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DISCRETE SUBGROUPS OF LIE GROUPS

BY

G. D. MOSTOW

1. Introduction

Ever since Felix KLEIN's Erlanger program, it has been natural to wonder on which spaces a Lie group can act transitively i.e., which spaces have the form G/C with G a Lie group and C a closed subgroup. In point of fact, the structure theory of a Lie group remained largely a question of the algebraic structure of its Lie algebra, until Herman WEYL's trilogy on the structure of semi-simple Lie groups, [32]. WEYL generalized HURWITZ's use of integration on the classical orthogonal and unitary groups to arbitrary compact Lie groups and his global viewpoint led to the first essential appearance of the one dimensional Betti number of a Lie group. WEYL's effective use of his theorem that a compact semi-simple group is covered only a finite number of times by its simply connected covering group led É. CARTAN to investigate the higher Betti numbers of semi-simple Lie groups, and to suggest to his student DERHAM that he prove in his doctoral dissertation the now-famous DeRham Theorems.

The topological structure of spaces G/C falls into two distinct types according as C has a finite or infinite number of connected components.

The finite case : Suppose that G and C each have a finite number of connected components. Then there is a "covariant" fibration $\pi : G/C \to KC/C \approx K/K \cap C$ with euclidean fiber and with K a maximal compact subgroup of G such that the fibers are permuted transitively by the actions of K. In particular, G/C and $K/K \cap C$ have the same homotopy type and the topology of such spaces reduces to the study of factor spaces of compact Lie groups. There is much detailed information available about such spaces (*cf.* [1], [4], [5]).

The infinite case : Suppose that C has an infinite number of connected components. We consider the case that G is the set of real points of a linear algebraic group.

Let C^0 denote the connected component of the identity in C and let H denote the normalizer of C^0 in G. Then H has a finite number of connected components and $C \subset H$. Thus we get the topological fibering

$$1 \to H/C \to G/C \to H/G \to 1$$

and in addition

$$H/C = H/C^0 \Big/ C/C^0.$$

That is, G/C is the total space of a fiber bundle whose base space has a covariant fibering as in Case 1 and whose fiber is the factor space of the group H/C^0 divided by the discrete subgroup C/C^0. Thus spaces G/C with C having an infinite number of connected components are related to orbit spaces of discrete subgroups of Lie groups.

É. Cartan in [6] lists the two dimensional manifolds on which a connected Lie group acts transitively as the sphere, projective plane, Möbius strip, plane, cylinder, and torus. Actually, the Klein bottle should be added to make the list complete (*cf.* [14]).

The general question of describing all quotients G/C is largely a question of how discrete subgroups of Lie groups operate on spaces G/C with C connected.

In this paper, I shall not attempt to survey all the work since Cartan's time on discrete subgroups of Lie groups. Rather I shall restrict myself to some recent developments on *lattice* subgroups, i.e., discrete subgroups Γ of a Lie group G with G/Γ having finite Haar measure.

2. Arithmetic subgroups

Let A be a connected algebraic matrix group defined over the field $\mathbf{Q}$ of rational numbers. By a $\mathbf{Q}$-character χ of A we mean a 1-dimensional representation $A \to GL(1)$ defined over $\mathbf{Q}$. Let ${}^1A$ denote the intersection of the kernels of all $\mathbf{Q}$-characters of A and let

$$({}^1A)(\mathbf{Z}) = {}^1A \cap SL(n, \mathbf{Z}).$$

By a theorem of Borel–Harish Chandra, $({}^1A)(\mathbf{Z})$ is a lattice in ${}^1A(\mathbf{R})$ (*cf.* [3]). The subgroup ${}^1A(\mathbf{Z})$ is an example of an arithmetic lattice.

Suppose in addition, that A is an algebraic group defined over $\mathbf{Q}$ and

$${}^1A(\mathbf{R})^0 = G_1 \times G_2 \times \cdots \times G_r \times G_{r+1} \times \cdots \times G_S,$$

where G_i is non-compact for $i = 1, \ldots, r$, and compact for $i = r+1, \ldots, s$. Set

$$\begin{aligned} G &= G_1 \times \cdots \times G_r \times G_{i_1} \times \cdots \times G_{i_t} && (r+1 \le i_1 < i_2 \cdots < i_t < s) \\ K &= \textstyle\prod_i G_i && (r+1 \le i \le s,\ i \notin \{i_1, \ldots, i_t\}). \end{aligned}$$

Then G is non-compact, K is compact, and

$$(2.1) \qquad {}^1A(\mathbf{R})^0 = G \times K$$

Let π denote the projection of ${}^1A(\mathbf{R})^0$ into G with respect to K. Since K is compact, π is a proper map. Set

$$\Gamma = \pi(A(\mathbf{Z}) \cap {}^1A(\mathbf{R})^0).$$

Definition. — Any subgroup Γ' of G commensurable with Γ (i.e., $\Gamma \cap \Gamma'$ has finite index in both Γ and Γ') is called an arithmetic lattice (or arithmetic subgroup) of the Lie group G.

We shall be concerned below with proving arithmeticity of a lattice in an adjoint group G. That will require finding a compact group K, and an algebraic group A with a prescribed $\mathbf{Q}$-structure such that the above situation is reproduced.

Example. Let k be an algebraic number field with $[k : \mathbf{Q}] = n$. Let A denote the set of all invertible elements in $\mathcal{A} = k \otimes_{\mathbf{Q}} \mathbf{C}$, operating on the algebra $\mathcal{A}$ by left multiplications, and with k as the given $\mathbf{Q}$-structure of the underlying vector space of $\mathcal{A}$. Then

$$\mathcal{A}(\mathbf{R}) = k \otimes_{\mathbf{Q}} \mathbf{R} = \mathbf{R}^{r_1} \oplus \mathbf{C}^{r_2}$$

where r_1 is the number of real places of k, r_2 is the number of complex places of k, and $r_1 + 2r_2 = n$. [Recall a complex *place* is complex conjugate pair of monomorphisms $\sigma : k \to \mathbf{C}$]. The group $A(R)$ of invertible elements in $\mathcal{A}(R)$ is therefore

$$A(\mathbf{R}) = (\mathbf{R}^\times)^{r_1} \times (\mathbf{C}^\times)^{r_2} = (\mathbf{R}^\times)^{r_1}(\mathbf{R}^+)^{r_2}T^{r_2}$$

where $(\ )^\times$ indicates the set of non-zero elements in $(\ ), \mathbf{R}^+$ denotes the multiplicative group of positive reals, and T denotes the multiplicative group of complex numbers of modulus one. Furthermore, since all $\mathbf{Q}$-characters of $\mathcal{A}$ are powers of the norm, we get that ${}^1A(\mathbf{R})^0$, the topologically connected component of the identity element of ${}^1A(\mathbf{R})$, is ${}^1A(\mathbf{R})^0 \approx (\mathbf{R}^+)^{r_1+r_2-1} \times T^{r_2}$. Since ${}^1A(\mathbf{Z})$ is a lattice subgroup of ${}^1A(\mathbf{R})$, we find that the rank of the abelian group ${}^1A(\mathbf{Z})$ is $r_1 + r_2 - 1$ — the classical result of Dirichlet on the rank of the group of units in the algebraic number field k.

