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Between distributions and hyperfunctions

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BETWEEN DISTRIBUTIONS AND HYPERFUNCTIONS

By

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1. Introduction. The space $\mathcal{D}'(X)$ of Schwartz distributions in an open subset X of $\mathbf{R}^n$ is by definition the space of continuous linear functionals on $C_0^\infty(X)$. A larger space is obtained if $C_0^\infty(X)$ is replaced by a dense subset with a stronger topology, such as the space of functions of compact support in a non-quasianalytic Denjoy-Carleman class of functions. (See section 2 below for definitions.) This leads essentially to the distribution spaces discussed by Beurling [2] (see also Björck [3]).

In the quasianalytic case this definition breaks down. However, dropping the condition of compact support one can always consider the dual as an analogue of the dual $\mathcal{E}'(X)$ of $C^\infty(X)$. The largest space of its kind is then the space $\mathcal{A}'(\mathbf{R}^n)$ of analytic functionals carried by compact subsets of $\mathbf{R}^n$; this is dual to the real analytic functions. Martineau [5] has shown how one can define the hyperfunctions of Sato [6] starting from the properties of $\mathcal{A}'(\mathbf{R}^n)$. The first point is to prove that every element in $\mathcal{A}'(\mathbf{R}^n)$ has a unique minimal carrier, the support. For any open set $X\subset\mathbf{R}^n$ the space of hyperfunctions in X can then be defined so that its elements are locally equal to those in $\mathcal{A}'(\mathbf{R}^n)$. We shall here use the analogous definition for any Denjoy-Carleman class.

In sections 2 and 3 we shall give the basic definitions and discuss the notion of support for the dual $\mathcal{E}_L'$ of any Denjoy-Carleman class C^L. Sections 4 and 5 are then devoted to the non-quasianalytic and the quasianalytic cases respectively. The properties of the distribution spaces $\mathcal{D}_L'(X)$ in an open set $X\subset\mathbf{R}^n$ are then summed up in section 6. We show in particular that the sheaf of distributions is flabby precisely in the quasianalytic case. (Flabbiness means that all distributions can be extended to the whole space.) Another equivalent property is that every distributions with support in the union $K_1 \cup K_2$ of two compact sets is the sum of one with support in K_1 and one with support in K_2. These facts are of course well-known for hyperfunctions. What may be new is the equivalence with quasianalyticity.

2. Denjoy-Carleman classes. Let L_k be an increasing sequence of positive numbers such that $L_0=1$ and

$$k \leqq L_k, \quad L_{k+1} \leqq CL_k; \; k = 0, 1, \ldots; \tag{2.1}$$

for some constant C. If $X \subset \mathbf{R}^n$ is an open set we shall denote by $C^L(X)$ the set of all $u \in C^\infty(X)$ such that for every compact set $K \subset X$

$$|u|_{L,r,K} = \sup_{x\in K} \sup_{\alpha} (r/L_{|\alpha|})^{|\alpha|} |D^\alpha u(x)| < \infty, \tag{2.2}$$

for some $r = r_K > 0$. When $L_k=k+1$ this means that $C^L(X)$ is the set of real analytic functions in X, which is thus the smallest class considered. The class C^L with $L_k=(k+1)^a$, $a>1$, is the Gevrey class of order a. Leibniz' formula shows at once that C^L is a ring,

$$|uv|_{L,r/2,K} \leqq |u|_{L,r,K} |v|_{L,r,K}. \tag{2.3}$$

It is invariant under differentiation since

$$(L_{j+1})^{j+1} \leqq (CL_j)^{j+1} \leqq C^{2j+1} L_j^{\,j}, \tag{2.4}$$

which implies

$$|D_k u|_{L,r/C^2,K} \leqq C/r \; |u|_{L,r,K}. \tag{2.5}$$

By the Denjoy-Carleman theorem there are non-trivial functions $u \in C^L(X)$ of compact support if and only if

$$\sum 1/L_k < \infty. \tag{2.6}$$

The class is then called non-quasianalytic. In the opposite case a function in $C^L(X)$ vanishing in a neighborhood of a point x must also vanish in the component of x in X. As a substitute for C^L functions of compact support one can often use cutoff functions with the following properties:

LEMMA 2.1. *Let K be a compact subset of $\mathbf{R}^n$ and denote by K(t) the set of points at distance $\leqq$ t from K. For any integer $\nu>0$ one* can find $\chi \in C_0^\infty(K(\nu t))$ equal to 1 in a neighborhood of K such that $0 \leqq \chi \leqq 1$ and

$$|D^\alpha \chi| \leqq C^{|\alpha|} t^{-|\alpha|}, \quad |\alpha| \leqq \nu. \tag{2.7}$$

Here C depends only on the dimension n.

