Astérisque

VICTOR GUILLEMIN

# The integral geometry of line complexes and a theorem of Gelfand-Graev

*Astérisque*, tome S131 (1985), p. 135-149
<http://www.numdam.org/item?id=AST_1985__S131__135_0>

Société Mathématique de France
Astérisque, hors série, 1985, p. 135-149

# THE INTEGRAL GEOMETRY OF LINE COMPLEXES AND A THEOREM OF GELFAND-GRAEV

BY

Victor GUILLEMIN

## 1. Introduction

Let $P = \mathbf{C}\mathrm{P}^3$ be the complex three-dimensional projective space and let $G = \mathbf{C}\mathrm{G}(2,4)$ be the Grassmannian of complex two-dimensional subspaces of $\mathbf{C}^4$. To each point $p \in G$ corresponds a complex line $l_p$ in $P$. Given a smooth function, $f$, on $P$ we will show in § 2 how to define properly the line integral,

$$(1.1) \qquad \int_{l_p} f(\lambda) d\lambda\, d\overline{\lambda} \cdot = \widehat{f}(p).$$

A complex hypersurface, $S$, in $G$ is called *admissible* if there exists no smooth function, $f$, which is not identically zero but for which the line integrals, (1,1) are zero for all $p \in S$. In other words if $S$ is admissible, then, in principle, $f$ can be determined by its integrals over the lines, $l_p$, $p \in S$. In the 60's GELFAND and GRAEV settled the problem of characterizing which subvarieties, $S$, of $G$ have this property. We will describe their result (and, in fact, sketch a rough proof of it) in § 3. At first glance their result is rather puzzling : Admissibility turns out *not* to be a generic property of varieties. In fact very few $S$'s posses this property.

The purpose of this paper is to describe how this result can be used as the rationale for a method of constructing multi-branched analytic solutions of the wave equation on compactified Minkowski space with prescribed singularities. We will describe this method in § 4 and illustrate it with examples in §§ 5-6. Finally in § 7 we will describe an analogue of the Gelfand-Graev theorem for compactified Minkowski space.

## 2. The Gelfand line transform

Let $f : f(z,\bar{z})$ be a smooth function on $\mathbf{C}^2 - 0$ which is bihomogeneous of bidegree $(-2,-2)$; i.e.

$$f(\lambda z, \bar{\lambda}\bar{z}) = |\lambda|^{-4} f(z,\bar{z}) \tag{2.1}$$

for all $\lambda \in \mathbf{C}^*$. Let $dz = dz_1 \wedge dz_2$. Since $f$ is not in $\mathcal{L}^1$ the integral

$$\int f(z,\bar{z})\, dz\, d\bar{z} \tag{2.2}$$

diverges; however, one can still make sense of (2,2) as follows. Let

$$\Xi = z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$$

and let $\omega$ be the form of type 1-1:

$$\omega = \iota(\Xi)\iota(\bar{\Xi}) f dz \wedge d\bar{z}. \tag{2.3}$$

This form has nice properties with respect to the principle fibration : $\mathbf{C}^2 - 0 \xrightarrow{\pi} \mathbf{CP}^1$. Namely it vanishes when restricted to fibers; and, by (2,1), it is invariant under the action of the structure group, $\mathbf{C}^*$. Thus there exists a form, $\mu$, of type 1-1 on $\mathbf{CP}_1$ such that

$$\omega = \pi^* \mu.$$

We define (2.2) to be the integral

$$\int_{\mathbf{CP}^1} \mu. \tag{2.4}$$

It is clear that we can formulate the definition, (2.4), in a coordinate-free way. If $V$ is a complex vector space of dimension 2, $f$ a smooth bihomogeneous function on $V - 0$ of bidegree $(-2,-2)$ and $\Omega$ an element of $\wedge^{2,2}(V^*)$ then the integral

$$\int_V f\Omega \tag{2.5}$$

is well-defined (independent of coordinates).

