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APPLICATION OF THE CONJECTURE ON THE MANIN OBSTRUCTION TO VARIOUS DIOPHANTINE PROBLEMS*

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Let X be a rational projective smooth surface over a number field k and $X(k_v) \neq \emptyset$ for any completion k_v of k. Let $\mathrm{Br}\, X = H^2_{et}(X, G_m) = \{$classes of Azumaya algebras over $X\}$ denote the Brauer-Grothendiek group of X.

Applying to X [10, Ch. VI] we obtain

<u>the Manin obstruction</u>: If for any $(P_v) \in \prod_{v \in \Omega} X(k_v)$ there exists $A \in \mathrm{Br}\, X$ such that $\sum_{v \in \Omega} \mathrm{inv}_v A(P_v) \neq 0$, then $X(k) = \emptyset$. (Here $A(P_v) \in \mathrm{Br}\, k_v$ denotes the specialization of A at $P_v \in V(k_v)$ ([10, Ch. VI, p. 222]), $\mathrm{inv}_v : \mathrm{Br}\, k_v \hookrightarrow \mathbb{Q}/\mathbb{Z}$ is the local invariant and Ω consists of all places of k).

Assuming that the Manin obstruction to the Hasse principle is the only one for certain families of cubic surfaces, we can solve for some cubic varieties the following problems:

(1) Does the Hasse principle hold for cubic three-folds?

(2) (Cassels-Swinerton-Dyer conjecture): Let $f(x_0, \ldots, x_n)$ be a cubic form with coefficients in a field k. Suppose f has a nontrivial solution in an algebraic extension K/k of degree prime to 3. Then f also has a non-trivial solution in k.

(3) Is it true that every smooth cubic surface over $\mathbb{Q}$ has a point over some abelian extension of $\mathbb{Q}$? (This is a particular case of Artin's conjecture: the maximal abelian extension of $\mathbb{Q}$ is a C_1-field.)

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REMARK: The question: "Is the Manin obstruction to the Hasse principle for rational surfaces the only one?" was first raised by Colliot-Thélène and Sansuc in [5], and some support given via [4], [6], [7].

In view of the assumptions in Theorems 1 and 2 below, it makes sense to reproduce here the following result from [6]:

FACT: Let a diagonal cubic surface $V \subset \mathbb{P}^3_{\mathbb{Q}}$ be given by an equation $ax^3 + by^3 + cz^3 + dt^3 = 0$, where $a,b,c,d \in \mathbb{Z}$ and x,y,z,t are homogeneous coordinates. Then for any $|a|,|b|,|c|,|d| < 100$ one of the following conditions holds:

(1) $V(\mathbb{Q}) \neq \emptyset$;

(2) there exists a prime p such that $V(\mathbb{Q}_p) = \emptyset$;

(3) for any prime p, $V(\mathbb{Q}_p) \neq \emptyset$, $V(\mathbb{Q}) = \emptyset$ and the Manin obstruction is non-empty.

Now we come to our results in this paper:

THEOREM 1: Let $W \subset \mathbb{P}^4_{\mathbb{Q}}$ be a cubic three-fold given by an equation $ax^3 + by^3 + cz^3 + f(t,u) = 0$, where $a,b,c \in \mathbb{Z}$, x,y,z,t,u are homogeneous coordinates and $f(t,u)$ is a homogeneous form of degree 3 in variables t, u with coefficients in $\mathbb{Z}$.

Let us assume that the Manin obstruction to the Hasse principle is the only one for diagonal cubic surfaces over $\mathbb{Q}$.

Then, if W is not a cone and $W(\mathbb{Q}_v) \neq \emptyset$ for any completion $\mathbb{Q}_v$ of $\mathbb{Q}$, W contains $\mathbb{Q}$-rational points.

REMARK: For $f(t,u) = dt^3 + eu^3$, $d,e \in \mathbb{Z}$, this theorem was proved in [6] and here we only slightly modify the corresponding proof in [6].

PROOF: We will use the following result from [6].

LEMMA: Let $V \subset \mathbb{P}^3_{\mathbb{Q}}$ be a diagonal cubic surface given by an equation $ax^3 + by^3 + cz^3 + dt^3 = 0$, where $a,b,c,d \in \mathbb{Z}$ and x,y,z,t are homogeneous coordinates. Let V have points everywhere locally and let there exist a prime p such that $abc \neq 0 \bmod p$, $d = 0 \bmod p$ and $d \neq 0 \bmod p^3$. Then the Manin obstruction vanishes for V.

