V. G. KAC
D. H. PETERSON

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DEFINING RELATIONS OF CERTAIN INFINITE DIMENSIONAL GROUPS

BY

V.G. KAC and D.H. PETERSON

In our papers [8], [4], [5], we began a systematic study of the "smallest" group $G(A)$ associated to a Kac-Moody algebra and of its "unitary form" $K(A)$. The groups $G(A)$ and $K(A)$ are connected simply-connected topological groups, in general infinite-dimensional. A complex semisimple (resp. compact) connected simply-connected Lie group G (resp. K), and a certain central extension by $\mathbf{C}^\times$ (resp. S^1) of the group of polynomial maps of $\mathbf{C}^\times$ into G (resp. S^1 into K), provide the simplest examples of such groups $G(A)$ (resp. $K(A)$).

In the present paper, we define the groups $G(A)$ axiomatically, without reference to the corresponding Kac-Moody algebras. We then give a detailed exposition of the structure theory of the group $G(A)$ sketched in [8]. For that, we develop a theory of "refined Tits systems" (§ 3), which are groups satisfying certain axioms which describe the groups $G(A)$ more adequately than the axioms of usual Tits systems. In a similar, axiomatic fashion, we study the groups $K(A)$.

The second objective of the paper is to establish presentation theorems for the groups $G(A)$ and $K(A)$. In fact, both are special cases of a general presentation theorem for certain subgroups of a group with the structure of a Tits system (THEOREM A). The presentation theorem for $G(A)$ states that this group is an amalgamated product of its "standard parabolic subgroups of rank ≤ 2 " (this follows also from a theorem of TITS [9]). On the other hand, one can reduce the problem of explicit presentation of $G(A)$ to that of the "Borel subgroup" of $G(A)$ in terms of its generating 1–parameter subgroups. We solve the latter problem in the rank 2 case (PROPOSITIONS 3.5 and 4.3) and state a conjecture in the general case. As an application

(COROLLARY 3.5), we generalize a theorem of NAGAO [9].

The presentation of $K(A)$ is especially simple and elegant (THEOREM B). It is achieved by decomposing $K(A)$ into a disjoint union of "cells", which also provides a solution to the word problem. Loosely speaking, our presentation is a "real-analytic" continuation of a presentation of an extension of a certain Coxeter group $W(A)$ by a power of $\mathbf{Z}/2\mathbf{Z}$. More precisely, we show that $K(A)$ is an amalgamated product of compact groups of semisimple rank one and two, and moreover, write the relations among the subgroups of rank one explicitly.

The "cellular decomposition" of $K(A)$ mentioned above may be regarded as an algebraic fact underlying the cellular decomposition of the associated flag variety. This decomposition plays a key role in our forthcoming work on the topological structure of the groups $K(A)$ [7].*

A weaker form of the presentation theorem for compact groups was obtained in [2] by making use of a topological argument, which does not generalize to the infinite-dimensional situation. THEOREM B shows that the definition of $K(A)$ given in [2] coincides with ours.

THEOREM B was presented at the conference "Combinatorics and algebraic groups" in Oberwolfach in June 1983 and in a lecture course by the first author at the University of Paris in the fall of 1983. After writing this paper, we learned about the paper [13], where a presentation theorem for compact Lie groups is proved by a similar method.

It is a pleasure to acknowledge the two main sources of inspiration during our work on this paper : the book of STEINBERG [10] and the lectures [12] by and discussions with TITS.

1. Coxeter systems

Let S be a finite set, and let $(m_{s,t})_{s,t\in S}$ be a *Coxeter matrix* on S, i.e., a symmetric matrix of non-negative integers such that $m_{s,t} = 1$ if and only if $s = t$. Let W be the associated *Coxeter group*, i.e., W is the group on generators S with defining relations

$$(st)^{m_{s,t}} = 1 \text{ for } s,t \in S.$$

(Note that for $s = t$, this relation gives $s^2 = 1$.) The pair (W, S) is called a *Coxeter system*. If J is a subset of S, then W_J denotes the subgroup of W generated by J.

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* A description of some of the results of this work is contained in the paper of the first author *Constructing groups associated to infinite-dimensional Lie algebras*, MSRI publications # 4, Springer-Verlag, 1985.

Given $w \in W$, an expression $w = s_1 \cdots s_k$, where $s_1, \ldots, s_k \in S$, is called *reduced* if k is minimal possible, and one writes $l(w) = k$.

The following two operations on words on S are called *elementary* :

(E1) delete a consecutive subword ss;

(E2) replace a consecutive subword $sts \cdots$ ($m_{s,t}$ factors) by $tst \cdots$ ($m_{s,t}$ factors).

Now we can state the first crucial lemma of the paper.

LEMMA 1.1. — *Any two words on S representing the same element of W can be transformed to a common word by elementary operations.*

Proof. — This follows from [1, Ch. IV, § 1, n° 1.5, PROPOSITION 4 and LEMMA 4]. ∎

COROLLARY 1.1. — *If R and R' are reduced expressions of an element of a Coxeter group W, then R' can be obtained from R by elementary operations of the form* (E2). ∎

Let $A = (a_{s,t})_{s,t \in S}$ be a *generalized Cartan matrix*, i.e. $a_{s,s} = 2$, $a_{s,t}$ is a non-positive integer for $s \neq t$, and $a_{s,t} = 0$ implies $a_{t,s} = 0$. Put $m^A_{s,s} = 1$ and, for distinct $s, t \in S$, put $m^A_{s,t} = 2, 3, 4, 6$ or 0 according as $a_{s,t}a_{t,s} = 0, 1, 2, 3$ or ≥ 4. Let $(W(A), S)$ be the Coxeter system associated to the Coxeter matrix $(m^A_{s,t})$.

Let Q and Q^v be free abelian groups on symbols α_s and $\alpha^v_s, s \in S$, respectively. Define a bilinear pairing $Q \times Q^v \to \mathbf{Z}$ by $\langle \alpha_t, \alpha^v_s \rangle = a_{s,t}$.

LEMMA 1.2. — *The formulas*

$$s \cdot \alpha_t = \alpha_t - a_{s,t}\alpha_s \ ; \qquad s \cdot \alpha^v_t = \alpha^v_t - a_{t,s}\alpha^v_s \tag{1.1}$$

define faithful actions of the group $W(A)$ by automorphisms of Q and Q^v respecting the pairing $\langle \ , \ \rangle$.

Proof. — See e.g. [3, PROPOSITION 3.13]. ∎

Remark. — If every off-diagonal entry of a Coxeter matrix is $2, 3, 4, 6$ or 0, then the associated Coxeter group is called *crystallographic* since then, by LEMMA 1.2, it has a faithful reflection representation by integral matrices (the converse is also true). These are precisely the Coxeter groups appearing in the sequel as the Weyl groups of certain infinite-dimensional groups $G(A)$; the lattices Q and Q^v will apear as the root and coroot lattices of the group $G(A)$. The Coxeter system $(W(A), S)$ and its action on Q (or Q^v) determines the group $G(A)$ uniquely.

2. The group $G(A)$

Let $A = (a_{s,s'})_{s,s' \in S}$ be a generalized Cartan matrix. We associate to A a group $G(A)$ as follows.

For $t \in \mathbf{C}^\times$ and $u \in \mathbf{C}$, introduce the following elements of $SL_2(\mathbf{C})$:

$$h(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, \quad x(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}, \quad y(u) = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}.$$

Let ϵ denote the compact involution of $SL_2(\mathbf{C})$, i.e. $\epsilon(a) = {}^t\overline{a}^{-1}$, so that the fixed point set of ϵ is SU_2.

The following axioms (G1), (G2) and (G3) determine, up to a unique isomorphism, a group $G(A)$ and homomorphisms $\varphi_s : SL_2(\mathbf{C}) \to G(A)$ for $s \in S$. Here and further on, $\varphi_s(h(t))$, $\varphi_s(x(u))$ and $\varphi_s(y(u))$ are denoted by $h_s(t)$, $x_s(u)$ and $y_s(u)$, for short.

(G1) There exists a faithful $G(A)$-module (V, π) over $\mathbf{C}$ such that each $SL_2(\mathbf{C})$-module $(V, \pi \circ \varphi_s)$ is a direct sum of rational finite-dimensional submodules.

(G2) a) $h_s(t)x_{s'}(u)h_s(t)^{-1} = x_{s'}(t^{a_{s,s'}}u)$ and
$h_s(t)y_{s'}(u)h_s(t)^{-1} = y_{s'}(t^{-a_{s,s'}}u)$
for all $s, s' \in S$, $t \in \mathbf{C}^\times$ and $u \in \mathbf{C}$;

b) $x_s(u)y_{s'}(v) = y_{s'}(v)x_s(u)$
for all distinct $s, s' \in S$ and all $u, v \in \mathbf{C}$.

(G3) If a group G and homomorphisms $\varphi'_s : SL_2(\mathbf{C}) \to G (s \in S)$ satisfy (G1) and (G2), then there exists a unique homomorphism $\psi : G(A) \to G$ such that $\varphi'_s = \psi \circ \varphi_s$ for all $s \in S$.

Put $G_s = \varphi_s(SL_2(\mathbf{C}), s \in S$. It follows from the axioms that the subgroups G_s, $s \in S$, generate the group $G(A)$. Put $H_s = \{h_s(t) | t \in \mathbf{C}^\times\}$, and let H be the subgroup of $G(A)$ generated by the subgroups H_s. Since the $x(u)$ and $y(u)$ generate $SL_2(\mathbf{C})$, (G2a) implies

$$(2.1) \qquad h_s(t)\varphi_{s'}\begin{pmatrix} a & b \\ c & d \end{pmatrix} h_s(t)^{-1} = \varphi_{s'}\begin{pmatrix} a & t^{a_{s,s'}}b \\ t^{-a_{s,s'}}c & d \end{pmatrix}.$$

In particular, H is abelian.

In order to proceed, we need a digression on Kac-Moody algebras.

Recall that the *Kac-Moody algebra* $\mathbf{g}'(A)$ associated to a generalized Cartan matrix A is the Lie algebra on generators e_s, f_s, α_s^v, $s \in S$, with the following defining relations :

(g1) $[\alpha_s^v, e_t] = a_{s,t}e_t$; $[\alpha_s^v, f_t] = -a_{s,t}f_t$; $[e_s, f_t] = 0$ if $s \neq t$;

(g2) $[e_s, f_s] = \alpha_s^v$; $[\alpha_s^v, \alpha_t^v] = 0$;

(g3) $(\operatorname{ad} e_s)^{1-a_{s,t}} e_t = 0, \quad (\operatorname{ad} f_s)^{1-a_{s,t}} f_t = 0 \quad \text{if } s \neq t.$

Then the α_s^v are linearly independent [3, Chapter 1] and the group $W(A)$ acting on the *coroot lattice* $Q^v = \sum_{s \in S} \mathbf{Z}\alpha_s^v$ by (1.1) is called the *Weyl group* of $\mathbf{g}'(A)$. For brevity, we write, W_J for $W(A)_J$ if $J \subset S$.

The Lie algebra $\mathbf{g}'(A)$ admits a gradation $\mathbf{g}'(A) = \bigoplus_{\alpha \in Q} \mathbf{g}_\alpha$ by the free abelian group Q on symbols α_s, $s \in S$, which is called the *root lattice*, such that $\mathbf{g}_0 = \bigoplus_s \mathbf{C}\alpha_s^v$, $\mathbf{g}_{\alpha_s} = \mathbf{C}e_s$ and $\mathbf{g}_{-\alpha_s} = \mathbf{C}f_s$ [3, Chapter 1]. The height of $\sum_s k_s\alpha_s \in Q$ is $\sum_s k_s$.

Let $\Delta = \{\alpha \in Q \mid \mathbf{g}_\alpha \neq 0, \alpha \neq 0\}$ be the set of *roots* of $\mathbf{g}'(A)$; it is $W(A)$-invariant [3, Chapter 3]. Put $\mathbf{Z}_+ = \{0,1,2,\dots\}$ and $Q_+ = \sum_s \mathbf{Z}_+\alpha_s \subset Q$. Elements of $\Delta_+ := Q_+ \cap \Delta$ are called *positive* roots. One knows that $\Delta = \Delta_+ \sqcup -\Delta_+$ ($\sqcup$ denotes a disjoint union). Elements of $\Delta^{\mathrm{re}} := \{w \cdot \alpha_s \mid w \in W(A), s \in S\}$ are called *real roots.* Put $\Delta_+^{\mathrm{re}} = \Delta^{\mathrm{re}} \cap \Delta_+$; then $\Delta^{\mathrm{re}} = \Delta_+^{\mathrm{re}} \sqcup -\Delta_+^{\mathrm{re}}$ (see [3, Chapters 1 and 5] for details).

In § 4, we will need

LEMMA 2.1.

(a) *If $w \in W(A)$ and $w \neq 1$, then there exists $s \in S$ such that $w \cdot \alpha_s \in -\Delta_+^{\mathrm{re}}$*

(b) *If J is a subset of S, then*

$$\bigcap_{w \in W_J} w \cdot \Delta_+^{\mathrm{re}} = \Delta_+^{\mathrm{re}} \setminus \sum_{s \in J} \mathbf{Z}\alpha_s.$$

(c) *If $s \in S$, then the set $\Phi_s := \{\beta \in \Delta_+^{\mathrm{re}} \setminus \mathbf{Z}\alpha_s \mid \langle \alpha_s^v, \beta \rangle \geq 0\}$ satisfies the following two properties.*

(i) $\Delta_+^{\mathrm{re}} = \Phi_s \cup (s \cdot \Phi_s) \cup \{\alpha_s\}$;

(ii) *if $\beta \in \Phi_s$, then*
$\Delta_+ \cap (\beta + \mathbf{Z}_+\beta + \mathbf{Z}_+\alpha_s) = \Phi_s \cap \{\beta, \beta + \alpha_s\}$.

Proof. — (a) is proved e.g. in [3, LEMMA 3.11]. Since $\langle \alpha_s, w \cdot \alpha_t^v \rangle > 0 \Leftrightarrow \langle w \cdot \alpha_t, \alpha_s^v \rangle > 0$ for all $s, t \in S$ and $w \in W(A)$ by [6, p. 139], the argument proving [8, LEMMA 1] proves (c). (These arguments are reproduced also in [3, 2nd ed., Exercise 5.19].)

To prove (b), first note that, for any $\beta \in Q$, $\beta + \sum_{s \in J} \mathbf{Z}\alpha_s$ is W_J-invariant. Hence if $\beta \in Q$ and $W_J \cdot \beta$ intersects Q_+ and $-Q_+$, then $\beta \in \sum_{s \in J} \mathbf{Z}\alpha_s$. This shows that $\Delta_+^{\mathrm{re}} \setminus \sum_{s \in J} \mathbf{Z}\alpha_s$ is W_J-invariant, so that $\Delta_+^{\mathrm{re}} \setminus \sum_{s \in J} \mathbf{Z}\alpha_s \subset \bigcap_{w \in W_J} w \cdot \Delta_+^{\mathrm{re}}$. Conversely, if $\beta \in \bigcap_{w \in W_J} w \cdot \Delta_+^{\mathrm{re}}$, choose $\gamma \in W_J \cdot \beta$ of minimal height. Then $\gamma \in \Delta_+^{\mathrm{re}}$, and $\langle \gamma, \alpha_s^v \rangle \leq 0$ for all $s \in J$ since $s \cdot \gamma = \gamma - \langle \gamma, \alpha_s^v \rangle \alpha_s$. If also $\gamma \in \sum_{s \in J} \mathbf{Z}\alpha_s$, then $\gamma \in \sum_{s \in J} \mathbf{Z}_+\alpha_s$ forces $\langle \gamma, \alpha_s^v \rangle \leq 0$ for all $s \in S \setminus J$, since $\langle \alpha_t, \alpha_s^v \rangle \leq 0$ for all distinct $s, t \in S$, so that $\langle \gamma, \alpha_s^v \rangle \leq 0$ for all $s \in S$, which by [3, PROPOSITION 5.1e] contradicts $\gamma \in \Delta_+^{\mathrm{re}}$. This proves (b). ∎

A complex $G(A)$-module (V,π) is called *differentiable* if the $SL_2(\mathbf{C})$-modules $(V,\pi\circ\varphi_s)$ are direct sums of rational finite-dimensional submodules. Given such a module, we have a module $(V,d\pi)$ over $\mathbf{g}'(A)$ defined by :

$$d\pi(e_s)=\frac{d}{du}\pi(x_s(u))\Big|_{u=0},\qquad d\pi(f_s)=\frac{d}{du}\pi(y_s(u))\Big|_{u=0},$$
$$d\pi(\alpha_s^v)=\frac{d}{dt}\pi(h_s(t))\Big|_{t=1}.$$

To check this, we have to show that the relations (g1)–(g3) are annihilated by π. Indeed, (g1) follows from (G2); the first part of (g2) is standard and the second part is clear from (2.1); (g3) follows from (g1) and (g2) by [4, LEMMA 1.1]. Moreover, the $\mathbf{g}'(A)$-module $(V,d\pi)$ is *integrable* (in the terminology of [8]), i.e. all $d\pi(e_s)$ and $d\pi(f_s)$ are locally nilpotent. Conversely, an integrable $\mathbf{g}'(A)$-module $(V,d\pi)$ gives rise to a unique differentiable G(A)-module (V,π) satisfying $\pi(x_s(u))=\exp d\pi(ue_s)$, $\pi(y_s(u))=\exp d\pi(uf_s)$, $u\in\mathbf{C}$. It follows that the definition of the group G(A) by axioms (G1)–(G3) coincides with that of [8].

If $s,t\in S$ and $a_{s,t}=a_{t,s}=0$, then (g1) and (g3) show that e_s and f_s commute with e_t and f_t, and therefore G_s and G_t commute.

The adjoint $\mathbf{g}'(A)$-module $(\mathbf{g}'(A),\mathrm{ad})$ gives rise to the adjoint $G(A)$-module $(\mathbf{g}'(A),\mathrm{Ad})$, which is related to a differentiable $G(A)$-module (V,π) by

(2.2) $$d\pi(\mathrm{Ad}(g)x)=\pi(g)d\pi(x)(g)^{-1}\text{ for } g\in G(A),\ x\in\mathbf{g}'(A).$$

This follows from the well-known formula $(\exp d\pi(a))d\pi(x)(\exp -d\pi(a))=d\pi((\exp\mathrm{ad}\,a)x)$, for any elements x and a of a Lie algebra and any of its modules $d\pi$ such that $\mathrm{ad}\,a$ and $d\pi(a)$ are locally nilpotent (see e.g. [3, (3.8.1)]).

It is convenient to introduce an exponential map exp from certain subset of $\mathbf{g}'(A)$ into $G(A)$, as follows. Let $x\in\mathbf{g}'(A)$ be such that $d\pi(x)$ is locally-finite for every integrable $\mathbf{g}'(A)$-module $(V,d\pi)$. If there exists $g\in G(A)$ such that $\pi(g)=\exp d\pi(x)$ for every integrable $\mathbf{g}'(A)$-module $(V,d\pi)$, we write : $g=\exp x$. It is shown in [8] that exp is defined on the set of all ad-locally-finite elements of $\mathbf{g}'(A)$ (but we will not use this fact). Note that $x_s(u)=\exp ue_s$, $y_s(u)=\exp uf_s$ and $h_s(e^u)=\exp u\alpha_s^v$ for all $s\in S$ and $u\in\mathbf{C}$. It follows from (2.2) that

(2.3) $$g(\exp x)g^{-1}=\exp(\mathrm{Ad}(g)x),\quad g\in G(A).$$

Using integrable highest weight $\mathbf{g}'(A)$-modules, one easily deduces as in [8] the following

LEMMA 2.2.