In this example, the lattice ${}^1A(\mathbf{Z})$ is even co-compact in ${}^1A(\mathbf{R})$. In general, if $A(\mathbf{Z})$ is a lattice in $A(\mathbf{R})$ with A an algebraic $\mathbf{Q}$-group

(2.2) *$A(\mathbf{R})/A(\mathbf{Z})$ is compact if and only if $A(\mathbf{Z})$ or equivalently, $A(\mathbf{Q})$ has no unipotent elements other than in the unipotent radical of A.*

This criterion was conjectured by GODEMENT and proved independently in [3] and [22].

This criterion is very useful in case the algebraic $\mathbf{Q}$-group A is $\mathbf{Q}$-simple. For in that case, there is an absolutely simple group, B defined over a number field k such that

(2.3)
$$A = \operatorname*{Restr}_{k|Q} B = \prod_\sigma B^\sigma,$$
$$A(\mathbf{Z}) \approx B(\theta) \qquad \text{(imbedded diagonally in } \prod_\sigma B^\sigma),$$

where σ ranges over all the distinct monomorphisms $k \longrightarrow \mathbf{C}$ and θ is the ring of (algebraic) integers in k. The set of complex places occur in pairs $\{\sigma, \bar\sigma\}$ and it is known that $(B^\sigma \times B^{\bar\sigma})(\mathbf{R}) \cong B(\mathbf{C})$, a complex simple group which is never compact. Hence, if any factor $B^\sigma(\mathbf{R})$ occurring in (2.3) is compact, the place σ must be a real place. If $B^\sigma(\mathbf{R})$ is compact for all but one place σ, then all σ are real, i.e., k is a totally real field.

(2.4) *If $B^\sigma(\mathbf{R})$ is compact for some real place σ, then $A(\mathbf{R})/A(\mathbf{Z})$ is compact.*

For if $u \in A(\mathbf{Z})$ is a unipotent element, then the projection u^σ of u in the factor $B^\sigma(\mathbf{R})$ is unipotent. Since $B^\sigma(\mathbf{R})$ is compact, it follows that ${}^\sigma u = 1$. Hence $u = 1$. Now (2.4) follows from the Godement criterion.

In particular, if G is a simple real Lie group and Γ is an arithmetic lattice in G, then a compact factor K may be required in (2.1) only if Γ is co-compact, i.e., if G/Γ is compact.

3. Arithmeticity of lattices

Classically, lattices were constructed in $SL(2, \mathbf{R})$ in a variety of ways. However, only the arithmetic construction seemed readily available for arbitrary classical groups. A. SELBERG was the first to launch an investigation of this question and conjectured : Apart from some exceptions, non co-compact lattices in simple real Lie groups are arithmetic. Later I. PIATETSKY-SHAPIRO modified the conjecture to :

If the $\mathbf{R}$-rank of a semi-simple real Lie group G is greater than 1, then any irreducible lattice Γ in G is arithmetic. An *irreducible* lattice Γ in a semi-simple linear Lie group G is one such that for every proper normal subgroup N of positive dimension, ΓN is topologically dense in G. The $\mathbf{R}$-*rank* of a Lie group G is the maximum dimension of a subgroup D that is diagonalizable over $\mathbf{R}$; i.e., D is conjugate to a diagonal matrix group with real coefficients only.

In 1973, the Selberg-Piatetsky-Shapiro conjecture was proved by G.A. MARGULIS (*cf.* [13]) for non co-compact lattices and in 1974 for co-compact lattices. The 1974 proof can in fact be extended to apply to arbitrary lattices.

We devote the remainder of this paper to results on construction of lattices by methods other than the arithmetic definition. In view of Margulis' remarkable theorem, one can hope to find non-arithmetic lattices only in the simple $\mathbf{R}$-rank 1 real Lie groups; these are, up to local isomorphisms, $SO(n,1)$, $SU(n,1)$, $Sp(n,1)$, and a form of F_4. They act by isometries on the hyperbolic spaces

$$\mathbf{Rh}^n, \quad \mathbf{Ch}^n, \quad \mathbf{Hh}^n, \quad \mathbf{Oh}^2$$

over the reals, complex numbers, quaternions, and octonions of dimension n, $2n$, $4n$, 16 respectively.

4. Constructions on $O(n,1)$

In $\mathbf{Rh}^2$, there is the classical construction of the group Γ generated by reflection in the sides of a geodesic triangle Δ whose angles are $\frac{\pi}{l}$, $\frac{\pi}{m}$, $\frac{\pi}{n}$ with l,m,n non-negative integers; the orientation preserving subgroup of index 2 in Γ is called a *triangle* group and is denoted $[l,m,n]$.

If we delete from the hyperbolic plane $\mathbf{Rh}^2$ the fixed point sets of all the reflections in the group generated by the reflections in the faces of Δ, it can be proved that Δ is a connected component of the remaining space R and all the connected components of R are permuted transitively by Γ. Thus Δ is a fundamental domain for Γ. The group Γ is a discrete subgroup of the isometry group of $\mathbf{Rh}^2$, the latter being $PO(2,1) = O(2,1)/(\pm 1)$.

A similar construction works in real hyperbolic n-space $\mathbf{Rh}^n$.

