.Odlyzko and H.J.J.te Riele [6] succeeded 1983 in finding values $T^*=\gamma_{2000}$ and v with $v\approx-1.4\cdot10^{64}$ such that

(7) $K_1(v,T^*) \approx 1.0615$

thereby disproving (1). (For other results and the history of the problem see also [6].) However, this method is completely ineffective, one obtains no contradiction for any concrete X if (1) is substituted by

(1') $\max_{1\leq x\leq X} |M(x)|/\sqrt{x} \leq 1.$

The main difficulty is that by classical methods it is not possible to derive from (1') any of the assertions (2)-(4), not even in a finite form, that is, for zeros satisfying

(8) $|\gamma_\nu|<f(X)$ where $\lim_{X\to\infty} f(X) = \infty$.

Before the disproof of the Mertens conjecture the present author could show [7] that (1') implies a weakened form of (2), namely that all zeros have

(2') $$|\beta_\nu-\tfrac{1}{2}|<\frac{3\log\gamma_\nu+c_1}{\log X},$$

where c_1 is an explicit constant, for which some calculations yield $c_1=\log 6$. In Section 3 we give a simplified version of this (see Theorem A).

The aim of this work is to show that if we consider the mean value

(5') $$K_2(v,T,k) = 2\sum_{0<\gamma_\nu<T} e^{-k\gamma_\nu^2}\frac{\cos(\gamma_\nu v-\pi\psi_\nu)}{|\rho_\nu\zeta'(\rho_\nu)|}$$

in place of (5) then we can dispense with (3) and (4), and it is enough to know that

(3') $\rho_\nu=\frac{1}{2}+i\gamma_\nu$ are simple and $|\gamma_{\nu+1}-\gamma_\nu|>9\cdot10^{-4}$ for $|\gamma_\nu|<1.1\cdot10^6$

which was verified by the computation of Rosser, Yohe and Schoenfeld [8]. Our result is

<u>THEOREM 1</u>. <u>If there exists a</u> $v\in[e^7,e^{5\cdot10^4}]$ <u>with</u>

(9) $|K_2(v,1.4\cdot10^4,1.5\cdot10^{-6})|>1+e^{-40}$

<u>then (1') is false for</u> $X=e^{v+\sqrt{v}}$.

A good candidate for v is naturally any positive value of v in the given range for which $|K_1(v,T^*)|>1$. Such a value $v_o\approx3.2097\cdot10^{64}$ was found during the computations of Odlyzko and te Riele. The author is deeply indebted to Prof.te Riele who showed

(7') $K_2(v_o,1.4\cdot10^4,1.5\cdot10^{-6}) = -1.00223....$

Theorem 1 and (7') imply

THEOREM 2. (Odlyzko-Pintz-te Riele) We have

(10) $\max\limits_{x\leq X}|M(x)|/\sqrt{x}>1$ for $X=\exp(3.21\cdot 10^{64})$.

Finally we remark that using other methods the author could show that (1') implies (2)-(4) for zeros with

(8') $|\gamma_\nu|<c_2\log^{1/10}X$ with explicit c_2

and this makes possible to show for any v,T and $\varepsilon>0$

(11) $\max\limits_{x\leq X}|M(x)/\sqrt{x}| > |K_1(v,T)|-\varepsilon$ for $X>c(v,T,\varepsilon)$

with an explicit $c(v,T,\varepsilon)$ thereby furnishing another effective disproof of Mertens conjecture which does not need any special numerical facts beyond the crucial relation (7).

2. Preliminary Lemmas

In the following we prove some lemmas (or sketch the proof of them) about the ζ-function. They are contained in standard books but we need here all error terms with explicit constants. We use always the notations $s=\sigma+it$, $\rho=\beta+i\gamma$ for non-trivial zeros of $\zeta(s)$ and we denote by Θ a number with $|\Theta|\leq 1$, not necessarily the same at each appearance, further we denote tg $1=1.55...$ by c_0.

LEMMA 1. For $\sigma=-1$ we have $|\zeta(s)|\leq\frac{2}{3}|s|^{3/2}$.