(a) *The homomorphism* $(\mathbf{C}^\times)^S \to G(A)$ *defined by* $(t_s)_{s\in S} \mapsto \prod_s h_s(t_s)$ *is an isomorphism onto* H.

(b) *The homomorphisms* φ_s *are injective*

(c) $G_s \cap G_{s'} = \{1\}$ *for* $s \neq s'$. ∎

Put $H_+ = \{\prod_s h_s(t_s) \mid (t_s)_{s\in S} \in \mathbf{R}_+^S\}$, where $\mathbf{R}_+$ denotes the multiplicative group of positive real numbers, and put $T = \{\prod_s h_s(t_s) \mid (t_s)_{s\in S} \in (S^1)^S\}$, where S^1 denotes the unit circle. The homomorphism of LEMMA 2.2(a) induces isomorphisms : $\mathbf{R}_+^S \xrightarrow{\sim} H_+$, $(S^1)^S \xrightarrow{\sim} T$. Note that $H = H_+ \times T$.

Put $\tilde{s} = \varphi_s\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $s \in S$; we have

$$\tilde{s}^2 = h_s(-1). \tag{2.4}$$

Recall formula (1.1). One knows that [3, LEMMA 3.8] :

$$\mathrm{Ad}(\tilde{s})\mathbf{g}_\alpha = \mathbf{g}_{s\cdot\alpha}; \quad \mathrm{Ad}(h)\mathbf{g}_\alpha = \mathbf{g}_\alpha \text{ for } h \in H. \tag{2.5}$$

Using (2.1), we have

$$\tilde{s}' h_s(t)\tilde{s}'^{-1} = h_s(t)h_{s'}(t^{-a_{s,s'}}) \text{ for } t \in \mathbf{C}^\times. \tag{2.6}$$

Another useful relation, obtained by calculating in $SL_2(\mathbf{C})$, is

$$y_s(t) = x_s(t^{-1})\tilde{s}h_s(-t)x_s(t^{-1}), \text{ for } t \in \mathbf{C}^\times. \tag{2.7}$$

LEMMA 2.3. — *If* $s \neq s'$, *then*

$$\tilde{s}\tilde{s}'\tilde{s}\cdots = \tilde{s}'\tilde{s}\tilde{s}'\ldots \qquad (m^A_{s,s'} \textit{ factors on each side}). \tag{2.8}$$

Proof ([11]). — We may assume that $m^A_{s,s'} \neq 0$. Let g and g' denote the left- and right-hand sides of (2.8). Then, putting $t = s$ or s' according as $m^A_{s,s'}$ is odd or even, we obtain, using (2.3) and (2.5) :

$$gG_tg^{-1} = G_{s'}.$$

(We also use the fact that $SL_2(\mathbf{C})$ is generated by the $x(u)$ and $y(u)$.) Therefore we have :

$$g'g^{-1} = \tilde{s}'g\tilde{t}^{-1}g^{-1} \in \tilde{s}'gG_tg^{-1} = \tilde{s}'G_{s'} = G_{s'}.$$

Interchanging s and s', we get $g'g^{-1} \in G_S$. By LEMMA 2.2(c), it follows that $g'g^{-1} = 1$. ∎

Remark. — If we take

$$\tilde{s} = \varphi_s \begin{pmatrix} 0 & t_s \\ -t_s^{-1} & 0 \end{pmatrix},$$

where the $t_s \in \mathbf{C}^\times$ are arbitrary, LEMMA 2.3 and its proof remain valid.

Let N be the subgroup of $G(A)$ generated by H and all the $\tilde{s}$, $s \in S$. Then H is a normal subgroup of N by (2.6). The group $W = N/H$ is called the *Weyl group* of $G(A)$.

PROPOSITION 2.1. — *There exists a unique isomorphism of W onto $W(A)$ taking $\tilde{s}H$ to s for all $s \in S$.*

Proof. — (2.5) and LEMMA 1.2 show that there exists a unique homomorphism from W to $W(A)$ taking $\tilde{s}H$ to s for all $s \in S$. Formulas (2.4) and (2.8) show that there exists a unique homomorphism from $W(A)$ to W taking s to $\tilde{s}H$ for all $s \in S$. ∎

Using PROPOSITION 2.1, we identify S with a subset of W by identifying $s \in S$ with the coset $\tilde{s}H \in N/H = W$. In the same way, we sometimes also identify $W(A)$ and W.

COROLLARY 2.1.

(a) *(W, S) is a Coxeter system with Coxeter matrix $(m^A_{s,s'})_{s,s' \in S}$.*

(b) *N is the group on generators $\tilde{s}(s \in S)$ and $h_s(t)$ $(s \in S$ and $t \in \mathbf{C}^\times)$ with defining relations :*

(N1) $h_s(t)h_s(t') = h_s(tt')$;
(N2) $h_s(t)h_{s'}(t') = h_{s'}(t')h_s(t)$;
(N3) $\tilde{s}'h_s(t)\tilde{s}'^{-1} = h_s(t)h_{s'}(t^{-a_{s,s'}})$;
(N4) $\tilde{s}^2 = h_s(-1)$;
(N5) $\tilde{s}\tilde{s}'\tilde{s}\cdots = \tilde{s}'\tilde{s}\tilde{s}'\cdots$ *($m^A_{s,s'}$ factors on each side).*

Proof. — (a) is immediate from PROPOSITION 2.1. Let N_0 be the group with the generators and relations in (b), and let H_0 be the abelian normal subgroup of N_0 generated by the $h_s(t)$, $s \in S$ and $t \in \mathbf{C}^\times$. Since the relations $(N1 - N5)$ hold in N by formulas (2.1), (2.4), (2.6) and (2.8), there exists a homomorphism μ of N_0 onto N mapping the generators to the corresponding elements of N. By $(N1)$, $(N2)$ and LEMMA 2.2(a), there exists a homomorphism φ of H onto H_0 such that $\mu \circ \varphi = \mathrm{id}_H$. Hence, $H_0 \cap \ker\mu = \{1\}$. But $H_0 = \mu^{-1}(H)$ by (a), so that $\ker\mu \subset H_0$. Hence, $\ker\mu = \{1\}$, proving (b). ∎

COROLLARY 2.2. — *The centralizer of H in N is H.*

Proof. — H is clearly an abelian normal subgroup of N. Since $\mathbf{C}$ is an infinite field, the corollary now follows from PROPOSITION 2.1, LEMMA 1.2 and formula (2.6). ∎

Let $\widetilde{W}$ be the subgroup of N generated by the $\tilde{s}$, $s \in S$, and let $H_{(2)}$ be the subgroup of H generated by the $\tilde{s}^2 = h_s(-1)$, $s \in S$. (Note that $\widetilde{W}$ is the fixed point set in N of the involution of $G(A)$ defined by $x_s(u) \leftrightarrow y_s(-u)$.)

COROLLARY 2.3.

(a) $H_{(2)} = \{h \in H | h^2 = 1\}$, *and the inclusion* $\widetilde{W} \subset N$ *induces an isomorphism from* $\widetilde{W}/H_{(2)}$ *onto* $W = N/H$.

(b) *There exists a unique map* $w \mapsto \tilde{w}$ *from* W *into* $\widetilde{W}$ *satisfying*

(i) $\tilde{1} = 1$;

(ii) $\tilde{s} = \varphi_s\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *for all* $s \in S$;

(iii) $\widetilde{ww'} = \tilde{w}\tilde{w}'$ *if* $w, w' \in W$ *and* $l(ww') = l(w) + l(w')$.

If $\psi : \widetilde{W} \to W$ *is the canonical map, then* $w \mapsto \tilde{w}$ *is a well-defined section of the map* ψ.

Proof. — $H_{(2)} = \{h \in H | h^2 = 1\}$ by LEMMA 2.2(a). By PROPOSITION 2.1 and LEMMA 2.3, $N = \widetilde{W}H$ and $\widetilde{W} \cap H$ is generated by the $\widetilde{W}$-conjugates of the $\tilde{s}^2$, $s \in S$. (a) follows. (b) follows from LEMMA 2.3 and COROLLARY 1.1. ∎

COROLLARY 2.4. — $\widetilde{W}$ *is the group on generators* $\tilde{s}$, $s \in S$, *with defining relations :*

(n1) $\tilde{t}\tilde{s}^2\tilde{t}^{-1} = \tilde{s}^2\tilde{t}^{-2a_{s,t}}$.

(n2) $\tilde{s}\tilde{t}\tilde{s}\cdots = \tilde{t}\tilde{s}\tilde{t}\cdots$ *($m^A_{s,t}$ factors on each side).*

Proof. — For $s \in S$, put $h_s = \tilde{s}^2$. Then (n1) and (n2) imply :

(m1) $h_s^2 = 1$;

(m2) $h_s h_t = h_t h_s$;

(m3) $\tilde{t}h_s\tilde{t}^{-1} = h_s h_t^{-a_{s,t}}$;

(m4) $\tilde{s}^2 = h_s$;

(m5) $\tilde{s}\tilde{t}\tilde{s}\cdots = \tilde{t}\tilde{s}\tilde{t}\cdots$ ($m_{s,t}$ factors on each side).

Indeed, (m3), (m4) and (m5) are clear, and (m1) follows from (n1) with $t = s$. To check (m2), write $h_t h_s h_t^{-1} = \tilde{t}(\tilde{t}h_s\tilde{t}^{-1})\tilde{t}^{-1} = \tilde{t}(h_s h_t^{-a_{s,t}})\tilde{t}^{-1} = (\tilde{t}h_s\tilde{t}^{-1})(\tilde{t}h_t^{-a_{s,t}}\tilde{t}^{-1}) = (h_s h_t^{-a_{s,t}})(h_t^{-a_{s,t}} h_t^{2a_{s,t}}) = h_s$ by (m3) and (m4). This verifies (m2).

The rest of the proof is essentially the same as that of COROLLARY 2.1(b). (One uses $-1 \neq 1$ in $\mathbf{C}^{\times}$ to construct the analogue of φ.) ∎

Introduce the 1-parameter subgroups $U_{\alpha_s} = \{x_s(u) \mid u \in \mathbf{C}\}$, $s \in S$, of $G(A)$. For a real root $\alpha = w \cdot \alpha_s$, take $n \in N$ such that $w = nH$ and put $U_\alpha = nU_{\alpha_s}n^{-1}$. We have $U_\alpha = n(\exp \mathbf{g}_{\alpha_s})u^{-1} = \exp(\mathrm{Ad}(n)\mathbf{g}_{\alpha_s}) = \exp \mathbf{g}_{w \cdot \alpha_s} = \exp \mathbf{g}_\alpha$; hence, the 1-parameter group U_α depends only on α. Note that $U_{-\alpha_s} = \{y_s(u) \mid u \in \mathbf{C}\}$.We have :

(2.9) $$nU_\alpha n^{-1} = U_{w \cdot \alpha} \text{ for } n \in N,\ w \in nH,\ \alpha \in \Delta^{\mathrm{re}}.$$

Recall that $\Delta^{\mathrm{re}} = \Delta^{\mathrm{re}}_+ \sqcup -\Delta^{\mathrm{re}}_+$. Let U_+ (resp. U_-) be the subgroup of G generated by the subgroups U_α (resp. $U_{-\alpha}$), $\alpha \in \Delta^{\mathrm{re}}_+$. (This definition is due to TITS [12]). These subgroups are analogues of maximal unipotent subgroups of reductive algebraic groups; they play an important role in the structure theory of the groups $G(A)$, which we will discuss in §§ 3 and 4.

Finally, it is clear from the axioms (G1)–(G3) that there exists a unique involution ω of $G(A)$ such that $\varphi_s \circ \epsilon = \omega \circ \varphi_s$ for all $s \in S$ (recall that ϵ is the compact involution of $SL_2(\mathbf{C})$). We call ω the *compact involution* of $G(A)$. It is clear that the subgroups G_s and H are stable under ω and that $\widetilde{W}$ is pointwise fixed by ω. Furthermore, $\omega(U_\alpha) = U_{-\alpha}$ for all $\alpha \in \Delta^{\mathrm{re}}$, and therefore $\omega(U_+) = U_-$.

Remark. — $\mathbf{g}'(A)$ can be characterized by axioms similar to (G1)–(G3). Also, the category of all integrable $\mathbf{g}'(A)$-modules and all $\mathbf{g}'(A)$-module homomorphisms is isomorphic in the obvious way to the category of all differentiable $G(A)$-modules over $\mathbf{C}$ and all $G(A)$-module homomorphisms, and this isomorphism is compatible with tensor products, etc.

3. Refined Tits Systems.

We call a 6-tuple (G, N, U_+, U_-, H, S) a *refined Tits system* if the following axioms hold* :

(RT1) G is a group, and N, U_+ and U_- are subgroups of G; G is generated by N and U_+; H is a normal subgroup of N; H normalizes U_+ and U_-; S is a subset of $W := N/H$; S generates W; $s^2 = 1$ for all $s \in S$.

For a subgroup M of G and $w = nH \in W$, we write wM for nM and Mw for Mn if $M \supset H$, and M^w for $n^{-1}Mn$ if H normalizes M.

(RT2) For $s \in S$, put $U_s = U_+ \cap U_-^s$. If $s \in S$ and $w \in W$, then :

* The reader may compare this definition with that of a split BN-pair, extensively used in finite group theory.

(a) $U_s^s \setminus \{1\} \subset U_s H s U_s$; $U_s^s \neq \{1\}$.
(b) $U_s^w \subset U_+$ or $U_s^w \subset U_-$.
(c) $U_+ = U_s(U_+ \cap U_+^s)$.

(RT3) If $u_- \in U_-$, $n \in N$, $u_+ \in U_+$ and $u_- n u_+ = 1$, then $u_- = n = u_+ = 1$.

Throughout this section, we assume only that (G, N, U_+, U_-, H, S) is a refined Tits system. We will show in § 4 that $(G(A), \ldots)$ is a refined Tits system.

Let B be the subgroup of G generated by H and U_+, so that $B = H \propto U_+$ by (RT1,3).

Remark. — If (G, N, U_+, U_-, H, S) is a refined Tits system, and if M is a subgroup of G such that $U_s \cup U_s^s \subset M$ for all $s \in S$, and M is generated by $N \cap M$ and $U_+ \cap M$, then $(M, N \cap M, U_+ \cap M, U_- \cap M, H \cap M, S_M)$ is a refined Tits system, where S_M corresponds to S under the isomorphism $(N \cap M)/(H \cap M) \xrightarrow{\sim} N/H$ induced by the inclusion $N \cap M \subset N$. In particular, the subgroup of G generated by the U_s and U_s^s, and the subgroup of G generated by N and the U_s, satisfy these conditions.

LEMMA 3.1.
(a) $B \cap N = H$.
(b) *If $s \in S$, then $sBs \neq B$.*
(c) *Let $s \in S$ and $w \in W$. Then :*
  (i) *Exactly one of the following holds :*
  $U_s^w \subset U_+$ *and* $U_s^{sw} \subset U_-$;
  $U_s^{sw} \subset U_+$ *and* $U_s^w \subset U_-$.
  (ii) $sBw \subset BswU_s^w$ *and* $sBw \subset Bsw \cup BwU_s^{sw}$.

Proof. — (a) follows from (RT3).

To prove (b), note that $U_s \cap sBs = (U_s^s \cap B)^s \subset (U_- \cap B)^s = \{1\} \not\supset U_s = U_s \cap B$.

To prove c(i), note that U_s^w is contained in exactly one of U_+ and U_-, and U_s^{sw} is contained in exactly one of U_+ and U_-. But by (RT2a), $U_s^w U_s^{sw} U_s^w \cap N \neq \{1\}$. Since $U_- \cap N = \{1\} = U_+ \cap N$ by (RT3), U_s^w and U_s^{sw} cannot both be contained in U_- or in U_+. This proves c(i).

To prove c(ii), we write $sBw = s[(U_+ \cap U_+^s) H U_s]w = (U_+ \cap U_+^s) H sw U_s^w \subset BswU_s^w$ and $sBw = (U_+ \cap U_+^s) U_s^s H sw \subset (U_+ \cap U_+^s)(\{1\} \cup U_s H s U_s) H sw \subset Bsw \cup BwU_s^{sw}$. ∎

LEMMA 3.1 shows that (G, B, N, S) is a Tits system (see [1] for the definition). The following are some well-known properties of Tits systems [1] :

(3.1) $G = \coprod_{w \in W} BwB$ (*Bruhat decomposition*);

(3.2) (W, S) is a Coxeter system;

(3.3) $l(sw) > l(w) \Leftrightarrow sBw \subset BswB$ for $s \in S$ and $w \in W$.

(3.4) $P_J := BW_JB$ is a subgroup of G for any $J \subset S$, and any subgroup of G containing B is of this form.

Since (W,S) is a Coxeter system by (3.2), we have its Coxeter matrix $(m_{s,t})_{s,t\in S}$ ($m_{s,t}$ is the non-negative integer satisfying $m_{s,t}\mathbf{Z} = \{n \in \mathbf{Z} \mid (st)^n = 1\}$).

The groups P_J of (3.4) are called standard *parabolic* subgroups of G. We sometimes write P_s for $P_{\{s\}}$, $s \in S$; these are called *minimal* standard parabolics. Note that for any $J \subset S$, $(P_J, W_J, H, U_+, U_- \cap P_J, H, J)$ is a refined Tits system.

COROLLARY 3.1. — *The normalizer of U_+ in G is B.*

Proof. — The normalizer, say P, of U_+ in G clearly contains B. If $P \neq B$, then $sH \subset P$ for some $s \in S$ by (3.4), so that sH also normalizes $B = HU_+$. This contradicts $sBs \neq B$ from LEMMA 3.1. ∎

Let I be a set, and let $(M_i)_{i\in I}$ be an indexed set of groups. For $i,j \in I$, let $M_{\{i,j\}}$ be a group and let $\varphi_{ij} = M_{\{i,j\}} \to M_i$ be a homomorphism. (Note that $M_{\{i,j\}} = M_{\{j,i\}}$.) The amalgamated product of the φ_{ij} is a pair $(M, (\varphi_i)_{i\in I})$, unique up to a unique isomorphism, satisfying :

(AP1) M is a group, and the $\varphi_i : M_i \to M$ are homomorphisms satisfying $\varphi_i \circ \varphi_{ij} = \varphi_j \circ \varphi_{ji}$ for all $i,j \in I$.

(AP2) If L is a group and if $\psi_i : M_i \to L$, $i \in I$, are homomorphisms satisfying $\psi_i \circ \varphi_{ij} = \psi_j \circ \varphi_{ji}$ for all $i,j \in I$, then there exists a unique homomorphism $\psi : M \to L$ satisfying $\psi_i = \psi \circ \varphi_i$ for all $i \in I$.

If the M_i are subgroups of a group F and φ_{ij} is the inclusion $M_i \cap M_j \subset M_i$ for all $i,j \in I$, then we say that the group M defined above is the *amalgamated product* of the M_i. If, moreover, the canonical homomorphism $\psi : M \to F$ defined by (AP2) is bijective, then we say that F is the *amalgamated product of its subgroups M_i*.

We say that a subgroup $M$of G is *W-graded* if, putting $M_w = M \cap BwB$, we have for all $w, w' \in W$:

$$M_{ww'} = M_w M_{w'} \quad \text{if} \quad l(ww') = l(w) + l(w'). \tag{3.5}$$

The next two results hold for arbitrary Tits systems.

