Astérisque

CHARLES FEFFERMAN
C. ROBIN GRAHAM

Conformal invariants

Astérisque, tome S131 (1985), p. 95-116
<http://www.numdam.org/item?id=AST_1985__S131__95_0>

© Société mathématique de France, 1985, tous droits réservés.

L'accès aux archives de la collection « Astérisque » (http://smf4.emath.fr/Publications/Asterisque/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Société Mathématique de France
Astérisque, hors série, 1985, p. 95–116

CONFORMAL INVARIANTS

BY

Charles FEFFERMAN (*) and C. Robin GRAHAM (**)

ABSTRACT. — This paper presents a construction of local invariants of a conformal structure on a Riemannian manifold. The method is to associate to the given conformal manifold M a Ricci flat Riemannian manifold $\widetilde{G}$ of two higher dimensions in such a way that invariants of the Riemannian metric on $\widetilde{G}$ give rise to conformal invariants on M. The questions of existence and uniqueness of the metric on $\widetilde{G}$ reduce to a characteristic initial value problem for Einstein's equations.

Let $g = \sum_{ij} g_{ij}(x)dx^i dx^j$ be a metric defined on a coordinate patch in $\mathbf{R}^n$. We want to write down all conformally invariant expressions in the $g_{ij}(x)$ and their derivatives of all orders. To start with, we ask for scalar conformal invariants. Thus, a conformal invariant $P(g)$ is a polynomial in $(\det g_{ij})^{-1}$ and the derivatives $\partial^\alpha g_{ij}$, satisfying two invariance properties :

(a) If g and g' are isometric, then $P(g) = P(g')$.

(b) If $g = \lambda(x)g'$ for a positive smooth function $\lambda(\cdot)$, then $P(g) = \lambda^{\text{power}}P(g')$.

Many authors have given examples of conformal invariants. We believe we have found the complete list of conformal invariants in odd dimensions. Our construction is rooted in the attempt to generalize to arbitrary conformal metrics the following elementary discussion of the sphere.

Recall that $O(n+1,1)$ acts conformally on $S^n \subset \mathbf{R}^{n+1}$ by linear fractional transformations. In order to see this, introduce projective coordinates $(\xi_0, \xi_1, \ldots, \xi_{n+1})$ so that the sphere $\sum_1^{n+1} x_k^2 - 1 = 0$ goes over to

$$G = \left\{ \sum_1^{n+1} \xi_k^2 - \xi_0^2 = 0 \right\}$$

(*) Research of the first author supported by a National Science Foundation grant.

(**) The second author held a visiting position at Princeton University while this work was being carried out. He is most grateful to the mathematics departement there for its support and hospitality.

under $x_k = \xi_k/\xi_0$ $(1 \le k \le n+1)$. When restricted to G, the flat Lorentz metric

$$\widetilde{g} = \sum_1^{n+1} d\xi_k^2 - d\xi_0^2$$

takes the form $\widetilde{g}|_G = \xi_0^2 \sum_1^{n+1} dx_k^2$. Hence any linear transformation of the ξ's preserving $\widetilde{g}$ will induce a conformal linear-fractional self-map of the unit sphere. Alternatively, the conformal self-maps of the sphere arise from isometries of the ball with its hyperbolic Poincaré metric

$$g^+ = \left(1-|x|^2\right)^{-1} \sum dx_i^2 + \left(1-|x|^2\right)^{-2} \left(\sum x_i dx_i\right)^2$$

by restriction to the boundary.

We seek to associate to any conformal metric g a Lorentz metric and a Poincaré metric analogous to $\widetilde{g}$, g^+. The resulting partial differential equations have essentially unique formal power series solutions, and from these we read off our conformal invariants. Proving that our procedure generates all possible conformal invariants is an unsolved problem of algebra, to which we hope to return. We have some reason to expect a positive answer, by analogy with [12].

I. Background

This work originated in an attempt to continue the program of [12] expressing the Bergman kernel of a strictly pseudoconvex domain in $\mathbf{C}^n$ asymptotically near the boundary in terms of local invariants of the biholomorphic geometry of the boundary of the domain. This geometry is closely related to conformal geometry; on the one hand it can be considered a complex analogue of conformal geometry as the unitary group is a complex analogue of the orthogonal group. But the relation is deeper than mere analogy : on a circle bundle over the boundary of a strictly pseudoconvex domain (or more generally, over a non-degenerate CR manifold) there is a conformal structure invariantly determined by the biholomorphic, or CR, structure of the base [11]. Moreover, all local CR-invariant data (i.e., curvature) of the base can be recovered from conformally invariant data on the circle bundle [5]. Hence one can hope to better understand CR geometry such as is needed for the Bergman kernel by studying conformal geometry. As the Bergman kernel is a scalar object, our primary interest is in constructing and understanding scalar CR and conformal invariants. Unfortunately, the geometric problem of interest for application to the asymptotic expansion of the Bergman kernel, that of constructing all scalar CR invariants, remains unsolved at present.

However our researches have led to some new ideas in conformal geometry which will be outlined here.

First we discuss scalar Riemannian invariants and define the scalar conformal invariants of interest. As the conformal structures on the circle bundles mentioned above are Lorentzian and it is also of interest to consider other signatures, metrics of general fixed signature (p,q), are allowed where $p+q=n$ is the dimension of the space, and such metrics are called Riemannian. A Riemannian metric g of signature (p,q) may be expressed in a local coordinate system as $g = g_{ij}dx^i dx^j$. By a scalar Riemannian invariant is meant a polynomial P in the variables

$$g_{ij,\alpha} = \left(\frac{\partial}{\partial x}\right)^{\alpha} g_{ij}, \quad |\alpha| \geq 0,$$

and $(\det g)^{-1}$, which is coordinate free in the sense that the value of P is independent of the coordinate system used to express and differentiate g. Analogously one speaks of tensor-valued or differential form-valued Riemannian invariants if each component of the tensor or form in a given coordinate system is a polynomial in the above variables and if under a change of coordinates the invariant transforms as a tensor or form. See [1] for a careful elaboration of this definition. Of course the most basic Riemannian invariant is the Riemannian curvature tensor $\mathbf{R}$, with components

$$R_{ijkl} = \frac{1}{2}\left[g_{ik,jl} + g_{jl,ik} - g_{jk,il} - g_{il,jk}\right] + g_{pq}\left[\Gamma^p_{ik}\Gamma^q_{jl} - \Gamma^p_{il}\Gamma^q_{jk}\right],$$

where $\Gamma^k_{ij} = \frac{1}{2}g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l})$ is the Christoffel symbol and $g^{ij} = (g_{ij})^{-1}$. The simplest scalar Riemannian invariant is the scalar curvature $S = g^{il}g^{jk}R_{ijkl}$. Other scalar Riemannian invariants can be easily constructed out of R and its iterated covariant derivatives $\nabla^l R$ by taking tensor products and contracting : $\operatorname{tr}(\nabla^{l_1}R \otimes \cdots \otimes \nabla^{l_r}R)$ is a scalar Riemannian invariant, called a Weyl invariant, for any choice of $l_i \geq 0$ and any pairing of the indices with respect to which the trace is taken. By utilizing a geodesic normal coordinate system one sees that any scalar Riemannian invariant may be expressed as a polynomial in the components of R and its covariant derivatives, and classical invariant theory identifies all such invariant polynomials. This leads to the conclusion that every scalar Riemannian invariant is a linear combination of Weyl invariants.