For a proof se e.g. [4, section 1.4]; one just takes χ as the convolution of the characteristic function of $K(t\nu/2)$ and ν convolution factors $\psi(x/t)t^{-n}$ where $0 \leqq \psi \in C_0^\infty(\{x;\ |x|<\frac{1}{2}\})$ and $\int \psi\, dx = 1$. The point is that one can then let all derivatives act on different factors $\psi(x/t)$.

3. The space $\mathcal{E}'_L(X)$. Let $K \subset \mathbf{R}^n$ be a compact set. The space $\mathcal{E}'(K)$ of Schwartz distributions supported by K consists of the linear forms u on $C^\infty(\mathbf{R}^n)$ such that for every neighborhood X of K we have for some C and N

$$|u(\varphi)| \leqq C \sum_{|\alpha| \leqq N} \sup_X |D^\alpha \varphi|, \quad \varphi \in C^\infty(\mathbf{R}^n).$$

It suffices to have such a functional defined for all polynomials φ, for they are dense in $C^\infty(\mathbf{R}^n)$. The following is therefore an analogue for the class C^L.

DEFINITION 3.1. *Let K be a compact set in $\mathbf{R}^n$. Then $\mathcal{E}'_L(K)$ is the space of linear forms u on the space of polynomials φ in $\mathbf{R}^n$ such that for every neighborhood X of K and every $r>0$ we have*

$$|u(\varphi)| \leqq C_{r,X} |\varphi|_{L,r,X}. \tag{3.1}$$

For any $X \subset \mathbf{R}^n$ we denote by $\mathcal{E}'_L(X)$ the union of $\mathcal{E}'_L(K)$ for all compact subsets K of X. When $L_k = k+1$ we also write $\mathcal{A}'(K)$ instead of $\mathcal{E}'_L(K)$ for the space of analytic functionals carried by K.

Note that $\mathcal{E}'_{L_1} \supset \mathcal{E}'_{L_2}$ if $L_1 \leqq CL_2$ for some C. In particular, $\mathcal{E}'_L \subset \mathcal{A}'$ for every L.

It follows from (3.1) that there is a unique linear extension of $u(\varphi)$ satisfying (3.1) in the set $\mathcal{A}$ of entire analytic functions. In fact, if $\varphi \in \mathcal{A}$ the partial sums of the Taylor series converge on any compact subset of $\mathbf{C}^n$. This implies convergence in the norm $|\ |_{L,r,X}$ for any $r>0$ and any bounded X. To extend the definition to a more reasonable set of test functions we prove:

PROPOSITION 3.2. <u>Let</u> $Y\subset\subset X$ <u>and let</u> $\varphi\in C^L(X)$. <u>Then there is a sequence of entire functions</u> φ_j <u>such that for sufficiently small</u> $r>0$

$$|\varphi-\varphi_j|_{L,r,Y}\to 0 \text{ as } j\to\infty. \tag{3.2}$$

PROOF. Choose $\chi\in C_0^1(X)$ with $0 \leqq \chi \leqq 1$ and $\chi = 1$ in a neighborhood of $\overline{Y}$, and set

$$\varphi_j(x) = \int E_j(x-y)\chi(y)\varphi(y)dy;\ E_j(x)=(j/\pi)^{n/2}\ e^{-j\langle x,x\rangle}.$$