Consider now a bihomogeneous function, $f$, on $\mathbf{C}^4-0$ of bidegree $(-2,-2)$. Given a point $p \in G$, let $V$ be the complex 2-dimensional subspaces of $\mathbf{C}^4$ represented by $p$. We will define the *line transform, $\widehat{f}$, of $f$ at $p$* as follows. By definition it will be an element of the space

$$\Lambda^{2,2}(V^*)^*. \tag{2.6}$$

Notice that an element of (2,6) is defined by describing how it pairs with an element, $\Omega$, of $\Lambda^{2,2}(V^*)$. For $\widehat{f}(p)$ the answer is given by the integral (2,5); i.e. by definition :

$$\langle \widehat{f}(p), \Omega \rangle = \int_V f\Omega. \tag{2.7}$$

Functions on $\mathbf{C}^4 - 0$ which are bihomogeneous of bidegree $(-2,-2)$ can be regarded as sections of a line bundle, $\mathcal{L} \to P$. We will denote by $\mathcal{M}$ the line bundle on $G$ whose fiber at $p$ is (2.6). With this notation we can regard the line transform described above as an integral operator

$$R : \Gamma(\mathcal{L}) \to \Gamma(\mathcal{M}), \quad Rf = \widehat{f}. \tag{2.8}$$

It is not hard to show that $R$ is injective and to describe its range using the representation theory of $SL(4,\mathbf{C})$. We prefer here to give a more elementary description of its range. Let $U_0$ be the open subset of $G$ consisting of all points $p \in G$ for which the restriction of $dz_1 \wedge dz_2$ to $V$ is non-zero. (As above $V$ is the 2-dimensional subspace of $\mathbf{C}^4$ represented by $p$). Then, $V$ can be described by linear equations of the form

$$z_3 = az_1 + bz_2,$$
$$z_4 = cz_1 + dz_2,$$

where $a, b, c$ and $d$ depend on $V$. In fact $a, b, c$ and $d$ are coordinate functions on $U_0$, and $dz_1 \wedge dz_2$ provides one with a trivialization of $\mathcal{M}$ over $U_0$; so for $p \in U_0$

$$\begin{aligned} \widehat{f}(p) &= \widehat{f}(a,b,c,d) \\ &= \int f(z_1, z_2, az_1 + bz_2, cz_1 + dz_2)\, dz_1\, dz_2\, d\bar{z}_1\, d\bar{z}_2. \end{aligned} \tag{2.9}$$

Differentiating under the integral sign one obtains

$$\Delta_0 f = \Big(\frac{\partial}{\partial a}\frac{\partial}{\partial d} - \frac{\partial}{\partial b}\frac{\partial}{\partial c}\Big) f = 0. \tag*{(2.10)$_0$}$$

Similarly

$$(2.11)_0 \qquad \overline{\Delta}_0 f = \Big(\frac{\partial}{\partial \bar{a}}\frac{\partial}{\partial \bar{d}} - \frac{\partial}{\partial \bar{b}}\frac{\partial}{\partial \bar{c}}\Big) f = 0.$$

More generally given a decomposible element $\nu$, of $\Lambda^2(\mathbf{C}^4)^*$ let $U_\nu$ be the open subset of $G$ consisting of all points, $p$, for which the restriction of $\nu$ to $V$ is non-zero. Then $\nu$ defines a trivialization of $\mathcal{M}$ over $U_\nu$; and, with respect to this trivialization, there exists second order differential operators, $\Delta_\nu$ and $\overline{\Delta}_\nu$, analogous to $(2.10)_0$ and $(2.11)_0$, such that

$$(2.10)_\nu \qquad \Delta_\nu f = 0$$

and

$$(2.11)_\nu \qquad \overline{\Delta}_\nu f = 0$$

on $U_\nu$. Let $\mathcal{M}_1$ be the line bundle over $G$ whose fiber at $p$ is $\Lambda^{2,2}(V^*) \otimes \Lambda^2(V^*)^* \otimes \Lambda^2(C^4/V)^*$, and let $\overline{\mathcal{M}}_1$ be its complex conjugate. Patching together the $\Delta_\nu$'s and $\overline{\Delta}_\nu$'s one gets intrinsically defined second order differential operators

$$(2.12) \qquad \Delta : \Gamma(\mathcal{M}) \to \Gamma(\mathcal{M}_1)$$

and

$$(2.13) \qquad \overline{\Delta} : \Gamma(\mathcal{M}) \to \Gamma(\overline{\mathcal{M}}_1)$$

such that $\Delta\widehat{f} = \overline{\Delta}\widehat{f} = 0$. This proves one half of the following proposition.

