MARK A. PINSKY

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BROWNIAN MOTION ON A SMALL GEODESIC BALL

by

Mark A. Pinsky

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1. Introduction

Let $\{X_t, t \geq 0\}$ be the Brownian motion process of a Riemannian manifold (M,g). The exit time from the geodesic ball centered at $m \in M$ is defined by

$$T_\varepsilon = \inf\{t > 0: d(X_t, m) = \varepsilon\}$$

where $d(\cdot,\cdot)$ is the distance function defined by g.

In a previous paper [4] we studied the mean exit time $E_m(T_\varepsilon)$ and obtained three non-zero terms of the asymptotic expansion when $\varepsilon \downarrow 0$. This was used to prove the following stochastic characterization of the Euclidean space (R^n, g_0): If for each $m \in M$, $E_m(T_\varepsilon) = \varepsilon^2/2n + O(\varepsilon^8)$ when $\varepsilon \downarrow 0$, then (M,g) is locally isometric to (R^n, g_0) provided $n < 6$. In case $n = 6$, we provided an example of a non-flat symmetric Riemannian manifold whose asymptotic expansion is $\varepsilon^2/2n + O(\varepsilon^{10})$ when $\varepsilon \downarrow 0$.

In this paper we shall extend our analysis to the second moment $E_m(T_\varepsilon^2)$, $m \in M$, $\varepsilon \downarrow 0$. By combining the previous techniques with the "stochastic Taylor formula" we obtain a three-term asymptotic expansion for the second moment, given at the end of section 4. As a by-product we have the following characterization of Euclidean space (R^n, g_0) valid in any dimension $n < \infty$: If for each $m \in M$, $E_m(T_\varepsilon) = \text{const.}\ \varepsilon^2 + O(\varepsilon^8)$ and $E_m(T_\varepsilon^2) = \text{const.}\ \varepsilon^4 + O(\varepsilon^{10})$ when $\varepsilon \downarrow 0$, then (M,g) is locally isometric to (R^n, g_0). Similar characterizations are obtained for any space of constant curvature.

The present work, which could be formulated in non-stochastic terms, may be viewed as complementary to the general theory of semi-martingales on manifolds as formulated by Laurent Schwartz [5]. In particular our stochastic Taylor formula (proposition 2.1 below) is a consequence of the martingale formulation of diffusion processes.

2. Notations and Definitions

Let (M,g) be an n-dimensional Riemannian manifold. We use the following notations.

$\bar{M}_m$ is the tangent space at $m \in M$.

$B_m(\varepsilon)$ is the ball of radius ε in M with center at $m \in M$.

$\bar{B}_m(\varepsilon)$ is the ball of radius ε in $\bar{M}_m$ with center at $0 \in \bar{M}_m$

$\exp_m$ is the exponential mapping (which is defined on all of $\bar{M}_m$ in case M is complete; otherwise it is a mapping) from $\bar{B}_m(\varepsilon)$ to $B_m(\varepsilon)$ for sufficiently small $\varepsilon > 0$.

Φ_ε is the mapping on functions defined by

$$(\Phi_\varepsilon f)(\exp_m \varepsilon x) = f(x);$$

Φ_ε maps from $C^\infty(\bar{B}_m(1))$ to $C^\infty(B_m(\varepsilon))$ for sufficiently small $\varepsilon > 0$.

Δ is the Laplace-Beltrami operator of the Riemannian manifold:

$$\Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i}\left(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial x_j}\right) \text{ where } g^{ij} = (g^{-1})^{ij},\ g = \det(g_{ij}).$$

The following result, which will be used repeatedly, was proved in [4].

<u>Proposition 2.0</u>: <u>There exist second order differential operators</u> $(\Delta_{-2}, \Delta_0, \Delta_1, \ldots)$ <u>on</u> $C^\infty(\bar{M}_m)$ <u>such that for each</u> $N \geq 0$ <u>and each</u> $f \in C^\infty(\bar{M}_m)$

$$\Phi_\varepsilon^{-1} \Delta \Phi_\varepsilon f = \varepsilon^{-2} \Delta_{-2} f + \sum_{j=0}^{N} \varepsilon^j \Delta_j f + O(\varepsilon^{N+1}) \qquad (\varepsilon \downarrow 0). \tag{2.1}$$

