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FAMILIES OF DIRAC OPERATORS WITH APPLICATIONS TO PHYSICS

BY

I.M. SINGER*

1. Introduction

This celebration of Élie CARTAN's work attests to the pervasiveness of his ideas in modern geometry, Lie groups, infinite groups, and systems of partial differential equations. In physics, geometric ideas are an integral part of mechanics, electromagnetism, and relativity. And symmetry groups have illuminated much of quantum mechanics and elementary particle physics. Currently, the set of connections on a principal bundle (vector potentials or Yang-Mills fields) is a fruitful unifying principal in physics. The quantization of such fields, particularly when the group of the bundle is non-abelian, presents serious difficulties and is not well understood. Some aspects of these problems, however, do lend themselves to a mathematical treatment, and in this talk I will try to give an exposition, for mathematicians, of one such topic-chiral anomalies. My remarks are based on joint work with several authors [1,2,4] and I am indebted to many physicists for illuminating discussions on the problems of quantizing gauge fields, particularly D. FRIEDAN, R. JACKIW and E. WITTEN.

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2. The Space of Connections

Let P denote a principal G bundle with base M, a connected compact oriented Riemannian spin manifold. We assume G is a compact Lie group. The special cases of $M = S^n$ and $G = SU(N)$ are of particular interest. Let $\mathcal{A}$ denote the set of all connections on P. An element $A \in \mathcal{A}$ is a 1-form on P with values in the Lie algebre $\mathbf{g}$ of G, equivariant under the action of G on P and $\mathbf{g}$. Let $F(A)$ denote the curvature 2-form of A and let $S(A)$ be the L^2-norm square of $F(A)$:

$$S(A) = \|F(A)\|^2 = \int_M |F(A)|^2$$

with $|F(A)|$ the norm of $F(A)$ considered as an element of $\mathbf{g} \otimes (\Lambda^2)$, which has an inner product inherited from $\mathbf{g}$ and the metric on M. We have chosen an G-invariant metric on $\mathbf{g}$ (unique up to a scale factor if G is simple).

Let $\mathcal{H}$ denote the group of automorphisms of P, i.e., the subgroup of diffeomorphisms commuting with the action of G. Since $\mathcal{H}$ preserves fibers, there is a homorphism $\varphi \to \overline{\varphi}$ of $\mathcal{H}$ into $\mathrm{Diff}(M)$, the group of diffeomorphisms on M. The kernel of this homomorphism is called the group of gauge transformation and is denoted by $\mathcal{G}$. It is the set of automorphisms of P leaving each fiber fixed. When $P = M \times G$, the product bundle, $\mathcal{G}$ can be identified with $[\varphi : M \to G]$. If G acts on the right on P, $p \to pg^{-1}$ then $\varphi \to \mathcal{G}$ acts on the left :

$$p = (m, g) \to \tilde{\varphi}(p) = (m, \varphi(m)g)$$

where $\tilde{\varphi}$ is the automorphism induced by the function $\varphi : M \to G$. Since vector fields on M can be lifted to P, the image of $\mathcal{H}$ in $\mathrm{Diff}(M)$ contains the identity component $\mathrm{Diff}_0(M)$.

$\mathcal{A}$ is an affine space, since the convex combination of two connections is a connection. Choosing a fixed connection B makes $\mathcal{A}$ into a linear space, i.e. $\mathcal{A} = B + \Lambda^1 \otimes \mathbf{g}$ where $\Lambda^1 \otimes \mathbf{g}$ is the set of 1-forms on P with values in $\mathbf{g}$, equivariant under G and zero on the fibers of P. When P is the product bundle explicitly displayed as a product $M \times G$, it is natural to choose B as the flat connection of the product decomposition.

The action of $\mathcal{H}$ on P induces an action of $\mathcal{H}$ on $\mathcal{A}$ which we denote by $\varphi \cdot A$. It has the following property. If γ is a smooth curve in M from m_1 to m_2 and if $P_\gamma(A)$ denotes parallel transport from m_1 to m_2 (as a map from the fiber F_{m_1} of P to F_{m_2}). Then $P_{\overline{\varphi}\cdot\gamma}(\varphi \cdot A) = \varphi P_\gamma(A)\varphi^{-1}$. Here $\overline{\varphi} \cdot \gamma$ is the transform of γ under the diffeomorphism $\overline{\varphi}$ induced by φ. We conclude :

If $\varphi \in \mathcal{G}$ with $\varphi(p) = p$ for some $p \in P$ and $\varphi \cdot A = A$ for some $A \in \mathcal{A}$, then $\varphi = I \in \mathcal{G}$.

For, under the hypothesis, each curve beginning at m gives $P_\gamma(A) = \varphi P_\gamma(A)$; thus φ is the identity on the fiber over the end point of γ. Since M is arcwise connected, $\varphi = I$ everywhere. Let $\mathcal{G}_m$ denote the subgroup of $\mathcal{G}$ whose restriction to F_m is the identity. Our remark proves that $\mathcal{G}_m$ acts on A without fixed points. One can show that $\mathcal{A}$ is a principal bundle with group $\mathcal{G}_m$ over the orbit space $\mathcal{A}/\mathcal{G}_m$. One can also show that when $M = S^r$ or when $\dim M \leq 4$, that $\mathcal{G}_m$ is weakly homotopic to $[\varphi : M \to G;\ \varphi(m) = I]$. More generally, this holds when the obstruction to P being a product bundle can be localized to a point [22]. We have the exact sequence $1 \to \mathcal{G}_m \to \mathcal{G} \to G \to 1$, where $\mathcal{G} \to G$ is restriction of $\mathcal{G}$ to F_m and G is identified with the automorphisms of F_m. Finally, note that $F(\varphi A) = \varphi^{-1} F(A)\varphi$ and $S(\varphi \cdot A) = S(A)$.

It is deceptively simple to state the main problem in the quantization of gauge theories. Let f be a function on $\mathcal{A}$ invariant under gauge transformations. Evaluate

$$\frac{1}{n}\int f(A)\exp(-\|F(A)\|^2)\mathcal{D}A \qquad \text{where} \qquad n = \int \exp(-\|F(A)\|^2\mathcal{D}A.\dagger$$

If such integrals can be understood, i.e., evaluated, then there are standard procedure for deducing the physical consequences of the theory : the scattering matrix, masses, etc. for them. The problem is to make sense of the integral. There are two hidden parameters in the integral, the scale factor α in the invariant inner product on the Lie algebra $\mathbf{g}$, and the volume of the Riemannian manifold M. One interpretation of the integral is a perturbative one. Replace $\|F(A)\|^2$ in the integral by $\frac{1}{\alpha^2}\|F(A)\|^2$ and expand formally in α. The coefficients in the expansion can be infinite and these "infinities" are successfully removed by the renormalization program.