To prove Theorem 1, using the lemma, we must find $t_0, u_0 \in \mathbb{Z}$ such that for the cubic surface V_{t_0,u_0}, given by the equation $ax^3 + by^3 + cz^3 + f(t_0,u_0)t^3 = 0$ (where $a,b,c,f(t_0,u_0)$ are coefficients and x,y,z,t are homogeneous coordinates), the following holds:

(i) V_{t_0,u_0} has points everywhere locally

(ii) there exists a prime p such that $abc \neq 0 \bmod p$, $f(t_0,u_0) = 0 \bmod p$ and $f(t_0,u_0) \neq 0 \bmod p^3$.

Then by the lemma, $V_{t_0,u_0}(\mathbb{Q}) \neq \emptyset$, i.e. $W(\mathbb{Q}) \neq \emptyset$.

In order to construct V_{t_0,u_0} with properties (i) and (ii) let us choose for every prime $q | 3abc$ a point $(x_q, y_q, z_q, t_q, u_q) \in W(\mathbb{Z}_q)$ (with integer coordinates not all equal to $0 \bmod q$). Further we must find a prime p not dividing $3abc$ and $t_p, u_p \in \mathbb{Z}$ such that

$$(*) \qquad \begin{aligned} f(t_p,u_p) &= 0 \bmod p \\ f(t_p,u_p) &\neq 0 \bmod p^3 . \end{aligned}$$

Then $t_0, u_0 \in \mathbb{Z}$, such that $(t_0,u_0) = (t_q,u_q) \bmod q^n$ $(n \gg 0)$ for all $q | 3abc$ or $q = p$, will be the integers that we are looking for. (Indeed, (i) will follow from Hensel's lemma and (ii) from (*).)

Thus the following considerations, which show that a choice of p, t_p, u_p satisfying (*) is possible, complete the proof of Theorem 1.

First of all the fact that V is not a cone implies that $f \neq c(\ell(t,u))^3$, where $c \in \mathbb{Q}$ and $\ell(t,u)$ is a linear form in variables t,u over $\mathbb{Q}$. From this it follows that $f \neq c_p(\ell_p(t,u))^3 \bmod p$ for almost all p, where $c_p \in \mathbb{Q}_p$ and $\ell_p(t,u)$ is a linear form in variables t,u over $\mathbb{Q}_p$. (Indeed, this is easy to see if f is reducible over $\mathbb{Q}$ and the case of irreducible f it follows from the fact that a number field is ramified at only finitely many places.

Finally, by Tchebotarev's theorem there exist infinitely many primes p such that f acquires a <u>simple</u> zero modulo p, whose suitable lifting over $\mathbb{Z}_p$ (by Hensel's lemma) provides t_p, u_p

satisfying (*).

THEOREM 2: Let us assume that the Manin obstruction is the only one for diagonal cubic surfaces over $\mathbb{Q}$. Let $V \subset \mathbb{P}^3_{\mathbb{Q}}$ be given by an equation $ax^3 + by^3 + cz^3 + dt^3 = 0$, where $a,b,c,d \in \mathbf{Z}$ and x,y,z,t are homogeneous coordinates. Let $K/\mathbb{Q}$ be a finite extension such that $V(K) \neq \emptyset$ and $[K:\mathbb{Q}]$ prime to 3. Then $V(\mathbb{Q}) \neq \emptyset$.

PROOF: We will use

Fact 1: $V(\mathbb{Q}_v) \neq \emptyset$ for any completion $\mathbb{Q}_v$ of $\mathbb{Q}$.

This follows from the theorem by Coray [8] stating that the Cassels-Swinnerton-Dyer conjecture holds for cubic surfaces over local fields and from the observation that for any place v of $\mathbb{Q}$ there exists a place w of K lying above v such that $[K_w:\mathbb{Q}_v]$ is prime to 3.

The following result was proved in [6].

Fact 2: If there exists a rational prime $p \mid 3\,abcd$ such that V is not $\mathbb{Q}_p$-rational, then the Manin obstruction vanishes for $V/\mathbb{Q}$.