Now a conformal structure on a manifold is an equivalence class of Riemannian metrics, where two metrics are identified if one is a smooth positive multiple of the other, and a diffeomorphism between two Riemannian manifolds is conformal if it maps one metric to a smooth positive multiple of the other. Consequently we define a scalar conformal invariant of weight k to be a scalar Riemannian invariant $P(g)$ as above with the additional property that $P(\lambda g) = \lambda^{-k} P(g)$ for all smooth functions $\lambda > 0$. (This normalization is chosen so that scalar conformal invariants have positive integral weight.) Of course most Riemannian invariants, for instance R, are not conformally invariant, as $P(\lambda g)$ depends on derivatives of the conformal factor λ and not just λ itself. However a part of the curvature tensor is conformally invariant. Let $R_{jk} = g^{il} R_{ijkl}$ be the Ricci tensor so that $S = g^{jk} R_{jk}$, and define

$$A_{jk} = \frac{1}{n-2}\left[R_{jk} - \frac{S}{2(n-1)} g_{jk}\right].$$

(As all metrics in two dimensions are conformally flat, there are no nontrivial conformal invariants if the dimension n is 2. Hence we always assume $n \geq 3$.) Then the Weyl conformal curvature tensor W is defined by

$$W_{ijkl} = R_{ijkl} - \left[A_{jk} g_{il} + A_{il} g_{jk} - A_{jl} g_{ik} - A_{ik} g_{jl}\right],$$

and one has $W(\lambda g) = \lambda W(g)$. Thus W is a conformally invariant tensor of weight -1. W is of fundamental importance, e.g. its vanishing characterizes conformally flat spaces if $n \geq 4$. See [13], for example. As in the Riemannian case, once one has conformally invariant tensors, scalar invariants can be obtained by taking tensor products and contracting. All traces of W itself vanish, but $\|W\|^2 = W^{ijkl} W_{ijkl}$ is, for instance, a scalar conformal invariant.

If $n = 3$ the Weyl tensor W always vanishes, but there is a substitute tensor first introduced by COTTON [7], herein referred to as the Cotton tensor, defined by

$$C_{jkl} = \nabla_l A_{jk} - \nabla_k A_{jl}.$$

When $n = 3$, C is a conformally invariant tensor of weight 0 whose vanishing characterizes conformally flat spaces. If $n \geq 4$ we still define C by this formula, but C is no longer conformally invariant.

It seems that the only other conformally invariant tensor known classically is the Bach tensor [2]. We define B in any dimension by

$$B_{jk} = \nabla^l C_{jkl} + A^{il} W_{ijkl}.$$

It turns out that B is symmetric and trace-free. If $n = 4$, B is conformally invariant of weight 1, but it is not conformally invariant in any other dimension. Some of its importance in dimension 4 stems from the fact that $B = 0$ for a space which is conformally Einstein, i.e. which is conformally equivalent to a space of constant Ricci curvature.

Our goal is to find all scalar conformal invariants, in the sense that one knows all scalar Riemannian invariants as previously described. Unlike the Riemannian case, it is a nontrivial matter even to write down some more examples, never mind proving that you have them all; covariant differentiation not being a conformally invariant process, one can't simply differentiate the Weyl tensor. We have a conjectured solution to this problem in the odd dimensional case, and in the even dimensional case can construct scalar invariants involving derivatives of the metric of order up to n.

Before describing our construction we mention some alternate approaches to the problem. The method of É. CARTAN associates to a manifold with a conformal structure a principal bundle with connection over that manifold whose fiber is a parabolic subgroup of $O(p+1,q+1)$. (See [15]). All conformally invariant information is carried in the curvature of the connection and derivatives of the curvature with respect to the parallelism on the principal bundle. Thus the problem of finding scalar conformal invariants reduces to a problem in the representation theory of the fiber group. However the requisite representation theory is little understood so that, at least at present, this approach does not help much, even in writing down new examples of scalar conformal invariants.

An approach in four dimensions for metrics of Lorentz signature which seems more fruitful is the method of local twistor transport, of PENROSE and DIGHTON [18], [9]. Local twistor transport is a conformally invariant calculus of covariant differentiation and as such can be used to differentiate the Weyl and Cotton tensors viewed together as a curvature twistor, to come up with further conformally invariant twistors, spinors and tensors; hence scalars too. SPARLING [21] has extended portions of this program to general signature and dimension. The relationship between this approach and our method is not clear.

Finally some other examples, constructions, and analysis of conformally invariant tensors are given in [10], [22].

II. Riemannian Structures Associated to a Conformal Structure

The method to be employed to construct conformal invariants is to invariantly associate to a manifold with a conformal structure another manifold with a Riemannian structure, so that Riemannian invariants, which are plentiful, give rise to conformal invariants. Two problems defining such Riemannian structures will be discussed, the meat of the matter reducing to proving existence and uniqueness for these problems. As previously discussed, the choice of these particular problems is motivated by the model case of the sphere. The first metric we call the ambient metric, which has signature $(p+1,q+1)$ on a space of dimension $n+2$ and generalizes the

Lorentz metric

$$\widetilde{g} = \sum_1^{n+1} d\xi_k^2 - d\xi_0^2.$$

The second is related to the first and is a singular metric g^+ of signature $(p+1, q)$ on a space of dimension $n+1$, generalizing the Poincaré metric

$$g^+ = \left(1 - |x|^2\right)^{-1} \sum dx_1^2 + \left(1 - |x|^2\right)^{-2} \left(\sum x_i, dx_i\right)^2$$

on the ball $\mathbf{B}^{n+1}$. Recall that g^+ may also be written

$$g^+ = \frac{4 \sum dy_i^2}{\left(1 - |y|^2\right)^2}$$

upon making the change of radial parameter $|x| = 2|y|/(1+|y|^2)$. It is actually this latter representation of g^+ that we use as the model for the general Poincaré metric.

Let M be a manifold with conformal structure $[g]$, g being some representative of the conformal class. We begin by constructing invariantly the ambient space, in which the ambient metric will live, g is of course not determined by the conformal structure, but rather at each point $x \in M$ the conformal structure determines a ray $\{t^2 g(x) : t > 0\}$ of quadratic forms. Taken together these rays form a ray subbundle

$$G = \{(x, t^2 g(x)) : x \in M,\ t > 0\}$$

of the bundle of symmetric 2–tensors on M, S^2T^*M, and one has the natural projection $\pi : G \to M$. G is called the metric bundle and a section of G is just a representative of the conformal structure. The conformal structure itself lives invariantly as a tautological symmetric 2–tensor g_0 on G, defined for $(x, \bar{g}) \in G$ and $X, Y \in T_{(x,\bar{g})}G$ by

$$g_0(X, Y) = \bar{g}(\pi_* X, \pi_* Y).$$

In addition there is a notion of homogeneity on G : for $s > 0$, $\delta_s(x, \bar{g}) = (x, s^2\bar{g})$ defines dilations $\delta_s : G \to G$; this action of $\mathbf{R}_+$, the multiplicative group of positive real numbers, gives G the structure of a $\mathbf{R}_+$- principal bundle over M. Also, g_0 is homogeneous of degree 2 : $\delta_s^* g_0 = s^2 g_0$. Now the ambient metric will live in the space $\widetilde{G} = G \times I$, where $I = (-1, 1) \subset \mathbf{R}$. We identify G with its image under the inclusion ι defined by $G \ni p \xrightarrow{\iota} (p, 0) \in \widetilde{G}$, and the dilations δ_s on G extend naturally to $\widetilde{G}$. Our interest is solely local, so we work only over a coordinate patch in M and restrict attention to a small neighborhood of G in $\widetilde{G}$. Also all constructions will be determined only up to a homogeneous, or $\mathbf{R}_+$-equivariant, diffeomorphism of $\widetilde{G}$ fixing G, so in particular, the projection $\pi : G \to M$ does not have an invariant extension to $\widetilde{G}$. It is useful to keep in mind the sphere, for which G is the upper half of the light cone and $\widetilde{G}$ a suitable neighborhood thereof.