THEOREM A.

(a) *Any W-graded subgroup M of G is the amalgamated product of its intersections with the P_J, $|J| \leq 2$.*

(b) *G and N are W-graded subgroups of G. If L is a W-graded subgroup of G, and if M is a subgroup of G satisfying $M(L\cap B) = L$, then M is a W-graded subgroup of G.*

(c) *Let L be a W-graded subgroup of G, and let Z_s, $s \in S$, be subsets of G such that $L \cap BsB = Z_s(L \cap B)$ for all $s \in S$. Let M be a subgroup of L containing the Z_s. Then M is a W-graded subgroup of G, and $M \cap BsB = Z_s(M \cap B)$ for all $s \in S$. For $s,t \in S$ and $z_1 \in Z_s$, $z_2 \in Z_t$, $z_3 \in Z_s, \ldots$, choose $z'_1 \in Z_t$, $z'_2 \in Z_s$, $z'_3 \in Z_t, \ldots$ and $b \in M \cap B$ such that*

$$z_1 z_2 z_3 \cdots = (z'_1 z'_2 z'_3 \cdots) b \quad (m_{s,t} \text{ factors } z \text{ on each side}). \tag{3.6}$$

Then M is the amalgamated product of $M \cap B$ and the $M \cap P_s$, $s \in S$, modulo the relations (3.6).

Proof. — Let L be a W-graded subgroup of G, put $B_L = L \cap B$, and let the Z_s, $s \in S$, be subsets of G satisfying $L \cap BsB = Z_s B_L$. Note that $L \cap P_s = Z_s B_L \cup B_L \supset B_L Z_s$. Since L is W-graded, we have $L \cap Bs_1 \cdots s_k B = (L \cap Bs_1 B) \cdots (L \cap Bs_k B) = (Z_{s_1} B_L) \cdots (Z_{s_k} B_L) = Z_{s_1} \cdots Z_{s_k} B_L$ for every reduced expression $s_1 \cdots s_k$. In particular, B_L and the Z_s generate L. Choose relations (3.6) as in (c) (with $M = L$), and let $\widetilde{L}$ be the amalgamated product of B_L and the $L \cap P_s$, $s \in S$, modulo the chosen relations. We may regard B_L and the Z_s as subsets of $\widetilde{L}$. We clearly have :

(i) B_L is a subgroup of $\widetilde{L}$.
(ii) Z_s, $s \in S$, is a subset of $\widetilde{L}$.
(iii) B_L and the Z_s generate $\widetilde{L}$.
(iv) For all $s \in S$, $B_L \cup Z_s B_L$ $(= L \cap P_s)$ is a subgroup of $\widetilde{L}$.
(v) For all $s,t \in S$, $Z_s Z_t Z_s \cdots B_L = Z_t Z_s Z_t \cdots B_L$
($m_{s,t}$ factors Z on each side).

Using LEMMA 1.1, we deduce that for every $g \in \widetilde{L}$, there exists a reduced expression $s_1 \cdots s_k$, where $s_1, \ldots, s_k \in S$, such that $g \in Z_{s_1} \cdots Z_{s_k} B_L$. Now let $\psi : \widetilde{L} \to L$ be the canonical surjective homomorphism defined by (AP2). If $\psi(g) = 1$, then by (3.1), $\psi(g) \in Bs_1 \cdots s_k B$ forces $k = 0$ and hence $g \in B_L$. Since ψ is the identity on B_L, we deduce that $g = 1$. Hence, ψ is bijective. This verifies (a) and also the case $M = L$ of (c).

We now prove (b). By (3.2) and (3.3), G and N are W-graded subgroups of G. Now let L be a W-graded subgroup of G and let M be a subgroup of G satisfying $M(L \cap B) = L$. For $w \in W$, put $L_w = L \cap BwB$ and $M_w = M \cap BwB$. Then, if $w, w' \in W$ and $l(ww') = l(w) + l(w')$, we have

$$\begin{aligned} M_{ww'}(L \cap B) = L_{ww'} &= L_w L_{w'} = M_w (L \cap B) L_{w'} \\ &= M_w L_{w'} = M_w M_{w'} (L \cap B), \end{aligned}$$

and hence

$$\begin{aligned} M_{ww'} &= M_{ww'}(M \cap B) = M_{ww'}(L \cap B) \cap M \\ &= M_w M_{w'} (L \cap B) \cap M = M_w M_{w'} (M \cap B) = M_w M_{w'}. \end{aligned}$$

This verifies (b). (c) follows from (b) and the special case $M = L$ of (c). ∎

Remark. — For $M = G$, TITS (see [9]) has proved a stronger version of (a) : G is the amalgamated product of N, B and the P_s. Actually, Tits defined the groups associated to $\mathbf{g}'(A)$ in this way [12]. Our results imply that our group $G(A)$ is isomorphic to his "minimal" group. In [12] one can find also a discussion of the relationship of these groups to that considered by other authors.

If X and $Y_1, \dots, Y_k$ are subsets of G, we write $X = Y_1 \cdots Y_k$ [unique] if $(g_1, \dots, g_k) \mapsto g_1 \cdots g_k$ defines a bijection from $Y_1 \times \cdots \times Y_k$ onto X.

The following crucial statement is a generalization of a theorem of STEINBERG [10, THEOREM 15].

PROPOSITION 3.1. — *If $w, w' \in W$ satisfy $l(ww') = l(w) + l(w')$, and if X, Y are subsets of G satisfying $BwB = XB$ [unique] and $Bw'B = YB$ [unique], then $Bww'B = XYB$ [unique].*

Proof. — Fix subsets X_s of G, $s \in S$, such that $BsB = X_sB$ [unique]. First, consider the case $w = s \in S$. Then by (3.3), we have $Bsw'B = (BsB)(Bw'B) = X_sBw'B = X_sYB$. To prove uniqueness, suppose $xyb = x'y'b'$, where $x, x' \in X_s$, $y, y' \in Y$, $b, b' \in B$. If $(x')^{-1}x \in BsB$, then, by (3.3), $y'b' = (x')^{-1}xyb \in Bsw'B$, which is impossible since $y'b' \in Bw'B$ (the decomposition (3.1) is disjoint). Hence, by (3.4), the only possibility is that $x'^{-1}x \in B$. It follows that $x \in x'B$ and hence $x = x'$. But then $yb = y'b'$ and hence $y = y'$, $b = b'$. (This argument is due to STEINBERG [10].)

Now, fix $w \in W$. Taking a reduced expression $w = s_1 \cdots s_k$, we deduce by induction on k from what has already been proved :

$$Bs_1s_2 \cdots s_kw'B = (Bs_1B)(Bs_2 \cdots s_kw'B) = X_{s_1}X_{s_2} \cdots X_{s_k}YB[\textit{unique}].$$

Put $X' = X_{s_1} \cdots X_{s_k}$ for short; we have proved $Bww'B = X'YB$ [unique] for any choice of Y. We have : $Bww'B = X'YB = X'(BYB) = (X'B)YB = (XB)YB = X(BYB) = XYB$. To prove uniqueness for any choice of X, we show :

$$(3.7) \qquad z, z' \in BwB \quad \text{and} \quad zBw'B \cap z'Bw'B \neq \emptyset \Rightarrow zB = z'B.$$

Indeed, write $z = xb$, $z' = x'b'$, where $x, x' \in X'$ and $b, b' \in B$. Then $xBw'B \cap x'Bw'B \neq \emptyset$, hence $xYB \cap x'YB \neq \emptyset$, hence $x = x'$ and (3.7) is proved.

If now $x, x' \in X$ but $xYB \cap x'YB \neq \emptyset$, then $xB = x'B$ from (3.7), so $x = x'$, which implies the uniqueness in question. ∎

LEMMA 3.2. — *The following three assertions on $s \in S$ and $w \in W$ are equivalent :*

(i) $U_s^w \subset U_+$;
(ii) $U_s^{sw} \subset U_-$;
(iii) $l(sw) > l(w)$.

Proof. — By LEMMA 3.1(c) and (3.3) we have : $U_s^w \subset U_+ \Rightarrow sBw \subset BswB \Rightarrow l(sw) > l(w) \Rightarrow sBsw \not\subset BwB \Rightarrow U_s^{sw} \not\subset U_+ \Rightarrow U_s^{sw} \subset U_- \Rightarrow U_s^w \subset U_+$. ∎

For $s \in S$, let G_s be the subgroup of G generated by U_s and U_s^s.

COROLLARY 3.2. — *If $s, t \in S$ and $w \in W$, then :*
(i) $U_s^w = U_t \Leftrightarrow wt = sw$ *and* $l(sw) > l(w)$;
(ii) $U_s^w = U_t^t \Leftrightarrow wt = sw$ *and* $l(sw) < l(w)$;
(iii) $\{U_s^w, U_s^{sw}\} = \{U_t, U_t^t\} \Leftrightarrow wt = sw$;
(iv) $G_s^w = G_t \Leftrightarrow wt = sw$.

Proof. — If $wt = sw$ and $l(sw) > l(w)$, then $U_s^w \subset U_+$ and $U_s^{wt} = U_s^{sw} \subset U_-$ by LEMMA 3.2, so that $U_s^w \subset (U_+ \cap U_-^t) = U_t$; since $U_t^{w^{-1}} \subset U_s$ by symmetry, we get $U_s^w = U_t$. Now suppose that $U_s^w = U_t$. Then $U_s^w \subset U_+$ and $U_s^{wt} \subset U_-$, so that $l(sw) > l(w)$ and $l(swt) < l(wt)$ by LEMMA 3.2. By [1], we deduce that $wt = sw$.

This proves (i); (ii) follows from (i), and (iii) follows from (i) and (ii). (iv) follows from (iii) since $\{U_s^w, U_s^{sw}\} = \{G_s^w \cap U_+, G_s^w \cap U_-\}$ and $\{U_t, U_t^t\} = \{G_t \cap U_+, G_t \cap U_-\}$. ∎

We now prove analogues of several of the results of § 2 for arbitrary refined Tits systems.

COROLLARY 3.3.

(a) *Let $s, t \in S$, and assume that $G_s \cap G_t = \{1\}$. Choose $\tilde{s} \in G_s \cap sH$ and $\tilde{t} \in G_t \cap tH$. Then*

$$\tilde{s}\tilde{t}\tilde{s}\cdots = \tilde{t}\tilde{s}\tilde{t}\cdots (m_{s,t} \text{ factors on each side }). \tag{3.8}$$

(b) *Assume that $G_s \cap G_t = 1$ whenever $s, t \in S$ and $m_{s,t} \geq 2$, and choose elements $\tilde{s}$ of $G_s \cap sH$, $s \in S$. Let $\widetilde{W}$ be a subgroup of N containing the $\tilde{s}$, $s \in S$. Then :*

(i) *There exists a function $w \to \tilde{w}$ from W into $\widetilde{W}$ satisfying : $\tilde{1} = 1$; $\tilde{s}$, $s \in S$, is as selected; $\widetilde{ww'} = \tilde{w}\tilde{w}'$ if $w, w' \in W$ and $l(ww') = l(w) + l(w')$; $\tilde{w}H = w$ for all $w \in W$.*

(ii) *$\tilde{W}$ is the amalgamated product of its subgroups $\widetilde{W} \cap B = \widetilde{W} \cap H$ and $\widetilde{W} \cap P_s = \widetilde{W} \cap (H \cup sH)$, $s \in S$, modulo the relations (3.8).*

Proof. — To prove (a), let g and g' be the left-hand and right-hand sides of (3.8), respectively, and put $w = sts\cdots(m_{s,t}$ factors) and $r = w^{-1}tw$.

Using $s^2 = t^2 = (st)^{m_{s,t}} = 1$, we have : $r = s$ or $r = t$, so that $r \in S$, and $\tilde{t}g = g'\tilde{r}$. Using COROLLARY 3.2, we have $gg'^{-1} = g\tilde{r}g^{-1}\tilde{t}^{-1} \in gG_r g^{-1}G_t = G_r^{w^{-1}}G_t = G_tG_t = G_t$ and, similarly, $g'g^{-1} \in G_s$. Hence, $g'g^{-1} \in G_s \cap G_t = \{1\}$, so that $g = g'$, proving (a). b(i) follows from (a) and COROLLARY 1.1, b(i) follows from (a) and THEOREM A. ▮

PROPOSITION 3.2.

(a) $G = \coprod_{n \in N} U_+ n U_+$ (Bruhat decomposition).

(b) *If* $w \in W$, *then* $U_+ wB = U_+(wH)(U_+ \cap U_-^w)$ [unique].

(c) $G = U_+U_-N$.

(d) *If* $w, w' \in W$ *satisfy* $l(ww') = l(w) + l(w')$, *then :*

(i) $U_- \cap U_+^{ww'} = (U_- \cap U_+^w)^{w'}(U_- \cap U_+^{w'})$ [unique] ;

(ii) $U_+ \cap U_-^{ww'} = (U_+ \cap U_-^w)^{w'}(U_+ \cap U_-^{w'})$ [unique] ;

(iii) $U_+ \cap U_+^{w'} = (U_+ \cap U_-^w)^{w'}(U_+ \cap U_+^{ww'})$ [unique].

.

Proof. — By the axioms, we have $B^sB = U_s^sB$ [unique] for each $s \in S$. By repeated use of PROPOSITION 3.1, we deduce that if $l(w) = k$ and $w = s_1 \cdots s_k$, where $s_1, \ldots, s_k \in S$, then $B^wB = U_{s_1}^{s_1 \cdots s_k}U_{s_2}^{s_2 \cdots s_k} \cdots U_{s_k}^{s_k}B$ [unique]. But $U_{s_1}^{s_1 \cdots s_k}U_{s_2}^{s_2 \cdots s_k} \cdots U_{s_k}^{s_k} \subset U_- \cap U_+^w$ by LEMMA 3.2, and $(U_- \cap U_+^w)B \subset B^wB$. Since $U_- \cap B = \{1\}$, we deduce that

$$U_- \cap U_+^w = U_{s_1}^{s_1 \cdots s_k}U_{s_2}^{s_2 \cdots s_k} \cdots U_{s_k}^{s_k} \quad \text{[unique]}$$

and

$$B^wB = (U_- \cap U_+^w)B \text{ [unique]}. \tag{3.8.1.}$$

The first equality applied to ww' implies d(i), and (3.8.1) applied to w^{-1} implies (b) by taking inverses. By applying d (i) to $w'^{-1}w^{-1}$, taking inverses and conjugating by ww', we obtain d(ii).

By induction on $l(w)$, we next prove

$$U_+^w = (U_+^w \cap U_-)(U_+^w \cap U_+) \text{ [unique]}. \tag{3.8.2}$$

We may assume $w \neq 1$. Choose $s \in S$ such that $l(sw) < l(w)$. Then $U_+ \subset U_sU_+^s$ by (RT2), so that $U_+^w \subset U_s^wU_+^{sw}$. Since $U_s^w \subset U_-$ by LEMMA 3.2, the induction hypothesis gives $U_+^w \subset U_-U_+$. Therefore,

$$U_+^w \cap B = (U_+^w \cap U_-U_+) \cap B = U_+^w \cap (U_-U_+ \cap B) = U_+^w \cap U_+,$$

the last equality by (RT3). Since $U_+^w \subset B^wB$, (3.8.2) now follows from (3.8.1).

We now prove d (iii). Using (3.8.2) applied to w'^{-1} and w, we obtain $U_+ = (U_+ \cap U_-^{w'})(U_+ \cap U_+^{w'})$ and $U_+^{ww'} = (U_+^{ww'} \cap U_-^{w'})(U_+^{ww'} \cap U_+^{w'})$. Since $U_-^{w'} \cap U_+^{w'} = \{1\}$, we deduce

$$U_+ \cap U_+^{ww'} = (U_+ \cap U_+^{ww'} \cap U_-^{w'})(U_+ \cap U_+^{ww'} \cap U_+^{w'}).$$

But $U_+^{ww'} \cap U_-^{w'} \subset U_-$ by d(i), so that the first factor $U_+ \cap U_+^{ww'} \cap U_-^{w'}$ is $\{1\}$; therefore, $U_+ \cap U_+^{ww'} = U_+ \cap U_+^{ww'} \cap U_+^{w'}$, i.e., $U_+ \cap U_+^{ww'} \subset U_+^{w'}$. By (3.8.2) applied to $(ww')^{-1}$ and d(ii), we have

$$\begin{aligned} U_+ &= (U_+ \cap U_-^{ww'})(U_+ \cap U_+^{ww'}) \text{ [unique]} \\ &= (U_+ \cap U_-^{w})^{w'}(U_+ \cap U_-^{w'})(U_+ \cap U_+^{ww'}) \text{ [unique]}. \end{aligned}$$

Since the first and third factors are contained in $U_+^{w'}$, and the second factor intersects $U_+^{w'}$ in $\{1\}$, we obtain d(iii).

By (3.8.2) applied to $w = nH$, we have $U_+nU_+ \subset nU_-U_+$ for all $n \in N$. If $n, n' \in N$ and $U_+nU_+ \cap U_+n'U_+ \neq \emptyset$, then $n' \in U_+nU_+ \subset nU_-U_+$ and so $n' = n$ by (RT3). Using (3.1), we deduce (a) and $G = NU_-U_+$. (c) follows by taking inverses. ∎

COROLLARY 3.4. — $\bigcap_{w\in W} U_-^w = \{1\}$.

Proof. — Suppose $u \in \bigcap_{w\in W} U_-^w$. By (3.1) and PROPOSITION 3.2 (b) write $u = u_+u_-n$, where $u_+ \in U_+$, $n \in N$ and $u_- \in U_- \cap nU_+n^{-1}$. Then $[U_-(nun^{-1})^{-1}]nu_+ = 1$, and $nun^{-1} \in U_-$ by assumption, so that by (RT3), $u_- = nun^{-1}$ and $n = 1$. Since $u_- \in U_- \cap nU_+n^{-1} = U_- \cap U_+ = \{1\}$, we have $u = 1$. ∎

PROPOSITION 3.3.

(a) $G = \coprod_{n\in N} U_-nU_+$ (Birkhoff decomposition).

(b) *If* $w \in W$, *then* $U_-wB = U_-(wH)(U_+ \cap U_+^w)$ [unique].

(c) $G = U_-U_+N$.

Proof. — If $s \in S$ and $w \in W$, then $sBw \subset BswU_- \cup BwU_-$ by LEMMA 3.1 (c). We conclude that U_+NU_- is stable under left multiplication by N and U_+ and hence equals G. Hence, $G = G^{-1} = U_-NU_+$. By (3.8.2) applied to $w = n^{-1}H$, we have $U_-nU_+ \subset U_-U_+n$ for all $n \in N$. If $n, n' \in N$ and $U_-nU_+ \cap U_-n'U_+ \neq \emptyset$, then $n' \in U_-nU_+ \subset U_-U_+n$ and so $n' = n$ by (RT3). Using (3.1), we deduce (a) and (c). (b) follows from (3.8.2) applied to w^{-1} and (RT3). ∎

PROPOSITION 3.4. — *U_- is generated by its subgroups U_s^w, where $s \in S$ and $w \in W$ are such that $l(sw) < l(w)$.*

Proof. — Let U' be the subgroup of U_- generated by these U_s^w. Then $G = U'NU_+$ by the argument proving PROPOSITION 3.3(a). (We also use LEMMA 3.2 here.)

Hence, $U' \subset U_- \subset U'NU_+$, which implies $U_- = U'$ by (RT3). ∎

We now determine the structure of U_- in certain cases.