In view of Fact 2 we can assume further that for all primes $p \mid 3\,abcd$ V is $\mathbb{Q}_p$-rational. Then for any completion K_w of K, $A \in \mathrm{Br}(V \otimes K_w)$ and $P, P' \in V(K_w)$, $A(P) = A(P')$ (for w at which $V \otimes K_w$ has bad reduction this follows from K_w-rationality of V and for "good" places of K - from [1]). Therefore (since $V(K) \neq \emptyset$) for any $A' \in \mathrm{Br}(V \otimes K)$ and $(P_w) \in \prod_w V(K_w)$ we have

$$(*) \qquad \sum_w \mathrm{inv}_w A'(P_w) = 0 .$$

This implies for any $(P_v) \in \prod_v V(\mathbb{Q}_v)$ and $A \in \mathrm{Br}\, V$:

$$(**) \qquad [K:\mathbb{Q}]\Big(\sum \mathrm{inv}_v A(P_v)\Big) = \sum_v \sum_{w \mid v} \mathrm{inv}_w \mathrm{Res}_{\mathbb{Q}_v/K_w}(A(P_v))$$

$$= \sum_w \mathrm{inv}_w (\mathrm{Res}_{\mathbb{Q}/K} A)(P_w)$$

$$= 0 .$$

Here (P_w) is the image of (P_v) under the diagonal map $\prod_v V(\mathbb{Q}_v) \longrightarrow \prod_w V(K_w)$, $\mathrm{Res}_{\mathbb{Q}_v/K_w} : \mathrm{Br}\,\mathbb{Q}_v \longrightarrow \mathrm{Br}\,K_w$ and $\mathrm{Res}_{\mathbb{Q}/K} : \mathrm{Br}\,V \longrightarrow \mathrm{Br}(V \otimes K)$ are restriction maps. (The first equality in (**) follows from local class-field theory ([2], Proposition 2, p. 133) and the last one from (*)),

By [6] for any $A \in \mathrm{Br}\,V$, $3A \in \mathrm{Br}\,\mathbb{Q}$ and, since $[K:\mathbb{Q}]$ is prime to 3, (**) implies that $\sum_v \mathrm{inv}_v A(P_v) = 0$. Thus the Manin obstruction vanishes for V, proving the theorem.

<u>THEOREM 3</u>: Let V be a non-singular cubic surface over a number field k. For any finite extension of fields K/k such that $V(K) = \emptyset$ and $V \otimes K$ has points everywhere locally, let us assume that the Manin obstruction is non-empty for $V \otimes K$. Then V contains a rational point in some abelian extension of k.

<u>PROOF</u>: For all but finitely many places v of k, $V(k_v) \neq \emptyset$ (k_v = completion of k at v). By Lang's theorem [9] (stating that the maximal unramified extension of a completion k_v of k is C_1) there exist cyclic extensions K_w/k_v such that $V(K_w) \neq \emptyset$ for all "bad" places v (for which $V(k_v) = \emptyset$) and by Grunwald-Wang's theorem ([1], p. 103) we can find an abelian extension K/k such that for all "bad" places v of k, $K \otimes_k k_v \simeq$ product of copies of K_w/k_v.

Thus, we can choose a finite abelian extension K/k such that $V(K_w) \neq \emptyset$ for all completions K_w of K.

Further, let L/K be a finite Galois extension such that all lines of V are defined over L. Then ([10, Ch. VI])

$$H^1(\mathrm{Gal}(L/K), \mathrm{Pic}(V \otimes L)) \simeq H^1(\mathrm{Gal}(\bar{k}/K), \mathrm{Pic}\,\bar{V}) = (\mathrm{Br}\,V \otimes K)/\mathrm{Br}\,K$$

(here $\bar{k}$ is an algebraic closure of k and $\bar{V} = V \otimes \bar{k}$).

Let $d = \#((\mathrm{Br}(V \otimes K))/\mathrm{Br}\,K)$. We can find a finite abelian extension M/K such that $d \mid [M:K]$, $M \cap L = K$ and M/k is an abelian extension. Indeed, let l_i $(i \in I)$ be the set of <u>cyclic</u> proper extensions of k which are included in L. For each i, choose a finite place v_i such that $l_i \otimes_k k_{v_i} = l_{i,v_i}$ is a <u>field</u> (possible by Tchebotarev). Now choose a place v not among those v_i and choose a cyclic extension l_v/k_v of degree d. Using Grunwald-Wang's theorem ([1, p. 103]), one finds an abelian extension M'/k such that $M' \otimes_k k_{v_i}$ = product of copies of k_{v_i} and $M \otimes_k k_v$ = product of copies of l_v. Now $M' \cap L$ must be k: otherwise, this abelian extension of k would contain a proper cyclic extension l_i of k, hence we would have $k \subset l_i \subset M'$ for some i, and this contradicts the local behaviour at v_i. Finally, we can set $M = M' \cdot K$. Now the theorem follows from

<u>LEMMA</u>: The Manin obstruction vanishes for $V_M = V \otimes_k M$.