3. The Dirac Operator Coupled to Vector Potentials

Let S denote the complex spin bundle over M and let $\partial\!\!\!/$ denote the Dirac operator on C^∞ sections of S. If $\{e_j\}_{j=1}^n$ is a local orthonormal frame field, then $\partial\!\!\!/ = \sum_{j=1}^n \gamma_j D_{e_j}$ where γ_{e_j} is Clifford multiplication by e_j and D_{e_j} is the Riemannian covariant differential in the e_j direction. If $n = \dim M$ is even, then $S = S_+ \oplus S_-$ where $S_\pm$ are the ± 1 (or $\pm i$) eigenvalues of $\gamma_1 \cdot \gamma_2 \ldots \gamma_{n'}$, the spinors of positive and negative chirality respectively. Now $\partial\!\!\!/ = \partial\!\!\!/(+)^* + \partial\!\!\!/(+)$ where

$$\partial\!\!\!/(+) : C^\infty(S_+) \to C^\infty(S_-).$$

† See for example L.D. FADEEV and A.A. SLAVNOV, Introduction to Quantum Field Theory, Benjamin/Cummings (1980) or C. ITZYKSON end J. ZUBER, Quantum Field Theory, McGraw Hill (1980).

Suppose r is a unitary representation of G on $\mathbf{C}^N$ (usually the identity representation when $G = SU(N)$). Let $E = P \underset{G}{\times} C^N$ be the associated vector bundle; for each $A \in$ we have the covariant differential

$$D(A) : C^\infty(E) \to C^\infty(E \otimes \Lambda^1).$$

Now define the Dirac operator coupled to $A, \not{\partial}_A$, mapping

$$C^\infty(S \otimes E) \to C^\infty(S \otimes E)$$

as the covariant differential

$$D \otimes I + I \otimes D_A : C^\infty(S \otimes E) \to C^\infty(S \otimes E \times \Lambda^1)$$

followed by Clifford multiplication. In a local frame,

$$\not{\partial}_A = \sum (\gamma_j \otimes I)(D_{e_j} \otimes I + I \otimes D_j(A)).$$

It is not hard to show that

$$\not{\partial}_{tA_1+(1-t)A_0} = t\not{\partial}_{A_1} + (1-t)\not{\partial} A_0$$

for $A_0, A_1 \in \mathcal{A}$ and t a real number. If $\tau \in C^\infty(\Lambda^1 \otimes \mathbf{g})$, then $\not{\partial}_{A+\tau} = \not{\partial}_A + \not{\tau}$ where

$$\not{\tau} = \sum_{i=1}^{n} \gamma_j \otimes dr(\tau_j)$$

for an orthonormal frame $\{e_j\}_{j=1}^m$. Also, $\not{\partial}_{\varphi \cdot A} = \widehat{\varphi}^{-1} \not{\partial}_A \widehat{\varphi}$ where $\widehat{\varphi} = I \otimes r \circ \varphi$ acting only on the second factor in $S \otimes E$, and $\varphi \in \mathcal{G}$. Put another way, sections of E are equivariant functions from P to $\mathbf{C}^N$, and $\mathcal{G}$ transforms the class to itself.

Because the Dirac operator depends on the metric ρ of M, it is not covariant under $\varphi \in \mathcal{H}$, for the diffeomorphism $\overline{\varphi}$ need not be an isometry. Let $\varphi \cdot \rho$ be the metric ρ transformed by $\overline{\varphi}$ and let $\not{\partial}_{\rho,A}$ indicate the dependence of the Dirac operator on ρ as well as A. Suppose $\overline{\varphi} \in \text{Diff}(M)$ can be covered by a transformation φ' of the spin bundle relative to ρ, mapping $S(\rho)$ to $S(\overline{\varphi} \cdot \rho)$. Then $\not{\partial}_{\varphi \cdot \rho, \varphi \cdot A} = \widetilde{\varphi}^{-1} \not{\partial}_{\rho,A} \widetilde{\varphi}$ where now $\widetilde{\varphi} = \varphi' \otimes r \circ \varphi$.

When gauge fields are coupled to gravity, one is led to the study of $\mathcal{H}$, the automorphism group of P and to the family of operators $\not{\partial}_{\rho,A}$. One can either vary metrics, vector potentials or both [2]. We focus mainly on a fixed metric and variable A in this paper.

One of the functions f whose integral one wants to evaluate — in the sense of the previous section — is the determinant of $\not{\partial}_A, \det \not{\partial}_A$. Shortly, we will discuss determinants of elliptic operators. For the moment, let us assume it is well defined. What is needed is gauge invariance, to be expected since $\not{\partial}_{\varphi \cdot A} = \widetilde{\varphi}^{-1} \not{\partial}_A \widetilde{\varphi}$. When $\not{\partial}_A$ operates on all spinors, invariance is indeed the case. However, in some physical theories, the relevant operator is

$$\not{\partial}_A(+) : C^\infty(S^+ \otimes E) \to C^\infty(S^- \otimes E).$$

Since $\not{\partial}_A(+)$ maps one space to another, the meaning of the determinant is unclear. It turns out one cannot in general define a gauge invariant determinant when G is non-abelian; the obstruction to doing so is called a chiral anomaly. We will interpret it as an element of $H^1(\mathcal{G}_m, Z)$.

4. Determinants

We use the zeta function method to define determinants [13, 19]. If L is a positive definite self-adjoint elliptic operator on sections of a vector bundle over a compact manifold, then L has pure point spectrum $\{\lambda_j\}$. One notes that purely formally

$$\frac{d}{ds}\bigg|_{s=0} \left(\sum \lambda_j^{-s}\right) = -\sum \log \lambda_j.$$

Let $\varsigma_L(s) = \operatorname{tr}(L^{-s})$. One can show [20] that $\varsigma_L(s)$ (i) is holomorphic in a half plane $\Re s > \dim M/\text{order of } L$ (ii) has a meromorphic extension to the entire s-plane, and (iii) $s = 0$ is not a pole. It is natural, then, to define

$$\det L = \exp -\frac{d}{ds}\bigg|_{s=0} \varsigma_L(s),$$

where $\varsigma_L(s)$ denotes the meromorphic extensions as well.

Since

$$\varsigma_L(S) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{tr}(e^{-tL})\, dt = \frac{h(s)}{(1/s) + g(s)} = \frac{sh(s)}{1 + sg(s)},$$

$$\begin{aligned} \frac{d}{ds}\bigg|_{s=0} \varsigma_L(s) &= \frac{d}{ds}\bigg|_{s=0} (sh(s)) - sh(s)\bigg|_{s=0} g(0) \\ &= (h(s) - \varsigma_L(0)/s)\bigg|_{s=0} - \varsigma_L(0) g(0), \end{aligned}$$

where $g(0)$ is Euler's constant and

$$h(s) = \int_0^\infty t^{s-1} \operatorname{tr}(e^{-tL})\, dt.$$

So

$$-\log\det L\text{``}=\text{''}\int_0^\infty \operatorname{tr}(e^{-tL})\,dt/t-\varsigma_L(0)/s\,\Big|_{s=0}-\varsigma_L(0)g(0).$$

One can interpret this as meaning that $-\log\det L$ equals $\int_{\epsilon\downarrow 0}^\infty \operatorname{tr}(e^{-tL})dt/t$, where one has to subtract a pole as $\epsilon\to 0$.