Since the value 2/3 is not very important we only sketch the proof. From the functional equation we have for $s=-1+it$

$$|\zeta(-1+it)| = \frac{|\Gamma(1+\frac{it}{2})|}{|\Gamma(-\frac{1}{2}+i\frac{t}{2})|}|\zeta(2-it)|\pi^{-\frac{3}{2}}$$

and from Stirling's formula one can derive for $|t|\geq 2$

$$\mathrm{Re}\{\log\Gamma(1+\tfrac{it}{2})-\log\Gamma(-\tfrac{1}{2}+i\tfrac{t}{2})\}\leq\tfrac{3}{2}\log|s|+\tfrac{3}{2}-\log 2$$

further for $|t|\leq 2$ one can show $|\Gamma(1+\frac{it}{2})|\leq|\Gamma(-\frac{1}{2}+i\frac{t}{2})||s|$, and so we obtain Lemma 1.

LEMMA 2. (Von Mangoldt [4]). For $c_0 \ll h \ll T-4$ we have

$$\sum_{T-h\ll\gamma\ll T+h} 1 \ll h \log T.$$

LEMMA 3. For $0\leq\sigma\leq 2$, $t\geq 4$ we have

$$\frac{\zeta'}{\zeta}(s) = \sum_{|\gamma-t|<c_0} \frac{1}{s-\rho}+\Theta(4 \log t+6).$$

Proof. From (2.12.7) of Titchmarsh [9] we obtain

$$\frac{\zeta'}{\zeta}(\sigma+it)-\frac{\zeta'}{\zeta}(2+it)=\frac{\sigma-2}{(\sigma+it-1)(1+it)}-\frac{1}{2}\left\{\frac{\Gamma'}{\Gamma}\left(1+\frac{\sigma}{2}+i\frac{t}{2}\right)-\frac{\Gamma'}{\Gamma}\left(2+i\frac{t}{2}\right)\right\}+$$

$$+\sum_{\rho}\left(\frac{1}{\sigma+it-\rho}-\frac{1}{2+it-\rho}\right).$$

The first term on the right hand side is clearly $2\Theta t^{-2}$ whilst using Stirling's formula for $\Gamma'/\Gamma(s)$ we obtain

$$\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}+1\right)=\log\left(1+\frac{s}{2}\right)-\frac{1}{s+2}-\int_0^{\infty}\frac{1/2-\{x\}}{(x+1+s/2)^2}dx$$

$$=\log\left(1+\frac{s}{2}\right)+\frac{\Theta}{t}+\frac{\Theta}{2}$$

and by $\log(1+z)=z+\Theta|z|^2$ for $|z|\leq 1/2$ we have

$$\log\left(1+\frac{\sigma}{2}+\frac{it}{2}\right)-\log\left(2+\frac{it}{2}\right)=\frac{\sigma-2}{4+it}+\Theta\frac{4}{t^2}=\frac{3}{4}\Theta \text{ for } |t|\geq 4.$$

Further, by Lemma 2 and the symmetric situation of the zeros we have with some calculations

$$\sum_{k=1}^{\infty}\ \sum_{(2k-1)c_0\leq|\gamma-t|\leq(2k+1)c_0}\frac{2-\sigma}{(t-\gamma)^2} \leq \sum_{k=1}^{\infty} 2c_0\log(t+2kc_0)\frac{3/2}{(2k-1)^2c_0^2} \leq$$

$$\leq 3c_0^{-1}\log t\cdot\frac{4}{5}\zeta(2)+3c_0^{-1}\sum_{1}^{\infty}\frac{\log 2kc_0}{(2k-1)^2},$$

$$\left|\sum_{|\gamma-t|<c_0}\frac{1}{2+it-\rho}\right| \leq \frac{3}{4}c_0 \log t,$$

and these, together with $|\zeta'(2)/\zeta(2)|<3/5$, imply Lemma 3.

LEMMA 4. For $0\leq\sigma\leq 2$, $t\geq 4$ we have $\log\zeta(s) = \sum_{|\gamma-t|<c_0} \log(s-\rho)+\Theta(11 \log t+14)$.