PROPOSITION 3.5.

(a) *If $s \in S$, $w \in W$ and $l(w^{-1}sw) = 2l(w) + 1$, then*

$$U_- \cap U_+^{sw} \subset Bw^{-1}swB \cup (U_- \cap U_+^w).$$

(b) *If $|S| = 2$ and $s \in S$, then*

$$U_-^{(s)} := U_- \cap \Big(\bigcup_{\substack{w \in W \\ l(w) > l(ws)}} U_+^w \Big)$$

is a subgroup of U_-.

(c) *If $S = \{s,t\}$ and $m_{s,t} = 0$, so that W is an infinite dihedral group, then U_- is the free product of its subgroups $U_-^{(s)}$ and $U_-^{(t)}$ defined in* (b).

Proof. — In the situation of (a), write $w = s_1 \cdots s_k$, where $k = l(w)$. Then we have, by PROPOSITION 3.2 d(i) applied to sw and by (RT2) :

$$\begin{aligned} U_-^{w^{-1}} \cap (U_+^s \setminus U_+) &= (U_-^{w^{-1}} \cap U_+)(U_s^s \setminus \{1\}) \\ &\subset B(U_s H s U s) \\ &\subset BsB, \text{ and hence, by (3,3),} \\ U_- \cap (U_+^{sw} \setminus U_+^w) &\subset w^{-1}BsBw \subset Bw^{-1}swB. \end{aligned}$$

This proves (a).

We now prove (b). Let $S = \{s,t\}$. If $m_{s,t} \neq 0$, we put $w_0 = sts \cdots (m_{s,t}$ factors). Using PROPOSITION 3.4, we then deduce that $U_+^{w_0} \supset U_-$ and hence that $U_-^{(s)} = U_-^{(t)} = U_-$. If $m_{s,t} = 0$, then it is easy to check that for $n = 1, 2, 3, \ldots$, there exists a unique $w_n \in W$ satisfying $l(w_n) = n > l(w_n s)$, and by using PROPOSITION 3.2 d(i) that $U_- \cap U_+^{w_n} \subset U_- \cap U_+^{w_{n+1}}$, so that $U_-^{(s)}$ is an increasing union of subgroups of U_- and hence is a subgroup of U_-. This proves (b).

To prove (c), note that, by using PROPOSITION 3.4, $U_-^{(s)}$ and $U_-^{(t)}$ generate U_-. For $r \in S$, put $W^{(r)} = \{w \in W \mid l(rw) = l(wr) < l(w)\}$; then $U_-^{(r)} \setminus \{1\} \subset \bigcup_{w \in W^{(r)}} BwB$ by using (a). Moreover, it is easy to check that if

$w_1 \in W^{(s)}$, $w_2 \in W^{(t)}$, $w_3 \in W^{(s)}, \ldots$, then $l(w_1 \cdots w_n) = l(w_1) + \cdots + l(w_n)$ for $n = 1, 2, 3, \ldots$ Hence, by (3.1) and (3.3), if $u_1 \in U_-^{(s)}$, $u_2 \in U_-^{(t)}$, $u_3 \in U_-^{(s)}, \ldots$ and $u_1, u_2, u_3, \ldots \neq 1$, then $u_1 u_2 \cdots u_n \neq 1$ for $n = 1, 2, \ldots$ Similarly, $u_2 u_3 \cdots u_{n+1} \neq 1$ for $n = 1, 2, \ldots$ This proves (c). ∎

Conjecture. — U_- is the amalgamated product of its subgroups $U_- \cap U_+^w$, $w \in W$. (PROPOSITION 3.5(c) confirms this when W is an infinite dihedral group; the conjecture is trivial when W is finite.)

We can now prove a generalization of a theorem of NAGAO [9] :

COROLLARY 3.5. — *Assume that $S = \{s, t\}$ and $m_{s,t} = 0$, (so that W is an infinite dihedral group), and that $U_- = U_s^s \propto (U_- \cap U_-^s)$. Then the "opposite minimal parabolic" $P_s^- := HG_s \propto (U_- \cap U_-^s)$ is the amalgamated product of its subgroups HG_s and $HU_-^{(s)}$ (defined in* PROPOSITION 3.5 (b)).

Proof. — Put $U_1 = U_-^{(s)} \cap U_-^s$. Clearly, H normalizes U_1 and $U_s^s \cap U_1 = \{1\}$. LEMMA 3.2 and the assumption $U_- = U_s^s \propto (U_- \cap U_-^s)$ imply that U_s^s normalizes U_1. PROPOSITION 3.2(d) shows that $U_-^{(s)} = U_s^s U_1$ and that $U_1^s = U_-^{(t)}$. We therefore obtain :

$$U_-^{(s)} = U_s^s \propto U_1, \text{ and } H \text{ normalizes } U_1. \tag{3.9}$$

$$U_-^{(t)} = U_1^s. \tag{3.10}$$

By using (RT2a), we obtain :

$$HG_s = HU_s^s \cup U_s^s s H U_s^s. \tag{3.11}$$

Now, let $\widetilde{P}_s^-$ be the amalgamated product of the subgroups HG_s and $HU_-^{(s)}$ of P_s^-, and let $\Psi : \widetilde{P}_s^- \to P_s^-$ be the canonical map. Identifying HG_s and $HU_-^{(s)}$ with subgroups of $\widetilde{P}_s^-$, let F be the subgroup of $\widetilde{P}_s^-$ generated by $\bigcup_{g \in HG_s} g U_1 g^{-1}$. Fixing $n \in sH$, (3.9) and (3.11) imply that F is generated by U_1 and $\bigcup_{u \in U_s^s} un U_1 (un)^{-1}$. Let $\widetilde{U}_-$ be the subgroup of $\widetilde{P}_s^-$ generated by U_s^s and F. Using (3.9), we see that $\widetilde{U}_-$ is generated by $U_-^{(s)}$ and nU_1n^{-1}. Clearly, $\Psi = \mathrm{id}$ on $U_-^{(s)}$, and Ψ maps nU_1n^{-1} isomorphically onto $U_-^{(t)}$ by (3.10). Hence, by PROPOSITION 3.5 (c), Ψ maps $\widetilde{U}_-$ isomorphically onto U_-. Since also $\Psi = \mathrm{id}$ on HG_s, we see that Ψ is surjective. By using (3.11) and $\widetilde{P}_s^- = HG_s F$, we have

$$\begin{aligned} \widetilde{P}_s^- &= H\widetilde{U}_- \cup U_s^s n H \widetilde{U}_- \\ &= H\widetilde{U}_- \cup nHU_s\widetilde{U}_- \subset (HG_s \cap NU_+)\widetilde{U}_-. \end{aligned}$$

If $g \in \widetilde{P}_s^-$ and $\Psi(g) = 1$, write $g = g'u$, where $g' \in HG_s \cap NU_+$ and $u \in \widetilde{U}_-$. Since $\Psi = \mathrm{id}$ on HG_s and $\Psi(\widetilde{U}_-) \subset U_-$, we have $1 = \Psi(g) = \Psi(g')\Psi(u) = g'\Psi(u)$ and hence $g' = \Psi(u) = 1$ by (RT3). But Ψ is injective on $\widetilde{U}_-$. Therefore, $u = 1$ and so $g = 1$. This shows that Ψ is injective. ∎

Let k be a field, NAGAO's theorem states that $SL_2(k[t^{-1}])$ is the amalgamated product of its subgroups

$$SL_2(k) \text{ and } \left\{ g \in SL_2(k[t^{-1}]) \;\middle|\; g = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\}.$$

We deduce this result from COROLLARY 3.5, as follows. Put

$$G = SL_2(k((t))), \qquad H = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a \in k^x \right\},$$

$$U_+ = \left\{ g \in SL_2(k(t)) \;\middle|\; g(t=0) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\},$$

$$U_- = \left\{ g \in SL_2(k[t^{-1}]) \;\middle|\; g(t=\infty) = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \right\}.$$

Let N be the subgroup of G generated

$$\text{by } H \text{ and } n_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad n_2 = \begin{pmatrix} 0 & t^{-1} \\ -t & 0 \end{pmatrix}.$$

Put $S = \{n_1H, n_2H\} \subset N/H = W$ and $s = n_1H \in S$. It is easy to check that (G, N, U_+, U_-, H, S) is a refined Tits system. (To check (RT3), one notes that $n \in N$ and $U_- \cap nU_+n^{-1} = \{1\}$ imply $n \in H$.) Since $U_- = U_s^s \propto (U_- \cap U_-^s)$, and since W is an infinite dihedral group, COROLLARY 3.5 applies. The conclusion is NAGAO's theorem.

Remark. — In the example above, it is easy to check that G is generated by N and U_+ by using the fact that $k((t))$ is a field. The corresponding fact for $k[t, t^{-1}]$ may be proved by using the density of $k[t, t^{-1}]$ in $k((t))$ and the fact that U_+ is an open subgroup. Furthermore, using the involution $t \to t^{-1}$ of $k[t, t^{-1}]$, we deduce by using PROPOSITION 3.4 the well-known fact that $SL_2(k[t, t^{-1}])$ is generated by its subgroups

$$SL_2(k) \text{ and } \left\{ \begin{pmatrix} a & bt^{-1} \\ ct & d \end{pmatrix} \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(k) \right\}.$$

Define a map $\theta : U_-HU_+ \to H$ by $\theta(u_-hu_+) = h$.

PROPOSITION 3.6. — *If $w, w' \in W$ and if $l(ww') = l(w) + l(w')$, then*

$$\theta(n'^{-1}gn'g') = n'^{-1}\theta(g)n'\theta(g') \tag{3.12}$$

for all $g \in B^wB$, $g' \in B^{w'}B$ *and* $n' \in w'H$.

Proof. — First, we prove (3.12) for $g' = 1$. By (3.8.1), write $g = u_-hu_+$, where $u_- \in U_- \cap U_+^w$, $h \in H$ and $u_+ \in U_+$. By (3.8.2), write $n'^{-1}u_+n' = u'_-u'_+$, where $u'_- \in U_-$ and $u'_+ \in U_+$. By PROPOSITION 3.2 d (i), $(U_- \cap U_+^w)^{w'} \subset U_-$, so $n'^{-1}u_-n' \in U_-$. It follows that

$$\begin{aligned} n'^{-1}gn' &= ((n'^{-1}u_-n')((n'^{-1}hn')u'_-(n'^{-1}hn')^{-1}))(n'^{-1}hn')u'_+ \\ &\in U_-(n'^{-1}hn')U_+, \end{aligned}$$

and hence $\theta(n'^{-1}gn') = n'^{-1}hn' = n'^{-1}\theta(g)n'$.

The proof of (3.12) for arbitrary $g' \in B^{w'}B$ follows by a straightforward calculation. Write $g' = n'^{-1}bn'b'$, where $b, b' \in B$. Then

$$\begin{aligned} \theta(n'^{-1}gn'g') &= \theta(n'^{-1}(gb)n'b') = \theta(n'^{-1}(gb)n')\theta(b') \\ &= n'^{-1}\theta(gb)n'\theta(b') = n'^{-1}\theta(g)\theta(b)n'\theta(b') \\ &= n'^{-1}\theta(g)n'(n'^{-1}\theta(b)n')\theta(b') \\ &= n'^{-1}\theta(g)n'\theta(n'^{-1}bn')\theta(b') \\ &= n'^{-1}\theta(g)n'\theta(n'^{-1}bn'b') \\ &= n'^{-1}\theta(g)n'\theta(g'). \quad \blacksquare \end{aligned}$$

PROPOSITION 3.7. — *Let* K *be a subgroup of* G *satisfying* $K \cap U_+ = \{1\}$, *and put* $T = \theta(K \cap B)$. *Let* H_+ *be a normal subgroup of* N, *and assume that* $H = H_+T$[unique]. *Assume that* $U_s \subset KBs$ *for all* $s \in S$. *Assume that* $w \mapsto \tilde{w}$ *is a map from* W *to* N *satisfying :* $s = \tilde{s}H$ *for all* $s \in S$; $\tilde{1} = 1$; $\widetilde{ww'} = \tilde{w}\tilde{w}'$ *for all* $w, w' \in W$ *such that* $l(ww') = l(w) + l(w')$. *For* $w \in W$, *put*

$$Z_w = \{k \in K \cap BwB \mid \theta(\tilde{w}^{-1}k) \in H_+\}. \tag{3.13}$$

Then :

(a) (i) $G = KH_+U_+$ [unique];

(ii) *for all* $w \in W$, $BwB = Z_wB$ [unique];

(iii) *for all* $w, w' \in W$ *such that* $l(ww') = l(w) + l(w')$, $Z_{ww'} = Z_wZ_{w'}$ [unique].

(b) *For* $s, t \in S$ *and* $m_{s,t}$ *elements* $z_1 \in Z_s$, $z_2 \in Z_t$, $z_3 \in Z_s, \ldots$, *there exists a unique sequence of* $m_{s,t}$ *elements* $z'_1 \in Z_t$, $z'_2 \in Z_s$, $z'_3 \in Z_t, \ldots$ *satisfying*

$$z_1z_2z_3\cdots = z'_1z'_2z'_3\cdots (m_{s,t} \textit{ factors on each side }). \tag{3.14}$$

Furthermore, K is the amalgamated product of its subgroups $K \cap B$ and $K \cap P_s$, $s \in S$, modulo the relations (3.14).

Proof. — For $s \in S$, we have $BsB = U_s sB \subset KBssB = KB$, and hence $G = KB$ by (3.1) and PROPOSITION 3.1. But $B = TH_+U_+ = TU_+H_+ = (K \cap B)U_+H_+ = (K \cap B)H_+U_+$. Hence, $G = KH_+U_+$. If $k, k' \in K$, $h, h' \in H_+$, $u, u' \in U_+$ and $khu = k'h'u'$, then $k^{-1}k' \in K \cap B$ and $\theta(k^{-1}k') = hh'^{-1} \in H_+$. Since $\theta(K \cap B) \cap H_+ = \{1\}$, we conclude that $h = h'$ and hence $k^{-1}k' = h'uu'^{-1}h'^{-1} \in K \cap U_+ = \{1\}$, so that $k = k'$ and $u = u'$. This proves a(i).

To prove a (ii), fix $w \in W$. If $k \in K \cap BwB$, choose $t \in K \cap B$ such that $\theta(\tilde{w}^{-1}k) \in H_+\theta(t)$. Then $k = (kt^{-1})t \in Z_wB$. Using a (i), we deduce that $BwB = KB \cap BwB = Z_wB$. Now suppose that $z, z' \in Z_w$, $b, b' \in B$ and $zb = z'b'$. Put $g = z^{-1}z' = bb'^{-1} \in K \cap B$, so that $\theta(\tilde{w}^{-1}z') = \theta(\tilde{w}^{-1}zg) = \theta(\tilde{w}^{-1}z)\theta(g)$. Hence, $\theta(g) = \theta(\tilde{w}^{-1}z)^{-1}\theta(\tilde{w}^{-1}z') \in T \cap H_+ = \{1\}$, so that $g \in U_+ \cap K = \{1\}$. This shows that $z = z'$ and $b = b'$, verifying a (ii).

To prove a (iii), fix $w, w' \in W$ such that $l(ww') = l(w) + l(w')$. We claim that $Z_wZ_{w'} \subset Z_{ww'}$. To verify this, let $k \in Z_w$ and $k' \in Z_{w'}$. Then

$$\begin{aligned} kk' \in Z_wZ_{w'} &\subset (K \cap BwB)(K \cap Bw'B) \\ &\subset K \cap (BwB)(Bw'B) = K \cap Bww'B, \end{aligned}$$

and also

$$\begin{aligned} \theta(\widetilde{ww'}^{-1}kk' = \theta((\tilde{w}\tilde{w}')^{-1}kk') &= \theta(\tilde{w}'^{-1}(\tilde{w}^{-1}k)\tilde{w}'(\tilde{w}'^{-1}k')) \\ &= \tilde{w}'^{-1}\theta(\tilde{w}^{-1}k)\tilde{w}'\theta(\tilde{w}'^{-1}k') \\ &\in \tilde{w}'^{-1}H_+\tilde{w}'H_+ = H_+, \end{aligned}$$

the third equality by PROPOSITION 3.6. This proves the claim. We have : $Z_wZ_{w'} \subset Z_{ww'}$; $BwB = Z_wB$ [unique] and $Bw'B = Z_{w'}B$ [unique]; $Bww'B = Z_{ww'}B$[unique]. Using PROPOSITION 3.1, we deduce a(iii).

(b) follows from (a) and THEOREM A. ∎

4. $G(A)$ is a refined Tits system.

Fix a generalized Cartan matrix A. Let $G(A)$ be the corresponding group, defined in § 2. Recall the subgroups N, U_+, U_- and H of $G(A)$, the Weyl group $W = N/H$ and the subset S of W, introduced in § 2.

For $s \in S$, put $U_{(s)} = U_{\alpha_s} (= \exp \mathfrak{g}_{\alpha_s})$ for short. We keep the "exponential" notation M^w of § 3. We shall see that $U_{(s)} = U_+ \cap U_-^s$.

PROPOSITION 4.1.

(a) *$G(A)$ is generated by N and U_+. The group H is a normal subgroup of N; it normalizes U_+ and U_-. The set S generates W, and $s^2 = 1$ for all $s \in S$.*

(b) *If $s \in S$ and $w \in W$, then :*

- (i) *$U_{(s)}$ is a subgroup of $U_+ \cap U_-^s$, and H normalizes $U_{(s)}$.*
- (ii) *$U_{(s)} \neq \{1\}$.*
- (iii) *$U_{(s)}^s \setminus \{1\} \subset U_{(s)} H s U_{(s)}$.*
- (iv) *$U_{(s)}^w \subset U_+$ or $U_{(s)}^w \subset U_-$.*
- (v) *$U_+ \subset U_{(s)} U_+^s$.*

(c)
- (i) *If $w \in W$ and $w \neq 1$, then $U_{(s)}^w \subset U_-$ for some $s \in S$.*
- (ii) *If $u_- \in U_-$, $h \in H$, $u_+ \in U_+$ and $u_- h u_+ = 1$, then $u_- = h = u_+ = 1$.*

Before proving PROPOSITION 4.1 we use it to deduce :

PROPOSITION 4.2. — *$(G(A), N, U_+, U_-, H, S)$ is a refined Tits system, and $U_{(s)} = U_+ \cap U_-^s$ for all $s \in S$.*

Proof. — (RT1) follows from PROPOSITION 4.1 (a). By PROPOSITION 4.1 c(ii), $U_- \cap U_+ = \{1\}$. Hence, by PROPOSITION 4.1 b(i,v), $U_{(s)} = U_+ \cap U_-^s$, which is U_s from § 3. (RT2) now follows from PROPOSITION 4.1 (b). To prove (RT3), suppose that $u_- \in U_-$, $n \in N$, $u_+ \in U_+$ and $u_- n u_+ = 1$. Then, since $U_- \cap U_+ = \{1\}$ by PROPOSITION 4.1 c(ii), we have

$$\{1\} = U_- \cap (u_- n u_+) U_+ (u_- n u_+)^{-1} = u_- (U_- \cap n U_+ n^{-1}) u_-^{-1},$$

so that $U_- \cap n U_+ n^{-1} = \{1\}$. By PROPOSITION 4.1 b(ii) and c(i), this forces $n \in H$. Now $u_- = n = u_+ = 1$ follows from PROPOSITION 4.1 c(ii), proving (RT3). ∎

Parts (a) and b(i, iv) of PROPOSITION 4.1 are clear. Part b(ii) is clear since $\mathrm{Ad}(x_s(1)) f_s = f_s + \alpha_s^\vee - e_s \neq f_s$. Part b(iii) follows from formula (2.7), and part c(i) follows from LEMMA 2.1 (a), PROPOSITION 2.1 and formula (2.9). To prove parts b(v) and c(ii), we need some constructions.