There are three conditions to be imposed to determine the ambient metric $\widetilde{g}$, of signature $(p+1, q+1)$, on $\widetilde{G}$. They are :

1) $\delta_s^* \widetilde{g} = s^2 \widetilde{g}$, $s > 0$;   2) $\iota^* \widetilde{g} = g_0$;   3) $\mathrm{Ric}(\widetilde{g}) = 0$.

The first two conditions are quite natural, especially upon consideration of the model case.For the model case $\widetilde{g}$ is actually flat, but this is of course too much to expect in general. So we try instead only to require that $\widetilde{g}$ be Ricci flat, giving rise to 3). Notice that 2) already forces $\widetilde{g}$ to have signature $(p+1, q+1)$. In fact, as a form on TG, g_0 is degenerate, since the vertical vector is orthogonal to everything. So on TG, $\widetilde{g}$ must have signature $(p, q, 1)$, and the conclusion follows by linear algebra. One cannot expect to have absolute uniqueness for the problem of finding a metric $\widetilde{g}$ satisfying 1) – 3). In fact, if $\Phi : \widetilde{G} \to \widetilde{G}$ is any diffeomorphism which fixes G and which commutes with the dilations, then $\Phi^* \widetilde{g}$ is a solution whenever $\widetilde{g}$ is.

We study the problem of the existence and uniqueness of a solution $\widetilde{g}$ to 1) – 3) on the formal power series level as a Cauchy problem, 2) being the initial condition and 3) the equation to be satisfied. Curiously, it turns out that the results depend decisively on whether the dimension n of M is even or odd. Our main theorem is :

THEOREM 2.1.

a) *n odd. Up to a $\mathbf{R}_+$-equivariant diffeomorphism fixing G, there is a unique formal power series solution $\widetilde{g}$ to* 1) – 3). *If the conformal structure on M is real analytic (i.e. has a real analytic representative), then this formal power series converges so that $\widetilde{g}$ actually exists in a neighborhood of G.*

b) *n even. There are conformal structures for which there is no formal power series solution of* 1) – 3). *However, if* 3) *is replaced by :*

3′) *Along G, all components of* $\mathrm{Ric}(\widetilde{g})$ *vanish to order $(n-4)/2$ and the components tangential to G vanish to order $(n-2)/2$,*

then there is a formal power series solution for $\widetilde{g}$ uniquely determined up to addition of terms vanishing to order $n/2$ and up to a $\mathbf{R}_+$-equivariant diffeomorphism fixing G.

As with the complex Monge-Ampère equation [11], [6], [17], one can continue the solution to higher order when n is even by including log terms in the expansion for $\widetilde{g}$. Working out the exact form and meaning of these higher asymptotics is of great interest.

Next consider the problem for the general version of the Poincaré metric.

We begin by defining conformal infinity, essentially following the formulation of LeBrun [16] of this concept extant for some time in the physics literature. Let N be a manifold with boundary and suppose that a conformal structure $[g]$ is given on bN. Let $r \in C^\infty(\overline{N})$ satisfy $r > 0$ in $N, r = 0$ and $dr \neq 0$ on bN. A Riemannian metric g^+ on $N \sim bN$ is said to have $[g]$ as conformal infinity if for some number $m > 0, r^m g^+$ has a smooth extension to $\overline{N}$, and when restricted to TbN, $r^m g^+ \in [g]$. This definition is independent of the choice of defining function r. If $[g]$ has signature $(p,q), g^+$ can possibly have signature either $(p, q+1)$ or $(p+1, q)$.

Let now M^n be a manifold with conformal structure $[g]$ of signature (p,q) and let $M^+ = M \times [0,1]$. Identify M with $bM^+ = M \times \{0\}$. The problem to be solved by the Poincaré metric g^+, a metric of signature $(p+1,q)$ on $M^+ \sim M$, is :

1) g^+ has $[g]$ as conformal infinity
2) $\mathrm{Ric}(g^+) = -ng^+$.

The choice of the particular constant n in 2) is an arbitrary convenient normalization.

Like the problem for the ambient metric, this problem has a gauge invariance : $\Phi^* g^+$ is a solution whenever g^+ is,where Φ is a diffeomorphism of M^+ fixing M.

As a local Cauchy problem this problem is underdetermined. Another initial condition is needed in order to get a unique solution, the nonuniqueness imposed by the gauge invariance of the problem notwithstanding. When $n = 3$ such a condition was studied by LeBrun [16], assuming that an orientation on M is fixed, giving rise to an orientation on M^+ inducing the given orientation on M. LeBrun's additional condition is that g^+ should be self-dual, i.e. that the anti-self-dual part of the Weyl tensor of g^+ should vanish. Using twistor methods, he showed when $n = 3$ that if $[g]$ is real analytic, then in some neighborhood of M in M^+ and up to diffeomorphism, there is an unique analytic self-dual solution g^+ to 1), 2). His work has influenced ours, especially by stimulating us to understand the relationship between his result and our ideas concerning Poincaré metrics.

Our Poincaré metric is distinguished by another condition. As LeBrun pointed out, for any metric g^+ satisfying 1) and 2), the order m of the singularity in the definition of conformal infinity must be 2. Some further computation shows that such a metric can in an appropriate coordinate system $(x^1, \dots, x^n, r)$ be written as

$$g^+ = r^{-2}\left[dr^2 + \sum_{i,j=1}^{n} g^+_{ij}(x,r)dx^i dx^j\right]. \tag{2.2}$$

Here r is a defining function for M in M^+ as above and $(x^1, \dots, x^n)$ forms a coordinate system on M. Our additional condition for g^+ is :

3) When written in the form (2.2), g^+_{ij} is an even function of r, $1 \le i, j \le n$.

Although it is not obvious, it is nonetheless true that this condition is independent of the choice of coordinate system (x, r) used to write g^+ in the form (2.2). It would be nice to have an invariant way of stating this condition, but we have not succeeded in finding one.

The analogue of THEOREM 2.1 for the Poincaré metric is :

THEOREM 2.3.

a) n *odd. Up to a diffeormorphism fixing* M, *there is a unique formal power series solution* g^+ *to* 1) − 3). *If* $[g]$ *is real analytic, then the power series converges so that* g^+ *exists and* $r^2 g^+$ *is analytic up to the boundary.*

b) n *even. There are conformal structures for which there is no formal power solution of* 1) − 3). *However, if* 2) *is replaced by :*

2′) *Along* M, *the components of* $\mathrm{Ric}(g^+) + ng^+$ *vanish to order* $n - 2$,

then there is a formal power series solution for g^+ *uniquely determined up to addition of terms vanishing to order* $n - 2$ *and up to a diffeomorphism fixing* M.