Δ_j <u>maps polynomials of degree</u> k <u>to polynomials of degree</u> $k+j$. <u>In any normal coordinate chart</u> $(x_1, \ldots, x_n)$ <u>we have</u>

(2.2)
$$\Delta_{-2}f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

$$\Delta_0 f = (1/3) \sum_{i,a,j,b=1}^{n} R_{iajb} x_a x_b \frac{\partial^2 f}{\partial x_i \partial x_j} - (2/3) \sum_{i,a=1}^{n} \rho_{ia} x_a \frac{\partial f}{\partial x_i} .$$

Here R_{iajb} is the Riemann tensor and $\rho_{ij} = \sum_{a=1}^{n} R_{iaja}$ is the Ricci tensor at $m \in M$.

Let (X_t, P_x) be the Brownian motion process with infinitesimal generator Δ. For each $m \in M$ let T_ε be the exit time from the geodesic ball $B_m(\varepsilon)$. To study the moments of T_ε we invoke the following "stochastic Taylor formula."

Proposition 2.1 [1,2]: Let (X_t, P_x) be a Feller-Markov process with infinitesimal generator A. Let T be a stopping time with $E_x(T^{N+1})$ finite and let f be a function in the domain of A^{N+1}. Then

$$f(x) - E_x f(X_T) = \sum_{k=1}^{N} \frac{(-1)^k}{k!} E_x\left\{T^k A^k f(X_T)\right\} + \frac{(-1)^{N+1}}{N!} E_x\left\{\int_0^T u^N A^{N+1} f(X_u)\, du\right\}$$

(If $N = 0$ the sum is empty and we have the Dynkin formula $E_x f(X_T) - f(x) = E_x\left\{\int_0^T Af(X_u)\, du\right\}$.)

Corollary 2.2: Let T_ε be the exit time from the geodesic ball $B_m(\varepsilon)$ and let $u_0 = 1$, $u_k(x) = (1/k!) E_x(T_\varepsilon^k)$ for $k \geq 1$. Then in the interior of $B_m(\varepsilon)$ we have $\Delta u_k = -u_{k-1}$ $(k = 1,2,\ldots)$ and on the boundary we have $u_k = 0$ $(k = 1,2,\ldots)$. In particular $\Delta^k u_N = (-1)^k u_{N-k}$, $0 \leq k \leq N$, $N \geq 1$.

Proof: Let $\bar{u}_0 = 1$ and let $\bar{u}_k$ be the classical solution of the elliptic problem $\Delta \bar{u}_k = -\bar{u}_{k-1}$ with $\bar{u}_k = 0$ on the boundary of $B_m(\varepsilon)$. Taking $T = \min(R, T_\varepsilon)$ and $f = \bar{u}_{N+1}$ in the proposition 2.1 we have

$$\bar{u}_{N+1}(x) - E_x \bar{u}_{N+1}(X_T) = \sum_{k=1}^{N} \frac{1}{k!} E_x\left\{T^k \bar{u}_{N-k+1}(X_T)\right\} + \frac{1}{(N+1)!} E_x(T^{N+1})$$

Thus

$$\frac{1}{(N+1)!} E_x(T^{N+1}) \leq 2|\bar{u}_{N+1}|_\infty + \sum_{k=1}^{N} \frac{|\bar{u}_{N-k+1}|_\infty}{k!} E_x(T^k)$$

Letting $R \to \infty$ in this inequality and using induction we see that $E_x(T_\varepsilon^{N+1})$ is finite. Taking $T = T_\varepsilon$ above yields

$$\bar{u}_{N+1}(x) - 0 = \frac{1}{(N+1)!} E_x(T_\varepsilon^{N+1}) = u_{N+1}(x)$$

This completes the necessary identification.