Henceforth, $U(\mathbf{g})$ denotes the universal enveloping algebra of a Lie algebra $\mathbf{g}$. The use of $U(\mathbf{n}_+)$ to investigate U_+, exploited below, was one of the ingredients of Tits [12].

Recall that the Kac-Moody algebra $\mathbf{g}'(A)$ has a triangular decomposition $\mathbf{g}'(A) = \mathbf{n}_- + \mathbf{g}_0 + \mathbf{n}_+$, where

$$\mathbf{n}_\pm = \bigoplus_{\alpha\in\Delta_+} \mathbf{g}_{\pm\alpha} = \bigoplus_{\alpha\in Q_+\setminus\{0\}} \mathbf{g}_{\pm\alpha}.$$

We complete the universal enveloping algebra $U(\mathbf{n}_+)$ with respect to its induced algebra gradation, obtaining an algebra $\widehat{U(\mathbf{n}_+)}$ consisting of all formal sums $\sum_{\alpha=Q_+} u_\alpha$, where $u_\alpha \in U(\mathbf{n}_+)_\alpha$. Let $\widetilde{U(\mathbf{n}_+)}$ be the subalgebra of $\widehat{U(\mathbf{n}_+)}$ consisting of all such formal sums $\sum_{\alpha\in Q_+} u_\alpha$ satisfying the following condition : If $(V, d\pi)$ is an integrable $\mathbf{g}'(A)$-module and $v \in V$, then $d\pi(u_\alpha)v = 0$ for all but a finite number of $\alpha \in Q_+$. Such a $(V, d\pi)$ then becomes a $\widetilde{U(\mathbf{n}_+)}$-module $(V, \tilde{\pi})$ by : $\tilde{\pi}(\sum u_\alpha)v = \sum d\pi(u_\alpha)v$.

For $\alpha \in \Delta_+^{\mathrm{re}}$, define a map $\widetilde{\exp} : \mathbf{g}_\alpha \to \widetilde{U(\mathbf{n}_+)}$ by :

$$\widetilde{\exp}\, x = \sum_{n=0}^{\infty} (n!)^{-1} x^n.$$

Let $\widetilde{U}_+$ be the subset of $\widetilde{U(\mathbf{n}_+)}$ generated by the $\widetilde{\exp}\, \mathbf{g}_\alpha$, $\alpha \in \Delta_+^{\mathrm{re}}$, under multiplication, so that $\widetilde{U}_+$ is a group under multiplication with identity 1.

LEMMA 4.1. — *There exists a unique surjective homomorphism* Ψ : $\widetilde{U}_+ \to U_+$ *such that* $\tilde{\pi} = \pi \circ \Psi$ *for every integrable* $\mathbf{g}'(A)$*-module* $(V, d\pi)$. *We have* $\exp = \Psi \circ \widetilde{\exp}$ *on* $\mathbf{g}_\alpha$ *for every* $\alpha \in \Delta_+^{\mathrm{re}}$.

Proof. — Let $(V, d\pi)$ be an integrable $\mathbf{g}'(A)$-module such that the associated $G(A)$-module (V, π) is faithful. Clearly, we have $\tilde{\pi}(\widetilde{\exp}\, x) = \pi(\exp x)$ for all $x \in \mathbf{g}_\alpha$, $\alpha \in \Delta_+^{\mathrm{re}}$. Hence, $\tilde{\pi}(\widetilde{U}_+) = \pi(U_+)$. Since π is injective on U_+, we conclude that there exists a unique map $\Psi : \widetilde{U}_+ \to U_+$ such that $\tilde{\pi} = \pi \circ \Psi$; clearly, Ψ is a surjective homomorphism, and $\exp = \Psi \circ \widetilde{\exp}$ on every $\mathbf{g}_\alpha$. If $(V', d\pi')$ is another integrable $\mathbf{g}'(A)$-module, then the same reasoning applied to $(V \oplus V', d\pi \oplus d\pi')$ yields a homomorphism $\Psi_0 : \widetilde{U}_+ \to U_+$ satisfying $\tilde{\pi} \oplus \tilde{\pi}' = (\pi \oplus \pi') \circ \Psi_0$, i.e., $\tilde{\pi} = \pi \circ \Psi_0$ and $\tilde{\pi}' = \pi' \circ \Psi_0$. Then $\Psi_0 = \Psi$ by the first equality and the uniqueness of Ψ, so that $\tilde{\pi}' = \pi' \circ \Psi$ by the second one. ∎

For $s \in S$, put

$$Y_s^\pm = \bigcup_\alpha U_\alpha,$$

where α runs over $\Delta_+^{\mathrm{re}} \setminus \{\alpha_s\}$ with $\pm\langle\alpha, \alpha_s^v\rangle \geq 0$.

LEMMA 4.2. — *Let $s \in S$. Then :*
(a) $Y_s^\pm = nY_s^\pm n^{-1}$ *for all* $n \in sH$.
(b) U_+ *is generated by* $U_{(s)}$, Y_s^+ *and* Y_s^-.
(c) $uzu^{-1}z^{-1} \in Y_s^+$ *for all* $u \in U_{(s)}$ *and* $z \in Y_s^+$.

Proof. — (a) and (b) are clear. If $u = \exp a \in U_{(s)}$ and $z = \exp b \in Y_s^+$, then $(\operatorname{ad} a)^2 b = 0 = (\operatorname{ad} b)^2$ a by LEMMA 2.1 (c). We have :

$$\begin{aligned} uzu^{-1} &= \Psi(\widetilde{\exp}\, a)\Psi(\widetilde{\exp}\, b)\Psi(\widetilde{\exp}\ -a) \\ &= \Psi\big((\widetilde{\exp}\, a)(\widetilde{\exp}\, b)(\widetilde{\exp}\ -a)\big) \\ &= \Psi\Big(\sum_{n=0}^{\infty}(n!)^{-1}x^n\Big), \end{aligned}$$

where $x = (\exp \operatorname{ad} a)b = b + [a,b]$. Since x and b commute, we get

$$\begin{aligned} uzu^{-1}z^{-1} &= \Psi\Big(\sum_{n=0}^{\infty}(n!)^{-1}x^n\Big)\Psi\Big(\sum_{m=0}^{\infty}(m!)^{-1}(-b)^m\Big) \\ &= \Psi\Big(\sum_{n=0}^{\infty}(n!)^{-1}x^n\sum_{m=0}^{\infty}(m!)^{-1}(-b)^m\Big) \\ &= \Psi\Big(\sum_{n=0}^{\infty}(n!)^{-1}(x-b)^n\Big) \\ &= \Psi\Big(\sum_{n=0}^{\infty}(n!)^{-1}[a,b]^n\Big). \end{aligned}$$

Since $\exp[a,b] \in Y_s^+$ by LEMMA 2.1 (c), we get $uzu^{-1}z^{-1} = \Psi(\widetilde{\exp}[a,b]) = \exp[a,b] \in Y_s^+$. This proves (c). ∎

COROLLARY 4.1. — *Let $s \in S$, and let $U^{(s)}$ be the subgroup of U_+ generated by $\{uzu^{-1} \mid u \in U_{(s)}, z \in Y_s^+ \cup Y_s^-\}$. Then $U_+ = U_{(s)}U^{(s)}$, $U_{(s)}$ normalizes $U^{(s)}$, and $H \cup sH$ normalizes $U^{(s)}$.*

Proof. — By LEMMA 4.2 (a,b), $U_+ = U_{(s)}U^{(s)}$, and $U_{(s)}$ and H normalize $U^{(s)}$. Thus, it suffices to show that if $u \in U_{(s)}$ and $z \in Y_s^+ \cup Y_s^-$, then there exists $n \in sH$ such that $nuzu^{-1}n^{-1} \in U^{(s)}$. If $z \in Y_s^+$, then $uzu^{-1} \in Y_s^+Y_s^+$ by LEMMA 4.2 (c), and hence $nuzu^{-1}n^{-1} \in Y_s^-Y_s^- \subset U^{(s)}$ for all $n \in sH$ by LEMMA 4.2 (a). If $u = 1$ and $z \in Y_s^-$, then $nuzu^{-1}n^{-1} = nzn^{-1} \in Y_s^+ \subset U^{(s)}$ for all $n \in sH$. Finally, suppose $u \neq 1$ and $z \in Y_s^-$. By using PROPOSITION 4.1 b(iii), choose $n \in sH$ and $u_1, u_2 \in U_{(s)}$ such that $nu = u_1 n u_2 n^{-1}$. Then

$$\begin{aligned} nuzu^{-1}n^{-1} = u_1 n u_2 n^{-1} z n u_2^{-1} n^{-1} u_1^{-1} &\in u_1 n u_2 Y_s^+ u_2^{-1} n^{-1} u_1^{-1} \\ &\subset u_1 n Y_s^+ Y_s^+ n^{-1} u_1^{-1} \\ &\subset u_1 Y_s^- Y_s^- u_1^{-1} \subset U^{(s)}. \quad \blacksquare \end{aligned}$$

A $\mathbf{g}'(A)$-module $(V, d\pi)$ is called *Q-graded* if there is a vector space decomposition $V = \bigoplus_{\beta \in Q} V_\beta$ satisfying $d\pi(\mathbf{g}_\alpha)V_\beta \subset V_{\alpha+\beta}$.

LEMMA 4.3. — *There exists a Q-graded integrable $\mathbf{g}'(A)$-module V which is a faithful $U(\mathbf{n}_+)$-module.*

Proof. — One can take for V the direct sum of all integrable lowest weight $\mathbf{g}'(A)$-modules. In more detail, given $\Lambda = (\lambda_s)_{s \in S} \in \mathbf{Z}_+^S$, define a 1- dimensional $U(\mathbf{g}_0 + \mathbf{n}_-)$-module $\mathbf{C}v_\Lambda$ by $\alpha_s^\vee(V_\Lambda) = -\lambda_s v_\Lambda$, $\mathbf{n} - (v_\Lambda) = 0$. Let

$$M^*(\Lambda) = U(\mathbf{g}'(A)) \otimes_{U(\mathbf{g}_0 + \mathbf{n}_-)} \mathbf{C}v_\Lambda,$$

regarded as a Q-graded $\mathbf{g}'(A)$-module, where the action is defined by left multiplication and the Q-gradation is induced from that of $U(\mathbf{g}'(A))$ by putting deg $v_\Lambda = 0$. Then it is easy to see that the Q-graded $\mathbf{g}'(A)$-module $L^*(\Lambda) = M^*(\Lambda)/\sum_s U(\mathbf{n}_+) e_s{}^{\lambda_s+1}(v_\Lambda)$ is integrable (*cf.* [3, LEMMA 3.4]). We put

$$V = \bigoplus_{\Lambda \in \mathbf{Z}_+^S} L^*(\Lambda).$$

If $u \in U(\mathbf{n}_+)_\beta$, $u \neq 0$ and $u(v_\Lambda) = 0$ in $L^*(\Lambda)$, then $\beta - (\lambda_s + 1)\alpha_s \in Q_+$ for some $s \in S$. It follows that V is a faithful $U(\mathbf{n}_+)$-module. ∎

We say that a subgroup F of $G(A)$ is *graded* if $u_- \in U_-, h \in H$, $u_+ \in U_+$ and $u_- h u_+ \in F$ imply $u_-, h, u_+ \in F$.

LEMMA 4.4. — *Let (V, π) be a Q-graded integrable $\mathbf{g}'(A)$-module. Then :*

(a) $\ker \pi$ *is a graded subgroup of* $G(A)$.

(b) *If V is a faithful $U(\mathbf{n}_+)$-module, then V is a faithful $\widetilde{U(\mathbf{n}_+)}$-module.*

Proof. — If $u \in U_+$ and $v \in V_\beta$, then

$$\pi(u)v - v \in \sum_{\alpha \in Q_+ \setminus \{0\}} V_{\beta+\alpha},$$

so that U_+ is "upper triangular" on V. Similarly, $H = \exp g_0$ is "diagonal" on V and U_- is "lower triangular" on V. If now $u_- \in U_-$, $h \in H$, $u_+ \in U_+$

and $u_-hu_+ \in \ker\pi$, then, for all $v \in V_\beta$, $\beta \in Q$, we have

$$\pi(u_+)v - v = \pi(h^{-1}u_-^{-1})v - v$$
$$\in \Big(\sum_{\alpha\in Q_+\setminus\{0\}} V_{\beta+\alpha}\Big) \cap \Big(\sum_{\alpha\in -Q_+} V_{\beta+\alpha}\Big) = (0).$$

Hence, $\pi(u_+) = 1$, so that $u_+ \in \ker\pi$ and, similarly, $u_- \in \ker\pi$ and so finally $h \in \ker\pi$.(a) follows. (b) is clear. ∎

COROLLARY 4.2.

(a) *The homomorphism* Ψ *of* LEMMA 4.1 *is an isomorphism from* $\widetilde{U}_+$ *onto* U_+.

(b) *If* $u_- \in U_-$, $h \in H$, $u_+ \in U_+$ *and* $u_-hu_+ = 1$, *then* $u_- = h = u_+ = 1$.

(c) *If* $s \in S$, *then* $U_{(s)} \neq \{1\}$ *and* $U_+ = U_{(s)} \propto (U_+ \cap U_+^s)$.

Proof. — (a) is clear from LEMMAS 4.1, 4.3 and 4.4(b). Suppose $u_- \in U_-$, $h \in H$, $u_+ \in U_+$ and $u_-hu_+ = 1$. By LEMMA 4.4(*a*), $u_+ \in \ker\pi$ for every Q-graded integrable $\mathbf{g}'(A)$-module $(V, d\pi)$; by LEMMAS 4.1, 4.3 and 4.4(b), this forces $u_+ = 1$. Similarly, by using the involution ω of $G(A)$, we conclude that $u_- = 1$. Hence, $h = 1$ also, proving (b). The first part of (c) follows from (a). Fix $s \in S$. Then

$$U_{(s)} \cap U_+^s \subset U_-^s \cap U_+^s = (U_- \cap U_+)^s = \{1\}$$

by using (b). By COROLLARY 4.1, $U_+ = U_{(s)}U^{(s)}, U^{(s)} \subset U_+ \cap U_+^s$, and $U_{(s)}$ normalizes $U^{(s)}$. Hence, $U_+ = U_{(s)} \propto U^{(s)}$ and $U^{(s)} = U_+ \cap U_+^s$. This proves (c).

Proof of the reminder of PROPOSITION 4.1 is immediate from COROLLARY 4.2. ∎

We shall henceforth use the results of § 3, applied to $G(A)$, without invoking PROPOSITION 4.2 each time.

PROPOSITION 4.3. — *Let* $A = \left(\begin{smallmatrix} 2 & -m \\ -n & 2 \end{smallmatrix}\right)$ *be a* 2×2 *matrix with* $m, n \in \mathbf{Z}_+$ *and* $mn \geq 4$. *Let* $(W(A), S)$ *be the associated Coxeter system, so that* $S = \{s, t\}$ *and* $m_{s,t} = 0$. *Put*

$$\Delta_+^s = \{(st)\}^k \cdot \alpha_s \mid k \in \mathbf{Z}_+\} \cup \{(st)^k s \cdot \alpha_t \mid k \in \mathbf{Z}_+\}$$

and

$$\Delta_+^t = \{(ts)^k \cdot \alpha_t \mid k \in \mathbf{Z}_+\} \cup \{(ts)^k t \cdot \alpha_s \mid k \in \mathbf{Z}_+\},$$

so that $\Delta_+^{re} = \Delta_+^s \sqcup \Delta_+^t$. *For* $r \in S$, *let* $U_+^{(r)}$ *be the subgroup of* $U_+ \subset G(A)$ *generated by the* U_α, $\alpha \in \Delta_+^r$. *Then* U_+ *is the free product of its subgroups* $U_+^{(s)}$ *and* $U_+^{(t)}$.

Proof. — Using the involution ω, this is clear from PROPOSITION 3.5(c).∎

Remarks. — (1) For $m = n = 2$, i.e. for the case $A_1^{(1)}$, PROPOSITION 4.3 was stated in [8, Example].

(2) For $m, n \geq 2$, each group $U_+^{(r)}$ from PROPOSITION 4.3 is the direct sum of its one-parameter subgroups U_α, $\alpha \in \Delta_+^r$; otherwise, each $U_+^{(r)}$ is a two-step nilpotent group.

(3) We *conjecture* that, in general, U_+ is the amalgamated product of its subgroups $U_+ \cap U_-^w$, $w \in W$. (This is a special case of the conjecture of § 3.)

We now explore some features of $G(A)$, which are related to the Q-gradation of $\mathfrak{g}'(A)$.

S is called *indecomposable* if, whenever J is a subset of S such that $J \neq \emptyset$ and $J \neq S$, there exist $s \in J$ and $t \in S \setminus J$ such that $st \neq ts$. (This corresponds to the indecomposability of A.) The following are general properties of Tits systems [1] :

(4.1) If S is indecomposable and F is a normal subgroup of $G(A)$, then $FB = B$ or $FB = G(A)$.

(4.2) The center of $G(A)$is contained in B.

We will also use the following special properties of $G(A)$.

(4.3) $G(A)$ is generated by the U_s and U_s^s, $s \in S$.

(4.4) $\bigcap_{w \in W} U_+^w = \{1\}$.

Indeed, (4.3) is clear, and (4.4) follows from COROLLARY 3.4 by using the involution ω.

We call a subgroup F of $G(A)$ *weakly graded* if $F \cap U_s^s B = (F \cap U_s^s)(F \cap B)$ for all $s \in S$. Note that every graded subgroup of $G(A)$ is weakly graded. Let C be the center of $G(A)$.

PROPOSITION 4.4.

(a) $C \subset H$.

(b) *Let F be a weakly graded normal subgroup of $G(A)$, and suppose that S is indecomposable. Then $F = G(A)$ or $F \subset C$.*

Proof. — $C \subset H$ follows from (4.2) and (4.4). Now let F be a weakly graded normal subgroup of $G(A)$, and assume that S is indecomposable. Suppose that $FB = B$. Then $F \subset B$ and hence, using (4.4), $F \subset \bigcap_{w \in W} B^w = H$. If $h \in F$ and $u \in U_+$, then $huh^{-1}u^{-1} \in F \cap U_+ = \{1\}$. Hence, h centralizes U_+; similarly, h centralizes U_-. (4.3) now shows that $F \subset C$. Now suppose that $FB \neq B$. Then $FB = G(A)$ by (4.1). Hence, for all $s \in S$,

$$U_s^s B = U_s^s B \cap FB = (U_s^s B \cap F)B = (U_s^s \cap F)(B \cap F)B = (U_s^s \cap F)B.$$

Since $U_s^s \cap B = \{1\}$, we conclude that $U_s^s \subset F$ and therefore $U_s \subset F$ for all $s \in S$. Hence, by (4.3), $F = G(A)$. ∎

We sometimes write $H(A)$ for H, $U_+(A)$ for U_+, etc., to emphasize the dependence on A.

COROLLARY 4.3.