When $n = 3$, our Poincaré metric differs in general from that of LeBrun. However it is possible to analyze his problem from the point of view of formal power series and thus give a new non-twistorial proof of his theorem. A perplexing question is whether there is an analogue of his problem in higher dimensions.

The problem of finding a Poincaré metric g^+ satisfying 1) − 3) is actually equivalent to the problem of finding an ambient metric $\widetilde{g}$ satisfying the corresponding 1) − 3). The procedure of constructing g^+ from $\widetilde{g}$ will be sketched here, and this process is easily seen to be reversible. Let T be the vector field on $\widetilde{G}$ given by infinitesimal dilation, so that

$$Tf(p) = \frac{d}{ds} f(\delta_s p)|_{s=0} \text{ for } f \in C^\infty(\widetilde{G}),\ p \in \widetilde{G}.$$

Now as has already been noted, $g_0(T, T) = 0$, so the function $\|T\|^2 = \widetilde{g}(T, T)$ is a smooth function on $\widetilde{G}$ which vanishes on G. It is not too hard to see that $\|T\|^2$ is homogeneous of degree 2 with respect to the dilations δ_s and vanishes to first order on G. Hence the surface $S = \{\|T\|^2 = -1\} \subset \widetilde{G}$ lies to one side of G, namely in $\widetilde{G}_- = \{\|T\|^2 < 0\}$, and each $\mathbf{R}_+$-orbit in $\widetilde{G}_-$ intersects S exactly once, thus giving rise to an identification of $\widetilde{G}_-/\mathbf{R}_+$ with S. But $\widetilde{G}_-/\mathbf{R}_+$ can also be identified with $M^+ \sim M$, so we can identify S and $M^+ \sim M$. Then g^+ is simply $i_S^* \widetilde{g}$ under this identification, where $i_S : S \to \widetilde{G}$

is inclusion. In fact it can be computed that $i_S^*\widetilde{g}$ satisfies condition 2) for a Poincaré metric and in an appropriate coordinate system (x, ρ) for M^+ induced from coordinates on $\widetilde{G}$, takes the from

$$i_S^*\widetilde{g} = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\sum_{ij=1}^{n} h_{ij}(x,\rho)dx^i dx^j. \tag{2.4}$$

Here $(x^1, \ldots, x^n)$ is a coordinate system on M, $\rho = 0$ on M, and

$$\sum_{i,j=1}^{a} h_{ij}(x,0)dx^i dx^j \in [g],$$

Thus introducing $r = \rho^{1/2}$, one obtains

$$i_S^*\widetilde{g} = r^{-2}\Big[dr^2 + \sum_{i,j=1}^{n} h_{ij}(x,r^2)dx^i dx^j\Big].$$

It follows that $i_S^*\widetilde{g}$ has $[g]$ as conformal infinity ans satisfies the evenness condition 3) as well, so must be the Poincaré metric g^+.

III. Applications to Conformal Invariants

It is an easy matter to construct scalar conformal invariants using the ambient metric $\widetilde{g}$. Since the Poincaré metric g^+ contains the same information as $\widetilde{g}$, in principle it should be possible to construct these invariants directly out of g^+ too, but it is less clear how to do this since g^+ is singular along M. In any case, let

$$P(\widetilde{g}) = \mathrm{tr}(\widetilde{\nabla}^{l_1}\widetilde{R} \otimes \cdots \otimes \widetilde{\nabla}^{l_r}\widetilde{R})$$

be any Weyl invariant of the Riemannian metric $\widetilde{g}$. Since $\widetilde{g}$ is homogeneous with respect to δ_s, the function $P(\widetilde{g})$ on $\widetilde{G}$ must also be homogeneous, the homogeneity degree depending upon the number of covariant differentiations and factors occuring in $P(\widetilde{g})$. Let now g be a representative of the conformal structure; equivalently a section of the metric bundle G. Associated to the Weyl invariant $P(\widetilde{g})$ and representative g, define a function $\widetilde{P}(g)$ on M by restricting $P(\widetilde{g})$ to G and pulling back by $g : \widetilde{P}(g) = g^*(P(\widetilde{g})|_G)$. Then $\widetilde{P}(g)$ is a scalar conformal invariant. In fact, the Taylor expansion of $\widetilde{g}$ along G is determined by that of a representative g and it turns out that the relationship is polynomial, so that $\widetilde{P}(g)$ is at least a scalar Riemannian invariant. However the homogeneity of $P(\widetilde{g})$ on G implies additionally that $\widetilde{P}(g)$ is conformally invariant, the weight of $\widetilde{P}$ being determined by the homogeneity degree of P. Hence, if n is odd, associated to every Weyl invariant on $\widetilde{G}$, there is an induced scalar conformal invariant on M. We call these invariants on M conformal Weyl invariants.

CONJECTURE 3.1. — *(n odd.) Every scalar conformal invariant is a linear combination of conformal Weyl invariants.*

When n is even this procedure still works except one is restricted by the ambiguity in $\widetilde{g}$ at order n/2. Consequently Weyl invariants on $\widetilde{G}$ make sense when restricted to G only if they do not involve too many covariant differentiations. In this case there is an analogous conjecture, applying only to scalar conformal invariants of sufficiently low weight. A proof of these conjectures will presumably involve doing invariant theory with respect to the parabolic subgroup of $O(p+1, q+1)$ mentioned earlier in the discussion of CARTAN's approach to conformal geometry.

It is also possible to compute scalar conformal invariants by this method. In principle this is clear since the Taylor expansion of $\widetilde{g}$ is determined on a purely formal basis, recursively, by solving the equation $\mathrm{Ric}(\widetilde{g}) = 0$ to higher and higher order, and the Riemannian invariants $P(\widetilde{g})$ are given by explicit formulae in the derivatives of $\widetilde{g}$. The simplest possible choices for $P(\widetilde{g})$ are the invariants $\mathrm{tr}(\widetilde{R} \otimes \cdots \otimes \widetilde{R})$ involving only the curvature tensor $\widetilde{R}$ of $\widetilde{g}$. Now $\widetilde{R}$ can be computed by carrying out the calculations just indicated. In order to express the result, let g be a representative for the conformal structure on M. g determines a fiber variable t on G by the requirement that $g_0 = t^2 \pi^* g$. Letting $(x^1, \ldots, x^n)$ denote a coordinate system in M, this is the same thing as calling $(t, x^1, \ldots, x^n)$ the coordinate of the point $(x, t^2 g(x)) \in G$. Finally these coordinates are suitably extended to $\widetilde{G}$ and completed by appending an appropriate variable ρ with $G = \{\rho = 0\}$. Thus $(t, x^1, \ldots, x^n, \rho)$ is our coordinate system on $\widetilde{G}$. Let $t = x^0$ and $\rho = x^{n+1} = x^m$, where we introduce $m = n+1$ so as to avoid writing $n+1$ as an index.