(4.1) *If Δ is a closed convex polyhedron bounded by $(n-1)$-faces lying on geodesic subspaces, and if the dihedral angles of Δ are $\pi/$integer, then the group Γ generated by reflections in the faces of Δ has Δ as a fundamental domain and is discrete in $PO(n,1)$.*

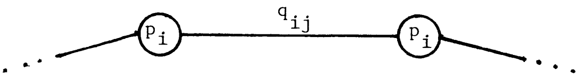
For $n = 3$, such polyhedra were studied by MAKAROV (*cf.* [10]), who showed that for some Δ, the resulting group Γ is a non-arithmetic lattice in $PO(3,1)$. Later E.B. VINBERG undertook a systematic study of such reflection groups, determining all geodesic n-simplices Δ in $\mathbf{R}\mathrm{h}^n$ of finite volume. In 1967 (*cf.* [29]) he determined all such simplices Δ for $n \leq 5$ and proved that there is no such compact simplex for $n > 5$. He found no non-arithmetic lattice generated by reflections in the faces of a simplex for $n > 5$.

Recently, Vinberg has proved the striking result.

If $n > 30$, *there is no compact simplex satisfying the condition in* $(4,1)$*; alternatively, there is no group* Γ *generated by reflections of real hyperbolic* n*-space having a compact fundamental domain.*

5. Groups generated by complex reflections in Ch^n

Coxeter diagrams for groups generated by reflections in real vector spaces are well-known (*cf.* N. BOURBAKI, Chapter V). Given a vector space V over the field $\mathbf{C}$ of complex numbers, a linear map $T : V \to V$ is called a *pseudo-reflection* if T–identity has a one-dimensional image. If in addition T is diagonalizable, we call T a $\mathbf{C}$-reflection. COXETER has introduced diagrams for finite groups generated by $\mathbf{C}$-reflections but they serve equally well for infinite groups. To each graph



with positive integer p_i assigned to node i, and positive integer q_{ij} assigned to a line segment joining nodes i, j, we associate a complex vector space $V = \bigoplus_i \mathbf{C}e_i$, and on V we define a hermitian form via

$$\langle e_i, e_i \rangle = 1 \qquad \text{for all } i,$$

$$|\langle e_i, e_j \rangle| = \begin{cases} 0 & \text{if } i,j \text{ is not joined by a line,} \\ \left(\dfrac{\cos\left(\frac{\pi}{p_i} - \frac{\pi}{p_j}\right) + \cos\frac{2\pi}{q_{ij}}}{2 \sin\frac{\pi}{p_i} \cdot \sin\frac{\pi}{p_j}} \right)^{1/2} & \text{otherwise.} \end{cases}$$

The hermitian form is still not specified until we give the argument of $\langle e_i, e_j \rangle$, but the resulting hermitian forms will all be equivalent if the graph has no loops; if there are loops, the values of $\arg\langle e_i, e_j \rangle$ are also specified. The hermitian form need not be definite.

Given V and the hermitian form as above, for each node i, define R_i by :

$$v \mapsto v + \left(e^{2\pi\sqrt{-1}/p_i} - 1\right)\langle v, e_i \rangle e_i.$$

It is easy to see that R_i preserves the hermitian form and is a **C**-reflection of order p_i. Let Γ denote the group generated by $\{R_i\ ;\ \text{all } i\}$.

Assume that $\arg\langle e_i, e_j\rangle$ *is so chosen that the hermitian form has signature (n plus signs, 1 minus sign) where* $n+1$ *is the number of nodes. Then* $\Gamma \subset U(n,1)$.

As a model for complex hyperbolic n-space $\mathbf{C}\mathrm{h}^n$, we may take the image in complex projective n-space $\mathbf{CP}^n$ of the cone

$$V^- = \{v \in V; \langle v, v\rangle < 0\}$$

under the mapping $\pi : V - 0 \to \mathbf{CP}^n$, distance being defined by

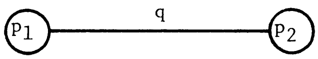
$$\cosh d(\pi(v), \pi(w)) = \frac{|\langle v, w\rangle|}{(\langle v, v\rangle\langle w, w\rangle)^{1/2}} \quad \text{for } v, w \in V^-.$$

Equivalently, the Riemannian metric is

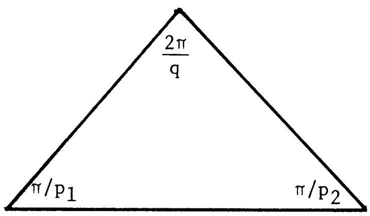
$$ds^2 = \frac{1}{\langle v, v\rangle^2}\begin{vmatrix} \langle dv, dv\rangle & \langle dv, v\rangle \\ \langle v, dv\rangle & \langle v, v\rangle \end{vmatrix}.$$

The connected component of the identity of the group of isometries of $\mathbf{C}\mathrm{h}^n$ is $PU(n,1)$, and the action of a **C**-reflection of V whose fixed point set in $\mathbf{C}\mathrm{h}^n$ is a subspace of complex codimension one in $\mathbf{C}\mathrm{h}^n$ is called a **C**-reflection of $\mathbf{C}\mathrm{h}^n$. The image of the group Γ above in $PU(n,1)$ is a group generated by **C**-reflections of $\mathbf{C}\mathrm{h}^n$.

In case $n = 1$, $\mathbf{C}\mathrm{h}^1 \approx \mathbf{R}\mathrm{h}^2$ and the groups obtained in this way include the triangle groups of § 4. The lattice of the Coxeter diagram



is the triangle group of $PU(1,1)$ corresponding to the geodesic triangle



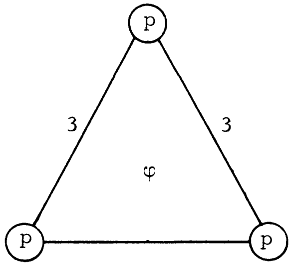
with p_1, p_2, q positive integers satisfying $1/p_1 + 1/p_2 + 2/q < 1$. Moreover, $(R_1R_2)^{q/2} = (R_2R_1)^{q/2} (= R_1R_2\cdots R_1$ if q is odd) and $(R_1R_2)^q = 1$ in

$PU(1,1)$. It follows that R_1 and R_2 are conjugate if q is odd, so that $p_1 = p_2$ if q is odd.

6. Non-arithmetic lattices in $PU(2,1)$

The first construction of non-arithmetic lattices in $PU(2,1)$ were obtained as groups generated by **C**-reflections **C**h^2 with Coxeter diagram, for special values of p and φ :

(6.1)



Here $V = \mathbf{C}e_1 + \mathbf{C}e_2 + \mathbf{C}e_3$, $\langle e_i, e_i \rangle = 1$,

$$\langle e_i, e_2 \rangle = \langle e_2, e_3 \rangle = \langle e_3, e_1 \rangle = -\frac{1}{2sin\frac{\pi}{p}}\varphi$$

where $|\varphi| = 1$. (*cf.* [18]).