The exact solution u_2 is not available for a general Riemannian manifold. Therefore, following [4] we shall construct an approximate solution v_2 in the form

$$v_2 = \Phi_\varepsilon(\varepsilon^4 g_0 + \varepsilon^6 g_2 + \varepsilon^7 g_3 + \varepsilon^8 g_4) \tag{2.3}$$

where g_0, g_2, g_3, g_4 are functions on $\bar{B}_m(1)$ satisfying

$$\Delta_{-2} g_0 = -f_0 \qquad g_0|\partial\bar{B}_m(1) = 0 \tag{2.4}$$

$$\Delta_{-2} g_2 + \Delta_0 g_0 = -f_2 \qquad g_2|\partial\bar{B}_m(1) = 0 \tag{2.5}$$

$$\Delta_{-2} g_3 + \Delta_1 g_0 = -f_3 \qquad g_3|\partial\bar{B}_m(1) = 0 \tag{2.6}$$

$$\Delta_{-2} g_4 + \Delta_0 g_2 + \Delta_2 g_0 = -f_4 \qquad g_4|\partial\bar{B}_m(1) = 0 \tag{2.7}$$

The functions f_0, f_2, f_3, f_4 are solutions of the following set of equations:

$$\Delta_{-2} f_0 = -1 \qquad f_0|\partial\bar{B}_m(1) = 0 \tag{2.8}$$

$$\Delta_{-2} f_2 + \Delta_0 f_0 = 0 \qquad f_2|\partial\bar{B}_m(1) = 0 \tag{2.9}$$

$$\Delta_{-2} f_3 + \Delta_1 f_0 = 0 \qquad f_3|\partial\bar{B}_m(1) = 0 \tag{2.10}$$

$$\Delta_{-2} f_4 + \Delta_0 f_2 + \Delta_2 f_0 = 0 \qquad f_4|\partial\bar{B}_m(1) = 0 \tag{2.11}$$

Letting $v_1 = \Phi_\varepsilon(\varepsilon^2 f_0 + \varepsilon^4 f_2 + \varepsilon^5 f_3 + \varepsilon^6 f_4)$ we have $\Delta v_2 = -v_1 + O(\varepsilon^8)$, $\Delta^2 v_2 = 1 + O(\varepsilon^6)$. Applying proposition 2.1 with $N = 1$, $f = v_2$ we have $v_2(p) = (\frac{1}{2})E_p(T^2(1 + O(\varepsilon^6))) = (\frac{1}{2})E_p(T^2) + O(\varepsilon^{10})$. To summarize, we have the following:

Proposition 2.3: The function v_2 defined by (2.3) - (2.7) satisfies $v_2|_{\partial B_m(\varepsilon)} = 0 = \Delta v_2|_{\partial B_m(\varepsilon)}$, $\Delta v_2 = -v_1 + O(\varepsilon^8)$, $\Delta^2 v_2 = 1 + O(\varepsilon^6)$, and $v_2(m) = \frac{1}{2}E_m(T_\varepsilon^2) + O(\varepsilon^{10})$ when $\varepsilon \downarrow 0$.

3. Determination of g_0, g_2

In this section we shall prove

Proposition 3.1. We have

$$g_0 = (1/2n)^2(1-r^2) - (1/8n(n+2))(1-r^4)$$

$$g_2 = \left(\rho - \frac{\tau r^2}{n}\right)\left[\frac{n+2}{6n^2(n+4)^2}(1-r^2) - \frac{n+3}{12n(n+2)(n+4)(n+6)}(1-r^4)\right]$$

$$+ \tau\left[\frac{1-r^2}{24n^3(n+2)} + \frac{1-r^4}{24n^3(n+2)} - \frac{1-r^6}{24n^2(n+2)(n+4)}\right]$$

where $\rho = \sum_{i,j=1}^{n} \rho_{ij}x_i x_j$ is the Ricci tensor, $r^2 = \sum_{i=1}^{n} x_i^2$ and $\tau = \sum_{i=1}^{n} \rho_{ii}$ is the scalar curvature.

Proof: Recall from the previous work [4]

$$f_0 = (1/2n)(1-r^2)$$

$$f_2 = \left(\rho - \frac{\tau r^2}{n}\right)\frac{1-r^2}{6n(n+4)} + \tau\,\frac{1-r^4}{12n^2(n+2)}$$

$$\Delta_{-2}(r^2) = 2n,\ \Delta_{-2}(r^4) = 4(n+2)r^2,\ \Delta_{-2}(r^6) = 6(n+4)r^4$$

$$\Delta_0(r^2) = -\frac{2}{3}\rho\ ,\ \Delta_0(r^4) = -\frac{4}{3}\rho r^2\ ,\ \Delta_0(r^6) = -2\rho r^4$$