PROPOSITION 3.2. — *On G, the components of the curvature tensor $\widetilde{R}$ of $\widetilde{g}$ are given by :*

i) $\widetilde{R}_{ijk0} = 0 \qquad 0 \le i, j, k \le n+1$

ii) $\widetilde{R}_{ijkl} = t^2 W_{ijkl} \qquad 1 \le i, j, k, l \le n$

iii) $\widetilde{R}_{ijkm} = t^2 C_{kij} \qquad 1 \le i, j, k \le n$

iv) $\widetilde{R}_{mijm} = (t^2/(n-4)) B_{ij} \qquad 1 \le i, j \le n, \quad n \neq 4,$

where W, C, B are the Weyl, Cotton, and Bach tensors for the representative g (see § I). Here $\widetilde{R}$ is evaluated at $p = (t, x) \in G$ and W, C, B at $x = \pi(p) \in M$.

These relations and the usual symmetries of the curvature tensor determine all the components of $\widetilde{R}$. Thus the curvature tensor $\widetilde{R}$ itself represents the classical Weyl, Cotton and Bach tensors in one invariant object. As follows from THEOREM 2.1 b), when $n = 4$ the second derivatives of $\widetilde{g}$ transverse to G are not determined by the equation; hence the component $\widetilde{R}_{mijm}$ in *iv*) is undetermined as well.

It is now possible to identify the conformal Weyl invariants arising from the invariants $\operatorname{tr}(\widetilde{R} \otimes \cdots \otimes \widetilde{R})$. Using PROPOSITION 3.2 and the fact that in the coordinates $(t, x, \rho)\widetilde{g}$ is given on G by $\widetilde{g} = 2tdt \cdot d\rho + g_0$, one obtains easily

PROPOSITION 3.3. — *The Weyl conformal invariant arising from the Riemannian invariant* $\operatorname{tr}(\widetilde{R} \otimes \cdots \otimes \widetilde{R})$ *is* $\operatorname{tr}(W \otimes \cdots \otimes W)$, *with the same pairing of the indices.*

This is somewhat disappointing. We already knew about the scalar conformal invariants $\operatorname{tr}(W \otimes \cdots \otimes W)$, so have obtained nothing new. Thus to get new scalar conformal invariants, Riemannian invariants involving $\widetilde{\nabla}\widetilde{R}$ must be considered. Note that when $n = 3$, $\operatorname{tr}(W \otimes \cdots \otimes W)$ always vanishes. Some lengthy calculations allow one to identify the Weyl conformal invariant arising from $\|\widetilde{\nabla}\widetilde{R}\|^2$. The result is

PROPOSITION 3.4. — *The Weyl conformal invariant arising from* $\|\widetilde{\nabla}\widetilde{R}\|^2$ *is*

$$\|V\|^2 + 16(W, U) + 16\|C\|^2,$$

where $W =$ Weyl tensor, $C =$ Cotton tensor,

$$V_{sijkl} = \nabla_s W_{ijkl} - g_{is}C_{jkl} + g_{js}C_{ikl} - g_{ks}C_{lij} + g_{ls}C_{kij},$$

and

$$U_{sjkl} = \nabla_s C_{jkl} + g^{pq}A_{sp}W_{qjkl}.$$

When $n = 3$ this invariant reduces to a multiple of $\|C\|^2$, which we already knew. But when $n \geq 4$ this is a new scalar conformal invariant.

As a component of the curvature tensor $\widetilde{R}$, the Bach tensor B enters only when $n \neq 4$, i.e. when B is not itself conformally invariant. However when $n = 4$, B enters naturally in the context of the ambient metric; namely as the obstruction to the existence of a formal power series solution for $\widetilde{g}$. This sheds some light on the relation between the Bach tensor and conformally Einstein metrics when $n = 4$, for if a conformal structure has an Einstein representative, then the equation $\operatorname{Ric}(\widetilde{g}) = 0$ reduces to an ordinary differential equation which can be seen to have formal power series solutions. Hence the Bach tensor must vanish. It turns out that there is an obstruction of exactly the same nature in any even dimension. Precisely, PROPOSITION 3.5. — *(n even.) There is a nontrivial symmetric trace-free 2-tensor which is conformally invariant of weight $(n-2)/2$, and involves derivatives of the metric through order n. This tensor is the obstruction to the existence of a formal power series solution for the ambient metric $\widetilde{g}$ in the sense that there is an infinite order formal solution for $\widetilde{g}$ above an open set in M if and only if this tensor vanishes in that set. It always vanishes for metrics which are conformally Einstein and when $n = 4$ it is the Bach tensor.*

One can now play the game of choosing all kinds of Riemannian invariant constructions on $\widetilde{G}$ and trying to interpret them suitably to make them pass to conformal invariants below. In this spirit, we indicate here how the conformally invariant Laplacian arises naturally from the Laplace-Beltrami operator for the ambient metric. Recall that the conformally invariant Laplacian is the differential operator

$$\Delta - \frac{n-2}{4(n-1)}S,$$

where $\Delta f = g^{ij}\nabla_i\nabla_j f$ is the usual Laplace-Beltrami operator for a Riemannian metric g. This operator is conformally invariant in the sense that if $g' = \mu g$ is a conformally related metric, then

$$\text{(3.6)} \qquad \Big(\Delta' - \frac{n-2}{4(n-1)}S'\Big)\Big(\mu^{-\frac{n-2}{4}}f\Big) = \mu^{-\frac{n+2}{4}}\Big(\Delta - \frac{n-2}{4(n-1)}S\Big)f.$$

This can be invariantly stated by letting G^* be the ray bundle dual to G, with fiber

$$G_x^* = \big\{g^* : G_x \to \mathbf{R}_+ \mid g^*(\lambda g) = \lambda g^*(g), \lambda > 0\big\},$$

and defining the G-valued α-densities by

$$|G|^\alpha = \big\{D : G^* \to \mathbf{R} \;\big|\; D(\lambda g^*) = \lambda^\alpha D(g^*),\ \lambda > 0\big\}, \quad \alpha \in \mathbf{R}.$$

Then the transformation law (3.6) implies that if

$$\gamma = \frac{n-2}{4} \quad \text{and} \quad D = f|g|^\gamma \in |G|^\gamma$$

for some representative (g) of $[g]$ and $f \in C^\infty(M)$, the definition

$$\Delta^c D = |g|^{\gamma+1}\Big(\Delta - \frac{n-2}{4(n-1)}S\Big)f$$

is independent of the decomposition of D as $D = f|g|^\gamma$ so invariantly defines the conformally invariant Laplacian $\Delta^c : |G|^\gamma \to |G|^{\gamma+1}$. Notice that by duality $D \in |G|^\alpha$ can also be thought of as a homogeneous function D^* of degree $-\alpha$ on $G : D^*(g) = D(g^*)$, where $g^* \in G^*$ is determined by the condition $g^*(g) = 1$. Thus in terms of the dilations δ_s, densities $D \in |G|^\alpha$ are represented as homogeneous functions on G of degree -2α.

Now the Laplace-Beltrami operator $\widetilde{\Delta}$ for the ambient metric is invariantly defined on scalar functions on $\widetilde{G}$, and to make conformally invariant sense of it the homogeneity must be respected, so one is led to compute $\widetilde{\Delta}h$, where h is homogeneous on $\widetilde{G}$ of degree β with respect to δ_s. It turns out that for exactly one value of β, namely $\beta = -2\gamma, \widetilde{\Delta}h|_G$ depends only upon $h|_G$. Furthermore, in this case $\widetilde{\Delta}h$ is homogeneous with respect to δ_s of degree $-2(\gamma+1)$, and under the identification above between homogeneous functions on G and G-valued densities, the induced operator is exactly Δ^c.