Proof. From Lemmas 2 and 3 we have

$$\log \frac{\zeta(\sigma+it)}{\zeta(2+it)} = \int_2^\sigma \frac{\zeta'}{\zeta}(\xi+it)d\xi = \int_2^\sigma \sum_{|\gamma-t|<c_o} \frac{1}{\xi+it-\rho} d\xi + \Theta(8 \log t+12)$$

$$= \sum_{|\gamma-t|<c_o} \log(s-\rho) - \sum_{|\gamma-t|<c_o} \log(2+it-\rho) + \Theta(8 \log t+12)$$

$$= \sum_{|\gamma-t|<c_o} \log(s-\rho) + \Theta\, c_o \log t \cdot \frac{3}{2} + \Theta(8 \log t+12).$$

This yields the following two lemmas valid for $0\leqslant\sigma\leqslant 2, t\geqslant 4$.

LEMMA 5. *If* $|s-\rho|\geqslant u^{-1}$ *for all zeros then*

$$\log|1/\zeta(s)| \leqslant c_o \log t \log u + 11 \log t + 14.$$

LEMMA 6. *Let us assume that all zeros* ρ_ν *with* $|\gamma_\nu-t|<c_o$ *are simple and satisfy* $|s-\rho_\nu|\geqslant u^{-1}$, $\min_{\nu\neq\mu}|\rho_\nu-\rho_\mu|=H^{-1}$. *Then*

$$\log|1/\zeta(s)| \leqslant \log u + c_o \log t \log(2H) + 11 \log t + 14.$$

Finally, we state without proof the following

LEMMA 7. $|\zeta(1/4+it)| \geqslant e^{-150}$ *for* $|t|\leqslant 4$.

3. The case when RH is false

The case when (RH) is false is treated by a simplified version of the Theorem of [7].

THEOREM A. *Let* $\zeta(\rho_o)=\zeta(\beta_o+i\gamma_o)=0$ *with* $\beta_o\geqslant 1/2$, $\gamma_o>0$. *Then*

$$D(Y) = \frac{1}{Y}\int_o^Y |M(x)|dx > \frac{Y^{\beta_o}}{\gamma_o^5} \quad \text{for } Y\geqslant\gamma_o^5.$$

Proof. Let $g(s) = \dfrac{s(s-1)\zeta(s)}{(s-\rho_o)(s+2)^6}$, $W(A) = \dfrac{1}{2\pi i}\int_{(2)} g(s)A^{s+1}ds \quad (A>0)$.

Integrating along the lines $\sigma=-1$ and $\sigma=B\to\infty$ we obtain

$$|W(A)| \leq \frac{1}{2\pi} \int_{(-1)} |g(s)|\,ds \leq \frac{1}{2\pi} \int_{(-1)} \frac{2|s|^2 \frac{2}{3}|s|^{3/2}}{|s|^6} |ds| \leq \frac{2}{3\pi} \int_{(-1)} \frac{|ds|}{|s|^2} = \frac{2}{3},$$

$$|W(A)| \leq A^{B+1} \cdot \frac{1}{2\pi} \int_{(B)} |g(s)|\,|ds| \leq A^{B+1} \to 0 \quad \text{if} \quad A<1 \quad \text{and} \quad B\to\infty.$$

Therefore we have

$$|U(Y)| := \frac{1}{Y} \left| \int_1^\infty M(x) W\left(\frac{Y}{x}\right) dx \right| \leq \frac{2}{3} \cdot \frac{1}{Y} \int_1^Y |M(x)|\,dx < \frac{2}{3} D(Y).$$

On the other hand interchanging the order of integrations

$$U(Y) = \frac{1}{2\pi i} \int_{(2)} Y^s g(s) \int_1^\infty \frac{M(x)}{x^{s+1}} dx\,ds = \frac{1}{2\pi i} \int_{(2)} \frac{s(s-1)\zeta(s)}{(s-\rho_o)(s+2)^6} \cdot \frac{Y^s ds}{s\zeta(s)}$$

$$= \frac{(\rho_o-1)Y^{\rho_o}}{(\rho_o+2)^6} + \frac{1}{2\pi i} \int_{(-1)} \frac{(s-1)Y^s ds}{(s-\rho_o)(s+2)^6} = \frac{(\rho_o-1)}{(\rho_o+2)^6} Y^{\rho_o} + \Theta Y^{-1},$$

and so $|U(Y)| > \frac{9}{10} \frac{Y^{\beta_o}}{\gamma_o^5} - \frac{1}{\gamma_o^5} > \frac{2}{3} Y^{\beta_o}$ which proves Theorem A.