$$\Delta_{-2}(\rho) = 2\tau,\ \Delta_{-2}(r^2\rho) = 2\tau r^2 + 2(n+4)\rho,\ \Delta_{-2}(r^4\rho) = 2\tau r^4 + 4(n+6)\rho r^2\ .$$

$$\Delta_0(\rho) = \frac{2}{3}(\rho\#R - 2\rho\circ\rho),\ \Delta_0(r^2\rho) = \frac{2r^2}{3}(\rho\#R - 2\rho\circ\rho) - \frac{2}{3}\rho^2\ ,$$

$$\Delta_0(r^4\rho) = \frac{2r^4}{3}(\rho\#R - 2\rho\circ\rho) - \frac{4}{3}\rho^2 r^2\ ,$$

where in the last two formulas we have used the fact that $\Delta_0(fg) = f\Delta_0 g + g\Delta_0 f$ if $f = f(r)$ is a radial function and g is arbitrary. A lengthy but straightforward computation then shows that $\Delta_{-2}g_0 = -f_0$, $\Delta_{-2}g_2 = -f_2 - \Delta_0 g_0$, as required. Clearly both g_0, g_2 satisfy the required boundary conditions.

4. Determination of $g_4(0)$

We introduce the Green's operator:

$$P: C^\infty(\bar{B}_m(1)) \longrightarrow C^\infty(\bar{B}_m(1))$$

defined uniquely by the properties that for all $f \in C^\infty(\bar{B}_m(1))$

$$\Delta_{-2}(Pf) + f = 0 \qquad \text{in } \bar{B}_m(1)$$

$$Pf = 0 \qquad \text{on } \partial\bar{B}_m(1) .$$

With this notation we have from (2.8) - (2.11)

$$f_0 = P1$$

$$f_2 = P\Delta_0 f_0$$

$$f_3 = P\Delta_1 f_0$$

$$f_4 = P\Delta_0 f_2 + P\Delta_2 f_0$$

Similarly equations (2.4) - (2.7) can be written in the form

$$g_0 = Pf_0$$

$$g_2 = Pf_2 + P\Delta_0 g_0$$

$$g_3 = Pf_3 + P\Delta_1 g_0$$

$$g_4 = Pf_4 + P\Delta_0 g_2 + P\Delta_2 g_0$$

$$= P^2\Delta_0 f_2 + P^2\Delta_2 f_0 + P\Delta_0 g_2 + P\Delta_2 g_0 .$$

Therefore to compute g_4 we must first compute $\Delta_0 f_2$, $\Delta_2 f_0$, $\Delta_0 g_2$, $\Delta_2 g_0$. To handle the terms $P\Delta_0 g_2$ and $P\Delta_2 g_0$ we may use lemma 6.3 of [4]. To handle the terms $P^2\Delta_0 f_2$ and $P^2\Delta_2 f_0$ we invoke the following lemma, where the integrals are normalized so that $\int_{S^{n-1}} d\theta = 1$

Lemma 4.1. Let j be the solution of the biharmonic Poisson equation $\Delta_{-2}^2 j = r^k g(\theta)$ in the unit ball $\bar{B}_m(1)$ and satisfying the boundary conditions $j = 0$ and $\Delta_{-2} j = 0$ on the boundary $\partial\bar{B}_m(1) = S^{n-1}$. Then

$$j(0) = \frac{n+k+4}{2(k+4)n(n+k)(n+k+2)} \int_{S^{n-1}} g(\theta)\,d\theta$$

Proof. Let $G(x,y)$ be the Green's function for the biharmonic equation $\Delta_{-2}^{2} G = \delta$ with the same boundary conditions. Then $j(x) = \int_{\bar{B}_m(1)} G(x,y)|y|^k g(y/|y|)\,dy$. Let $\bar{g} = \int_{S^{n-1}} g(\theta)\,d\theta$ be the mean value of g on the unit sphere. Then

$$j(0) = \int_{\bar{B}_m(1)} G(0,y)|y|^k[g(y/|y|)-\bar{g}] + \int_{\bar{B}_m(1)} G(0,y)|y|^k dy\ .$$

The first integral is zero, since $G(0,y) = G(|y|)$, a radial function. The second integral is the solution of the problem $\Delta_{-2}^{2} j = r^k \bar{g}$, which is directly computed as

$$j(r) = \frac{\bar{g}}{(k+2)(n+k)}\left[\frac{1-r^2}{2n} - \frac{1-r^{k+4}}{(k+4)(n+k+2)}\right]\ .$$

Thus

$$j(0) = \frac{\bar{g}}{(k+2)(n+k)}\left[\frac{1}{2n} - \frac{1}{(k+4)(n+k+2)}\right]$$

which is of the required form.