PROPOSITION. — *A section $g \in \Gamma(\mathcal{M})$ satisfies the equations*

$$(2.14) \qquad \Delta g = \overline{\Delta} g = 0$$

*if and only if $g = \widehat{f}$ for some section, $f$, of $\mathcal{L}$.*

We recall next that if $p \in G$ the cotangent space to $G$ at $p$ can be identified with

$$(2.15) \qquad \mathrm{Hom}(\mathbf{C}^4/V, V).$$

Let $\Sigma_p$ be the set of rank one elements in this space. Since $\mathbf{C}^4/V$ and $V$ are two-dimensional the set, $\Sigma_p$, is a quadratic cone inside $T_p^*$. This shows that $G$ is equipped with an intrinsic (complex) conformal structure such that $\Sigma_p$

is the cone of "light-like" rays at $p$. We will say more about this conformal structure in § 4.

Let $\Sigma$ be the fiber bundle over $G$ whose fiber at $p$ is $\Sigma_p$. We claim that *$\Sigma$ is the characteristic variety of the system of partial differential equations* (2.14). In fact let $a, b, c$ and $d$ be the coordinate functions on $U_0$ described above and let $\alpha, \beta, \gamma$ and $\delta$ be the dual cotangent coordinates. Then for $p \in U_0$

$$\Sigma_p = \{(\alpha, \beta, \gamma, \delta), \alpha\delta - \beta\gamma = 0\},$$

whereas

$$\sigma(\Delta_0)(\alpha, \beta, \gamma, \delta) = \alpha\delta - \beta\gamma$$

by (2.10).

## 3. Admissibility

Let $S$ be a complex hypersurface in $G$. One calls $S$ admissible if the integral transform

$$\Gamma(\mathcal{L}) \to \Gamma(\mathcal{M}1S), \quad f \to \widehat{f}1S$$

is injective. In [2], Gelfand et al. show that the following $S$'s are admissible :

*Example 1.* — Let $W$ be a non-singular curve in $P$ and let $S$ be the set of all points $p \in G$ such that $W$ and $l_p$ intersect.

*Example 2.* — Let $W$ be a non-singular surface in $P$ and let $S$ be the set of all points $p \in G$ such that $l_p$ has at least one point of tangency with $W$.

Their main result is the following converse statement :

THEOREM. — *If $S$ is admissible then near a generic point $S$ is locally as in example one or as in example two.*

We will sketch a proof of this below. We first claim :

LEMMA. — *For $S$ to be admissible it has to be characteristic with respect to the differential operator, $\Delta$.*

*"Proof".* — If $S$ were non-characteristic then the Cauchy problem

$$\Delta g = \overline{\Delta} g = 0, \quad g = 0 \text{ on } S \tag{2.1}$$

would be well-posed. But if $g$ is a non-trivial solution of (2.1) then, by the proposition in § 2, $g = \widehat{f}$ and $\widehat{f}1S = 0$. Contradiction.

Unfortunately, if $S$ is non-characteristic at a point, $p$, the Cauchy problem (3.1) is well-posed only in a small neighborhood of $p$; whereas, to get a contradiction, we need to find a non-trivial *global* solution of (3.1). Therefore, this "proof" is not completely convincing. There is a convincing proof involving (3.1); but we won't attempt to describe it here.

We next require some facts about the characteristic variety, $\Sigma$, of the differential operator, $\Delta$. Since $\Sigma$ is a co-isotropic subvariety of $T^*G$, it is equipped with a canonical null-foliation. We will show that this null-foliation is *fibrating* with $\mathbf{CP}^1$'s as fibers and $T^*P - 0$ as base.

*Proof.* — A typical element of $T^*P-0$ is of the form, $\xi \otimes x$, with $x \in \mathbf{C}^4-0$, $\xi \in (\mathbf{C}^4)^* - 0$, and $\langle x, \xi \rangle = 0$. Given a point, $p$, in $G$ let $V$ be the two-dimensional subspace of $\mathbf{C}^4$ represented by $p$. We will say that $p$ belongs to $\gamma_{x,\xi}$ if

$$x \in V \text{ and } \xi \in V^0. \tag{3.2}$$

The set, $\gamma_{x,\xi}$, defined by (3.2) is a complex line in $G$; and it is easy to see that the $\gamma_{x,\xi}$'s are exactly the light rays on $G$ associated with the canonical conformal structure. Q.E.D.