COROLLARY A. If for a given Y there exists a zero ρ_o with $\beta_o > \frac{1}{2} + \frac{5\log|\gamma_o|}{\log Y}$ then $D(Y) > \sqrt{Y}$.

4. Two mean values of M(x)

We shall investigate the following mean values of $M(x)$ $(u,k>0)$:

$$M_1(u) = \frac{e^{-k/4-u/2}}{2\sqrt{\pi k}} \int_{e^{u-2\sqrt{ku}}}^{e^{u+2\sqrt{ku}}} \frac{M(x)}{x} \exp\left(-\frac{(u-\log x)^2}{4k}\right) dx,$$

$$M_2(u) = \frac{e^{-k/4-u/2}}{2\sqrt{\pi k}} \int_1^\infty \frac{M(x)}{x} \exp\left(-\frac{(u-\log x)^2}{4k}\right) dx.$$

LEMMA A. If $|M(x)| \leq A\sqrt{x}$ for $|\log x - u| \leq 2\sqrt{ku}$ then $|M_1(u)| < A$.

Proof. $\frac{e^{-k/4-u/2}}{2\sqrt{\pi k}} \int_{e^{u-2\sqrt{ku}}}^{e^{u+2\sqrt{ku}}} \sqrt{x} \exp(-\frac{(u-\log x)^2}{4k}) \frac{dx}{x} = \frac{1}{2\sqrt{\pi k}} \int_{-2\sqrt{ku}}^{2\sqrt{ku}} e^{-\frac{y^2}{4k}+\frac{y}{2}-\frac{k}{4}} dy$

$$= \frac{1}{\sqrt{\pi}} \int_{-\sqrt{u}-\sqrt{k}/2}^{\sqrt{u}-\sqrt{k}/2} e^{-v^2} dv < \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} dv = 1.$$

LEMMA B. $|M_2(u)-M_1(u)| < 2e^{-u/4}$ if $k<1/4$, $u \geq 16$.

Proof. $\frac{e^{-k/4-u/2}}{2\sqrt{\pi k}} \int_{e^{u+2\sqrt{ku}}}^{\infty} \exp(-\frac{(u-\log x)^2}{4k}) dx = \frac{e^{3k/4+u/2}}{2\sqrt{\pi k}} \int_{2\sqrt{ku}}^{\infty} e^{-\frac{y^2}{4k}+y-k} dy$

$$= \frac{e^{3k/4+u/2}}{\sqrt{\pi}} \int_{\sqrt{u}-\sqrt{k}}^{\infty} e^{-v^2} dv < e^{3k/4+u/2-u-k+2\sqrt{ku}} < e^{-u/4},$$

and the same holds for the integral on $[1, e^{u-2\sqrt{ku}}]$.

LEMMA C. $M_2(u) = \frac{e^{-k/4-u/2}}{2\pi i} \int_{(2)} \frac{e^{ks^2+us}}{s\zeta(s)} ds.$

Proof. This follows from the identity $\frac{e^{-y^2/4k}}{2\sqrt{\pi k}} = \frac{1}{2\pi i} \int_{(2)} e^{ks^2+ys} ds.$

5. The case when RH is (approximately) true

In the following let $k=1.5\cdot 10^{-6}$ and let us suppose

(i) $M(x) \leq \sqrt{x}$ for $x \leq e^u$ where $e^{7-k} \leq u \leq e^{5\cdot 10^4}$.

Then Corollary A implies for every zero

(ii) $\beta_o \leq \frac{1}{2} + \frac{5 \log|\gamma_o|}{u}$.