It is tempting to speculate that other conformally invariant differential operators can be generated by this method; for example that the Laplacian on forms for the ambient metric gives rise to the conformally invariant operators on form-densities discovered by BRANSON [4]. Perhaps new conformally invariant objects can be found this way as well.

IV. Connection with Several Complex Variables

When the conformal structure under study is that on the circle bundle over the boundary of a strictly pseudoconvex domain in $\mathbf{C}^n$, the construction of the ambient metric was already carried out in [11], and in this case Einstein's equation $\mathrm{Ric}(\widetilde{g}) = 0$ reduces to a complex Monge-Ampère equation. In fact, knowledge of this special case was an important guide in guessing the existence of the ambient metric for a general conformal structure.

Actually, the conformal structure on the circle bundle was originally defined in [11] in terms of the ambient metric! Let $\Omega \subset\subset \mathbf{C}^n$ be smooth and strictly pseudoconvex (all considerations are local near a piece of $b\Omega$); then the relevant complex Monge-Ampère problem is to find a function $u > 0$ in Ω satifsfying $J(u) = 1$ with $u|_{b\Omega} = 0$, where

$$J(u) = (-1)^n \det \begin{pmatrix} u & u_{\bar{j}} \\ u_i & u_{i\bar{j}} \end{pmatrix} \qquad \text{and} \qquad u_i = \frac{\partial u}{\partial z^i}, \text{ etc.}$$

Associated to a solution u of this problem is a Kähler-Lorentz metric on $\mathbf{C}^* \times \Omega$, namely

$$ds^2 = \sum_{i,j=0}^{n} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} (- |z^0|^2 u(z)) dz^i d\bar{z}^j,$$

where $(z^0, z) \in \mathbf{C}^* \times \Omega$ and $\mathbf{C}^* = \mathbf{C} - \{0\}$. The minus sign is inserted so that this Hermitian metric has signature $(n, 1)$. One checks that ds^2 restricts to

$$M = S^1 \times b\Omega = \{(z^0, z) : |z^0| = 1, z \in b\Omega\}$$

to be non-degenerate with signature $(2n - 1, 1)$, and this restriction is by definition a representative g of the conformal structure $[g]$ on M. Clearly only the 2-jet of u along $b\Omega$ is of importance in the definition of $|g|$, and it can be shown that this 2-jet is independent of which solution of the Monge-Ampère problem was chosen and can even be computed by using not an exact solution to $J(u) = 1$ but only a solution to second order at $b\Omega$. Additionally it is locally determined by $b\Omega$. Thus $[g]$ also has these properties.

We next show that ds^2 is the ambient metric for $[g]$. First,

$$G = \mathbf{R}_+ \times M = \mathbf{R}_+ \times S^1 \times b\Omega \cong \mathbf{C}^* \times b\Omega,$$

and $\widetilde{G}$ can be identified with $\mathbf{C}^* \times \widetilde{\Omega}$, where $\widetilde{\Omega}$ is a collar neighborhood of $b\Omega$ in $\mathbf{C}^n$. Clearly ds^2 is homogeneous of degree 2, and its restriction to $G = \mathbf{C}^* \times b\Omega$ is g_0, by homogeneity and the fact that ds^2 restricts to the distinguished representative g when $|z^0| = 1$. Finally, by an identity from Kähler calculus, the components of the Ricci tensor $R_{i\bar{j}}$ of ds^2 are given by

$$r_{i\bar{j}} = -(\log(\det(-|z^0|^2 u(z))_{k\bar{l}}))_{i\bar{j}},$$

and since

$$\det(-|z^0|^2 u(z))_{k\bar{l}} = -|z^0|^{2n} J(u),$$

it follows that $\mathrm{Ric}(ds^2) = 0$. Thus ds^2 satisfies conditions 1) – 3) defining the ambient metric.

In this setting our results about and applications of the ambient metric are already familiar. The circle bundles $M = S^1 \times b\Omega$ all have even dimension, and the failure of the construction of formal smooth solutions to the complex Monge-Ampère equation has been known for several years [11]. Also our construction of scalar invariants from the ambient metric is exactly the procedure used in [12] to construct CR invariants. The unfortunate side of this is that we have obtained no new information or results about CR invariants as a consequence of this work.

There is a second interpretation of the complex Monge-Ampère equation which relates it to Poincaré metrics. Namely, a solution to $J(u) = 1, u|_{b\Omega} = 0$ has the property that the metric

$$d\sigma^2 = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial z^i \partial \bar{z}^j}\left(\log \frac{1}{u}\right) dz^i d\bar{z}^j$$

on Ω is a Kähler-Einstein metric of constant negative Ricci curvature (see [6]). We describe here the relationship of such a Cheng-Yau metric to the Poincaré metric for the conformal structure $[g]$ on $M = S^1 \times b\Omega$.

Recall that the ambient space $\widetilde{G}$ for $[g]$ can be identified with $\mathbf{C}^* \times \widetilde{\Omega}$— as we are interested now in the one-sided Poincaré metric and have access to the result of CHENG-YAU asserting the global existence of a solution of the complex Monge-Ampère equation, we focus attention on $\widetilde{G}_- \cong \mathbf{C}^* \times \Omega$. Using polar coordiantes $z^0 = re^{i\theta}$,

$$\widetilde{g} = ds^2 = \sum_{i,j=0}^{n} \frac{\partial^2}{\partial z^i \partial \bar{z}^j}(-|z^0|^2 u(z)) dz^i d\bar{z}^j$$

can be written

$$\widetilde{g} = -u(dr^2 + r^2 d\theta^2) - r dr du - r^2 d\theta d^c u - r^2 \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} dz^i d\bar{z}^j,$$

where $d^c u = i(\bar{\partial} - \partial)u$. Now the conformal Poincaré metric g^+ lives on $\widetilde{G}_-/\mathbf{R}_+ = S^1 \times \Omega$, and as described in § II, can be obtained by restricting $\widetilde{g}$ to the surface $S = \{\|T\|^2 = -1\}$, where T is infinitesimal dilation. In our coordinates $T = r(\partial/\partial r)$, so $S = \{ur^2 = 1\}$. Setting $r = u^{-1/2}$ in $\widetilde{g}$ and simplifying one finds that in these coordinates the Poincaré metric g^+ is given by

$$g^+ = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial z^i \partial \bar{z}^j}\left(\log \frac{1}{u}\right) dz^i d\bar{z}^j - \omega^2,$$

where

$$\omega = d\theta - \frac{1}{2} d^c \log \frac{1}{u}.$$

We view ω as a connection 1-form on the S^1-principal bundle $S^1 \times \Omega$, hence defining horizontal lifts of vector fields on Ω. Via this horizontal lifting, the Einstein metric g^+ on $S^1 \times \Omega$ thus induces the Kähler-Einstein metric $d\sigma^2$ on Ω, the curvature of ω meanwhile being $d\omega = -i\partial\bar{\partial}\log(1/u)$, a multiple of the Kähler form of $d\sigma^2$.

V. About the Proof of Theorem 2.1

In this section some of the ingredients in the proof of THEOREM 2.1 are discussed. The first difficulty is to deal with the gauge invariance of the problem. This is a familiar matter in the study of problems involving Einstein's equations; see [8] for an overview of results concerning noncharacteristic Cauchy problems for the Ricci curvature operator. We adopt the common technique of breaking the gauge invariance by introduction of special coordinate systems.