We will denote by

$$\pi : \Sigma \to T^*P - 0 \tag{3.3}$$

the null-fibration. Now let $S$ be a hypersurface in $G$ which is characteristic with respect to $\Delta$. Then its conormal vector at each point is "light-like"; so the conormal bundle

$$\Lambda = N^*S - 0$$

is contained in $\Sigma$. Since $\Lambda$ is Lagrangian this implies that *for every point in* $\Lambda$ *the leaf of the null-foliation passing through this point is also in* $\Lambda$. Therefore $\Lambda$ has to be of the form $\pi^{-1}(\Lambda_1)$ where $\Lambda_1$ is a Lagrangian submanifold of $T^*P - 0$. At "generic" points $\Lambda_1$ is locally of the form

$$\Lambda_1 = N^*W - 0,$$

where $W$, the projection of $\Lambda_1$ into $P$, is a submanifold of $P$. Therefore we have proved that

$$N^*S = \pi^{-1}(N^*W) \tag{3.4}$$

at "generic" points of $N^*S$. From (3.4) it is easy to deduce Gelfand's theorem in the following form

THEOREM. — *The hypersurface, $S$, consists of all points, $p \in G$, such that $l_p$ intersects $W$ non-transversally.*

## 4. The Penrose transform

The Penrose transform is the holomorphic analogue of line transform described in § 2. It was used by PENROSE and his collaborators to construct solutions of the wave equation on compactified Minkowski space. (See [3] and [4].) Before describing it we will review some facts about the geometry of compactified Minkowski space. A good reference for the material below is the survey article of WELLS, [5].

We have already observed that $G$ is equipped with a canonical (complex) conformal structure. It has three real forms, on which the induced (real) conformal structures are of type $(++++)$, $(++--)$ and $(+++-)$ respectively, and these are $S^4$, $\mathbf{R}G(2,4)$ and compactified Minkowski space, which we will denote by $M$. A good way to view $M$ as a submanifold of $G$ is as follows. Consider on $\mathbf{C}^4$ the Hermitian form

$$H(z) = |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 . \tag{4.1}$$

For each $p \in G$ let $V_p$ be the two-dimensional subspace of $\mathbf{C}^4$ represented by $p$. Then

$$M = \{p \in G,\ H = 0 \text{ on } V_p\}. \tag{4.2}$$

From this description of $M$ one sees easily that the group $SU(2,2)$ acts as conformality transformations on $M$.

We will now show how one can take a holomorphic function defined on an appropriate open subset of $P$ and convert it via the *Penrose transform* into a solution of the conformal wave equation on $M$. Incidentally the version of the Penrose transform which we will describe below is very close to the version which one finds in PENROSE's earlier papers. (See [4].) Later EASTWOOD, PENROSE and WELLS found a more elegant and general definition, involving sheaf cohomology, which we won't attempt to describe here. (See [1].)

To start with, let $f$ be a meromorphic function on $\mathbf{C}^2 - 0$ which is homogeneous of degree $-2$, i.e. satisfies

$$f(\lambda z) = \lambda^{-2} f(z)$$

for all $\lambda \in \mathbf{C}^*$. Let $\Xi$ be the vector field, $z_1\, \partial/(\partial z_1) + z_2\, \partial/\partial z_2)$, and let $\omega$ be the one form, $\iota(\Xi) f dz_1 \wedge dz_2$. As in § 2 $\omega$ is of the form $\omega = \pi^*\mu$, where $\mu$ is a meromorphic one-form on $\mathbf{C}\mathrm{P}^1$ and $\pi : \mathbf{C}^2 - 0 \to \mathbf{C}\mathrm{P}^1$ is the canonical projection. Given a contour, $\gamma$, on $\mathbf{C}\mathrm{P}^1$ not intersecting the poles of $\mu$, we will denote by $\mathrm{Res}_\gamma\, f dz_1 dz_2$ the integral

$$\mathrm{Res}_\gamma\, f\, dz_1\, dz_2 = \int_\gamma \mu. \tag{4.3}$$

It is clear that this definition is independent of the choice of coordinates, i.e. if $V$ is a two-dimensional complex vector space, $f$ a homogeneous meromorphic function on $V - 0$ of degree $-2$ and $\Omega$ an element of $\wedge^{2,0}(V^*)$, then for an appropriate contour, $\gamma$, on the projective space $\mathbf{P}V$, the residue

$$\text{Res}_\gamma f\Omega \tag{4.4}$$

is well-defined.