<u>PROOF</u>: First of all let us show that the restriction map $\mathrm{Res}_{K/M} : H^1(\mathrm{Gal}(\bar{k}/K), \mathrm{Pic}\,\bar{V}) \longrightarrow H^1(\mathrm{Gal}(\bar{k}/M), \mathrm{Pic}\,\bar{V})$ is an isomorphism.

We have the restriction map

$$\mathrm{Pic}\,V_L \longrightarrow \mathrm{Pic}\,V_{LM}$$

which is obviously an isomorphism of $\mathrm{Gal}(L/K) \simeq \mathrm{Gal}(LM/M)$ -modules ($V_L := V \otimes L$ and $V_{LM} := V \otimes LM$). Hence the top isomorphism in the following commutative diagram:

$$\begin{array}{ccc} H^1(\mathrm{Gal}(L/K), \mathrm{Pic}\,V_L) & \xrightarrow{\sim} & H^1(\mathrm{Gal}(LM/M), \mathrm{Pic}\,V_{LM}) \\ \downarrow & & \downarrow \\ H^1(\mathrm{Gal}(\bar{k}/K), \mathrm{Pic}\,\bar{V}) & \longrightarrow & H^1(\mathrm{Gal}(\bar{k}/M), \mathrm{Pic}\,\bar{V}) \end{array}$$

where the horizontal arrows are restriction maps and the vertical maps are inflation maps. From $\mathrm{Pic}\,V_L = \mathrm{Pic}\,\bar{V}$ (and similarly $\mathrm{Pic}\,V_{LM} = \mathrm{Pic}\,\bar{V}$), one concludes that the vertical maps are isomorphisms. Hence also the bottom horizontal map, whence $\mathrm{Br}\,V_K/\mathrm{Br}\,K \xrightarrow{\sim} \mathrm{Br}\,V_M/\mathrm{Br}\,M$. Therefore $\mathrm{Br}\,V_M = \mathrm{Res}_{K/M}(\mathrm{Br}(V \otimes K)) + \mathrm{Br}\,M$, where $\mathrm{Res}_{K/M} : \mathrm{Br}(V \otimes K) \longrightarrow \mathrm{Br}\,V_M$ is the restriction map.

Now let the diagonal map $\prod_v V(K_v) \longrightarrow \prod_w V(M_w)$ map some $(P_v) \in \prod_v V(K_v)$ to $(P_w) \in \prod_w V(M_w)$ and let $A = (\mathrm{Res}_{K/M} A') + a \in \mathrm{Br}\,V_M$, where $A' \in \mathrm{Br}\,V \otimes K$ and $a \in \mathrm{Br}\,M$. Then, applying the local class-field theory ([2, Proposition 2, p. 133]) we obtain:

$$\begin{aligned}\sum_w \mathrm{inv}_w A(P_w) &= \sum_w \mathrm{inv}_w [(\mathrm{Res}_{K/M} A') + a](P_w) \\ &= \sum_w \mathrm{inv}_w (\mathrm{Res}_{K/M} A')(P_w) \\ &= \sum_v \sum_{w|v} \mathrm{Res}_{K_v/M_w} A'(P_v) \\ &= [M:K] \sum_v \mathrm{inv}_v A'(P_v) \\ &= 0 \quad .\end{aligned}$$

(The last equality follows from this fact that $d \cdot A' \in \mathrm{Br}\, K$ and $d \mid [M:K]$.) This completes the proof of the lemma (and of Theorem 3).

REMARK: Assuming that the Manin obstruction is the only one for any rational smooth projective X/L $([L:\mathbb{Q}] < \infty)$, one can prove that given such an X over a numberfield k, $X(K) \neq \emptyset$ for some abelian extension K/k.

The proof of this fact closely follows to the proof of Theorem 3. In particular, in order to apply as above Lang's theorem, one can use the following Manin's conjecture (whose proof was recently completed by Colliot-Thélène): If X/K is a rational projective smooth surface and k is C_1, then $X(k) \neq \emptyset$.

Finally, I would like to mention the following generalization of Manin's conjecture, which was also proposed by Colliot-Thélène: (under the above assumptions on X) if the cohomological dimension of $k \leq 1$, then $X(k) \neq \emptyset$.

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