The special coordinate systems on $\widetilde{G}$ depend first on a representative

$$g = \sum_{i,j=1}^{n} g_{ij}(x)dx^i dx^j$$

for the conformal structure on M; $\{x\}$ here is any coordinate system on M. As previously described, g determines a fiber variable t on G so that

$$g_0 = t^2\pi^* g = t^2 \sum g_{ij}(x)dx^i dx^j.$$

If ρ is any function on $\widetilde{G}$ which is homogeneous of degree 0 and vanishes to first order on G, then by the defining conditions 1) and 2), any ambient metric $\widetilde{g}$ must at $\rho = 0$ have the form

$$\widetilde{g} = t^2 \sum g_{ij}(x)dx^i dx^j + \eta \cdot d\rho,$$

where η is a 1-form, homogeneous of degree 2, satisfying additionally

$$\eta\left(\frac{\partial}{\partial t}\right) \neq 0$$

by the nondegeneracy of $\widetilde{g}$. It is easily seen that the coordinates can be initially normalized so that $\eta = 2tdt$. Now the coordinate system is extended off of G by following the geodesics of $\widetilde{g}$ in the direction of $\partial/\partial\rho$. Thus one sees that every ambient metric $\widetilde{g}$ gives rise to a special coordinate system $(t, x\rho)$ in which

$$\widetilde{g} = 2tdtd\rho + t^2 \sum g_{ij} dx^i dx^j$$

at $\rho = 0$ and $\rho \to (t, x, \rho)$ are geodesics for $\widetilde{g}$. Hence THEOREM 2.1 reduces to an analysis of the following coordinate-dependent problem.

PROBLEM 5.1.

Given a metric $g = \sum_{i,j=1}^{n} g_{ij}(x)dx^i dx^j$ on $D^{\text{open}} \subset \mathbf{R}^n$, find a metric $\widetilde{g}$ near

$$\{(t, x, \rho) : t > 0, x \in D, \rho = 0\} \subset \mathbf{R}^{n+2}$$

satisfying :

a) $\widetilde{g}$ *is homogeneous of degree 2 in* t

b) $\widetilde{g} = t^2 \sum g_{ij}(x)dx^i dx^j + 2tdt \cdot d\rho$ *at* $\rho = 0$

c) $\rho \longrightarrow (t, x, \rho)$ *are geodesics for* $\widetilde{g}$

d) $\mathrm{Ric}(\widetilde{g}) = 0$.

By the reduction above, uniqueness for PROBLEM 5.1 implies uniqueness up to diffeomorphism in THEOREM 2.1.

Condition c) immediately translates into something more concrete. The ordinary differential equations defining geodesics reduce, by c), to first order partial differential equations for the coefficients of $\widetilde{g}$, which in conjunction with the initial condition b), can be integrated to give

$$\text{c}') \qquad \widetilde{g}\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial \rho}\Big) = t \quad \widetilde{g}\Big(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial \rho}\Big) = \widetilde{g}\Big(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \rho}\Big) = 0 \text{ all } \rho.$$

Hence one concludes that

$$\widetilde{g} = t^2 \sum_{i,j=1}^{n} g_{ij}(x,\rho)dx^i dx^j + \Big(a(x,\rho)dt + 2t\sum_{j=1}^{n} b_j(x,\rho)dx^j\Big) \cdot dt + 2tdtd\rho,$$

and the unknowns are g_{ij}, a, b_j with initial conditions

$$g_{ij}(x,0) = g_{ij}(x), a(x,0) = b_j(x,0) = 0.$$

Now a) – c) have been insured, leaving one with the dirty work of analyzing d). Since the restriction of $\widetilde{g}$ to the initial hypersurface $\rho = 0$ is degenerate, it follows that this Cauchy problem is characteristic; otherwise the Cauchy-Kowalewski Theorem would complete the story. This is also the reason that one can get away with prescribing only one piece of Cauchy data for a second order problem. At this point things get somewhat gory — one must do an inductive perturbation calculation to extend a solution of order s to order $s+1$, and thus must calculate the Ricci curvature of a metric of the special form of $\widetilde{g}$ well enough to carry this out. A complication is the following : at this stage the problem seems to be overdetermined, since the unknowns are g_{ij}, a, b_j, amounting to a symmetric 2-tensor on a space of dimension $n+1$, while the number of equations in $\mathrm{Ric}(\widetilde{g}) = 0$ is that of a symmetric 2-tensor on $\mathbf{R}^{n+2}$. The trick, again familiar from previous studies of Einstein's equations, is to recall that the Ricci curvature itself satisfies some partial differential equations — the contracted Bianchi identities. So one does not have to make all components of $\mathrm{Ric}(\widetilde{g})$ vanish, but only enough to solve for the unknowns g_{ij}, a, b_j. However, for particular s the equations degenerate so that the components one chooses to make vanish as well as the role of the Bianchi identities depend on s. The conclusion of this analysis for n odd is

THEOREM 5.2. — *(n odd.) There is a unique formal power series solution to Problem 5.1.*

It turns out that the solution $\tilde{g}$ takes an even more special form. Namely, $a = 2\rho$ and $b_j = 0$, so that

$$\tilde{g} = t^2 \Sigma g_{ij}(x,\rho) dx^i dx^j + 2\rho dt^2 + 2t dt d\rho.$$

Geometrically, this has the interpretation that the parameterized $\mathbf{R}_+$-orbits $s \to (st, x, \rho)$ are geodesics. It would be a substantial simplification if one knew this ahead of time, but we know of no way to derive this condition without actually proving THEOREM 5.2.

The equations being analyzed amount to a nonlinear Fuchsian system in several variables. In fact, in terms of the remaining unknowns $g_{ij}(x,\rho)$ in our reduced form for $\tilde{g}$, the equation can be written

$$\rho g_{ij,\rho\rho} - \rho g^{kl} g_{ik,\rho} g_{jl,\rho} + \frac{1}{2} \rho g^{kl} g_{kl,\rho} g_{ij,\rho} + \frac{2-n}{2} g_{ij,\rho} - \frac{1}{2} g^{kl} g_{kl,\rho} g_{ij} + \mathrm{Ric}_{ij}(g) = 0.$$

Here $\mathrm{Ric}_{ij}(g)$ refers to the Ricci curvature operator acting in the x-variables alone on the metric

$$\sum_{i,j=1}^{n} g_{ij}(x,\rho) dx^i dx^j$$

with fixed ρ, and $g_{ij,\rho} = (\partial/\partial\rho) g_{ij}$, etc. Letting $r = (n(n+1)/2)$ be the number of unknowns, this Fuchsian system has as characteristic exponents 0 with multiplicity r, $n/2$ with multiplicity $r-1$, and n with multiplicity 1. The characteristic exponent $n/2$ gives rise to the difference in even and odd dimensions — it is irrelevant on the formal power series level for n odd, but causes log terms for n even. The trickiest part of the proof of THEOREM 5.2 comes in showing that the characteristic exponent n does not give rise to log terms, and necessitates the previously mentioned interplay in the roles of the Bianchi identity and the Ricci curvature equation.

BAOUENDI-GOULAOUIC [3] have proved convergence of formal solutions of certain Fuchsian systems in the case in which no log terms enter. Thus once THEOREM 5.2 has been established, convergence follows from their work after one Taylor expands the solution beyond the last characteristic exponent and makes a change of variables so as to cast the problem in the framework of [3].