Now let $f$ be a meromorphic function on $\mathbf{C}^4 - 0$ which is homogeneous of degree $-2$ and let $W$ be the set of rays (in $P$) on which $f$ is singular. $W$ is an algebraic subvariety of $P$, but it need not be non-singular; so we will denote by $W_0$ the non-singular points of $W$ and by $W_1$ the curve of singular points. Let $S$ be the set of all points, $P \in G$, such that the line, $l_p$, either intersects $W_1$ or has a common point of tangency with $W_0$. Let $p$ be a point not on $S$ and let $V = V_p$ the subspace of $\mathbf{C}^4$ represented by $p$. We will define $\widehat{f}(p) \in \Lambda^{2,0}(V^*)^*$ by the formula

$$\langle \widehat{f}(p), \Omega \rangle = \text{Res}_\gamma f\Omega \tag{4.5}$$

for $\Omega \in \Lambda^{2,0}(v^*)$, $\gamma$ being a contour on the line $l_p = \mathbf{P}V$ avoiding points of $W \cap l_p$. Let $\mathcal{M}$ be the line bundle on $G$ with fiber

$$\Lambda^{2,0}(V^*)^* = \Lambda^{2,0}(V)$$

at $p$. If one varies the contour, $\gamma$, continuously with respect to $p$, one gets from (4.5) a multi-branched holomorphic section of $\mathcal{M}$ over $G - S$ which satisfies the holomorphic analogue of the wave equation discussed in § 2. By the theorem of Gelfand discussed in § 3, $S$ is *characteristic* with respect to the wave equation; so the Penrose transform can be regarded as a tool for constructing multi-branched holomorphic solutions of the wave equation on $G$ with singularities along a prescribed characteristic hypersurface. Restricted to $M$ these solutions often become single-valued with singularities along a prescribed *real* characteristic hypersurface. (See § 7). We won't attempt here to give a systematic description of these solutions; but, in the next couple of sections, we will illustrate this method by means of examples.

## 5. Characteristic hypersurfaces of the first kind

We saw in § 3 that there are two kinds of characteristic hypersurfaces in $G$. The first kind consists of all lines which pass through a fixed curve, and the second kind consists of all lines which have a common point of tangency with a fixed surface. In this section we will describe how to construct single-valued holomorphic solutions of the wave equation with singularities along characteristic hypersurfaces of the first kind.

Let $W$ be the algebraic curve in $\mathbf{C}\mathrm{P}^3$ defined by the equations

$$Q_1(z) = Q_2(z) = 0,$$

where $Q_1$ and $Q_2$ are homogeneous polynomials in $(z_1, z_2, z_3, z_4)$ with no common factor. Let the function, $f$, in (4.5) be of the form

$$f = Q_3/Q_1^{m_1}Q_2^{m_2}, \tag{5.1}$$

where $\deg Q_3 = m_1 \deg Q_1 + m_2 \deg Q_2 - 2$, and choose the contour, $\gamma$, in (4.5) so that it surrounds all the zeroes of $Q_1$ on the projective line, $l = \mathbf{P}\mathrm{V}$, but none of the zeroes of $Q_2$. Then the expression (4.5) is well-defined providing no point on $l$ is simultaneously a zero of $Q_1$ and $Q_2$; i.e. providing the line, $l$, doesn't intersect the curve, $W$. In other words, let $S$ be the characteristic hypersurface of the first kind consisting of all points, $p \in G$, for which the line, $l_p$, intersects $W$. Then, to each function of the form (5.1), there corresponds a holomorphic solution of the wave equation with singularities on $S$. Notice that this correspondence is *not* injective. If either $m_1$ or $m_2$ were equal to zero in (5.1), then the contour, $\gamma$, would surround *all* zeroes of $Q_1^{m_1}Q_2^{m_2}$; so the expression (4.5) would be identically zero. The most satisfactory way to describe this correspondence is in sheaf-theoretic terms : Let $\mathcal{L}_{\mathrm{can}}$ be the canonical line bundle of the projective space $P$ and let $\mathcal{L} = \mathcal{L}_{\mathrm{can}}^2$. Let $U_1$ and $U_2$ be the subsets of $P$ on which $Q_1$ and $Q_2$ are non-zero. Functions of the form (5.1) are identical with sections of $\mathcal{L}$ over $U_1 \cap U_2$ and functions of the form (5.1) with $m_2 = 0$ (respectively, $m_1 = 0$) are just sections of $\mathcal{L}$ over $U_1$ (respectively $U_2$). By MAYER-VICTORIS :