(a) *Let A' be an indecomposable generalized Cartan matrix, and let $\Psi : G(A') \to G(A)$ be a homomorphism such that $\Psi(U_\pm(A')) \subset U_\pm$ and $\Psi(H(A')) \subset H$. Then either* $\ker\Psi = G(A')$ *or else*

$$\ker\Psi \subset \text{Center}\,(G(A')) \subset H(A').$$

(b) *If J is a subset of S and $A_J = (a_{s,t})_{s,t\in J}$ is the corresponding principal submatrix of A, then the obvious homomorphism $G(A_J) \to G(A)$ is injective.*

Proof. — (a) follows from PROPOSITION 4.4, since $\ker\Psi$ is graded and hence weakly graded. Since the homomorphism of (b) is injective on H, (b) follows from (a). ∎

COROLLARY 4.4.

(a) *If $(V, d\pi)$ is a Q-graded integrable $\mathbf{g}'(A)$-module and if A is indecomposable, then* $\ker\pi = G(A)$ *or* $\ker\pi \subset C \subset H$ *for the corresponding $G(A)$-module.*

(b) *The direct sum of all irreducible highest weight modules with fundamental highest weights (see* [3, Chapter 10] *for the definition) is a faithful differentiable $G(A)$-module.*

Proof. — (a) follows from PROPOSITION 4.4, since $\ker\pi$ is graded and hence weakly graded. Since the module of (b) is a faithful H-module, (b) follows from (a). ∎

COROLLARY 4.5. — *Assume that the generalized Cartan matrix A is indecomposable and not of affine type, and let $(V, d\pi)$ be an integrable $\mathbf{g}'(A)$-module. Then* $\ker\pi = G(A)$ *or* $\ker\pi \subset C \subset H$ *for the corresponding $G(A)$-module.*

Sketch of proof. — Since A is not of affine type, there exist integers k_s, $s \in S$, such that $\alpha_{s'}(\sum_{s\in S} k_s\alpha_s^\vee) > 0$ for all $s' \in S$ [3, THEOREM 4.3]. For $t \in \mathbf{C}^\times$, put $h(t) = \prod_{s\in S} h_s(t)^{k_s}$. Define a $\mathbf{Z}$-gradation $\mathbf{g}'(A) = \oplus_{n\in\mathbf{Z}}\mathbf{g}_n$ by

$$\mathbf{g}_n = \{x \in \mathbf{g}'(A) \mid \text{Ad}(h(t))x = t^n x \quad \text{for all} \quad t \in \mathbf{C}^\times\}.$$

Now let $(V, d\pi)$ be an integrable $\mathbf{g}'(A)$-module. Define a $\mathbf{Z}$-gradation $V = \oplus_{n\in\mathbf{Z}} V_n$ by

$$V_n = \{v \in V \mid \pi(h(t))v = t^n v \quad \text{for all } t \in \mathbf{C}^\times\}.$$

These gradations are compatible, and by imitating the arguments proving LEMMA 4.4(a), one shows that $\ker \pi$ is a graded subgroup of $G(A)$. COROLLARY 4.5 now follows from PROPOSITION 4.4. ∎

COROLLARY 4.6. — Ad *is faithful on* U_+. *Moreover,* $\ker \mathrm{Ad} = C \subset H$.

Proof. — This is clear from PROPOSITION 4.4. ∎

Remark. — One may also prove the first part of COROLLARY 4.6 by defining a map log from U_+ to $\widehat{\mathfrak{n}}_+ \subset \widehat{U(\mathfrak{n}_+)}$ and noting that the center of $\mathfrak{g}'(A)$ is contained in $\mathfrak{g}_0$. However, this procedure is not valid over a field of positive characteristic, and also involves the Campbell-Hausdorff formula. For these reasons, we omit this approach here.

The following statement is clear from $(G2a)$ (we use that $t^2 \neq 1$ for some $t \in \mathbf{C}^\times$) :

(4.5) If $s \in S$, then the centralizer of H in U_s is $\{1\}$.

PROPOSITION 4.5.

(a) *Let F be a graded subgroup of $G(A)$ containing N such that $F \cap U_s = \{1\}$ for all $s \in S$. Then $F = N$.*

(b) *The normalizer of H in $G(A)$ is N.*

Proof. — We first deduce (b) from (a). Let $\widetilde{N}$ be the normalizer of H in $G(A)$. Then $\widetilde{N}$ contains N. Suppose $u_- \in U_-$, $h \in H$, $u_+ \in U_+$ and $u_- h u_+ \in \widetilde{N}$. Put $n = u_- h u_+$. If $h' \in H$, then

$$u_+ h' u_+^{-1} h'^{-1} = (u_- h)^{-1})(n h' n^{-1})(u_- h) h'^{-1} \in U_+ \cap H U_- = \{1\},$$

so that u_+ centralizes H and, similarly, u_- centralizes H. Along with (4.5), this verifies the hypotheses of (a) with $F = \widetilde{N}$. Hence, by (a), $\widetilde{N} = N$, proving (b).

We now prove (a). We first show that N normalizes $F \cap U_+$. Indeed, suppose that $s \in S$, $n \in sH$ and $u \in F \cap U_+$. By (3.8.2), write $nun^{-1} = u_1 u_2$, where $u_1 \in U_- \cap nU_+n^{-1}$ and $u_2 \in U_+$. Since $n, u \in F$ and F is graded, we obtain $u_1, u_2 \in F$. But then $n^{-1}u_1 n \in F \cap U_s = \{1\}$, so that $u_1 = 1$ and hence $nun^{-1} = u_2 \in F \cap U_+$. This shows that N normalizes $F \cap U_+$. Hence, $F \cap U_+ \subset \bigcap_{w \in W} U_+^w = \{1\}$ by (4.4). Now let $g \in F$. By PROPOSITION 3.2(a,b) write $g = u_+ u_- n$, where $n \in N$, $u_- \in U_- \cap nU_+n^{-1}$ and $u_+ \in U_+$. Since $g, n \in F$ and F is graded, we have $u_-, u_+ \in F$. Hence, $u_+, n^{-1}u_- n \in F \cap U_+ = \{1\}$, so that $g = n \in N$. This proves (a). ∎

COROLLARY 4.7. — *The centralizer of H in $G(A)$ is H.*

Proof. — This follows from PROPOSITION 4.5(b) and COROLLARY 2.2. ∎

We now discuss Levi decompositions of parabolics.

PROPOSITION 4.6. — *Let J be a subset of S, and put $M_J = P_J \cap \omega(P_J)$, $U_J = M_J \cap U_+$ and $U^J = \bigcap_{w \in W_J} U_+^w$. Then $P_J = M_J \propto U^J$ and, moreover :*

(a) *M_J is generated by H and the G_s, $s \in J$.*

(b) *U_J is generated by the U_α, $\alpha \in \Delta_+^{\mathrm{re}} \cap \sum_{s\in J} \mathbf{Z}\alpha_s$.*

(c) *U^J is the smallest normal subgroup of U_+ containing the U_α, $\alpha \in \Delta_+^{\mathrm{re}} \setminus \sum_{s\in J} \mathbf{Z}\alpha_s$.*

Proof. — Let $\widetilde{M}_J$, $\widetilde{U}_J$ be the subgroups asserted in (a), (b) and (c) to be M_J, U_J and U^J. Clearly, we have :

(4.6) $\quad U_+ = \widetilde{U}_J \widetilde{U}^J$.

(4.7) $\quad HW_J \subset \widetilde{M}_J \subset M_J$.

We shall prove the following assertions :

(4.8) $\quad \widetilde{U}_J \subset \widetilde{M}_J$.

(4.9) $\quad \widetilde{M}_J$ normalizes $\widetilde{U}^J$.

(4.10) $\quad M_J \cap U^J = \{1\}$.

We first show that these assertions suffice to validate the proposition.

Since $HW_J \subset \widetilde{M}_J$ by (4.7) and $\widetilde{U}^J \subset U_+$ by (4.6), (4.9) gives $\widetilde{U}^J \subset U^J$. By (4.6,7,8), $\widetilde{U}_J \subset U_J$; by (4.6), $U_J U^J \subset \widetilde{U}_J \widetilde{U}^J$; by (4.10), $U_J \cap U^J = \{1\}$. These yield :

(4.11) $\quad \widetilde{U}_J = U_J$ and $\widetilde{U}^J = U^J$.

By (4.6,7,8), $\widetilde{M}_J$ and $\widetilde{U}^J$ generate P_J; by (4.9), $\widetilde{M}_J$ normalizes $\widetilde{U}^J$; by (4.7), $M_J \subset M_J \subset P_J$; by (4.10, 11), $M_J \cap \widetilde{U}^J = \{1\}$. These yield :

(4.12) $P_J = \widetilde{M}_J \propto \widetilde{U}^J$ and $\widetilde{M}_J = M_J$.

The proposition follows from (4.11) and (4.12).

It remains to verify (4.8), (4.9) and (4.10). (4.8) follows from LEMMA 2.1(b). COROLLARY 3.6 applied to the refined Tits system (P_J, HW_J, U_+, $P_J \cap U_-$, H, J) implies $\bigcap_{w\in W_J} (P_J \cap U_-)^w = \{1\}$; applying ω, we deduce (4.10). Finally, we verify (4.9). Suppose $s \in J$, and put

$$X^{\pm} = \bigcup_{\alpha} U_\alpha,$$

where α runs over $\left(\Delta_+^{\mathrm{re}} \cap \sum_{t\in J} \mathbf{Z}\alpha_t\right) \setminus \{\alpha_s\}$ with $\pm\langle \alpha, \alpha_s^\vee \rangle \geq 0$ and

$$Y^{\pm} = \bigcup_{\alpha} U_\alpha,$$

where α runs over $\Delta_+^{\mathrm{re}} \setminus \sum_{t\in J} \mathbf{Z}\alpha_t$ with $\pm\langle\alpha, \alpha_s^v\rangle \geq 0$.

Let U_1 be the subgroup of U_+ generated by $\{uxu^{-1} \mid u \in U_s, x \in X^+ \cup X^-\}$ and let U_2 be the subgroup of U_+ generated by $\{uyu^{-1} \mid u \in U_s, y \in Y^+ \cup Y^-\}$. Using LEMMA 2.1 (c), the argument proving COROLLARY 4.1 shows that HG_s normalizes U_1 and U_2. Let U_3 be the subgroup of U_+ generated by $\{u_1u_2u_1^{-1} \mid u_1 \in U_1, u_2 \in U_2\}$; since U_s, U_1 and U_2 generate U_+, and since U_s normalizes U_1 and U_2, we deduce that U_3 is the smallest normal subgroup of U_+ containing U_2. Hence, $U_3 = \widetilde{U}^J$, so that HG_s normalizes $\widetilde{U}^J$. Varying $s \in J$, we obtain (4.9). ∎

Remark. — It is easy to show that, for all $j \in J$, P_J is the normalizer of U^J in $G(A)$ and M_J is the normalizer of M_J in P_J.

We conclude this section with some technical results about "finite-dimensional" subgroups of U_+.

PROPOSITION 4.7. — *Let $\alpha, \beta \in \Delta_+^{\mathrm{re}}$. Then the following assertions are equivalent :*

(a) $|(\mathbf{Z}_+\alpha + \mathbf{Z}_+\beta) \cap \Delta_+^{\mathrm{re}}| < \infty$.

(b) *For some $w \in W$, one has : $w \cdot \alpha$, $w \cdot \beta \in -\Delta_+^{\mathrm{re}}$.*

(c) *(U_α, U_β) is contained in the subgroup of U_+ generated by the U_γ, where $\gamma \in (\mathbf{Z}_+\alpha + \mathbf{Z}_+\beta) \cap \Delta_+^{\mathrm{re}}$ and $\gamma \neq \alpha, \beta$.*

Sketch of proof. — (We use here some notions defined e.g. in [3, Chapter 5]. First, suppose $\langle\alpha, \beta^v\rangle > 0$ and $\langle\beta, \alpha^v\rangle > 0$. Then (a) and (c) hold by LEMMA 2.1c(ii) and the argument proving LEMMA 4.2(c). We have $(1 - \langle\beta, \alpha^v\rangle\langle\alpha, \beta^v\rangle)\beta = \langle\beta, \alpha^v\rangle r_\beta \cdot \alpha + r_\alpha \cdot \beta$, hence $r_\alpha \cdot \beta < 0$ or $r_\beta \cdot \alpha < 0$. If $r_\alpha \cdot \beta < 0$ (resp. $r_\beta \cdot \alpha < 0$), then $w = r_\alpha$ (resp. $= r_\beta$) satisfies (b).

Now, suppose $\langle\alpha, \beta^v\rangle = 0 = \langle\beta, \alpha^v\rangle$. Then (a) and (c) hold by LEMMA 2.1c(ii) and the argument proving LEMMA 4.2(c), and $w = r_\alpha r_\beta$ satisfies (b).

By [6, p. 139] or [3, 2nd ed., Exercise 5.19], the only remaining case is $\langle\alpha, \beta^v\rangle < 0$ and $\langle\beta, \alpha^v\rangle < 0$. By using W, we may assume that $\beta = \alpha_s$ for some $s \in S$. If $\alpha - \alpha_s \in \Delta$, put $\gamma = \alpha - \alpha_s$; otherwise, put $\gamma = \alpha$. Then :

$$\beta, \gamma \in \Delta_+^{\mathrm{re}}\,; \quad \langle\beta, \gamma^v\rangle < 0 \quad \text{and} \quad \langle\gamma, \beta^v\rangle < 0\,; \quad \gamma - \beta \notin \Delta.$$

Put $T = \{1, 2\}$, and define a generalized Cartan matrix $B = (b_{t,u})_{t,u\in T}$ by $b_{11} = b_{22} = 2$, $b_{12} = \langle\gamma, \beta^v\rangle$, $b_{21} = \langle\beta, \gamma^v\rangle$. Let α_1, α_2 be the corresponding generators of the root lattice of the Kac-Moody algebra $\mathbf{g}'(B)$. One can show that there exists a homomorphism $\Psi : \mathbf{g}'(B) \to \mathbf{g}'(A)$ such that, if $k, l \in \mathbf{Z}$ and $\delta = k\beta + l\gamma$, $\tilde{\delta} = k\alpha_1 + l\alpha_2$, then $\delta \in \Delta_+^{\mathrm{re}} \Leftrightarrow \tilde{\delta} \in \Delta_+^{\mathrm{re}}(B)$, and $\Psi(\mathbf{g}'(B)_{\tilde{\delta}}) = \mathbf{g}'(A)_\delta$ if $\delta \in \Delta^{\mathrm{re}}$. Since the induced homomorphism from

$G(B)$ to $G(A)$ is injective on $U_+(B)$ by COROLLARY 4.3(a), and since the implication $(b) \Rightarrow (a)$ always holds by LEMMA 4.5(e) below, this reduces us to the following case :

A is a generalized Cartan matrix $\begin{pmatrix} 2 & -m \\ -n & 2 \end{pmatrix}$ where $m, n > 0$; $\beta = \alpha_1$; $\alpha = \alpha_2$ or $\alpha_2 + \alpha_1$; $\langle \alpha, \beta^v \rangle < 0$.

First, suppose $mn \geq 4$. Then the $(r_{\alpha_1} r_{\alpha_2})^k \cdot \alpha_1$, $k = 0, 1, 2, \ldots$, are distinct elements of $(\mathbf{Z}_+\alpha + \mathbf{Z}_+\beta) \cap \Delta_+^{\mathrm{re}}$, so that (a) is false and hence (b) is false. Moreover, in this case (c) is false by PROPOSITION 4.3.

Finally, suppose $mn \leq 3$. Then $W(A)$ is a finite dihedral group and $w_0(\Delta_+(A)) = -\Delta_+(A)$ for the longest element w_0 of $W(A)$. Therefore (b) holds, and hence (a) holds. One can show that (c) holds by using the theory of algebraic groups over $\mathbf{C}$, but we will give a self-contained argument instead. Put $w = r_{\alpha_2}$ if $\alpha = \alpha_2$ and $w = r_{\alpha_1} r_{\alpha_2} = r_{\alpha_2} r_\alpha$ if $\alpha = \alpha_1 + \alpha_2$. Using COROLLARY 4.2(c), one shows that : U_α normalizes $U_+^{r_{\alpha_2}} \cap U_+^{r_{\alpha_2} r_\alpha}$, and $U_\beta \subset U_+^{r_{\alpha_2}} \cap U_+^{r_{\alpha_2} r_\alpha} \subset U_+^w$ so that $(U_\alpha, U_\beta) \subset U_+^w$; U_β normalizes $U_+ \cap U_+^{r_{\alpha_1}}$, and $U_\alpha \subset U_+ \cap U_+^{r_{\alpha_1}}$, so that $(U_\alpha, U_\beta) \subset U_+^{r_{\alpha_1}}$. Therefore, $(U_\alpha, U_\beta) \subset U_+^{r_{\alpha_1}} \cap U_+^w$. But by using PROPOSITION 3.3(d), one sees that $U_+^{r_{\alpha_1}} \cap U_+^w$ is the subgroup defined in (c). Hence, (c) holds.

This verifies that in all cases, (a), (b) and (c) are true or false simultaneously. ∎

For $w \in W$, put $\Phi(w) = \Delta_+^{\mathrm{re}} \cap -w \cdot \Delta_+^{\mathrm{re}}$.

LEMMA 4.5. — *Let $w, w' \in W$ satisfy $l(ww') = l(w) + l(w')$. Then :*

(a) $\Phi(w) = \Delta_+^{\mathrm{re}} \cap \sum_{\alpha \in \Phi(w)} \mathbf{Z}_+\alpha$.

(b) *For $\alpha \in \Delta_+^{\mathrm{re}}$, $\alpha \in \Phi(w)$ if and only if $U_\alpha \subset U_+ \cap U_-^{w^{-1}}$.*

(c) $\Phi(1) = \emptyset$. *For $s \in S$, $\Phi(s) = \{\alpha_s\}$.*

(d) $\Phi(ww') = \Phi(w) \sqcup w \cdot \Phi(w')$.

(e) $|\Phi(w)| = l(w)$.

Proof. — Since Δ^{re} is W-invariant, Q_+ is a semigroup and $\Delta_+^{\mathrm{re}} = \Delta^{\mathrm{re}} \cap Q_+$, (a) is clear. We have $U_\alpha^w = U_{w^{-1} \cdot \alpha}$, $U_+ \cap U_- = \{1\}$, $\Delta^{\mathrm{re}} = \Delta_+^{\mathrm{re}} \sqcup -\Delta_+^{\mathrm{re}}$, and $U_\alpha \subset U_\pm \Leftrightarrow \alpha \in \pm\Delta_+^{\mathrm{re}}$ for $\alpha \in \Delta^{\mathrm{re}}$, so that (b) is clear. (c) is clear, and (e) follows from (c) and (d).

It is easy to deduce (d) from PROPOSITION 3.2(d). ∎

LEMMA 4.6 [10]. — *Let F be a group, and let $F_1, \ldots, F_k$ be subgroups of F satisfying : for $i = 1, \ldots, k$, $F_i F_{i+1} \cdots F_k$ is a normal subgroup of F; $F = F_1 F_2 \cdots F_k$* [unique]. *Then we have, for any permutation σ of $\{1, \ldots, k\}$, $F = F_{\sigma(1)} F_{\sigma(2)} \cdots F_{\sigma(k)}$* [unique]. ∎

PROPOSITION 4.8. — *Let Φ be a finite subset of Δ_+^{re} satisfying $\Phi =$*

$\Delta_+^{\mathrm{re}} \cap \sum_{\beta\in\Phi} \mathbf{Z}_+\beta$, *and let* $\beta_1,\ldots,\beta_n$ *be an enumeration of* Φ. *Then* $U = U_{\beta_1}\cdots U_{\beta_n}$ [unique], *where* U *is the subgroup of* U_+ *generated by the* U_{β_k}.