Of course there is a finite-order analogue of THEOREM 5.2 when n is even, from which THEOREM 2.1 b) follows. More conformally invariant information is carried in the log terms and higher asymptotics, although how to fully utilize this information is not yet evident. This is a good problem for future study. In particular, can all scalar conformal invariants (or even a conjecture for all) be constructed from an infinite order ambient metric $\tilde{g}$? For some results in this direction for the complex Monge-Ampère equation, see [14]. Another question is to prove convergence of the expansions with log terms in the analytic case. This is open even for the complex Monge-Ampère equation.

Of course, an obvious problem is to do the invariant theory and thus prove Conjecture 3.1. Initial investigations indicate that the case of low weight invariants is on the same order of difficulty as the invariant theory in [12], and that the higher weight case is substantially more complicated.

Then there is the original problem of constructing scalar CR invariants and analyzing the log term of the Bergman kernel. Our best guess here is, as above, to study the higher asymptotics of the ambient metric, although it is possible that other techniques could be used. Further insight into the relationship between conformal and CR geometry has been recently provided in the work of SPARLING [20] characterizing intrinsically those conformal structures which arise on circle bundles over CR manifolds.

All of the considerations of this work have been purely local. We close by posing a global problem of interest both for its own sake as a boundary value problem for a nonlinear Fuchsian system, and for potential geometric application. The problem is just the global version of the existence and uniqueness of the Poincaré metric. Precisely, let N^{n+1} be a manifold with boundary and suppose given a positive definite conformal structure $[g]$ on bN. Prove the existence, uniqueness up to diffeomorphism, and boundary regularity of a positive definite Riemannian metric g^+ on $N \sim bN$ having $[g]$ as conformal infinity and satisfying Einstein's equations $\mathrm{Ric}(g^+) = -ng^+$. This problem is the analogue in conformal geometry of the Cheng-Yau [6] metric in biholomorphic geometry, whose higher boundary asymptotics were established by LEE-MELROSE [17]. One expects that for n odd, $r^2 g^+ \in C^\infty(\overline{N})$, where r is a defining function for bN, while for n even g^+ will have a logarithmic asymptotic expansion at bN.

Note. — Since the above was written, we have discovered that in 1936 SCHOUTEN and HAANTJES [19] studied the problem of locally embedding a manifold M^n with conformal structure $[g]$ into a $(n+1)$-dimensional space with a projective structure satisfying certain conditions, so that the projective structure induces the given conformal structure on M. As they formulate it, this problem is equivalent to the problem of finding the ambient metric $\tilde{g}$, and versions of our THEOREM 2.1 and PROPOSITION 3.5 are contained in their work. SCHOUTEN-HAANTJES were apparently interested in these results for their own sake and did not consider applications to conformal invariants.

The formulation of SCHOUTEN-HAANTJES is of course also motivated by the model case of S^n, whose conformal geometry is induced from the projective geometry of $\mathbf{R}^{n+1}$ as we previously described. More generally, an embedding of M into a space with a projective structure can be realized in terms of our construction as follows. Even though the Poincaré metric g^+ is singular at the boundary it can be shown that the projective structure which it determines extends smoothly up to the boundary in the coordinates (x, ρ) used in (2.4). Additionally, this projective structure induces the given conformal structure $[g]$ on the boundary M, so that M has been embedded in the manner required.

REFERENCES

[1] ATIYAH (M.), BOTT (R.) and PATODI (V.K.). — On the heat equation and the index theorem, *Inv. Math.*, t. **19**, 1973, p. 279–330.

[2] BACH (R.). — Zur Weylschen Relativitätstheorie und der Weylschen Erweiterung des Krümmungstensorbegriffs, *Math. Z.*, t. **9**, 1921, p. 110–135.

[3] BAOUENDI (M.S.) and GOULAOUIC (C.). — Singular nonlinear Cauchy problems, *J. Diff. Eq.*, t. **22**, 1976, p. 268–291.

[4] BRANSON (T.). — Conformally covariant equations on differential forms, *Comm. in P.D.E.*, t. **7**, 1982, p. 393–431.

[5] BURNS (D.), DIEDERICH (K.) and SHNIDER (S.). — Distinguished curves in pseudoconvex boundaries, *Duke Math. J.*, t. **44**, 1977, p. 407–431.

[6] CHENG (S.-Y.) and YAU (S.-T.). — On the existence of a complete Kähler metric on non-compact complex manifolds and the regularity of Fefferman's equation, *Comm. Pure Appl. Math.*, t. **33**, 1980, p. 507–544.

[7] COTTON (E.). — Sur les variétés à trois dimensions, *Ann. Fac. Sc. Toulouse*, t. **1**, 1899, p. 385–438.

[8] DETURCK (D.). — The Cauchy problem for Lorentz metrics with prescribed Ricci curvature, *Comp. Math.*, t. **48**, 1983, p. 327–349.

[9] DIGHTON (K.). — An introduction to the theory of local twistors, *Inf. J. Theoretical Physics*, t. **11**, 1974, p. 31–43.

[10] DUPLESSIS (J.C.). — Polynomial conformal tensors, *Proc. Camb. Phil. Soc.*, t. **68**, 1970, p. 329–344.

[11] FEFFERMAN (C.). — Monge-Ampère equations, the Bergman kernel, and geometry of pseudoconvex domains, *Ann. Math.*, t. **103**, 1976, p. 395–416.

[12] FEFFERMAN (C.). — Parabolic invariant theory in complex analysis, *Adv. Math.*, t. **31**, 1979, p. 131–262.

[13] GOLDBERG (S.). — *Curvature and Homology*. — New York, Academic Press, 1962.

[14] GRAHAM (R.). — Higher asymptotics of the complex Monge-Ampère equation and scalar boundary invariants of strictly pseudoconvex domains, in preparation.

[15] KOBAYASHI (S.). — *Transformation Groups in Differential Geometry*. — Berlin, Springer-Verlag, 1972.

[16] LEBRUN (C.). — $\mathcal{H}$-Space with a cosmological constant, *Proc. Roy. Soc. London Ser. A*, t. **380**, 1982, p. 171–185.

[17] LEE (J.) and MELROSE (R.). — Boundary behaviour of the complex Monge-Ampère equation, *Acta Math.*, t. **148**, 1982, p. 159–192.

[18] PENROSE (R.) and MACCALLUM (M.A.H.). — Twistor theory : an approach to the quantisation of fields and space-time, *Physics Reports*, t. **6C**, 1973, p. 241–316.

[19] SCHOUTEN (J.A.) and HAANTJES (J.). — Beiträge zur allgemeinen (gekrümmten) konformen Differentialgeometrie I, II, *Math. Ann.*, t. **112**, 1936, p. 594–629, t. **113**,1936, p. 568–583.

[20] SPARLING (G.). — Twistor theory and the characterization of Fefferman's conformal structures, preprint.

[21] SPARLING (G.). — Personal communication.

[22] SZEKERES (P.). — Conformal tensors, *Proc. Roy. Soc. London Ser. A.*, t. **304**, 1968, p. 113–122.

Charles FEFFERMAN,
Department of Mathematics,
Princeton University,
Princeton, N.J. 08544, U.S.A.

C. Robin GRAHAM,
Department of Mathematics,
University of Washington,
Seattle, WA. 98195, U.S.A.