Set $t = \frac{1}{\pi} \arg \varphi^3$, and denote the image of the group Γ in $PU(2,1)$ by $\Gamma(p,t)$.

LEMMA 6.2. — *The hermitian form of the diagram above has signature* $(2+, 1-)$ *for* $|t| < 3(\frac{1}{2} - \frac{1}{p})$ *if* $3 \leq p \leq 5$ *and for* $|t| < \frac{1}{2} + \frac{1}{p}$ *for* $p \geq 6$.

The group generated by the two distinct reflections $\{R_i, R_j\}$ *is finite if and only if* $p \leq 5$.

The problem of proving that the subgroup $\Gamma(p,t)$ is discrete in $PU(2,1)$ raises the general question : How can one prove that a group is discrete? That question will occupy us is this section and the next.

One method of proving $\Gamma(p,t)$ discrete is to produce a fundamental domain for it. If Δ is a group acting by isometries on connected complete metric space X, and if P_0 is a point in X, we set for any $g \in \Delta$

(6.3)
$$g^+ = \{x \in X; d(x, p_0) \leq d(gx, p_0)\}$$
$$\widehat{g} = \{x \in X; d(x, p_0) = d(gx, p_0)\}$$

then $F = \bigcap_{g \in \Delta} g^+$ is a fundamental domain for Δ on X mod Δ_F if and only if Δ has a fundamental domain, and this occurs if and only if F has a non-empty interior, here Δ_F denotes $\{g \in \Delta,\ gF = F\}$.

Choose now $X = \mathbf{C}\mathrm{h}^2$, $P_0 = \pi(e_1 + e_2 + e_3)$, and choose Δ to be the subgroup Γ_{ij} if $\Gamma(p,t)$ generated by two reflections $\{R_i, R_j\}$ $(1 \leq i \neq j \leq 3)$. For $p \leq 5$, the resulting group Γ_{ij} is finite and has a fundamental domain $F_{ij} = \bigcap_{g \in \Gamma_{ij}} g^+$. Set $F = F_{12} \cap F_{23} \cap F_{31}$. The proof that $\Gamma(p,t)$ is discrete involves proving that the region F satisfies two conditions, the first for all (p,t) and the second only when $\Gamma(p,t)$ is discrete.

LEMMA 6.4. — *For each* (p,t) *with* $3 \leq p \leq 5$, $|t| < 3(\frac{1}{2} - \frac{1}{p})$, *the following condition holds :*

(CD1) *The* 3-*dimensional faces of* F *occur in pairs and for each pair* (F^+, F^-) *of* 3-*faces, there is a unique element* g *of* $\Gamma(p,t)$ *with* $gF^+ = F^-$ *;* g *lies in* $\Gamma_{12} \cup \Gamma_{23} \cup \Gamma_{31}$.

Notation. — If (F^+, F^-) is a pair of 3-faces of F with $gF^+ = F^-$ for $g \in \Gamma_{ij}$, we set $F^+ = \widetilde{g}, F^- = \widetilde{g}^{-1}$. We have (*cf.* (6.3)) $\widetilde{g} \subset \widehat{g}$ and $\widetilde{g}^{-1} \subset \widehat{g}^{-1}$.

Remark. — Unlike the case of spaces of constant curvature, the subset g^+ equidistant from p_0 and $g^{-1}p_0$ is *not* a geodesic subspace since $\mathbf{C}\mathrm{h}^2$ has no 3-dimensional geodesic subspaces.

Whenever condition (CD1) holds, one can form for any codimension 2-face e_0 of F, a circuit of images : $F, g_1F, g_1g_2F, g_1g_2g_3F, \ldots$, in the following way.

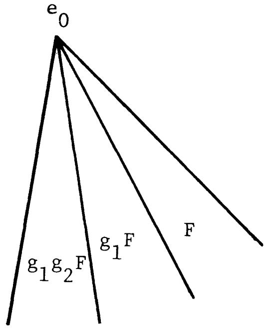
$$e_0 = \widetilde{g}_0 \cap \widetilde{g}^{-1} \quad \text{for unique} \quad g_0, g_1 \in \Gamma_{12} \cup \Gamma_{23} \cup \Gamma_{31}.$$

Then

$$e_0 \subset \widetilde{g}^{-1} = g_1\widetilde{g}_1 \quad \text{and} \quad g_1^{-1}e_0 = \widetilde{g}_1 \cap \widetilde{g}_2^{-1}$$

for a unique $g_2 \in \Gamma_{12} \cup \Gamma_{23} \cup \Gamma_{31}$. Repeating the argument,

$$g_2^{-1}g_1^{-1}e_0 = \widetilde{g}_2 \cap \widetilde{g}_3^{-1},\ g_3^{-1}g_2^{-1}g_1^{-1}e_0 = \widetilde{g}_3 \cap \widetilde{g}_4^{-1}, \text{ etc.}$$



Definition. — F satisfies condition (CD2) if and only if : For all k, interior $g_1g_2 \cdots g_kF \cap F \neq \emptyset$ implies $g_1g_2 \cdots g_kF = F$.

LEMMA 6.5. — *If condition* (CD2) *holds, then F is a fundamental domain for* $\Gamma(p,t)$ modulo Γ_F.

Proofs. — The proof of LEMMA 6.4 entails explicit determination of the 3-dimensional faces of F, including the precise location of all the vertices.

One begins by determining the 3-faces of F_{12}. In case $p = 5$ for example, F_{12} is the intersection of 600 half-spaces. A computer was used to determine that in actuality, the only faces of F_{12} lie on $\widehat{R}_1^{\pm 1}$, $\widehat{R}_2^{\pm 1}$, $(\widehat{R_1 R_2})^{\pm 1}$, $(\widehat{R_2 R_1})^{\pm 1}$, $(R_1 \widehat{R_2} R_1)^{\pm 1}$. Although the computations involve solving systems of non-linear equations, it turned out to be possible to verify condition (CD1) for all stated (p,t) by hand calculations in closed form (*cf.* [18]), albeit lengthy. The proof of LEMMA 6.5 is topological (*cf.* [18]).

LEMMA 6.6. — *For each* (p,t) *with* $3 \le p \le 5$, $|t| < 3(\frac{1}{2} - \frac{1}{p})$, *$F$ has finite volume.*

THEOREM 6.7. — *Assume* $3 \le p \le 5$, $0 < t < 3(\frac{1}{2} - \frac{1}{p})$.