$$\Gamma(U_1, \mathcal{L}) \oplus \Gamma(U_2, \mathcal{L}) \xrightarrow{\rho} \Gamma(U_1 \cap U_2, \mathcal{L}) \to H^1(U_1 \cup U_2, \mathcal{L}) \to 0, \tag{5.2}$$

and the image of $\rho$ is contained in the kernel of the Penrose transform; so the Penrose transform is actually a map of $H^1(P - W, \mathcal{L})$ into the space of holomorphic solutions of the wave equation with singularities on $S$. This is the way the Penrose transform is described in [1] (where it is shown, in addition, that it is bijective).

*Example.* — Let $Q_1 = z_1, Q_2 = z_2$ and $f = (z_1 z_2)^{-1}$.

In this example $S$ is the characteristic cone consisting of all points, $p \in G$, for which the line $l_p$ intersects the fixed line, $l$, defined by the equations $z_1 = z_2 = 0$. The apex of this cone is the point, $p_0$, represented by the line, $l$, itself.

Let $U = G - S$. Notice that $U$ consists of all points $p \in G$ with the property that the two-form $dz_1 \wedge dz_2$ doesn't vanish on the space $V = V_p$. Hence there is a natural trivialization of the line bundle, $\mathcal{M}$, over $U$; and, with respect to this trivialization, the solution of the wave equation associated with $f$ takes at $p$ the value

$$\int_\gamma \mu\,, \tag{5.3}$$

$\mu$ being the one form

$$\mu = \iota\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}\right)\frac{dz_1\,dz_2}{z_1 z_2} = \frac{dz}{z}\,,$$

where $z = z_2/z_1$ and $\gamma$ is a contour on the line, $l_p$, surrounding the point $z = 0$. However, it is clear that this integral is $2\pi i$ for all $p$; i.e. the Penrose transform, $\widehat{f}$, of $f$ is the constant function $\widehat{f} = 2\pi i$.

Next let $U'$ be the set of points $p \in G$ for which the two-form $dz_3 \wedge dz_4$, restricted to $V_p$, doesn't vanish. If $p \in U \cap U^1$ the subspace, $V_p$, of $\mathbf{C}^4$ can be described by a pair of equations of the form

$$\begin{aligned} z_1 &= az_3 + bz_4, \\ z_2 &= cz_3 + dz_4, \end{aligned}$$

and, restricted to $V_p$,

$$dz_1 \wedge dz_2 = \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} dz_3 \wedge dz_4. \tag{5.4}$$

The fact that neither of these restrictions is zero says that

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0.$$

As in § 2, $a$, $b$, $c$ and $d$ can be employed as coordinate functions on $U \cap U'$, and with respect to these coordinates, the transition function relating the trivializations of $\mathcal{M}$ given by $dz_3 \wedge dz_4$ and by $dz_1 \wedge dz_2$ is just the function

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

by (5.4). Therefore, in terms of the trivialization given by $dz_3 \wedge dz_4$, $\widehat{f}$ is equal to

$$\widehat{f}(a,b,c,d) = 2\pi i/(ad - bc)$$

on $U'$; i.e. $\widehat{f}$ is the so-called *elementary solution* of the wave equation :

$$\frac{\partial}{\partial a}\frac{\partial}{\partial d} - \frac{\partial}{\partial b}\frac{\partial}{\partial c}.$$

## 6. Characteristic hypersurfaces of the second kind : an example

Let $Q$ be an arbitrary non-degenerate quadratic form on $\mathbf{C}^4$. After making an appropriate change of coordinates we can assume that

$$Q(z) = z_1 z_2 + z_3 z_4. \tag{6.1}$$

Let $W$ be the quadratic surface in $P$ defined by $Q = 0$, and let $S$ be the characteristic hypersurface in $G$ associated with $W$. In other words $p \in S$ if and only if $l_p$ is tangent to $W$. Notice that for $p \in S$ either $l_p$ intersects $W$ in a single point or $l_p$ is entirely contained in $W$. Let $S_1$ be the set of points for which the second alternative holds. It is easy to see that $S_1$ is the singular locus of $S$ and is the disjoint union of two $\mathbf{CP}^1$'s (corresponding to the two rulings of $W$).