For small values of k, we have for example

$$k=0: \quad j(0) = \frac{(n+4)}{8n^2(n+2)}\bar{g}$$

$$k=2: \quad j(0) = \frac{(n+6)}{12n(n+2)(n+4)}\bar{g}$$

$$k=4: \quad j(0) = \frac{(n+8)}{16n(n+4)(n+6)}\bar{g}\ .$$

We also recall the following integral formulas which were used in [4] where integration is with respect to the normalized uniform surface measure on S^{n-1}.

Lemma 4.2 $$\int_{S^{n-1}} \left(\rho - \frac{\tau r^2}{n}\right) = \frac{2}{n(n+2)}\left(\|\rho\|^2 - \frac{\tau^2}{n}\right)$$

$$\int_{S^{n-1}} \rho \# R = \frac{\|\rho\|^2}{n}$$

$$\int_{S^{n-1}} \rho\circ\rho = \frac{\|\rho\|^2}{n}$$

$$\int_{S^{n-1}} R\#R = \frac{1}{n(n+2)}\left(\|\rho\|^2 + \frac{3}{2}\|R\|^2\right)$$

$$\int_{S^{n-1}} \nabla^2\rho = \frac{2}{n(n+2)}\Delta\tau$$

It is easily checked that this implies that $\int_{S^{n-1}} \Delta_0\left(\rho - \frac{\tau r^2}{n}\right)$ $= -(2/3n)\left(\|\rho\|^2 - \frac{\tau^2}{n}\right)$.

Computation of $P^2\Delta_2 f_0$: We have

$$\Delta_2 f_0 = (1/90n)(9\nabla^2\rho + 2R\#R)$$

Both of these terms are homogeneous with $k = 4$. Applying the above lemmas 4.1 and 4.2 we have

$$(P^2\Delta_2 f_0)(0) = \frac{n+8}{90\cdot 16n^2(n+4)(n+6)}\left[\frac{18}{n(n+2)}\Delta\tau + \frac{2}{n(n+2)}\left(\|\rho\|^2 + \frac{3}{2}\|R\|^2\right)\right]$$

Computation of $P\Delta_2 g_0$: We have

$$\Delta_2 g_0 = \frac{1}{90}(9\nabla^2\rho + 2R\#R)\left(\frac{1}{2n^2} - \frac{r^2}{2n(n+2)}\right)$$

which is a combination of terms with $k = 4$ and $k = 6$. Applying lemma 6.3 of [4] and lemma 4.2 above, we have

$$(P\Delta_2 g_0)(0) = \frac{n^2 + 20n + 48}{90\cdot 48n^2(n+2)(n+4)(n+6)}\left[\frac{18}{n(n+2)}\Delta\tau + \frac{2}{n(n+2)}\left(\|\rho\|^2 + \frac{3}{2}\|R\|^2\right)\right]$$

Computation of $P^2\Delta_0 f_2$: We have

$$\Delta_0 f_2 = \left(\rho - \frac{\tau r^2}{n}\right)\frac{\rho}{9n(n+4)} + \frac{(1-r^2)}{6n(n+4)}\left[\frac{2}{3}(\rho\#R - 2\rho\circ\rho) + \frac{2\tau\rho}{3n}\right] + \frac{\tau\rho r^2}{9n^2(n+2)}$$

which is a combination of terms with $k = 2$ and $k = 4$. Applying lemmas 4.1 and 4.2 we have

$$(P^2\Delta_0 f_2)(0) = -\frac{n^2 + 12n + 48}{432n^3(n+2)(n+4)^2(n+6)}\left(\|\rho\|^2 - \frac{\tau^2}{n}\right) + \frac{n+8}{144n^4(n+2)(n+4)(n+6)}\tau^2$$