(i) *Condition* (CD2) *holds for only* 17 *values of* (p,t).

(ii) *The lattice* $\Gamma(p,t)$ *is non arithmetic for the seven values :*

$(p,t) = (3,5/42)$, $(3,1/12)$, $(3,1/30)$, $(4,3/20)$, $(4,1/12)$, $(5,1/5)$, $(5,11/30)$.

(*cf.* [18], Theorem 17.3).

7. Construction of lattices in $PU(n,1)$ via monodromy, $n \le 5$

The results in this section are a summary of forthcoming paper with P. DELIGNE (*cf.* [7]).

Let P denote a complex line, let P^{n+3} denote the set of all $(n+3)$-tuples of points of P, and let M denote the subset of elements in p^{n+3} having $n+3$ distinct coordinates $(x_0, x_1, \ldots, x_{n+2})$. For any $m \in M$, let P_m denote the punctured line with punctures at the $n+3$ coordinates of m. Set

$$P_M = \{(x,m) \in P \times M; x \in P_m\}.$$

We shall consider a flat one dimensional vector bundle L on P_M characterized as follows :

Fix an $n+3$-tuple of complex $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n+2})$ such that

$$\begin{cases} \alpha_i \neq 1 \text{ for all } i \quad 0 \le i \le n+2, \\ \prod_{i=0}^{n+2} \alpha_i = 1. \end{cases}$$

(7.1) *L has monodromy α along each P_m.*

In other words, if we denote by L_m the restriction of L to P_m, and if we choose a base point $0 \in P_m$ and denote by $\widehat{P}_m$ the simply connected cover of P_m, then $L_m = \mathbf{C} \times_{\pi_1(P_m,0)} \widehat{P}_m$, that is, an element in $\pi_1(P_m, 0)$ represented by a positive loop around the i^{th} puncture of P_m effects multiplication by α_i on L_m. This determines the isomorphism class of L_m uniquely, but L_m is not determined up to a unique isomorphism. A similar remark holds for L. One can construct such flat bundles on P_M.

The projection $P_M \to M$ is locally on M a topological direct product. Hence $\{H^1(P_m, L_m); m \in M\}$ forms a flat vector bundle over M. To obtain a bundle *canonically* determined by the data α, we pass to the projective space $PH^1(P_m, L_m)$ (of 1-dimensional subspaces of $H^1(P_m, L_m)$) and consider the bundle of projective spaces over M

$$B(\alpha)_M = \{PH^1(P_m, L_m); m \in M\};$$

it is flat and hence $B(\alpha)_M = PH^1(P_0, L_0) \times_{\pi_1(M,0)} \widehat{P}_0$ where 0 is a base point in M. Let $\theta : \pi_1(M_1 0) \to PGL(H^1(P_0, L_0))$ denote the action of $\pi_1(M, 0)$ resulting from horizontal transport over M.

The group $\operatorname{Aut} P$, the fractional linear group, operates diagonally on P^{n+3}, hence on M and P_M. Set $Q = \operatorname{Aut} P \setminus M$. The effect of $\operatorname{Aut} P$ on the fibers of $B(\alpha)_M$ commutes with horizontal transport. Hence θ descends to a homomorphism, also denoted θ.

$$\theta : \pi_1(Q, 0) \to PGL(H^1(P_0, L_0))$$

and the flat fiber bundle $B(\alpha)_M$ descends to a flat bundle $B(\alpha)_Q$ over Q of projective spaces.

(7.2) *Definition.* — Γ_α = Image θ= the α-hypergeometric monodromy group.

The relation of the group Γ_α to hypergeometric functions requires looking at $H^1(P_0, L_0)$ from the point of view of de Rham theory.

For each i, choose μ_i so that

$$e^{2\mu\sqrt{-1}\mu_i} = \alpha_i, \qquad \sum_i \mu_i = 2 \quad (0 \le i \le n+2). \tag{7.3}$$

Take $P = \mathbf{C} \cup \{\infty\}$. Set $M_0 = \{m \in M;\ m_0 = 0,\ m_1 = 1,\ m_{d+2} = \infty\}$. Then $\operatorname{Aut} P \times M_0 \xrightarrow{\sim} M$ and $M_0 \xrightarrow{\sim} Q$ via the quotient map. Let z denote the projection $P \times M_0$ to P composed with the identity coordinate of P. z is then a coordinate on P_{M_0}, the pullback of P_M via $M_0 \to M$. For any

$m \in M_0$, let e denote a multivalued section of L_m i.e., a constant section of the product **C** bundle on the simply connected cover $\widehat{P}_m$. Set

$$\omega_\mu(m) = \prod_{i \neq n+2} (z - m_i)^{-\mu_i} e\, dz.$$

This defines a 1-form on P_m with coefficients in L_m and therefore a cohomology class in $H^1(P_m, L_m)$. In [7] we prove

(7.4) $H^1(P_m, L_m) \cong H^1_c(P_m, L_m)$ (compact supports) for all $m \in M$.

(7.5) $\dim H^1(P_m, L_m) = n + 1$.

(7.6) The cohomology class of $\omega_\mu(m)$ in $H^1(P_m, L_m)$ is non-zero.

(7.7) The map $m \to \omega_\mu(m)$, $m \in M_0$, defines a holomorphic section ω_μ of Q in the flat projective space fiber bundle $B(\alpha)_Q$.

Choose a base point $O \in Q$ and above it $\widehat{O} \in \widehat{Q}$, the simply connected cover of Q. Then the flat bundle section ω_μ of (7.7) defines a multivalued map, also denoted ω_μ, of Q to the fiber $PH^1(P_0, L_0)$, that is a $\pi_1(Q, 0)$-equivariant map $\widehat{\omega}_\mu$ of $\widehat{Q}$ to $PH^1(P_0, L_0)$.