A more heuristic approach is :

$$\log\det L=\operatorname{tr}\log L=-\frac{d}{ds}\Big|_{s=0}\operatorname{tr}(L^{-s}),$$

since $\log L=-\frac{d}{ds}|_{s=0}L^{-s}$. It is not hard to show that if $L=L_u$ is a smooth function of a parameter u, then

$$\text{(I)}\qquad \begin{aligned}\frac{d\det(L_u)}{du}/\det L_u&=-\frac{d}{du}\frac{d}{ds}\Big|_{s=0}\operatorname{tr}(L^{-s})\\&=\frac{d}{ds}\Big|_{s=0}\Big(s\operatorname{tr}\Big(L^{-(s+1)}\frac{dL}{du}\Big)\Big)\\&=\frac{d}{ds}\Big|_{s=0}\Big(s\operatorname{tr}\Big(L^{-s}\frac{dL}{du}/L\Big)\Big)\\&=\operatorname{tr}\Big(L^{-s}\frac{dL}{du}/L\Big)\Big|_{s=0}\end{aligned}$$

when the latter expression has no pole at the origin. Again, heuristically,

$$\frac{d}{du}\log\det L=\frac{d}{du}\operatorname{tr}\log L=\operatorname{tr}\Big(L^{-1}\frac{dL}{du}\Big).$$

The determinant can be extended to the case when L has a positive definite symbol, is invertible, but is not necessarily positive. Now all but a finite number of eigenvalues say $\lambda_1\dots\lambda_k$ lie in any positive cone about the positive axis [20]. For $j>k, \lambda_j^{-s}=e^{-s\log\lambda_j}$ is well defined using the cut along the negative real axes. One can define $\det L=\lambda_1\dots\lambda_k\det\widetilde{L}$ where

$$\log\det\widetilde{L}=-\frac{d}{ds}\Big|_{s=0}\sum_{j>\kappa}\lambda_j^{-s}.$$

It is easy to verify that det L is well defined, i.e. independent of the choice of k, and is smooth in u, if $L=L_u$ is. It is also easy to check that (I) holds in this case as well.

We are now able to discuss the determinant for the operators $\partial\!\!\!/_A(+) : C^\infty(S_+, \otimes E) \to C^\infty(S_- \otimes E)$. In the finite dimensional case, a linear transformation $T : V \to W$ gives $\det T : \Lambda^k(V) \to \Lambda^k(W)$ where $\dim V = \dim W = k$, so that $\Lambda^k(V)$ and $\Lambda^k(W)$ are one-dimensional spaces. If T depends on some parameter space X, and V, W are vector bundles over X, then $\det T$ gives a map from one line bundle $\Lambda^k(V)$ over X to a second line bundle $\Lambda^k(W)$ over X. If T is always nonsingular, the two line bundles over X are isomorphic via $\det T$. Otherwise, the ratio of these line bundles will be determined by the kernels and cokernels of T. This point of view will be exploited in section 7.*

The more prosaic approach in defining the determinant is to identify $C^\infty(S_- \otimes E)$ with $C^\infty(S_+ \otimes E)$ via a *fixed* operator $\partial\!\!\!/^*_B(+)$. Since we are interested in the determinant as a function on a fixed orbit, we choose a point A on the orbit and let $B = A$. We investigate the operators

$$\partial\!\!\!/^*_A(+)\partial\!\!\!/_{\varphi\cdot A}(+) = \partial\!\!\!/_A(-)\partial\!\!\!/_{\varphi\cdot A}(+)$$

which we denote by L_φ. The operators L_φ are elliptic with symbol $\xi \to |\xi|^2$ so that our previous discussion holds, providing $\partial\!\!\!/_A(+)$ and therefore $\partial\!\!\!/_{\varphi\cdot A}(+)$ is an isomorphism. In particular we must assume that index $\partial\!\!\!/_A(+) = 0$. As a consequence, for $M = S^{2n}$, P is a trivial bundle, and for generic $A \in \mathcal{A}$, $\partial\!\!\!/_A$ is an isomorphism. (There is no serious difficulty in the more general case. Suppose index $\partial\!\!\!/_A(+) > 0$. For generic $A \in \mathcal{A}$, $\ker\partial\!\!\!/_A(-) = 0$. Then on a fixed orbit $\ker\partial\!\!\!/_{\varphi\cdot A}(+) = \widetilde{\varphi}(\ker\partial\!\!\!/_A(+))$, a finite dimensional vector bundle on the orbit. L_φ is non-singular on the orthogonal compliment of $\ker\partial\!\!\!/_{\varphi\cdot A}(+)$, and a nonvanishing determinant is obtained.).

Consider then the function $\mathrm{DET} : \mathcal{G}_m \to \mathbf{C} - 0$ given by

$$\mathrm{DET}(\varphi) = \det L_\varphi = \det\partial\!\!\!/_A(-)\partial\!\!\!/_{\varphi A}(+) = \det(\partial\!\!\!/_A(-)\widetilde{\varphi}^{-1}\partial\!\!\!/_A(+)\widetilde{\varphi}).$$

An element of $H^1\mathcal{G}_m, Z)$ is obtained by pulling back the generator of $H^1(\mathbf{C} - 0, Z)$ via DET. In terms of differential forms, the generator of $H^1(\mathbf{C} - 0, Z)$ is $\frac{1}{2\pi i}\frac{dz}{z}$; so its image in $H^1(\mathcal{G}_m, Z)$ can be represented by the differential form

$$\omega = \frac{1}{2\pi i}\frac{d_{\mathcal{G}}\mathrm{DET}}{\mathrm{DET}}.$$

If one naively assumes that the determinant is multiplicative, then $\widetilde{\varphi}$ and $\widetilde{\varphi}^{-1}$ will cancel and DET would be a constant. That doesn't happen; however, it is important to know whether the cohomology class is nontrivial. In terms

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* See QUILLEN [17] for a discussion of determinant line bundles. He computes det $\overline{\partial}$, where $\overline{\partial}$ is the Cauchy-Riemann operator on a Riemann surface M coupled to holomorphic structures of a vector bundle E over M.

of forms, is ω exact, i.e., is log DET well defined? For physics one needs not only exactness of ω, but also that log DET should be a 'local' function of $\varphi \cdot A$ (See Section 9.)