Proof. — We may assume by using W that $\alpha_s \in \Phi$ for some $s \in S$. Let $\gamma_1 = \alpha_s, \gamma_2, \ldots, \gamma_n$ be an enumeration of Φ such that the height of γ_{i-1} is at most that of γ_i, $2 \le i \le n$. By PROPOSITION 4.7, $U = U_{\gamma_1}\cdots U_{\gamma_n}$, and $U_{\gamma_k}\cdots U_{\gamma_n}$ is a normal subgroup of U for $k = 1,\ldots,n$.

Put $U' = U_{\gamma_2}\cdots U_{\gamma_n}$. Since $U_{\gamma_1} \cap U' \subset U_s \cap U_+^s = \{1\}$ and since $U = U_{\gamma_1}U'$, we obtain $U = U_{\gamma_1}U'$ [unique]. By induction on n,

$$U' = U_{\gamma_2}\cdots U_{\gamma_n} \text{ [unique]}.$$

Hence,

$$U = U_{\gamma_1}U_{\gamma_2}\cdots U_{\gamma_n} \text{ [unique]}.$$

Now we apply LEMMA 4.6. ∎

COROLLARY 4.8. — *If* $w \in W$, *then*

$$U_+ \cap U_-^{w^{-1}} = U_{\beta_1}\cdots U_{\beta_n} \text{ [unique]}$$

for any enumeration $\beta_1,\ldots,\beta_n$ *of* $\Phi(w)$.

Proof. — We proceed by induction on $l(w)$, the cases $l(w) \le 1$ being trivial. Choose $s \in S$ such that $l(sw) < l(w)$. Then $U_+ \cap U_-^{w^{-1}} = (U_+ \cap U_-^s)(U_+ \cap U_-^{(sw)^{-1}})^s$ by PROPOSITION 3.2(d). By the induction hypothesis, $U_+ \cap U_-^{(sw)^{-1}}$ is generated by the U_β, $\beta \in \Phi(sw)$, and hence $(U_+ \cap U_-^{(sw)^{-1}})^s$ is generated by the U_β, $\beta \in s\cdot\Phi(sw)$. Since $U_+ \cap U_-^s = U_{\alpha_s}$, we conclude that $U_+ \cap U_-^{w^{-1}}$ is generated by $\{\alpha_s\} \cup s\cdot\Phi(sw)$, which equals $\Phi(w)$ by LEMMA 4.5. LEMMA 4.5(a) and PROPOSITION 4.8 complete the proof. ∎

5. The group $K(A)$

Recall the involution ω of $G(A)$ from § 2, and let $K(A)$ be the fixed-point set of ω. We shall give explicit generators and relations for $K(A)$.

Let $D = \{u \in \mathbf{C} \mid |u| \le 1\}$ be the closed unit disc, let $S^1 = \{t \in \mathbf{C} \mid |t| = 1\}$ be the unit circle and let $\mathring{D} = D \setminus S^1$. For $u \in D$, put

$$z(u) = \begin{pmatrix} u & (1-|u|^2)^{1/2} \\ -(1-|u|^2)^{1/2} & \overline{u} \end{pmatrix} \in SU_2.$$

Note that $z(t) = h(t)$ if $t \in S^1$ (*cf.* § 2).

For $s \in S$, $u \in D$ and $t \in S^1$, put $z_s(u) = \varphi_s(z(u))$ and $h_s(t) = \varphi_s(h(t))$. For $s \in S$, put $K_s = K \cap G_s$. Note that $z_s(u) \in K_s = \varphi_s(SU_2)$ and $z_s(0) = \tilde{s}$ (*cf.* § 2). Recall the subgroups H_+ of $G(A)$ and T of $K(A)$ introduced in § 2.

PROPOSITION 5.1.

(a) $G(A) = K(A)H_+U_+$ [unique] *(Iwasawa decomposition).*

(b) $K(A)$ *is generated by the* K_s, $s \in S$.

(c) *If* $w = s_1 \cdots s_k$ *is a reduced expression and* $g \in K(A) \cap BwB$, *then there exist unique* $u_1, \ldots, u_k \in \mathring{D}$ *and* $t \in T$ *such that*

$$g = z_{s_1}(u_1) \cdots z_{s_k}(u_k)t.$$

(d) *For all* $s, t \in S$, *there exists a unique map* $\Gamma_{s,t} : \mathring{D}^{m_{s,t}} \to (\mathring{D})^{m_{s,t}}$ *such that if* $u = (u_1, u_2 \cdots) \in (\mathring{D})^{m_{s,t}}$ *and* $\Gamma_{s,t}(u) = v = (v_1, v_2, \cdots) \in (\mathring{D})^{m_{s,t}}$, *then*

$$z_s(u_1)z_t(u_2)z_s(u_3) \cdots = z_t(v_1)z_s(v_2)z_t(v_3) \cdots$$

(e) K *is the amalgamated product of its subgroups* $K \cap P_s$, $s \in S$, *modulo the relations in* (d).

Proof * . — We use PROPOSITION 3.7. If $h \in H$, $u_+ \in U_+$ and $hu_+ \in K(A)$, then $\omega(hu_+) = hu_+$ and hence $\omega(u_+)^{-1}(\omega(h)^{-1}h)u_+ = 1$. Since $\omega(u_+) \in U_-$, $\omega(h)^{-1}h \in H$ and $u_+ \in U_+$, we deduce that $\omega(h)^{-1}h = 1$ and $u_+ = 1$. Hence, $hu_+ = h \in H \cap K(A)$. Using LEMMA 2.2(a), it is easy to check that $H \cap K(A) = T$. Hence, $K(A) \cap U_+ = \{1\}$ and $T = \theta(K \cap B)$. Clearly, H_+ is a normal subgroup of N and $H = H_+T$ [unique]. If $u \in \mathbf{C}$, then $z(-(1+|u|^2)^{-1/2}u)^{-1}x(u)z(0)$ is of the form $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$. This shows that $U_s \subset KBs$ for all $s \in S$. By COROLLARY 2.3(b), there exists a unique map $w \to \tilde{w}$ from W to N satisfying : $\tilde{1} = 1$; $\tilde{s} = z_s(0)$ for all $s \in S$; $\widetilde{ww'} = \tilde{w}\tilde{w}'$ if $w, w' \in W$ and $l(ww') = l(w) + l(w')$.

This verifies the hypotheses of PROPOSITION 3.7 and shows that $T = K \cap B$, and $U_s \subset z_s(\mathring{D})Bs$ for all $s \in S$. Recall Z_w defined by (3.13). If $s \in S$, then : $BsB = U_s sB \subset (z_s(\mathring{D})Bs)sB = z_s(\mathring{D})B$; z_s defines an injection from $\mathring{D}$ into Z_s by an easy calculation; $BsB = Z_sB$ [unique] by PROPOSITION 3.7. Hence, z_s defines a bijection from $\mathring{D}$ onto Z_s for all $s \in S$. PROPOSITION 3.7 now shows that (a), (c), (d) and (e) hold, and that $K(A)$ is generated by T and the Z_s. Since, $Z_s \subset K_s$, and since T is generated by the $K_s \cap T$, (b) follows. ∎

Note the following corollary of PROPOSITION 5.1(c).

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* The proof of the Iwasawa decomposition is a straigthforward generalization of that of STEINBERG [10]. In the affine case this has been done in [14].

COROLLARY 5.1. — *For $J \subset S$, denote by K_J the subgroup of $K(A)$ generated by the K_s with $s \in J$. Then $K(A) \cap P_J = K_J T$.* ▮

We wish to determine the maps $\Gamma_{s,t}$ of PROPOSITION 5.1(d). Using COROLLARY 4.3(b), we see that $\Gamma_{s,t}$ depends only on $a_{s,t}$ and $a_{t,s}$. Clearly, $\Gamma_{s,t} \circ \Gamma_{t,s} = \text{id}$, and $\Gamma_{s,s} = \text{id}$. If $a_{s,t} = a_{t,s} = 0$, then G_s and G_t commute and so $\Gamma_{s,t}(\alpha, \beta) = (\beta, \alpha)$. If $m_{s,t} = 0$, then $\Gamma_{s,t}$ is trivial. If $a_{s,t} = -1$ and $a_{t,s} = -k$, $k = 1, 2$ or 3, we write Γ_k for $\Gamma_{s,t}$. We must calculate Γ_1, Γ_2 and Γ_3.

LEMMA 5.1.

(a) *If $S = \{1,2\}$ and A is the generalized Cartan matrix $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$, then $\mathbf{C}^3$ is a faithful $G(A)$-module by :*

$$\varphi_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} (x, y, z) = (ax + by, cx + dy, z)$$

and

$$\varphi_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} (x, y, z) = (x, ay + bz, cy + dz).$$

(b) *If $S = \{1,2\}$ and A is the generalized Cartan matrix $\left(\begin{smallmatrix} 2 & -1 \\ -2 & 2 \end{smallmatrix}\right)$, then $\mathbf{C}^4$ is a faithful $G(A)$-module by :*

$$\varphi_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} (x, y, z, w) = (x, ay + bw, z, cy + dw)$$

and

$$\varphi_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} (x, y, z, w) = (ax + by, cx + dy, dz - cw, -bz + aw).$$

Moreover,

$$\varphi_1 \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & \alpha + \beta j \end{pmatrix}$$

and

$$\varphi_2 \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \to \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

defines a faithful representation of $K(A)$ by quaternionic matrices.

Proof. — Using COROLLARY 4.4 and LEMMA 2.2(a), we see that the modules defined in the lemma are faithful. Let $\mathbf{H}$ be the associative $\mathbf{R}$-algebra of quaternions, with standard $\mathbf{R}$-basis 1, i, j, k, with $ij = k$, $jk = i$, $ki = j$, and $i^2 = j^2 = k^2 = -1$. $\mathbf{C}^4$ becomes a right $\mathbf{H}$-module under $(x,y,z,w)i = (xi,yi,zi,wi)$ and $(x,y,z,w)j = (\overline{z},\overline{w},-\overline{x},-\overline{y})$, which is free on generators $v_1 = (1,0,0,0)$ and $v_2 = (0,1,0,0)$. It is easy to check that $\varphi_1(SU_2)$ and $\varphi_2(SU_2)$ give $\mathbf{H}$-module endomorphisms of $\mathbf{C}^4$ under the action defined in (b). But

$$\sigma \longrightarrow \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix},$$

where $\sigma(v_i) = v_1 q_{1i} + v_2 q_{2i}$, defines an isomorphism from $\mathrm{End}_{\mathbf{H}}(\mathbf{C}^4)$ onto the ring of 2-by-2 matrices over $\mathbf{H}$. The lemma now follows from a calculation. ∎

COROLLARY 5.2. — *If* $\alpha_i \in \mathring{D}$ *and* $u_i = (1-|\alpha_i|^2)^{(1/2)}$, $1 \le i \le 4$, *then :*

(*a*)
$$\begin{aligned}(\beta_1,\beta_2,\beta_3) &= \Gamma_1(\alpha_1,\alpha_2,\alpha_3) \quad \text{if and only if :}\\ (1-|\beta_1|^2)^{-(1/2)}\beta_1 &= (u_2u_3)^{-1}(u_1\alpha_3 + \overline{\alpha}_1\alpha_2u_3),\\ \beta_2 &= \alpha_1\alpha_3 - u_1\alpha_2u_3,\\ (1-|\beta_3|^2)^{-(1/2)}\beta_3 &= (u_1u_2)^{-1}(\alpha_1u_3 + u_1\alpha_2\overline{\alpha}_3).\end{aligned}$$

(b) *Define* $A,B,C,D,E,F,G,H \in \mathbf{C}$ *by :*

$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha_1 + u_1 j \end{pmatrix} \begin{pmatrix} \alpha_2 & u_2 \\ -u_2 & \overline{\alpha}_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_3 + u_3 j \end{pmatrix} \begin{pmatrix} \alpha_4 & u_4 \\ -u_4 & \overline{\alpha}_4 \end{pmatrix} = \begin{pmatrix} A+Bj & C+Dj \\ E+Fj & G+Hj \end{pmatrix}$$

in $M_2(\mathbf{H})$.

Then $(\beta_1,\beta_2,\beta_3,\beta_4) = \Gamma_2(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ *if and only if :*

$$\begin{aligned}(1-|\beta_1|^2)^{-(1/2)}\beta_1 &= B^{-1}\overline{F},\\ (1-|\beta_2|^2)^{-(1/2)}\beta_2 &= (|B|^2+|F|^2)^{-1}(A\overline{B}+E\overline{F}),\\ (1-|\beta_1|^2)^{-(1/2)}\beta_3 &= B^{-1}(AF-BE),\\ (1-|\beta_2|^2)^{-(1/2)}\beta_4 &= (|B|^2+|F|^2)^{-1}(BG-CF).\end{aligned}$$

Proof. — LEMMA 5.1 and a calculation show that the given formulas hold if $(\beta_1,\cdots) = \Gamma_k(\alpha_1,\cdots)$. Since $(1-|\beta|^2)^{-1/2}\beta$ determines β for $|\beta| < 1$, the corollary follows. ∎

Unfortunately, a similar calculation of Γ_3, i.e. a matrix calculation for the exceptional 14-dimensional group G_2, seems difficult. As an alternative, we shall utilize the embedding of G_2 in D_4. For that, we apply to D_4 the following lemma* :

LEMMA 5.2. — *Let $\mathcal{A}$ be a group of permutations σ of S satisfying $a_{\sigma(s),\sigma(s')} = a_{s,s'}$, for all $s, s' \in S$. For $\sigma \in \mathcal{A}$ define an automorphism $\tilde{\sigma}$ of $G(A)$ by $\tilde{\sigma} \circ \varphi_s = \varphi_{\sigma(s)}$ for all $s \in S$ and an automorphism $\tilde{\sigma}$ of $W(A)$ by $\tilde{\sigma}(s) = \sigma(s)$. Let $G(A)^{\mathcal{A}}$ and $W(A)^{\mathcal{A}}$ be the corresponding fixed-point subgroups. Let $S/\mathcal{A}$ be the set of all orbits of $\mathcal{A}$ on S. Assume that if $t \in S/\mathcal{A}$, and s and s' are distinct elements of t, then $a_{s,s'} = 0$, so that G_s and $G_{s'}$ commute. For $t, u \in S/\mathcal{A}$, fix $s \in u$ and put $b_{t,u} = \sum_{r \in t} a_{r,s}$. Then $B = (b_{t,u})_{t,u \in S/\mathcal{A}}$ is a generalized Cartan matrix.*

Define homomorphisms $g \mapsto \overline{g}$ from $G(B)$ into $G(A)$ and $w \mapsto \overline{w}$ from $W(B)$ into $W(A)$ by :

$$\overline{\varphi_t(x)} = \prod_{s \in t} \varphi_s(x) \text{ for all } t \in S/\mathcal{A} \text{ and } x \in SL_2(\mathbf{C}) \;;$$

$$\overline{t} = \prod_{s \in t} s \text{ for all } t \in S/\mathcal{A}.$$

Then :

(a) *$g \mapsto \overline{g}$ is an isomorphism from $G(B)$ onto $G(A)^{\mathcal{A}}$.*

(b) *$w \mapsto \overline{w}$ is an isomorphism from $W(B)$ onto $W(A)^{\mathcal{A}}$. For any reduced expression for $w \in W(B)$, the corresponding expression for $\overline{w} \in W(A)^{\mathcal{A}}$ is reduced.*

Proof. — It is easy to check that B is a generalized Cartan matrix. We denote the homomorphisms $g \mapsto \overline{g}$ and $w \mapsto \overline{w}$ by Ψ. For any subset F of $G(A)$, we put $F^{\mathcal{A}} = F \cap G(A)^{\mathcal{A}}$. It is easy to check that Ψ is well-defined and that $\Psi(G(B)) \subset G(A)^{\mathcal{A}}$, $\Psi(U_\pm(B)) \subset U_\pm(A)^{\mathcal{A}}$. It is easy to check $\Psi(H(B)) \subset H(A)^{\mathcal{A}}$, $\Psi(N(B)) \subset N(A)^{\mathcal{A}}$, and that Ψ on $G(B)$ induces Ψ on $W(B)$. Using LEMMA 2.2(a), it is easy to see that $\Psi(H(B)) = H(A)^{\mathcal{A}}$, and using COROLLARY 4.3(a), it is easy to check that Ψ is injective on $G(B)$. Hence, Ψ is injective on $W(B)$.

If $w \in W(A)^{\mathcal{A}}$ and $w \neq 1$, choose $t \in S/\mathcal{A}$ such that $l(sw) < l(w)$ for some $s \in t$. Since $w \in W(A)^{\mathcal{A}}$, we deduce that $l(sw) < l(w)$ for all $s \in t$, so that $l(\overline{t}w) = l(w) - l(\overline{t}) = l(w) - |t|$ (here $|t|$ means Card (t)) by a standard fact about Coxeter groups [1].

By induction on $l(w)$, we deduce :

(5.1) If $w \in W(A)^{\mathcal{A}}$, then there exist $t_1, \ldots, t_n \in S/\mathcal{A}$ such that $w = \overline{t}_1 \cdots \overline{t}_n$ is a reduced expression.

* We use some arguments of [15] in the proof of this lemma.

We next prove :

(5.2) If $w \in W(A)^{\mathcal{A}}$, then $(U_+(A) \cap U_-(A)^w)^{\mathcal{A}} \subset \Psi(G(B))$.

If $w = 1$, (5.2) is clear. Suppose $w = \bar{t}$ for some $t \in S/\mathcal{A}$. Let $s_1, \ldots, s_m$ be an enumeration of t. If $g \in (U_+(A) \cap U_-(A)^w)^{\mathcal{A}}$, write

$$g = x_{s_1}(u_1) \cdots x_{s_m}(u_m)$$

by PROPOSITION 3.2(d), where $u_1, \ldots, u_m \in \mathbf{C}$ are determined by g. If $\sigma \in \mathcal{A}$, let τ be the permutation of $\{1, \ldots, m\}$ defined by $\sigma(s_i) = s_{\tau(i)}$. Then

$$\begin{aligned}\tilde{\sigma}(g) &= \tilde{\sigma}(x_{s_1}(u_1)) \cdots \tilde{\sigma}(x_{s_m}(u_m)) \\ &= x_{\sigma(s_1)}(u_1) \cdots x_{\sigma(s_m)}(u_m) \\ &= x_{s_1}(u_{\tau^{-1}(1)}) \cdots x_{s_m}(u_{\tau^{-1}(m)})\end{aligned}$$

since $G(A)_{s_1}, \ldots, G(A)_{s_m}$ commute. Since g determines the u_i, we must have $u_1 = u_{\tau^{-1}(1)}$. Varying σ, we conclude that $u_1 = \cdots = u_m$, so that $g = \Psi(x_t(u_1))$, verifying (5.2).