Let $\check{L}$ denote the dual bundle of L. Then there is a perfect pairing (*lf* denoting locally finite)

$$H^1(P_0, L_0) \times H_1^{lf}(P_0, \check{L}_0) \to C. \tag{7.8}$$

As a base of $H_1^{lf}(P_0, L_0)$, one can take $\{C_1, \ldots, C_{n+1}\}$ where C_i is an open-ended path in P_0 from O'_i to O'_{i+1}, where O' denotes the point in M_0 over 0. For convenience, we may choose $O'_i = i$, $0 \le i \le n+1$, and select each C_i as an interval. Then for any point $\widehat{q}$ in $\widehat{Q}$ above $q \in Q$, $\{\langle \widehat{\omega}_\mu(\widehat{q}), C_i \rangle\ ; i = 1, \ldots, n+1\}$ are the homogeneous coordinates of $\widehat{\omega}_\mu(\widehat{q})$, and we have

$$\begin{aligned} &\langle \widehat{\omega}_\mu(\widehat{q}), C_i \rangle \\ &\quad = \int_{\widehat{q}C_i} z^{-\mu_0}(z-1)^{-\mu_1}(z - m(2))^{-\mu_2} \ldots (z - m(n+1))^{-\mu_{n+1}}\, dz \end{aligned} \tag{7.9}$$

where m is the point of M_0 above q, and $\widehat{q}C_i$ is the horizontal transport of C_i corresponding to the path in $\widehat{Q}$ from $\widehat{O}$ to $\widehat{q}$.

For $n = 2$, PICARD takes the integral in (7.9) as the definition of a hypergeometric function of two variables (*cf.* [25]). For $n = 1$, the integral over the path $\widehat{q}(C_1 + C_2)$ reduces to Euler's integral formula for the hypergeometric series, which he was the first to define in 1778 (*cf.* [18]). Upon setting $z = u^{-1}$

$$\begin{aligned} \int_1^\infty z^{-\mu_0}(z-1)^{-\mu_1}(z-x)^{-\mu_2}\, dz &= \int_0^1 u^{b-1}(1-u)^{c-b-1}(1-ux)^{-a}\, du \\ &= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a, b, c; x) \end{aligned}$$

where $\mu_0 = c - a$, $\mu_1 = 1 + b - c$, $\mu_2 = a$,

$$F(a,b,c;x) = \sum_{k=0}^{\infty} \frac{(a,k)(b,k)}{(c,k)} \frac{x^k}{k!},$$

and $(a,k) = \prod_{i=0}^{k-1}(a+i)$.

Assume

$$|\alpha_i| = 1 \quad \text{for all } i. \tag{7.10}$$

Then the dual $\check{L}$ is isomorphic to the complex conjugate $\overline{L}$ and the pairing (7.8) together with (7.4) yields a Hermitian form in $H^1(P_0, L_0)$ invariant under the action of the monodromy group Γ_α. The hermitian form Ψ is unique up to a real scalar multiple. We can choose real multiple so as to make $\langle \omega_\mu(q), \omega_\mu(q) \rangle < 0$ for all $q \in Q$.

(7.11) *The hermitian form* $\langle \ , \ \rangle$ *has signature (n plus, 1 minus) if and only if* $0 < \mu_i < 1$, $\sum_i \mu_i = 2$ $(0 \leq i \leq n+2)$.

We assume hereafter that

$$0 < \mu_i < 1, \quad \sum_i \mu_i = 2 \quad (0 \leq i \leq n+2) \tag{7.12}$$

and we denote Γ_α by Γ_μ, where $\mu = (\mu_0, \ldots, \mu_{n+2})$.

We have therefore

$$\Gamma_\mu \subset PU(H^1(P_0, L_0); \Psi) \approx PU(n,1). \tag{7.13}$$

(7.14) *Definition.* — A sequence $\mu = (\mu_0, \ldots, \mu_{n+2})$ satisfies condition INT if and only if it satisfies (7.12) and in addition for all i, j with $0 \leq i \neq j \leq n+2$, $\mu_i + \mu_j < 1$ implies $(1 - \mu_i - \mu_j)^{-1}$ is an integer.

THEOREM 7.15. — *If* μ *satisfies* INT, *then* Γ_μ *is a lattice in* $PU(n,1)$.

The case $n = 1$ is proved by SCHWARZ in his seminal paper [28]. The case $n = 2$ was stated by PICARD in 1885 under additional hypothesis that $\mu_i + \mu_j < 1$ for all $i \neq j$ (*cf.* [25]) but his proof is incomplete (*cf.* [19]); later, he weakened the hypothesis, without proof, to a condition equivalent to : for all i, j with $0 \leq i \neq j \leq n+2$, $(1 - \mu_i - \mu_j)^{-1}$ is an integer (*cf.* [25b]).

In terms of the definitions made above, the proofs of SCHWARZ, PICARD, and DELIGNE-MOSTOW all take as starting point the multivalued hypergeometric map

$$\omega_\mu : Q \to PH^1(P_0, L_0).$$

If $n = 1$, Schwarz's case, $Q = P - \{0, 1, \infty\}$, the complex projective line with $0, 1, \infty$ deleted. SCHWARZ proves that ω_μ maps the upper half plane into a triangle T bounded by circular arcs. Condition INT implies that the angles of the triangle T are of the form $\frac{\pi}{l}$, $\frac{\pi}{m}$, $\frac{\pi}{n}$, with l, m, n integers. A fundamental domain for Γ_μ in this case is the quadrilateral obtained from combining T with its reflection in one of its edges. The lattice Γ_μ is a triangle subgroup of $PU(1,1)$.

In the case $n = 2$, $Q = P \times P$ minus the seven lines

$$x = \begin{cases} 0 \\ 1 \\ \infty \end{cases}, \qquad y = \begin{cases} 0 \\ 1 \\ \infty \end{cases}, \qquad x = y.$$

PICARD makes three dimensional slits in Q obtaining a simply connected domain, Q' and he argues in [25] that the image $\omega_\mu(Q')$ is a fundamental domain in the complex 2-ball if, in effect, for all $0 \leq i \neq j \leq 4$, $(1-\mu_i-\mu_j)^{-1}$ is a positive integer on zero. The gap in PICARD's proof can be explained in terms of LEMMA 6.5. The matching of the three-dimensional faces $\omega_\mu(Q')$, which arise in pairs from the slits in Q, assures the condition (CD1). Condition INT assures the condition (CD2) for all two dimensional faces which are fixed point sets of the **C**-reflections arising from monodromy of puncture i around puncture j, $0 \leq i \neq j \leq 4$. However, these two dimensional faces are *not all* the two-faces. PICARD is entirely silent on (CD2) for the remaining two-dimensional faces of $\omega_\mu(Q')$.

The proof in DELIGNE-MOSTOW [7] is based on a different strategy.

We construct for all μ satisfying (7.12) a Γ_μ-space $\widetilde{Q}_{st}$ and a Γ_μ-map $\widetilde{\omega}_\mu$ from $\widetilde{Q}_{st}$ to the complex n-ball, such that on the space $\widetilde{Q}_{st}$ the action of the group Γ_μ is obviously discontinuous. Then we prove : under hypothesis INT, the map $\widetilde{\omega}_\mu$ is a homeomorphism. It follows immediately that Γ_μ is discrete in $PU(n,1)$.