If f is an infinitesimal gauge transformation, i.e., an element of Lie algebra g of $\mathcal{G}_m$ thought of as a left invariant vector field on $\mathcal{G}_m$, then $\omega(f)$ at φ equals

$$\frac{d}{du}\bigg|_{u=0} \log\det \not{\partial}_A(-)\not{\partial}_{e^{uf}\varphi A(+)} = \operatorname{tr}\left(L_\varphi^{-s}L_\varphi^{-1}\frac{d}{du}\bigg|_{u=0} L_{e^{uf}\cdot\varphi}\right)\bigg|_{s=0}.$$

But

$$\frac{d}{du}\bigg|_{u=0} L_{e^{uf}\cdot\varphi} = \not{\partial}_A(-)\frac{d}{du}\bigg|_{u=0} e^{-uf}\not{\partial}_{\varphi A}(+)e^{uf} = \not{\partial}_A(-)[\not{\partial}_{\varphi_A}(+),\widetilde{f}]$$

where $\widetilde{f} = I \otimes r(f)$. Hence

$$\omega(f) = \operatorname{tr}\left(L_\varphi^{-s}(\not{\partial}_{\varphi A}(+))^{-1}[\not{\partial}_\varphi A(+),\widetilde{f}]\right)\bigg|_{s=0}.$$

A short computation gives $[\not{\partial}_{\varphi A}(+),\widetilde{f}] = D_{\varphi A}f$ so that

$$\omega(f) = \operatorname{tr}(L_\varphi^{-s}\not{\partial}_{\varphi A}(+)^{-1}D_{\varphi A}f)\bigg|_{s=0}.$$

Formally,

$$\omega(f) = \operatorname{tr}\left(\not{\partial}_{\varphi A}(+)^{-1}\frac{\delta(\not{\partial}_{\varphi A}(+))}{\delta f}\right) = \operatorname{tr}(\not{\partial}_{\varphi A}(+)^{-1}D_{\varphi A}(f)),$$

with the factor L_φ^{-s} regulating the operators not of trace class.

By definition, the one-form ω is closed. Associated to ω is another one-form which is not closed. For any $B \in \mathcal{A}$ with $\not{\partial}_B$ nonsingular, let

$$\widetilde{\omega}_B(f) = \frac{d}{dt}\bigg|_{t=0}\left(\det(\not{\partial}_B(-)\not{\partial}_{e^{tf}\cdot B}(+))\right)/\det(\not{\partial}_B(-)\not{\partial}_{e^{tf}\cdot B}(+)).$$

The restriction of $\widetilde{\omega}$ to an orbit and hence to $\mathcal{G}$ gives a 1-form on $\mathcal{G}$. It is not closed; however, it is easy to compute and agrees with ω at the identity of $\mathcal{G}$. Also $\widetilde{\omega}$ is invariant under left translation whereas, as we shall see, ω in general is not invariant.

While on the topic of determinants, it is perhaps worth pointing out that one can extend the definition to elliptic operators which are self-adjoint but whose symbols are not positive, like $\not{\partial}_A$. The eigenvalues of such an operator T can be positive $\{\lambda_j\}$ or negative $-\mu_j$. Again $\log\det A$ should equal

$$-\frac{d}{ds}\bigg|_{s=0}\left(\sum\lambda_j^{-s}+\sum(-1)^{-s}\mu_j^{-s}\right),$$

making a choice $(-1)^{-s}=e^{i\pi s}$. If

$$\varsigma_{|T|}(s)=\sum\lambda_j^{-s}+\sum\mu_j^{-s} \text{ and } \eta_T(s)=\sum\lambda_j^{-s}-\sum\mu_j^{-s}.$$

then

$$\log\det T=-\frac{d}{ds}\bigg|_{s=0}\left(\frac{\varsigma+\eta}{2}+e^{i\pi s}\left(\frac{\varsigma-\eta}{2}\right)\right)=-\{\varsigma'(0)+\frac{i\pi}{2}(\varsigma(0)-\eta(0))\}.$$

Hence we define $\det T = \det|T|\cdot\exp\big(i\pi/2(\eta_T(0)-\varsigma_{|T|}(0))\big)$. Note that $\varsigma_{T^2}(0)=\varsigma_{|T|}(0)$. If T is a self adjoint differential operator on an odd dimensional manifold, then $\varsigma_{T^2}(0)=0$ [20] and in that case we have $\det T = \det|T|e^{i\pi/2(\eta_T(0))}$.

For geometric operators which are nonsingular, $\eta_T(0)/2$ is a secondary characteristic class α associated to the operator, modulo integers [3]. Our formula becomes $\det T=\det|T|\cdot e^{i\pi\kappa}e^{i\pi\alpha}=\pm\det|T|e^{i\pi\alpha}$. The secondary characteristic classes α have appeared in the action for various physical theories. See [10, 14a]. The previous paragraph explaining how they arise from $\not{\partial}_A$ is my interpretation of a lecture of R. Jackiw. See [14b].

5. The Cohomology of $\mathcal{G}_m$

In the previous section, $H^1(\mathcal{G}_m,Z)$ as represented by DeRham cohomology became of interest. We now study $H^*(\mathcal{G}_m,Z)$ and for simplicity, we consider only the case $M=S^n$; $\mathcal{G}_m$ is then weakly homotopic to $[\varphi: S^n\to G,\ \varphi(m)=Id]=\Omega^n(G)$. Hence $\pi_k(\mathcal{G}_m)\cong\pi_{n+k}(G)$. In particular, $\pi_0(\mathcal{G}_m)=\pi_n(G)$ and $\pi_1(\mathcal{G}_m)=\pi_{n+1}(G)$. First consider the rational homotopy type of $\mathcal{G}_m$. Since torsion is killed, the rational homotopy of G is isomorphic (n even) to the rational homotopy of a product of odd spheres of appropriate dimensions, which are themselves rational Eilenberg-Maclane spaces. Therefore $\mathcal{G}_m$ is rationally homotopic to a product of spheres when n is even. The rational cohomology of $\mathcal{G}_m$ is a Grassman algebra with primitive generators in dimensions given by the spheres for n even, and a polynomial algebra when n is odd. In particular, for $G=SU(N)$, $H^*(\mathcal{G}_m,\mathbf{R})$ has primitive generators in dimensions $1,3,\dots,2k-1$ for $k\le N-\frac{n}{2}$ when n is even and in dimensions $0,2,\dots,2k$, for $k\le N-((n+1)/2)$ when n is odd.

(Two exceptions occur because $G \neq U(N)$; for $n = 0$, $H^1 = 0$ and for $n = 1, H^0 = 0$). Later we shall also need that $H^1(\mathcal{G}_m, \mathbf{R}) \neq 0$ for $n = 4l + 2$ and $G = SO(n)$.

Consider the map $\beta : M \times \mathcal{G}_m \to G$ given by $\beta(x, \varphi) = \varphi(x)$. Cohomology elements of $\mathcal{G}_m$ can be obtained by pulling back cohomology of G and integrating over cycles in M. In particular, the previous discussion yields.