In this section we will compute the Penrose transform of the function

$$f = 1/Q. \tag{6.2}$$

Before we do so, however, let's consider a somewhat simpler problem. Let $q = q(z_1, z_2)$ be a non-degenerate quadratic form on $\mathbf{C}^2 - 0$, and let's compute the residue

$$\mathrm{Res}_\gamma\,(dz_1\,dz_2/q),$$

where $\gamma$ is a contour on $\mathbf{CP}^1$ surrounding one of the zeroes of $q$. We can make a linear change of coordinates

$$\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = B \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{6.3}$$

so that $q(z_1, z_2) = w_1 w_2$. Moreover, if

$$q(z) = k_{11} z_1^2 + k_{12} z_1 z_2 + k_{21} z_2 z_1 + k_{22} z_2^2,$$

with $k_{12} = k_{21}$, and $J$ and $K$ are the matrices

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix},$$

then

$$BJB^t = K. \tag{6.4}$$

With this change of coordinates we get

$$\operatorname{Res}_\gamma (dz_1\, dz_2/q) = (\det B)^{-1} \operatorname{Res}_\gamma (dw_1\, dw_2/w_1 w_2).$$

In § 5 we showed that the residue on the right was just $2\pi\sqrt{-1}$; so we get the formula

$$\operatorname{Res}_\gamma (dz_1\, dz_2/q) = 2\pi (\det K)^{-1/2}. \tag{6.5}$$

since $-(\det B)^2 = \det K$ by (6.4).

Let's come back now to the problem of computing the Penrose transform of (6.2). Let $U$ be the subset of $G$ consisting of all points, $p$, for which the restriction of $dz_1 \wedge dz_2$ to $V_p$ is non-zero. If $p \in U$ the equations of $V_p$ are

$$\begin{aligned} z_3 &= az_1 + bz_2, \\ z_4 &= cz_1 + dz_2, \end{aligned}$$

and $a$, $b$, $c$ and $d$ can be employed as coordinate functions on $U$. Let $A$ be the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the quadratic form, $z_1 z_2 + z_3 z_4$, restricted to $V_p$, is of the form

$$(z_1, z_2)(J + AJA^t)(z_1, z_2)^t,$$

and, therefore, by (6.5), the Penrose transform of (6.2) is the function

$$\widehat{f} = \widehat{f}(a, b, c, d) = 2\pi \det(J + AJA^t)^{1/2}. \tag{6.6}$$

Affine Minkowski space sits inside of $U$ as the set of matrices

$$A = \begin{pmatrix} u & w \\ \overline{u} & v \end{pmatrix}$$

with $w$ complex and $u$ and $v$ real; so the restriction of (6.6) to affine Minkowski space is

$$2\pi[(\det A + 1)^2 + 4|w|^2]^{-1/2}, \tag{6.7}$$

or, in terms of the more familiar space-time coordinates,

$$u = (1/\sqrt{2})(x_0 + x_1), \qquad v = (1/\sqrt{2})(x_0 - x_1), \qquad w = (1/\sqrt{2})(x_2 + ix_3),$$

$$\widehat{f} = 2\pi h(x)^{-1/2}, \tag{6.8}$$

where

$$h(x) = \left[\frac{1}{2}(x_0^2 - x_1^2 - x_2^2 - x_3^2) + 1\right]^2 + 4(x_2^2 + x_3^2). \tag{6.9}$$

Notice that (6.9) is non-negative, so $\widehat{f}$ is *single-valued* on Minkowski space with singularities on the hyperbola

$$x_0^2 - x_1^2 = 2, \qquad x_2 = x_3 = 0.$$

*Remark.* — One gets a very similar formula for the Penrose transformation of the function

$$P/Q^k,$$

$P$ being a homogeneous polynomial in $z$ of degree $2k - 2$.