The construction of $\widetilde{Q}_{st}$ is based on a certain compactification Q_{sst} which is analogous to D. MUMFORD's "quotient variety" structure for the space $\operatorname{Aut} P \setminus P^{n+3}$ and depends on μ [24]. In case $n = 2$ and $\mu_i + \mu_j < 1$ for all $0 \leq i \neq j \leq 4$, the space Q_{sst} is the blowup of $P \times P$ at the three points $(0,0)$, $(1,1)$, (∞,∞). Here Q_{sst} has ten exceptional lines (i.e., of self intersection –1) lying above each of the seven lines

$$. \qquad x = \begin{cases} 0 \\ 1 \\ \infty \end{cases}, \qquad y = \begin{cases} 0 \\ 1 \\ \infty \end{cases}, \qquad x = y,$$

and the three blown up points.

THEOREM 7.16. — *If* $\mu = (\mu_0, \ldots, \mu_{n+2})$ *satisfies* INT, *then* $n \leq 5$.

8. Lattices generated by C-reflections : Part II, $n \leq 9$

The groups constructed in § 7 via monodromy are in fact generated by **C**-reflections : the monodromy of the i^{th} puncture in P_0 around the j^{th} puncture is a pseudo-reflection if $\mu_i + \mu_j \leq 1$ and is a **C**-reflection if $i \neq j$, $\mu_i + \mu_j < 1$ (*cf.* [7], § 9.1). It is natural to wonder about the relation, in case $n = 2$, of the subgroups $\Gamma(p,t)$ and Γ_μ of $PU(2,1)$. For p odd, the $\Gamma(p,t)$ are not commensurable with Γ_μ where μ satisfies INT. However, the family of μ for which Γ_μ is a lattice can be enlarged in the following way.

(8.1) *Definition.* — A sequence $\mu = (\mu_0, \ldots, \mu_{n+2})$ satisfies condition ΣINT if and only if it satisfies (7.12) and in addition for some subset $S_1 \subset \{0, 1, \ldots, n+2\}$.

$$\mu_i = \mu_j \quad \text{for all } i, j \in S_1$$

$$i \neq j \quad \text{and } \mu_i + \mu_j < 1 \quad \text{implies } (1 - \mu_i - \mu_j)^{-1} \in \begin{matrix} \tfrac{1}{2}\mathbf{Z} \text{ if } i, j, \in S_1 \\ \mathbf{Z}\text{, otherwise.} \end{matrix}$$

In [20], one proves

THEOREM 8.2. — *If μ satisfies ΣINT, then Γ_μ is a lattice in $PU(n,1)$.*

THEOREM 8.3. — *Let $n = 2$. Then there is a bijective correspondence between sequences μ satisfying ΣINT with $\mu_0 = \mu_1 = \mu_2$ and pairs (p,t), $3 \leq p$, with $|t| < 3(\frac{1}{2} - \frac{1}{p})$ if $p \leq 5$ and $|t| < \frac{1}{2} + \frac{3}{p}$ if $p \geq 6$.*

The correspondence is :

$$(p,t) \to \mu = (\frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{1}{4} + \frac{3}{2p} - \frac{1}{2}t, \frac{1}{4} + \frac{3}{2p} + \frac{1}{2}t)$$

$$\mu \to (p,t) = ((\frac{1}{2} - \mu_0)^{-1}, \mu_4 - \mu_3).$$

Moreover
$$\mathrm{card}(\Gamma_\mu / \Gamma_\mu \cap \Gamma(p,t)) = 1 \text{ or } 3$$
$$\mathrm{card}(\Gamma(p,t) / \Gamma_\mu \cap \Gamma(p,t)) = 1 \text{ or } 6.$$

Note. — One can use this correspondence to obtain new presentations for the lattices Γ_μ.

Thus, the lattices $\Gamma(p,t)$ defined in 1978 by the construction described in § 6 are commensurable with those arising from the monodromy groups Γ_μ, even though not quite for the parameters considered by PICARD.

Picard's conditions and our condition INT are sufficient though not necessary for Γ_μ to be discrete in $PU(n,1)$. The sufficient condition ΣINT actually turns out to be necessary if $n > 1$. Namely, I have proved

THEOREM 8.4. — *Assume* $n > 1$ *and* $\mu = (\mu_0, \mu_1, \ldots, \mu_{n+2})$ *with* $0 < \mu_i < 1$, $\sum_i \mu_i = 2$ $(0 \le i \le n+2)$. *Then the following three conditions are equivalent*

(i) Γ_μ *is closed in* $PU(n,1)$.
(ii) Γ_μ *is a lattice in* $PU(n,1)$.
(iii) Γ_μ *satisfies condition* ΣINT.

See [21] for the proof.

THEOREM 8.5. — *If* $\mu = (\mu_0, \ldots, \mu_{n+2})$ *satisfies* ΣINT, *then* $n \le 9$.

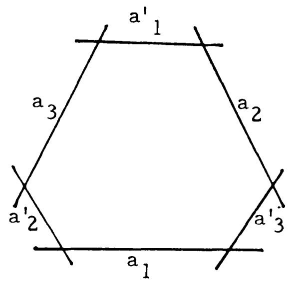
9. Negatively curved surfaces not covered by the ball

We take up again the groups $\Gamma(p,t)$ of § 6. Even when $\Gamma(p,t)$ is not discrete in $PU(2,1)$, it is still of considerable interest from the point of view of complex differential geometry.

In LEMMA 6.4 we have seen that the region F satisfies the condition (CD1) of matching 3-faces for all (p,t) with $3 \le p \le 5$. This enables one to piece together abutting images of F under $\Gamma(p,t)$, ignoring overlapping interiors, and so to arrive at a cell complex $Y(p,t)$ on which $\Gamma(p,t)$ operates by permuting the cells. If t is a rational number, it is proved in [18] that $Y(p,t)$ is in fact a non-singular complex analytic manifold. Upon dividing $Y(p,t)$ by a torsion-free subgroup Γ_0 of finite index in $\Gamma(p,t)$, one obtains a compact complex analytic manifold which is indeed a Kaehler manifold (*cf.* MOSTOW-SIU [23]).