THEOREM. — *For $M = S^n$ and $G = SU(N)$, primitive generators of $H^*(\mathcal{G}_m, \mathbf{R})$ are obtained as $\int_M \beta^*(\omega_j)$ where ω_j are primitive generators of $H^*(G, \mathbf{R})$ of degree $j = n+1, n+3, \ldots, 2N-1$ for n even and $j = n, n+2, \ldots, 2N-1$ for n odd.*

Since ω_j can be represented explicitly as two sided invariant differential forms on $SU(N)$, the theorem gives an explicit formula for the primitive generators of $H^*(\mathcal{G}_m, \mathbf{R})$. That is,

$$T(M \times \mathcal{G}_m, (x, \varphi)) = T(M, x) + T(\mathcal{G}_m, \varphi) = T(M, x) \oplus \mathbf{g}.$$

Then $d\beta(v, f) = \varphi^{-1} d\varphi(v) + f(x)$ and

$$\begin{aligned} &\beta^*(\omega_j)((v_1, f_1), \ldots, (v_r, f_r)) \\ &\quad = \omega_j(\varphi^{-1} d\varphi(v_1) + f_1(x), \ldots, \varphi^{-1} d\varphi(v_j) + f_j(x)) \\ &\quad = \sum \pm \omega_j(\varphi^{-1} d\varphi(v_{i_1}), \ldots, \varphi^{-1} d\varphi(v_{i_n}),\ f_{l_1}(x), \ldots, f_{l_{j-n}}(x)). \end{aligned}$$

Since $\varphi^{-1} d\varphi : T(M) \to su(N)$, let $(\varphi^{-1} d\varphi)^n : \Lambda^n T(M) \to gl(N)$ be its skew extension. That is,

$$\begin{aligned} &(\varphi^{-1} d\varphi)^n (v_1, \ldots, v_n) \\ &\quad = \frac{1}{n!} \sum_{\pi \in S_n} (-1)^\pi \varphi^{-1} d\varphi(v_{\pi(1)}) \varphi^{-1} d\varphi(v_{\pi(2)}) \ldots \varphi^{-1} d\varphi(v_{\pi(n)}). \end{aligned}$$

For consistency, let $g^{-1} dg$ denote the identification of $T(SU(N), e)$ with $su(N)$ and let $(g^{-1} dg)^j$ be the map of $\Lambda^j(su(N)) \to gl(N)$. Then the primitive integral generator ω_{2l-1} equals

$$\left(\left(\frac{1}{2\pi i} \right)^l ((l-1)!)^2 \operatorname{tr}(g^{-1} dg)^{2l-1} \right).$$

So, for example, the generator for $H^1(\mathcal{G}_m, Z)$ is the 1-form

$$f \to \int_{S^{2n}} \operatorname{tr}((\varphi^{-1} d\varphi)^{2n} f) / (2\pi i)^{n+1} (n!)^2,$$

$n \leq N - 1$, n even.

The formulas above for the primitive generators are a special case, the flat connection case, of a more general formula for the same generators involving any connection. See section 8.

We can now derive the Wess-Zumino Lagrangian from our description of $H^1(\mathcal{G}_m, Z)$. Let ω be the integral generator. Given $\varphi \in \mathcal{G}_m$ choose a smooth curve γ from the identity of $\mathcal{G}_m$ to φ. (Such a curve exists, for $\mathcal{G}_m$ is connected when n is even, $n \leq N-1$.) Let $\mathcal{L}(\varphi) = \int_\gamma \omega$. The functional $\mathcal{L}(\varphi)$ is well defined up to an integer — another choice of γ^1 gives

$$\int_\gamma \omega - \int_{\gamma^1} \omega = \int_c \omega,$$

with c a closed curve. But γ defines a homotopy of $\varphi : S^n(N)$ to the identity and

$$\int_\gamma \omega = \int_\gamma \int_{S^n} \omega_{n+1} = \int_{D^{n+1}} \omega_{n+1}$$

where ω_{n+1} is the primitive $n+1$ generator in $H^{n+1}(SU(N))$ and D^{n+1} is the image of $[0,1] \times S^n$ under $\tilde{\gamma} : (t,x) \to \gamma(t)(x)$, so that $\partial D^{n+1} = \varphi(S^n)$. Thus we obtain the usual description of the Wess-Zumino Lagrangian : For each $\varphi \in \mathcal{G}_m$, $\varphi(S^n)$ is homotopically trivial and hence is the boundary of a set D^{n+1}; let $\mathcal{L}(\varphi) = \int_{D^{n+1}} \omega_{n+1}$. The function $\varphi \to e^{2\pi i \mathcal{L}(\varphi)}$ is now well defined and its logarithmic differential represents the generator of $H^1(\mathcal{G}_m, A)$ [24,25].

6. A Family of Dirac Operators

In this section, we use the families index theorem to show that the DET function of section 4 represents the integral generator of $H^1(\mathcal{G}_m, Z)$.

Let $\mathcal{K}$ denote the group of all invertible bounded operators on a Hilbert space, of the form I + compact. We remind the reader that $\mathcal{K}$ is a closed subgroup of all bounded invertibles in the norm topology. Also, since compact operators can be approximated uniformly by ones with finite dimensional range, $\mathcal{K}$ is homotopically equivalent to $U(\infty)$. In particular,

$$\pi_j(\mathcal{K}) = \begin{cases} 0, & j \text{ even;} \\ Z, & j \text{ odd,} \end{cases}$$

and $H^*(\mathcal{K}, Z)$ is a Grassman algebra with one primitive generator in each odd dimension.

Let h be the map of $\mathcal{G}_m \to \mathcal{K} : h(\varphi) = \not{\partial}_A^{-1}(+)\not{\partial}_{\varphi \cdot A}(+)$. Since the symbols of $\not{\partial}_{\varphi A}$ and $\not{\partial}_A$ are the same, $h(\varphi)$ is indeed in $\mathcal{K}$; in fact $h(\varphi) = I + T$

where T is an operator of order -1. Let τ_φ be the element in $\Lambda^1 \otimes \mathbf{g}$ equal to $\varphi \cdot A - A$. Since $\partial\!\!\!/_{\varphi A}(+) - \partial\!\!\!/_A(+) = \tau\!\!\!/_\varphi, T = \partial\!\!\!/_A^{-1}(+)\tau\!\!\!/_\varphi$.

We now state a consequence of the families index theorem for the family $\{\partial_{\varphi \cdot A}(+)\}_{\varphi \in \mathcal{G}_m}$. In the next section our use of the family index will be discussed further.

THEOREM. — *The map h induces an isomorphism in homotopy groups π_j, for $M = S^n$, n even, $G = SU(N)$ and $j \leq 2N - 1 - n$. Consequently, the primitive cohomology generators in $H^*(\mathcal{K}, Z)$ pull back to the primitive generators in $H^*(\mathcal{G}_m, Z)$, for degree $\leq 2N - 1 - n$.*

Let $\mathcal{K}_t = [I + T \in \mathcal{K}; T$ of trace *class*$]$. Then the Lie algebra k_t, of $\mathcal{K}_t$ is the set of trace class operators and has an invariant linear functional, the trace. The cohomology of $\mathcal{K}_t$ (which is the same as $H^*(\mathcal{K})$) can be obtained as invariant skew forms on k_t, using the trace on products as in the finite dimensional case.