Computation of $P\Delta_0 g_2$: We have

$$\Delta_0 g_2 = \left(\rho - \frac{\tau r^2}{n}\right)\Delta_0\left[\frac{n+2}{6n^2(n+4)^2}(1-r^2) - \frac{n+3}{12n(n+2)(n+4)(n+6)}(1-r^4)\right]$$

$$+\left[\frac{n+2}{6n^2(n+4)^2}(1-r^2) - \frac{n+3}{12n(n+2)(n+4)(n+6)}(1-r^4)\right]\Delta_0\left(\rho - \frac{\tau r^2}{n}\right)$$

$$+\tau\Delta_0\left[\frac{1-r^2}{24n^3(n+2)} + \frac{1-r^4}{24n^3(n+2)} - \frac{1-r^6}{24n^2(n+2)(n+4)}\right]$$

$$= \left(\rho - \frac{\tau r^2}{n}\right)\left[\frac{\rho(n+2)}{9n^2(n+4)^2} - \frac{\rho r^2(n+3)}{9n(n+2)(n+4)(n+6)}\right]$$

$$+\left[\frac{n+2}{6n^2(n+4)^2}(1-r^2) - \frac{n+3}{12n(n+2)(n+4)(n+6)}(1-r^4)\right]\left[\frac{2}{3}(\rho\#R - 2\rho\circ\rho) + \frac{2\tau\rho}{3n}\right]$$

$$+\tau\left[\frac{\rho}{36n^3(n+2)} + \frac{\rho r^2}{18n^3(n+2)} - \frac{\rho r^4}{12n^2(n+2)(n+4)}\right]$$

which is a combination of terms with $k=4$ and $k=6$. Applying lemma 4.2 above and lemma 6.3 of [4] we have after some lengthy algebra

$$(P\Delta_0 g_2)(0) = -\frac{n^5 + 27n^4 + 290n^3 + 1312n^2 + 2784n + 2304}{432n^3(n+2)^2(n+4)(n+6)}\left(\|\rho\|^2 - \frac{\tau^2}{n}\right)$$

$$+\frac{5n^2 + 106n + 240}{864n^4(n+2)^2(n+4)(n+6)}\tau^2$$

These results are recorded in the table in the Appendix. We summarize the result in the following form.

<u>Theorem 4.3.</u> <u>For small</u> $\varepsilon > 0$

$$\tfrac{1}{2}E_m\left(T_\varepsilon^2\right) = c_0\varepsilon^4 + c_1\varepsilon^6\tau_m + \varepsilon^8\left[c_2\Delta\tau + c_3\tau^2 + c_4\|\rho\|^2 + c_5\|R\|^2\right]_m + O\left(\varepsilon^{10}\right)$$

<u>where the constants</u> $c_0, c_1, c_2, c_3, c_4, c_5$ <u>depend on the dimension</u> n. <u>In fact</u> $c_0 = g_0(0)$ <u>and</u> $c_1 = g_2(0)$ <u>given by proposition 3.1</u>; c_2, c_3, c_4, c_5 <u>are given in the appendix</u>. <u>Here</u> $\tau = \sum_{i=1}^{n}\rho_{ii}$ <u>is the scalar curvature</u>

and $\Delta\tau = \sum_{i=1}^{n} \nabla^2_{ii}\tau$ is the Laplacian of the scalar curvature. Also $\|R\| = \left\{\sum R^2_{ijk\ell}\right\}^{\frac{1}{2}}$ and $\|\rho\| = \left\{\sum \rho^2_{ij}\right\}^{\frac{1}{2}}$ are the lengths of the curvature tensor and the Ricci curvature.

5. Converse theorems

Theorem 5.1. Let (M,g) be a Riemannian manifold such that for all $m \in M$ we have $E_m(T_\varepsilon) = \text{const.}\ \varepsilon^2 + O(\varepsilon^8)$ and $E_m(T^2_\varepsilon) = \text{const.}\ \varepsilon^4 + O(\varepsilon^{10})$ when $\varepsilon \downarrow 0$. Then (M,g) is locally isometric to (R^n, g_0).