Now suppose $w \in W(A)^{\mathcal{A}}$, $w \neq 1$. By (5.1), choose $t \in S/\mathcal{A}$ such that $l(\bar{t}w) = l(w) - l(\bar{t})$. If $g \in (U_+(A) \cap U_-(A)^w)^{\mathcal{A}}$, use PROPOSITION 3.2(d) to write $g = g_1 g_2$, where $g_1 \in (U_+(A) \cap U_-(A)^{\bar{t}})^{\bar{t}w}$ and $g_2 \in U_+(A) \cap U_-(A)^{\bar{t}w}$. Using (5.1), choose $n \in N(B)$ such that $\Psi(n) \in \bar{t}wH(A)$, and put $g' = \Psi(n)g\Psi(n)^{-1}$, $g_1' = \Psi(n)g_1\Psi(n)^{-1}$ and $g_2' = \Psi(n)g_2\Psi(n)^{-1}$. Then $g' \in G(A)^{\mathcal{A}}$, $g' = g_1'g_2'$, $g_1' \in U_+(A)$ and $g_2' \in U_-(A)$. If $\sigma \in \mathcal{A}$, then $g' = \tilde{\sigma}(g') = \tilde{\sigma}(g_1')\tilde{\sigma}(g_2')$, where $\tilde{\sigma}(g_1') \in U_+(A)$ and $\tilde{\sigma}(g_2') \in U_-(A)$. Since $U_+(A) \cap U_-(A) = \{1\}$, we deduce that $\tilde{\sigma}(g_1') = g_1'$ and $\tilde{\sigma}(g_2') = g_2'$. Hence, $g_1' \in (U_+(A) \cap U_-(A)^{\bar{t}})^{\mathcal{A}} \subset \Psi(G(B))$. Similarly, by induction on $l(w)$, $g_2' \in \Psi(G(B))$ and hence $g \in \Psi(G(B))$. This proves (5.2).

We next prove :

(5.3) $$G(A)^{\mathcal{A}} \subset \Psi(G(B))U_+(A)^{\mathcal{A}}.$$

To avoid confusion, let B_+ denote the subgroup $H(A)U_+(A)$ of $G(A)$. Suppose $w \in W(A)$ and $G(A)^{\mathcal{A}} \cap B_+wB_+ \neq \emptyset$. Since $\tilde{\sigma}(B_+wB_+) = \tilde{\sigma}(B_+)\tilde{\sigma}(w)\tilde{\sigma}(B_+) = B_+\tilde{\sigma}(w)B_+$ for all $\sigma \in \mathcal{A}$, (3.1) forces $w \in W(A)^{\mathcal{A}}$. Using (5.1), choose $n \in N(B)$ such that $\Psi(n) \in wH(A)$. If $g \in G(A)^{\mathcal{A}} \cap B_+wB_+$, write $g = g_1\Psi(n)hg_2$, where $g_1 \in U_+(A) \cap U_-(A)^{w^{-1}}$, $h \in H(A)$, $g_2 \in U_+(A)$. As before, we deduce that $g_1 \in (U_+(A) \cap U_-(A)^{w^{-1}})^{\mathcal{A}}$, $h \in H(A)^{\mathcal{A}}$ and $g_2 \in U_+(A)^{\mathcal{A}}$. By (5.2), we have $g_1 \in \Psi(G(B))$, and also $h \in H(A)^{\mathcal{A}} = \Psi(H(B))$. Hence, $g \in \Psi(G(B))g_2 \subset \Psi(G(B))U_+(A)^{\mathcal{A}}$. By (3.1), this proves (5.3).

To prove (a), it remains to show that $U_+(A)^{\mathcal{A}} \subset \Psi(G(B))$. Let $g \in U_+(A)^{\mathcal{A}}$. Then $\omega(g) \in U_-(A)^{\mathcal{A}}$. By (5.3), choose $g' \in G(B)$ and $g'' \in U_+(A)$ such that $\omega(g) = \Psi(g')g''$. Write $g' = g_1 n g_2$, where $g_1 \in U_-(B)$, $n \in N(B)$ and $g_2 \in U_+(B)$. Then

$$(\omega(g)^{-1}\Psi(g_1))\Psi(n)(\Psi(g_2)g'') = 1$$

and hence, by (RT3), $\omega(g)^{-1}\Psi(g_1) = 1$. We conclude that

$$g = \omega^2(g) = \omega(\Psi(g_1)) = \Psi(\omega(g_1)).$$

This proves (a).

It remains to prove the assertion of (b) about reduced expressions. We need :

(5.4) There exists a function $\bar{l}$ on $W(B)$ such that $\bar{l}(t_1 \cdots t_n) = |t_1| + \cdots + |t_n|$ if $t_1 \cdots t_n$ is a reduced expression.

Indeed, by LEMMA 1.1, we need only to show that if $t, u \in S/\mathcal{A}$, then $|t| + |u| + |t| + \cdots = |u| + |t| + |u| + \cdots$ ($m^B_{t,u}$ summands on each side). If $m^B_{t,u}$ is even, this is clear. Suppose $t \neq u$ and $m^B_{t,u}$ is odd. Then since B is a generalized Cartan matrix, $b_{t,u} = -1 = b_{u,t}$. Hence, $a_{r,s} = 0$ or -1 for all $r \in t$ and $s \in u$, since otherwise $b_{t,u} = \sum_{r \in t} a_{r,s}$ would be less than -1. Similarly, $a_{s,r} = 0$ or -1 for all $r \in t$ and $s \in u$. Since A is a generalized Cartan matrix, we deduce that $a_{r,s} = a_{s,r}$ for all $r \in t$ and $s \in u$, and hence

$$-|u| = |u|\, b_{t,u} = \sum_{\substack{r \in t \\ s \in u}} a_{r,s} = \sum_{\substack{s \in u \\ r \in t}} a_{s,r} = |t|\, b_{u,t} = -|t|.$$

Therefore, $|t| + |u| + |t| + \cdots = |u| + |t| + |u| + \cdots$, proving (5.4).

Now let $t_1 \cdots t_n$ be a reduced expression. By (5.1), choose $t'_1, \ldots, t'_m \in S/\mathcal{A}$ such that $\bar{t}_1 \cdots \bar{t}_n = \bar{t}'_1 \cdots \bar{t}'_m$ and $\bar{t}'_1 \cdots \bar{t}'_m$ is a reduced expression. Since Ψ is injective on $W(B)$, we have $t_1 \cdots t_n = t'_1 \cdots t'_m$. Using LEMMA 1.1, we have :

$$\begin{aligned} |t'_1| + \cdots + |t'_m| &\geq \bar{l}(t'_1 \cdots t'_m) = \bar{l}(t_1 \cdots t_n) \\ &= |t_1| + \cdots + |t_n| \geq l(\bar{t}_1 \cdots \bar{t}_n) = l(\bar{t}'_1 \cdots \bar{t}'_m) = |t'_1| + \cdots + |t'_m|. \end{aligned}$$

Hence, $l(\bar{t}_1 \cdots \bar{t}_n) = |t_1| + \ldots + |t_n|$, so that $\bar{t}_1 \cdots \bar{t}_n$ is a reduced expression. This proves (b). ∎

COROLLARY 5.3. — *Let $k = 2$ or 3, let $S = \{0, 1, \ldots, k\}$, and let A be the generalized Cartan matrix $(a_{i,j})_{i,j \in S}$ defined by :*

$$a_{i,i} = 2 \text{ for } 0 \leq i \leq k\ ;$$
$$a_{0,i} = a_{i,0} = -1 \text{ for } 1 \leq i \leq k\ ;$$
$$a_{i,j} = a_{j,i} = 0 \text{ if } 1 \leq i < j \leq k.$$

Define maps $\tilde{z}_1 : \mathring{D} \to K(A)$ *and* $\tilde{z}_2 : \mathring{D} \to K(A)$ *by :*

$$\tilde{z}_1(u) = z_0(u), \quad \tilde{z}_2(u) = z_1(u)z_2(u)\cdots z_k(u).$$

Let $u_i, v_i \in \mathring{D}$, $1 \leq i \leq 2k$, *and put* $u = (u_1, \ldots, u_{2k})$, $v = (v_1, \ldots, v_{2k})$. *Then* $v = \Gamma_k(u)$ *if and only if*

$$\tilde{z}_1(u_1)\tilde{z}_2(u_2)\tilde{z}_1(u_3)\cdots\tilde{z}_2(u_{2k}) = \tilde{z}_2(v_1)\tilde{z}_1(v_2)\tilde{z}_2(v_3)\cdots\tilde{z}_1(v_{2k}).$$

Proof. — Let $\mathcal{A}$ be the group of all permutations of S fixing 0, and apply LEMMA 5.2(a). ∎

COROLLARY 5.4. — *Let* $k = 2$ *or* 3, *and put* $N = k(k+1)$. *Define maps* C, R *and* Γ *from* $\mathring{D}^N$ *to* $\mathring{D}^N$ *by :*

$$\begin{aligned} C(x_1, \ldots, x_N) &= (x_2, \ldots, x_N, x_1); \\ R(x_1, \ldots, x_N) &= (x_2, x_1, x_3, \ldots, x_N); \\ \Gamma(x_1, \ldots, x_N) &= (y_1, y_2, y_3, x_4, \ldots, x_N) \end{aligned}$$

if $(y_1, y_2, y_3) = \Gamma_1(x_1, x_2, x_3)$. *(We have* $\Gamma^2 = \mathrm{id}$.*)*
Define $i : \mathring{D}^{2k} \to \mathring{D}^N$ *and* $j : \mathring{D}^N \to \mathring{D}^{2k}$ *by :*

$$i(x_1, \ldots, x_4) = (x_1, x_2, x_2, x_3, x_4, x_4) \text{ and } j(y_1, \ldots, y_6) = (y_2, y_3, y_5, y_6)$$

if $k = 2$ *;*

$$i(x_1, \ldots, x_6) = (x_1, x_2, x_2, x_2, x_3, x_4, x_4, x_4, x_5, x_6, x_6, x_6)$$

and

$$j(y_1, \ldots, y_{12}) = (y_3, y_4, y_7, y_8, y_{11}, y_{12})$$

if $k = 3$.
Define $\widetilde{\Gamma}_k$ *by :*

$$\begin{aligned} \widetilde{\Gamma}_2 &= C\Gamma C^{-2}\Gamma CRC \,; \\ \widetilde{\Gamma}_3 &= F^{-1}E^{-2}FE^2B^{-1}F^{-1}EBF, \end{aligned}$$

where

$$B = C^{-2}\Gamma C^{-2}\Gamma C^4 \Gamma C^{-1}, \quad E = RC \quad \text{and} \quad F = C^4.$$

Then

$$\Gamma_k = jC^{-k}\, \widetilde{\Gamma}_k^k i.$$

Proof. — Let $S = \{0, \ldots, k\}$ and A be as in COROLLARY 5.3. If

$$i_1, \ldots, i_N \in S \quad \text{and} \quad x = (x_1, \ldots, x_N) \in \mathring{D}^N,$$

we put

$$z_{i_1,\ldots,i_N}(x) = z_{i_1}(x_1) \cdots z_{i_N}(x_N) \in K(A).$$

Suppose $k = 2$. It is easy to check that $y = C^{-2}\widetilde{\Gamma}_2^2(x) \Rightarrow z_{2,1,0,1,2,0}(y) = z_{0,1,2,0,1,2}(x)$. Since 210120 is a reduced expression in $W(A)$ by LEMMA 5.2(b), $z_{2,1,0,1,2,0}(y)$ determines y by PROPOSITION 5.1(c); hence, we obtain

$$z_{2,1,0,1,2,0}(y) = z_{0,1,2,0,1,2}(x) \Rightarrow y = C^{-2}\widetilde{\Gamma}_2^2(x).$$

Noting that $z_1(\alpha)z_2(\beta) = z_2(\beta)z_1(\alpha)$ for all $\alpha, \beta \in \mathring{D}$, the case $k = 2$ follows from COROLLARY 5.3.

For $k = 3$, the argument is similar, using
$y = C^{-3}\widetilde{\Gamma}_3^3(x) \Leftrightarrow z_{1,2,3,0,1,2,3,0,3,2,1,0}(y) = z_{0,1,2,3,0,3,2,1,0,1,2,3}(x)$. ∎

We will need :

LEMMA 5.3. — *SU_2 is the group on generators $z(\alpha)$, $\alpha \in D$, with defining relations (we put $h(t) = z(t)$ for $t \in S^1$):*

(a) *$h(t)h(t') = h(tt')$, where $t, t' \in S^1$.*

(b) *$h(t)z(\alpha) = z(t^2\alpha)h(t^{-1})$, where $t \in S^1$, $\alpha \in D$.*

(c) *$z(ic)h(t)z(ic)^{-1} = z(c^2t + (1-c^2)\bar{t})$, where $0 \leq c \leq 1$, $t \in S^1$, $\operatorname{Im} t \geq 0$.*

Proof. — Let K be the group on generators $z(\alpha)$, $\alpha \in D$, with the given relations. Since these relations hold in SU_2, and since every element of SU_2 is uniquely of one of the forms $h(t)$, $t \in S^1$, or $z(\alpha)h(t)$, $\alpha \in \mathring{D}$ and $t \in S^1$, it suffices to check that every element of K is of one of these forms. By (a) and (b), we need only do this for $z(\beta)z(\gamma)$, where $\beta, \gamma \in \mathring{D}$.

Define a homeomorphism (F, G) from $\mathring{D}$ onto $(0,1) \times \{t \in S^1 \mid \operatorname{Im} t > 0\}$ by requiring $\alpha = F(\alpha)G(\alpha) + (1 - F(\alpha))\overline{G(\alpha)}$ for all $\alpha \in \mathring{D}$. Define $H : S^1 \to \mathbf{R}$ by $H(t) = F(t\beta) - F(\bar{t}\gamma)$. Since $F(\alpha) + F(-\alpha) = 1$ for all $\alpha \in \mathring{D}$, we have $H(1) + H(-1) = 0$, so that, by the continuity of H, $H(t'^2) = 0$ for some $t' \in S^1$. Put $t_1' = G(t'^2\beta)$ and $t_2' = G(\overline{t'}^2\gamma)$. If $\operatorname{Im} t_1't_2' \geq 0$, we put $t = t'$, $t_1 = t_1'$, $t_2 = t_2'$; otherwise, we put $t = it'$, $t_1 = -\overline{t_1'}$, $t_2 = -\overline{t_2'}$. Put $c = F(t^2\beta)^{1/2}$. Then we have :

$$t, t_1, t_2 \in S^1 \; ; \quad \operatorname{Im} t_1 , \quad \operatorname{Im} t_2 , \quad \operatorname{Im} t_1t_2 \geq 0 \; ;$$
$$0 \leq c \leq 1 \; ; \quad \beta = \bar{t}^2(c^2t_1 + (1-c^2)\bar{t}_1) , \quad \gamma = t^2(c^2t_2 + (1-c^2)\bar{t}_2).$$

Put $\alpha = c^2 t_1 t_2 + (1-c^2)\bar{t}_1\bar{t}_2$. Then (a), (b) and (c) imply :

$$\begin{aligned} h(t)z(\beta)z(\gamma)h(\bar{t}) &= z(t^2\beta)z(\bar{t}^2\gamma) \\ &= z(c^2t_1 + (1-c^2)\bar{t}_1)z(c^2t_2 + (1-c^2)\bar{t}_2) \\ &= [z(ic)h(t_1)z(ic)^{-1}][z(ic)h(t_2)z(ic)^{-1}] \\ &= z(ic)h(t_1t_2)z(ic)^{-1} = z(\alpha). \end{aligned}$$

Hence,

$$z(\beta)z(\gamma) = h(\bar{t})z(\alpha)h(t) = z(\bar{t}^2\alpha)h(t^2),$$

and hence also $z(\beta)z(\gamma) = h(\alpha)$ if $\alpha \in S^1$. This brings $z(\beta)z(\gamma)$ to the required form. ∎

THEOREM B. — *$K(A)$ is the group on generators $z_s(u)$, $s \in S$ and $u \in D$, with defining relations (we put $h_s(t) = z_s(t)$ if $t \in S^1$) :*

(K1) $h_s(t)h_s(t') = h_s(tt')$ *if* : $s \in S\,;\, t, t' \in S^1$.

(K2) $z_s(ic)h_s(t)z_s(ic)^{-1} = z_s(c^2t + (1-c^2\bar{t})$ *if* : $s \in S\,;\, 0 \le c \le 1\,;\, t \in S^1$, $\operatorname{Im} t \ge 0$.

(K3) $h_s(t)z_{s'}(u) = z_{s'}(t^{a_{s,s'}}u)h_{s'}(t^{-a_{s,s'}})h_s(t)$ *if* : $s, s' \in S\,;\, t \in S^1\,;$ $u \in D$.

(K4) $z_s(u)z_{s'}(v) = z_{s'}(v)z_s(u)$ *if* : $s, s' \in S$, $m^A_{s,s'} = 2\,;\, u, v \in D$.

(K5) $z_s(u_1)z_{s'}(u_2)z_s(u_3)\cdots = z_{s'}(v_1)z_s(v_2)z_{s'}(v_3)\cdots$ *($m^A_{s,s'}$ factors on each side) if* $s, s' \in S$, $a_{s,s'} = -1$, $a_{s',s} = -k\,;\, 1 \le k \le 3\,;\, (v_1, \ldots, v_{m^A_{s,s'}}) = \Gamma_k(u_1, \ldots, u_{m^A_{s,s'}})$, *and* Γ_1, Γ_2 *and* Γ_3 *are as defined in* COROLLARIES 5.2 *and* 5.4.

Proof. — Let $\widetilde{K(A)}$ be the group on the given generators with the given defining relations. We write $\widetilde{z_s(u)}$ and $\widetilde{h_s(t)}$ for the generators of $\widetilde{K(A)}$, to avoid confusion. Relations (K1) and (K2) hold in K(A) due to LEMMA 5.3; relations (K3) hold thanks to (2.1); relations (K4) are clear; relations (K5) hold thanks to COROLLARIES 5.2 and 5.4. Hence, there exists a unique homomorphism $\Psi = \widetilde{K(A)} \to K(A)$ such that $\Psi(\widetilde{z_s(u)}) = z_s(u)$ for all $s \in S$ and $u \in D$.

For $s \in S$, LEMMA 5.3 and LEMMA 2.2(b) show that there exists a unique homomorphism $\tau_s : K_s \to \widetilde{K(A)}$ satisfying $\tau_s(z_s(u)) = \widetilde{z_s(u)}$ for all $u \in D$ (here we use (K1), (K2) and (K3)). By LEMMA 2.2(a), there exists a unique homomorphism $\tau : T \to \widetilde{K(A)}$ satisfying $\tau(h_s(t)) = \widetilde{h_s(t)}$ for all $s \in S$ and $t \in S^1$ (here we use (K1) and (K3) for $u \in S^1$). Clearly, $\tau_s = \tau$ on $K_s \cap T = \{h_s(t) \mid t \in S^1\}$, and $\tau(h)\tau_s(g)\tau(h)^{-1} = \tau_s(hgh^{-1})$ for all $h \in T$, $s \in S$ and $g \in K_s$ by (K3). Hence, for $s \in S$, there exists a homomorphism $\overline{\tau}_s : TK_s \to \widetilde{K(A)}$ extending τ and τ_s.

Let $\widehat{K(A)}$ be the amalgamated product of the $K \cap P_s = TK_s$, $s \in S$. Then there exists a unique homomorphism $\widehat{\tau} : \widehat{K(A)} \to \widetilde{K(A)}$ such that, for all $s \in S$, $\widehat{\tau} \in \overline{\tau}_s$ on TK_s. By PROPOSITION 5.1(e) and relations (K4) and (K5), $\widehat{\tau}$ induces a homomorphism $\Phi : K(A) \to \widetilde{K(A)}$. It is easy to check that Φ and Ψ are mutually inverse. This proves the theorem. ∎

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Victor KAC, Dale PETERSON,
Department of Mathematics,
Massachusetts Institute of Technology,
Cambridge MA 02139, U.S.A.