If the codimension-2 condition (CD2) fails it fails on one of six complex geodesic lines in $\mathbf{Ch}^2$ which schematically we may draw

(9.1)



These six lines may be taken as passing through two dimensional faces of the region F, and the figure is invariant under the cyclic permutation of axes $e_1 \to e_2 \to e_3$ in the complex vector space V of § 6. Furthermore, all the intersections are pictured, and are at right angles.

For certain values of t (an infinite number), one gets condition (CD2) satisfied for the a_i' but not for the $a_i (i = 1,2,3)$. The orbit of the a_i in $Y(p,t)$ are all disjoint. In joint work with SIU (*cf.* [23]) one could use this fact to introduce a $\Gamma(p,t)$-invariant Kaehler-metric on $Y(p,t)$ with strictly negative sectional curvature. The resulting quotient $R = Y(p,t)/\Gamma_0$ is thus a compact Kaehler manifold of strictly negative sectional curvature.

One computes its characteristic classes and finds (*cf.* [23], p. 360)

$$\frac{c_1^2}{c_2} = \frac{\frac{3}{8}\left(\frac{3}{2} - \frac{6}{p} + \frac{6}{2} - 2t^2\right) + \frac{5(m-1)}{4p}\left(\frac{1}{2} - \frac{1}{p} + t\right)}{\frac{1}{3} - \frac{1}{p} + \frac{1}{N} - \left(\frac{1}{2} - \frac{1}{p}\right)\left(\frac{1}{\rho} + \frac{1}{\sigma}\right) + \frac{1}{\rho\sigma}} \tag{9.2}$$

where m is the branching order of $Y(p,t) \to \mathbf{C}\mathrm{h}^2$ along a_i,

$$\rho\left(\frac{1}{2} - \frac{1}{p} + t\right) = 2m, \qquad \sigma\left(\frac{1}{2} - \frac{1}{p} - t\right) = 2$$

$$N = 24\left(\frac{p}{6-p}\right)^2.$$

The ratio $c_1^2/c_2 = 3$ if and only if $m = 1$, in which case $Y(p,t) \approx \mathbf{C}\mathrm{h}^2$. For $m \neq 1$, one gets a strict-negatively curved Kaehler manifold with $c_1^2/c_2 \neq 3$; thus its universal covering manifold is not biholomorphic to the ball – contrary to what had been previously conjectured. This construction also gave the first known example of negatively curved Riemannian fourfold not diffeomorphic to a locally symmetric manifold.

In view of THEOREM 8.4, the spaces $Y(p,t)$ constructed above are merely special cases of the space $\widetilde{Q}_{st}$ of § 8, and it should be possible to construct spaces such as R above as ramified covers of the compact space Q_{sst} of § 7. However, in 1981, even before THEOREM 8.4 was discovered, SIU and I considered branched Galois covering R of the complex two manifold Q_*, the blow up of $P \times P$ at $(0,0)$, $(1,1)$, (∞,∞), with R ramifying over each of the ten exceptional lines.

The multivalued hypergeometric map ω_μ of § 8 maps Q_* to the ball with each exceptional line of Q_* going to the Γ_μ orbit of the fixed point set X_{ij} of the monodromy moving puncture i around puncture j, $0 \leq i \neq j \leq 4$; we denote the exceptional line of Q_* that the multivalued map ω_μ sends to X_{ij} by E_{ij}. We denote by *area* E_{ij} the area in X_{ij} of a slit domain in E_{ij}; inasmuch as the monodromy group Γ_μ operates by isometries in the ball, area E_{ij} is well defined.

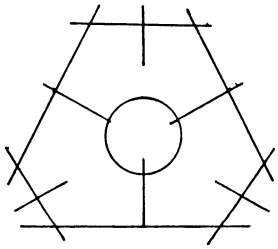
Then, writing (ij) for the unordered pair $0 \le i \ne j \le 4$ and writing $1 - \mu_i - \mu_j = (l_{ij}/k_{ij})$ in lowest terms, we have

$$\text{(9.3)} \qquad \frac{c_1^2(R)}{c_2(R)} = \frac{5 + \sum_{(ij)} \left(1 - \frac{1}{k_{ij}}\right)\left(4 + \frac{l_{ij}}{k_{ij}} + \frac{1}{k_{ij}}\right)\left(\frac{1}{2} - \frac{l_{ij}}{k_{ij}}\right) - \sum_{(ij)(rs)} \left(1 - \frac{1}{k_{ij}}\right)\left(1 - \frac{l_{rs}}{k_{rs}} + \frac{1}{k_{rs}}\right)}{7 + \sum_{(ij)} \left(1 - \frac{1}{k_{ij}}\right) - \sum_{(ij)(rs)} \left(1 - \frac{1}{k_{ij}k_{rs}}\right)}$$

with i, j, r, s distinct

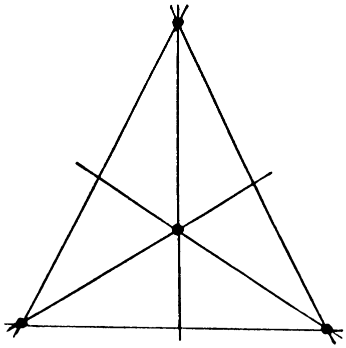
Schematically, the family of 10 lines $\{E_{(ij)}; 0 \le i \ne j < 4\}$ can be pictured as

(9.4)



This family can be obtained by fixing four points in complex projective 2-space P^2, taking the arrangement of all six lines through them, and blowing up the four three-fold intersections, thereby getting

(9.5)



ten exceptional lines in the blowup. This fact has led F. HIRZEBRUCH to consider more general line arrangements and the surfaces resulting from ramified covers (*cf.* [9]) and obtains in this way surfaces not covered by the ball as well as covered by the ball.

10. Some open problems

We turn to the question : is the Margulis theorem on arithmeticity of lattices sharp? SELBERG's original conjecture was less definite than the hypothesis that the group have-**R**-rank greater than one. All of the examples so far suggest the :

CONJECTURE 1. — *Except in low dimensions, lattices in* **R***-rank 1 groups are arithmetic*

At present, the only known exceptions are $PO(n,1)$ for $n \leq 5$ and $PU(n,1)$ for $n \leq 3$ (up to local isomorphism).

A related problem is

CONJECTURE 2. — *Except in low dimensions, there are no lattices in* $PU(n,1)$ *generated by complex reflections*

Conjecture 2 would imply that in higher dimensions, one cannot get the ball as the universally branched cover of a simply connected compact variety with prescribed ramification data along **C**-codimension 1 subvarieties. At present, examples are known only for $n \leq 9$ (*cf.* [20]).

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