Let $h^{-1}dh$ be the mapping of $T(\mathcal{G}_m, \varphi) \to \mathbf{k}$, the Lie algebra of $\mathcal{K}$, induced by h. Note that $\mathbf{k}$ the set of compact operators is closed under multiplication, and hence we can extend the map to a map $(h^{-1}dh)^r : \Lambda^r(\mathcal{G}_m, \varphi) \to \mathbf{k}$ as before. Since h is of order 0 and dh is of order -1, $(h^{-1}dh)^r$ is of order $-r$, i.e. maps $\Lambda^r(\mathcal{G}_m, \varphi)$ into operators of order $-r$. When $r > n$, $(h^{-1}dh)^r(\Lambda^r(\mathcal{G}_m, \varphi))$ is of trace class.

THEOREM [1]. — *For $2l - 1 > n$, the primitive generator of degree $2l - 1$ has representative $h^*\omega_{2l-1}$, the pull back of the invariant $2l - 1$ skew form on k_t. Moreover, $h^*\omega_{2l-1}$ is invariant under left translation in $\mathcal{G}_m$.* †

If $\widetilde{h}(\varphi) = h(\psi\varphi)$, then

$$\widetilde{h}^{-1}d\widetilde{h} = \psi^{-1}h^{-1}dh\psi \qquad \text{and} \qquad (\widetilde{h}^{-1}d\widetilde{h})^r = \psi^{-1}(h^{-1}dh)^r\psi;$$

taking traces shows $h^*\omega_{2l-1}$ is left invariant.

When $2l - 1 \leq n$, then in fact one cannot represent the cohomology by invariant forms. For example, when $l = 1$, a closed left invariant 1-form is equivalent to the existence of a nonzero linear functional μ_1, on $\mathbf{g}$ such that $\mu_1([f, g]) = 0$ for all $f, g \in \mathbf{g}$. We leave the reader to prove that no such linear functional exists for any M and $G = SU(N)$.

In the range $2l - 1 \leq n$, the continuous cohomology of $\mathcal{G}_m$ cannot be represented by the Lie algebra cohomology. However, as has been pointed out to me by A. CONNES and D. QUILLEN, there is a long exact sequence

† Since these forms are left invariant, they can be represented by the cohomology of g, the Lie algebra of $\mathcal{G}_m$. When $G = \mathrm{Gl}(N)$, g is a full matrix algebra and its cohomology is isomorphic to the cyclic cohomology of A. CONNES [9, 15]. The theorem above also follows from this identification.

relating these two cohomologies with the cohomology of the discrete group $\mathcal{G}_m$. In particular, one has $H^k(g) \to H^k_{\text{top}}(\mathcal{G}) \to H^{k+1}_{\text{discrete}}(\mathcal{G})$.

For the generator of the first cohomology, one can write formally $h^*\omega_1 = \operatorname{tr}(h^{-1}dh)$. So at $\varphi \in \mathcal{G}_m$, $(h^*\omega_1)(f)$ is supposed to equal

$$\operatorname{tr}\left(\partial^{-1}_{\varphi A}(+)\frac{\delta \not{\partial}_{\varphi A}(+)}{\delta f}\right) = \operatorname{tr}\left(\not{\partial}_{\varphi A}(+)^{-1}[\not{\partial}_{\varphi A}(+), \tilde{f}]\right).$$

Note that this is precisely the formula obtained in section 4 for the variation of log DET.

Since the trace does not exist, we have to regulate, i.e. smooth out h so its range lies in $\mathcal{K}_t$. Let $h_s(\varphi) = I + (\not{\partial}_A(-)\not{\partial}_{\varphi\cdot A}(+))^{-s}\not{\partial}_A^{-1}(+)\not{f}_\varphi$. Now dh_s is of order $-(2s+1)$ so that $(h_s^{-1}dh_s)^r$ is of order $-r(2s+1)$.

THEOREM [1]. — *The maps h_s, $Rls > (n-1)/2$ are homotopic to h. The forms $h_s^*(\omega_{2l-1})$ are well defined and represent the primitive generators of $H^*(\mathcal{G}_m, Z)$ for $1 \leq l \leq 2N-1-n$. In particular the 1-form* $\operatorname{tr}(h_s^{-1}dh_s)$ *can be analytically continued to $s = 0$ and represents the same class as $\omega = 1/(2\pi i)d_{\mathcal{G}}$*DET/DET *of section 4. The* DET *function represents a generator of $H^1(\mathcal{G}_m, Z)$.*

We remark that the theorem above identifies DET as a 1-cohomology generator without computation for $M = S^{2n}$. A somewhat different argument can be found in [14].

7. Dirac Operators Indexed by $\mathcal{A}/\mathcal{G}_m$

We remarked in section 4 that two k-dimensional vector bundles V and W over a manifold and an element $T \in \operatorname{Hom}(V, W)$ gives $\det T \in \operatorname{Hom}(\Lambda^k V, \Lambda^k W)$. The kernels and cokernels of T measure the extent to which the line bundle $\Lambda^k(V)/\Lambda^k(W)$ is nontrivial. This point of view is the one adopted in [4] to measure the obstruction to defining a gauge invariant determinant of $\not{\partial}_A(+) : C^\infty(S_+ \otimes E) \to C^\infty(S_- \otimes E)$ for all $A \in \mathcal{A}$). Our purpose in this section is to relate this viewpoint with the one in the previous section.

As described in [4] where n is even, the covariance $\not{\partial}_{\varphi\cdot A}(+) = \varphi^{-1}\not{\partial}_A\varphi(+)$ gives a family of elliptic operators $\not{\partial}$ indexed by $\mathcal{A}/\mathcal{G}_m$. Its analytic index $\operatorname{Ind}\not{\partial}$ is an element of $K(\mathcal{A}/\mathcal{G}_m)$, and is formally the virtual vector bundle of the kernels of $\not{\partial}_A(+)$ minus the cokernels of $\not{\partial}_A(-)$. The index theorem for families describes $\operatorname{Ind}\not{\partial}$ topologically; in particular, it gives a formula for the Chern character of $\operatorname{Ind}\not{\partial}$.

The first Chern class, $c_1(\operatorname{Ind}\not{\partial})$, has the following interpretation. To each element of $K(X)$ is associated a well defined line bundle, its determinant line bundle : if V of dimension k and W of dimension l are vector bundles over X, then $\det(V-W) = \Lambda^k(V)/\Lambda^l(W)$. In these terms, $c_1(\mathrm{Ind}\,\partial\!\!\!/) = c_1(\det(\mathrm{Ind}\,\partial\!\!\!/))$. In a way, $\det(\mathrm{Ind}\,\partial\!\!\!/)$ measures the obstruction to defining an analytic $\mathcal{G}$-invariant $\det \partial\!\!\!/_A$.