Let us transform the way of integration in Lemma C onto the line which consists of $L_1, L_1', L_2, L_2', L_3$ and their reflection on the real axis, where

$L_1 = \{s; \sigma = \frac{1}{2} + \frac{5 \log t+2}{u}\}$, $L_1' = \{s = \sigma + iT_1, \sigma - \frac{1}{2} \in [\frac{1}{u}, \frac{5 \log T_1+2}{u}]\}$,

$L_2 = \{s = \frac{1}{2} + \frac{1}{u} + it;\ t \in [T_o, T_1]\}$, $L_2' = \{s = \sigma + iT_o, \sigma \in [\frac{1}{4}, \frac{1}{2} + \frac{1}{u}]\}$,

$L_3 = \{s = \frac{1}{4} + it;\ t \in [0, T_o]\}$, $T_o = 1.4\cdot 10^4$, $T_1 = 10^6$;

here T_o satisfies $|\gamma-T_o|\geqslant u^{-1}$ for every zero. Then by (3') and (ii) we have for every ρ

(iii) $|s-\rho|\geqslant u^{-1}$ if $s\in L$

(iv) the conditions of Lemma 6 hold for $s\in L_2\cup L_2'$ with $H=\frac{10^4}{9}$.

LEMMA D. $M_2(u) = \frac{1}{2\pi i}\int_{(L)} \frac{e^{k(s^2-1/4)+u(s-1/2)}}{s\zeta(s)} ds+K_2(u+k,T_o,k)$.

Proof. Transforming the way of integration in Lemma C onto L we obtain the following sum of residues:

$$\sum_{|\gamma|<T_o} \frac{e^{k(\rho^2-1/4)+u(\rho-1/2)}}{\rho\zeta'(\rho)} = 2\sum_{0<\gamma_\nu<T} |a_{\rho_\nu}|e^{-k\gamma_\nu^2}\mathrm{Re}\, e^{-i\pi\psi_\nu+ik\gamma_\nu+iu\gamma_\nu}.$$

LEMMA E. $\left|\int_{(L)} \frac{e^{k(s^2-1/4)+u(s-1/2)}}{s\zeta(s)}ds\right|\leqslant 5e^{-40}$.

Proof. By (iii) and Lemma 5 we have

$$\left|\int_{(L_1)}\right|\leqslant \int_{L_1} \frac{1}{t^2}\exp\{k(\sigma^2-t^2)+6\log t+2+c_o\cdot 5\cdot 10^4\log t+11\log t+14\}dt$$

$$\leqslant \max_{t\geqslant 10^6} \exp\{k(\log^2 t-t^2)-10^5\log t\}\leqslant \exp(-10^5)$$

and the same holds for the integral on L_1' too. Let

$$I_n=\{s\in L_2;\ \frac{n}{u}\leqslant\min_\rho|\gamma-t|<\frac{n+1}{u}\}\ \text{for}\ 1\leqslant n\leqslant 10^3 u.$$

Then $|I_n|<2u^{-1}\sum_{\gamma<T_1+10^4} 1<4\cdot 10^6u^{-1}$ and by (iii), (iv) and Lemma 6 we have

$$\left|\int_{I_n}\right|\leqslant|I_n|\max_{T_o\leqslant t\leqslant T_1}\exp\{k(1-t^2)+1+14+\log t(10+c_o\log\frac{10^4}{4})\}\frac{u}{n}$$

$$\leqslant 4\cdot 10^6n^{-1}\exp\{-k\,T_o^2+16+23\log T_o\}<e^{-43}n^{-1}.$$

So we have

$$\left|\int_{L_2}\right|\leqslant e^{-43}\sum_{n<10^3u} n^{-1}<e^{-43}(\log 5\cdot 10^7+1)<e^{-40}$$

and the same holds for the integral on L_2'. Finally we have for $4\leqslant t\leqslant T_o$ from Lemma 5

$$|\zeta^{-1}(s)| \leq e^{10(c_0 \log 4+11)+14} < e^{150} \quad (\text{if } s \in L_3),$$

and so, using also Lemma 7 we get

$$|\int_{L_3}| \leq T_0 e^{-u/4} 4e^{150} < e^{-100}. \qquad \text{Q.E.D.}$$

6. Proof of Theorem 1

Lemmas A-E imply that if $k=1.5\cdot 10^{-6}$, $e^{7-k} \leq u \leq 5\cdot 10^4$ and $|M(x)| \leq \sqrt{x}$ for $x \leq e^{u+2\sqrt{ku}}$ then $|M_1(u)|<1$, $|M_2(u)|<1+e^{-270}$, and finally $|K_2(u+k,T_0,k)|<1+e^{-270}+(5/2\pi)e^{-40}<1+e^{-40}$. This proves the theorem, since $e^{u+2\sqrt{ku}}<e^{u+k+\sqrt{u+k}}$.

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