## 7. Characteristic hypersurfaces in compact Minkowski space

There is an interesting analogue of the theorem proved in § 3 for compact Minkowski space. For $p \in M$ let $\Sigma_p$ be the light cone in $T_p^* - 0$ and let $\Sigma$ be the fiber bundle over $M$ whose fiber at $p$ is $\Sigma_p$. Since $\Sigma$ is of codimension 1 as a submanifold of $T^*M - 0$, it is co-isotropic and hence is equipped with a null-foliation. We will show that this null-foliation is fibrating with $\mathbf{RP}^1$'s as fibers and an extremely interesting symplectic manifold as base. Recall that in § 4 we introduced the Hermitian form

$$H(z) = |z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 \tag{7.1}$$

on $\mathbf{C}^4$. Let $P^+$ (respectively $N$) be the set of points in $P$ where $H$ is positive (respectively zero). It is not hard to show that $P$ is a non-degenerate complex domain in the sense that at each point, $p$, on its boundary, $N$, the Levi form at $p$ is non-degenerate. Moreover, at each point, $p$, of $N$ there is an "inward pointing" real covector $\xi \in T_p^*N - 0$ annihilated by the tangential Cauchy-Riemann vectors at $p$. Let $\tau$ be the fiber bundle over $N$ whose fiber at $p$ is the ray

$$\tau_p = \{\lambda\xi, \lambda > 0\}. \tag{7.2}$$

Since the Levi form is non-degenerate $\tau$ is a symplectic submanifold of $T^*N - 0$.

THEOREM. — *The null-foliation of $\Sigma$ is fibrating with $\mathbf{R}P^1$'s as fibers and $\tau$ as base.*

*Proof.* — Given $x \in N$ let $\gamma_x$ be the set of all points $p \in M$ such that the complex line, $l_p$, in $N$, associated with $p$, contains the point $x$. It is fairly obvious that $\gamma_x$ is an $\mathbf{R}P^1$. On the other hand it is not hard to see that $\gamma_x$ is a light ray with respect to the conformal Lorentzian structure on $M$ and that all light rays are of this form. (See for instance § 3 of [5].) Q.E.D.

Let $\pi : \Sigma \to \tau$ be the null-fibration. Given a characteristic hypersurface, $S$, in $M$ let $\Lambda$ be its conormal bundle. To say that $S$ is characteristic is equivalent to saying that $\Lambda$ is contained in $\Sigma$; so $\Lambda$ must necessarily be of the form

$$\Lambda = \pi^{-1}(\Lambda_1), \tag{7.3}$$

where $\Lambda_1$ is a conic Lagrangian submanifold of $\tau$. Since the fiber, $\tau_p$, of $\tau$ above $p \in N$ consists of the single ray (7.2), $\Lambda_1$ is completely determined by its projection, $W$, on $N$. Moreover, since $N$ is the projectivization of a conic symplectic manifold it has an intrinsic contact structure, and $\Lambda_1$ is Lagrangian in $\tau$ if and only if $W$ is Legendrian in $N$. We leave for the reader to show that (7.3) translates into the following statement :

THEOREM. — *Let $S$ be a characteristic (light-like) hypersurface in $M$. Then there exists a unique Legendrian submanifold, $W$, in $N$ such that*

$$S = \{p \in M,\ l_p \text{ intersects } W\}. \tag{7.4}$$

*Conversely if $W$ is a Legendrian submanifold of $N$ the set (7.4) is a characteristic hypersurface in $M$.*

REFERENCES

[1] EASTWOOD (M.), PENROSE (R.) and WELLS, JR. (R.O.). — Cohomology and massless fields, *Communications in Math. Phys.*, t. **78**, 1981, p. 305–351.

[2] GELFAND (I.M.), GRAEV (M.I.) and VILENKIN (N.Ya.). — *Generalized Functions*, vol. 5. — New York, Academic Press, 1966.

[3] HUGHSTON (L.P.) and WARD (R.S.). — *Advances in Twistor Theory*. — London, Pitman, 1979.

[4] PENROSE (R.) and MACCALLUM (M.A.H.). — Twistor theory : an approach to the quantization of fields and space-time, *Phys. Rep.*, t. **6**C, 1974, p. 241–316.

[5] WELLS (R.O.). — Complex manifolds and mathematical physics, *Bull. A.M.S.*, t. **1,2**, 1981, p. 296–336.

Victor GUILLEMIN
Department of Mathematics,
Massachusetts Institute of Technology,
Cambridge, Massachusetts 02139, U.S.A.