The cohomology class $\mathrm{ch}(\mathrm{Ind}\,\partial\!\!\!/)$ lies in $H^{\mathrm{even}}(\mathcal{A}/\mathcal{G}_m, \mathbf{R})$ and can be anti-transgressed as in the finite dimensional case to $H^{\mathrm{odd}}(\mathcal{G}_m, \mathbf{R})$. For $M = S^{2n}$, $G = SU(N)$ and in the appropriate range, using the Chern classes, one gets the primitive generators for $H^*(\mathcal{G}_m, \mathbf{R})$, as described in the previous section. In fact, using secondary characteristic classes, one obtains explicit formulas for the primitive generators as a function of the orbit. We remind the reader that to each invariant polynomial p, the characteristic differential form $p(F_A)$ is exact on P, equaling $d\alpha_{A,p}$ with $\alpha_{A,p}$ given explicitly say as in [7]. For $P = G \times S$, the primitive generator in $H^{2k-1}(\mathcal{G}_m, A)$ is then given by the differential form at

$$\varphi \in \mathcal{G}_m : f_1, \ldots, f_{2k-1} \to \int_{S^{2n}} i(f_1) \ldots i(f_{2k-1}) \alpha_{\varphi A,p}$$

where p is the polynomial of degree $k+n$ representing the $(n+k)^{th}$ Chern class. When $A = 0$, the present formula reduces to the previous one of section 5. For arbitrary P, a more general formula can be found in [4].

Let $\mathcal{V}$ denote the invertible operators on a Hilbert space (in our case, the L_2 completion of $C^\infty(S_+ \otimes E)$). It is well known that $\widetilde{\mathcal{V}} = \mathcal{V}/\mathcal{K}$ is isomorphic to the space $\mathcal{F}$ of Fredholm operators of index 0 modulo compact operators. Also, V is the component of the identity of the invertible elements in the algebra of all bounded operators modulo the ideal of compact operators.

We have two principal bundles with total spaces trivial.

$$\begin{array}{ccc} \mathcal{A} & \quad & \mathcal{V} \\ \downarrow{\scriptstyle \mathcal{G}_m} & & \downarrow{\scriptstyle \mathcal{K}} \\ \mathcal{A}/\mathcal{G}_m & & \widetilde{\mathcal{V}} \end{array}$$

The Dirac family $\partial\!\!\!/$ gives a map from $\mathcal{A}/\mathcal{G}_m$ to $\mathcal{V}$, a classifying space for K-theory (reduced). It induces a map $\widetilde{\partial\!\!\!/} : \Omega(\mathcal{A}/\mathcal{G}_m) \to \Omega(\widetilde{\mathcal{V}})$. But these loop spaces can be lifted to an orbit $\mathcal{G}_m \cdot A$ and $\mathcal{K}$ respectively, giving a map from $\mathcal{G}_m \cdot A$ to $\mathcal{K}$. The map h of section 6 gives a specific realization of this map. One consequence is the first theorem of that section.

8. The Odd Dimensional Case

The spacetime formulation of quantum field theory was used at the end of section 2 to state the main problem; there $\dim M$ is even. In the canonical formalism, M is space rather than space time and of odd dimension. Again assume $M = S^n$, $G = SU(N)$ and $N \geq (n+3)/2$. Since n is odd, $\mathcal{G}_m$ is not connected, except for $n = 1$; we restrict our attention to the component of the identity. The generator of $H^1(\mathcal{G}_m, Z)$ in the even dimensional case becomes the generator of $H^2(\mathcal{G}_m, Z)$ for n odd. We make several observations that may be relevant for the representation theory for $\mathcal{G}_m, n = 3$ as a generalization of the representation theory for loop groups ($n = 1$). These remarks are based on joint work with I. FRENKEL.

First, as in the previous section, the primitive operator in $H^{2k}(\mathcal{G}_m, Z)$ is given by

$$f_1, f_2, \ldots, f_{2k} \to \int_{S^{2n+1}} i(f_1) \ldots i(f_{2k}) \alpha_{\varphi A, p}$$

where p is the polynomial representing the $k+n+1$ Chern class.

When $n \geq 3$, the closed 2-form w representing the generator (as in section 5) cannot be chosen to be invariant under left translation, while for $n = 1$ it can. The reason is explained in section 6. Regulating traces, not necessary for $n = 1$, breaks invariance. If $f_i, i = 1, 2, 3$ are elements of g-left invariant vector fields on $\mathcal{G}_m$. Then in general

$$0 = dw(f_1, f_2, f_3) = -w([f_1, f_2], f_3) - \cdots + f_1 w(f_2, f_3) + \cdots.$$

When w is left invariant, one gets a closed 2-form in the ordinary Lie algebra cohomology and that gives a *central* 1-dimensional extension of g. However for $n \geq 3$, we need the actual De Rham cohomology, i.e., cohomology with coefficients in the ring of functions of $\mathcal{G}_m$. Let this abelian algebra of real functions be denoted by J. One obtains a noncentral extension $g \oplus J$ with g acting on J by left invariant fields and $[(f_1, 0), (f_2, 0)] = ([f_1, f_2], w(f_1, f_2))$.

The extension arises in the following way. As usual, the 2-form w representing an integral cohomology class is the first Chern class of a line bundle $\mathcal{L}$ over $\mathcal{G}_m$. One would like to extend left translation on $\mathcal{G}_m$ to be covered by automorphisms of $\mathcal{L}$. The automorphisms of $\mathcal{L}$ covering the identity map of $\mathcal{G}_m$ is the group of invertible unitary functions $J^{\cdot}$ whose Lie algebra is J. The group $\mathcal{G}_m$ of left translations cannot be lifted to $\mathcal{L}$, but the semidirect product $\mathcal{G}_m \times J^{\cdot}$ with Lie algebra $g + J$ above, can. For $n = 1$, the invariance of w implies that $g + J$ can be reduced to the subalgebra g + constants which *does* act on $\mathcal{L}$. In fact $\mathcal{G}_m$ has a holomorphic structure, $\mathcal{L}$ is a holomorphic line bundle and the group with Kac-Moody algebra $g + \mathbf{R}$ acts as holomorphic automorphisms of $\mathcal{L}$ [12, 21]. To us $g + \widetilde{J}$ seems the natural extension to the S^3 case of $\mathbf{g}+\mathbf{R}$ for the S^1 case. Where $\widetilde{J}$ is a subring of J invariant under an appropriate subgroup of $\mathcal{G}_m$.*

For n odd, the cohomology classes are again obtainable from families of Dirac operators. The spin bundle does not split and $\not{\partial}_A$ is self adjoint on $C^\infty(S \otimes E)$. Again because of the covariance, one gets an elliptic *self-adjoint* family indexed by $\mathcal{A}/\mathcal{G}_m$, thus a map $\not{\partial}$ from $\mathcal{A}/\mathcal{G}_m$ to $\mathcal{F}^1$, the space of self-adjoint Fredholm operators. The space $\mathcal{F}^1$ is classifying for $\mathcal{K}^1$ and is homotopic to $\Omega(\mathcal{F})$ [5]. Then

$$\mathcal{G}_m \sim \Omega(\mathcal{A}/\mathcal{G}_m) \xrightarrow{\widetilde{\not{\partial}}} \Omega(\mathcal{F}^1) = \Omega^2(\mathcal{F})$$

equaling $\mathcal{F}$ by Bott periodicity. The generator of $H^2(\mathcal{G}_m, Z)$ is obtained by pulling back the generator of $H^2(\mathcal{F}, Z)$.