Proof. From the first hypothesis and theorem 1.1 of [4] we have that for all $m \in M$, $\tau_m = 0$ and $\|R\|_m = \|\rho\|_m$. From the second hypothesis and theorem 4.3 above we have in addition that $c_4\|\rho\|^2_m + c_5\|R\|^2_m = 0$. This is possible for $\|R\|_m \neq 0$ if and only if $c_4 + c_5 = 0$. From the table of values in the Appendix this entails the equality

$$18(n+4)^2(n+6)(2n^2+25n+48) = 33n^5 + 792n^4 + 8292n^3 + 38208n + 69120$$

Multiplying out the left side it is seen that the left side is strictly greater than the right side for every $n \geq 1$. Therefore $c_4 + c_5 \neq 0$ and we must have $\|R\|_m = 0 = \|\rho\|_m$ and (M,g) is locally isometric to (R^n, g_0).

Theorem 5.2. Let (M,g) be a Riemannian manifold such that for all $m \in M$ we have $E_m^{(M,g)}(T_\varepsilon) - E_m^{(M_\lambda,g_\lambda)}(T_\varepsilon) = O(\varepsilon^8)$ and $E_m^{(M,g)}(T^2_\varepsilon) - E_m^{(M_\lambda,g_\lambda)}(T^2_\varepsilon) = O(\varepsilon^{10})$ when $\varepsilon \downarrow 0$ where (M_λ, g_λ) is a space of constant sectional curvature λ. Then (M,g) is locally isometric to (M_λ, g_λ).

Proof. From the first hypothesis and theorem 1.1 of [4] we have that for all $m \in M$

$$\tau_m = \tau(\lambda)$$

$$\|R\|^2_m - \|\rho\|^2_m = \|R(\lambda)\|^2 - \|\rho(\lambda)\|^2$$

where $\tau(\lambda)$, $R(\lambda)$, $\rho(\lambda)$ are the values for a space of constant sectional curvature. From the second hypothesis and theorem 4.3 above, we have further

$$c_4\|\rho\|_m^2 + c_5\|R\|_m^2 = c_4\|\rho(\lambda)\|^2 + c_5\|R(\lambda)\|^2$$

The proof of theorem 5.1 above shows that $c_4 + c_5 \neq 0$. Therefore the above equations uniquely determine the values $\|R\|_m^2 = \|R(\lambda)\|^2$, $\|\rho\|_m^2 = \|\rho(\lambda)\|^2$. It is well known that this implies that (M,g) has constant sectional curvature.

6. Appendix. Table of the coefficients of $g_4(0) = c_2 \Delta\tau + c_3 \tau^2 + c_4 \|\rho\|^2 + c_5 \|R\|^2$

coefficient of / in	$\Delta\tau$	$\|\rho\|^2$	$\|R\|^2$
$P^2 \Delta_0 f_2$	0	$-\dfrac{n^2 + 12n + 48}{432n^3 (n+2)(n+4)^2 (n+6)}$	0
$P^2 \Delta_2 f_0$	$\dfrac{n+8}{80n^3 (n+2)(n+4)(n+6)}$	$\dfrac{n+8}{720n^3 (n+2)(n+4)(n+6)}$	$\dfrac{n+8}{480n^3 (n+2)(n+4)(n+6)}$
$P \Delta_0 g_2$	0	$-\dfrac{n^5 + 27n^4 + 290n^3 + 1312n^2 + 2784n + 2304}{432n^3 (n+2)^2 (n+4)^3 (n+6)^2}$	0
$P \Delta_2 g_0$	$\dfrac{n^2 + 20n + 48}{240n^3 (n+2)^2 (n+4)(n+6)}$	$\dfrac{n^2 + 20n + 48}{2160n^3 (n+2)^2 (n+4)(n+6)}$	$\dfrac{n^2 + 20n + 48}{1440n^3 (n+2)^2 (n+4)(n+6)}$
TOTAL	$c_2 = \dfrac{2n^2 + 25n + 48}{120n^3 (n+2)^2 (n+4)(n+6)}$	$c_4 = -\dfrac{33n^5 + 792n^4 + 8292n^3 + 38208n^2 + 83520n + 69120}{12960n^3 (n+2)^2 (n+4)^3 (n+6)^2}$	$c_5 = \dfrac{2n^2 + 25n + 48}{720n^3 (n+2)^2 (n+4)(n+6)}$

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Mark A. PINSKY
Department of Mathematics
Northwestern University
Evanston, Illinois 60201
U.S.A.