9. Concluding Remarks

We mentioned in section 6 that to date the successful approach to the quantization of gauge fields is the perturbative one. And in fact anomalies have their origin in triangle Feynman graphs, representing terms in the perturbative expansion. However, for Quantum Chromodynamics, the non-abelian gauge theory which is supposed to express the strong force, the perturbation expansion is inadequate. The coupling constant is too large. In searching for a nonperturbative theory it makes sense, where possible, to redo results obtained perturbatively in a non-perturbative way, as we have now explained for the chiral anomaly.

The discussion above — and [2, 4] as well — is motivated by the search for a non-perturbative geometric theory. We have not had time here to discuss other applications of the family $\{\not{\partial}_A\}$, particularly the beautiful non-perturbative results in [23, 26]. The application we *have* discussed implies that the anomaly cannot be removed, because it is topological in nature.

However, the ordinary cohomology is inadequate for the purposes of physics. What is need is local cohomology [6]. Although we have concentrated on a given orbit, $d\det\not{\partial}_A(+)/\det\not{\partial}_A(+)$ is a well defined closed 1-form wherever $\not{\partial}_A(+)$ is invertible. Because Lagrangians are local functionals of fields, the physical question is not whether the 1-form is exact, but whether

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* After this paper was written, we became aware of a preprint by L. FADEEV, now published [10]. He arrives at the same conclusion (clearly earlier than us) that for $n=3$ the extension will depend on functions of $\mathcal{G}_m$. His cocycle differs from ours (p. 17) by an exact 1-form and he derives it from a physical point of view, rather than the mathematical one above. At the same time, we learned of the paper of T.R. RAMADAS [15]. He constructs the line bundle above and observes that $\mathcal{G}_m$ cannot be lifted to it.

it is the differential of a function f on $\widetilde{\mathcal{A}}$ which is local in A, i.e., a polynomial in A and its derivatives integrated over $M.\{\widetilde{\mathcal{A}} = [A \in \mathcal{A}; \not{\partial}_A$ is invertible$]\}$.

In the case we have discussed, the closed 1-form represents a generator of $H^1(\mathcal{G}_m, Z) \neq 0$. So naturally it is not exact in the more refined local sense. However, in the case of gravitational anomalies [2], for $M = S^{4k+2}$, it is expected but not known whether $H^1(\mathrm{Diff}_0(M), \mathbf{R}) = 0$. On the other hand, there is indeed a local gravitational anomaly; i.e. the local $H^1(\mathrm{Diff}_0(M), \mathbf{R})$ is not zero. (Private communication from O. ALVAREZ.) It seems natural, then, to suggest that local cohomology in the above sense be given some serious attention by mathematicians.

REFERENCES

[1] ALVAREZ (O.) and SINGER (I.M.). — Still another view of chiral anomalies, in preparation.

[2] ALVAREZ (O.), SINGER (I.M.) and ZUMINO (B.). — Gravitational Anomalies and the Family's Index Theorem, *Comm. Math. Phys.*, t. **96**, 1984, p. 409.

[3] ATIYAH (M.F.), PATODI (V.K.) and SINGER (I.M.). — Spectral asymmetry and Riemannian geometry II, *Math. Proc. Camb. Phil. Soc.*, t. **78**, 1975, p. 405.

[4] ATIYAH (M.F.) and SINGER (I.M.). — Dirac operators coupled to vector potentials, *Proc. Nat. Acad. Sci. USA*, t. **81**, 1984, p. 2597.

[5] ATIYAH (M.F.) and SINGER (I.M.). — Index Theory for skew-adjoint Fredholm Operators, *Pub. Math. IHES*, t. **37**, 1969, p. 305.

[6] BONORA (L.), COTTA and RAMUSINO (P.). — Some remarks on B.R.S. transformations, anomalies and the cohomology of the Lie algebra of the group of gauge transformations, *Comm. Math. Phys.*, t. **87**, 1983, p. 589.

[7] CHERN (S.S.). — *Geometry of Characteristic Classes, Appendix, Complex Manifolds without Potential Theory.* — Springer, N.Y. 1979.

[8] CHERN (S.S.) and SIMONS (J.). — Characteristic forms and geometric invariants, *Ann. of Math.*, t. **99**, 1974, p. 48.

[9] CONNES (A.). — Cohomologue cyclique et foncteurs Ext^n, *C. R. Acad. Sci. Paris*, t. **296**, 1983, p. 953.

[10] DESER (S.), JACKIW (R.) and TEMPLETON (S.). — Three Dimensional Massive Gauge Theories, *Ann. Phys. (NY)*, t. **140**, 1982, p. 372.

[11] FADEEV (L.). — Operational Anomalies and the Gauss law, *Phys. Lett.*, t. **14**B, 1984, p. 81.

[12] FRENKEL (I.B.) and KAC (V.). — Basic representations of affine Lie algebras and dual resonance models, *Invent. Math.*, t. **62**, 1980, p. 23.

[13] HAWKING (S.W.). — Zeta function regularization of path integrals in curved spacetime, *Comm. Math. Phys.*, t. **55**, 1977, p. 133.

[14] JACKIW (R.). — (a) Quantization of Physical Parameters, (b) Fractional Fermions, *Comments Nuclear and Particle Physics*, t. **13**, 1984, p. 15, (*Comments Nuclear and Particle Physics*, t. **13**, 1984, p. 141).

[15] LODAY (J.L.) and QUILLEN (D.). — Cyclic homology and the Lie algebra homology of matrices, preprint.

[16] LOTT (J.). — A proof of axial anomaly, Harvard preprint.

[17] QUILLEN (D.). — Determinants of Cauchy-Riemann Operators over a Riemann surface, IHES preprint, July 1984.

[18] RAMADAS (T.R.). — The Wess. Zumino term and fermionic solitons, *Comm. Math. Phys.*, t. **93**, 1984, p. 355.

[19] RAY (D.) and SINGER (I.M.). — R-torsion and the Laplacian on Riemannian Manifolds, *Adv. in Math.*, t. **7**, 1971, p. 145.

[20] SEELEY (R.T.). — Complex powers of an elliptic operator, *Amer. Math. Soc. Proc. Sym. Pure Math.*, t. **10**, 1967, p. 288.

[21] SEGAL (G.). — Unitary representations of some infinite dimensional groups, *Comm. Math. Phys.*, t. **80**, 1981, p. 301.

[22] SINGER (I.M.). — Some remarks on the Gribov ambiguity, *Comm. Math. Phys.*, t. **60**, 1978, p. 7.

[23] VAFA (C.) and WITTEN (E.). — Eigen value inequalities for fermions in gauge theories, preprint Joseph Henry Lab. Princeton Apr. 1984.

[24] WESS (J.) and ZUMINO (B.). — Consequences of Anomalous Ward Identities, *Phys. Lett.*, t. **37** B, 1971, p. 95.

[25] WITTEN (E.). — Global aspects of current algebra, *Nuc. Phys. B*, t. **223**, 1983, p. 422.

[26] WITTEN (E.). — An $SU(2)$ anomaly, *Phys. Lett.*, t. **117** B, 1982, p. 324.

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