By induction we obtain for any N

$$D_{i_N}\dots D_{i_1}(E_j*(\chi\varphi)) = E_j*(\chi D_{i_N}\dots D_{i_1}\varphi) + \sum_{\nu=1}^{N}\prod_{\nu<\mu\leqq N} D_{i_\mu} E_j*((D_{i_\nu}\chi)\prod_{1\leqq\mu<\nu} D_{i_\mu}\varphi). \tag{3.3}$$

We can write

$$E_j*(\chi D^\alpha\varphi)(x) - D^\alpha\varphi(x) = \int E_j(y)(\chi(x-y)D^\alpha\varphi(x-y)-D^\alpha\varphi(x))dy.$$

Choose $c>0$ so that $\chi(x-y)=1$ when $x\in Y$ and $|y|<c$, and let ρ be so small that $M = |\varphi|_{L,\rho,\operatorname{supp}\chi}<\infty$. By (2.4) we have

$$|D^\alpha\varphi(x-y)-D^\alpha\varphi(x)| \leqq n|y|(L_{|\alpha|+1}/\rho)^{|\alpha|+1}M \leqq$$
$$\leqq nC|y|/\rho\ (L_{|\alpha|}C^2/\rho)^{|\alpha|}M;\ x\in Y,\ |y|<c;$$

and

$$j^{\frac{1}{2}}\int |y|E_j(y)\ dy$$

is independent of j. Furthermore,

$$\int_{|y|>c} E_j(y)\ dy = O(e^{-c^2 j/2}),\ j\to\infty.$$

This proves that

$$\sup_Y |E_j*(\chi D^\alpha\varphi)-D^\alpha\varphi| \leqq C'(L_{|\alpha|}C^2/\rho)^{|\alpha|}M/j^{\frac{1}{2}}.$$

When $y\in\operatorname{supp} d\chi$ we have $\operatorname{Re}\langle z-y, z-y\rangle > 0$ for z in a complex neighborhood of $\overline{Y}$. Hence Cauchy's inequalities give for some $c>0$ if $x\in Y$

$$|D^\alpha E_j(x-y)| \leqq |\alpha|!c^{-|\alpha|}e^{-cj} \leqq L_{|\alpha|}{}^{|\alpha|}c^{-|\alpha|}e^{-cj}.$$

Using (3.3) we now obtain if $r<c$ and $rC^2<\rho$

$$|D^\alpha\varphi_j - D^\alpha\varphi|_{L,r,Y} \leqq C''M/j^{\frac{1}{2}},$$

which proves the proposition.

From Proposition 3.2 it follows at once that every $u\in\mathcal{E}'_L(K)$ can be uniquely extended to a linear functional on $C^L(X)$ for any neighborhood X of K. However, this is not very useful until we know that there is a unique minimal compact set K such that $u\in \mathcal{E}'_L(K)$. For the analytic class this follows from basic facts on the cohomology of the sheaf of germs of holomorphic functions (see Martineau [5]). An elementary proof using only properties of the corresponding Poisson integral can be found in [4, section 9.1]. We quote the result without repeating the proof.

THEOREM 3.3. *If* $u\in\mathcal{A}'(\mathbf{R}^n)$ *then there is a smallest compact set* $K\subset\mathbf{R}^n$ *such that* $u\in\mathcal{A}'(K)$; *it is called the support of* u.

If u is a Schwartz distribution of compact support this agrees with the usual definition. In fact, if X is a neighborhood of the Schwartz support K then

$$|u(\varphi)| \leqq C\sum_{|\alpha|\leqq N}\sup_X |D^\alpha\varphi| \leqq C_{L,r}|\varphi|_{L,r,X};\ \varphi\in C^\infty;$$

in particular $u\in\mathcal{A}'(K)$. On the other hand, if $u\in\mathcal{E}'(\mathbf{R}^n)\cap\mathcal{A}'(K)$ and $\varphi\in C_0^\infty(\mathbf{R}^n\setminus K)$, we obtain with the notation in the proof of Proposition 3.2

$$u(\varphi) = \lim_{j\to\infty} u(E_j*\varphi) = 0$$

since $E_j*\varphi\to\varphi$ in $C^\infty(\mathbf{R}^n)$ and

$$E_j*\varphi(x) = \int E_j(x-y)\varphi(y)\,dy \to 0$$

in a complex neighborhood of K when $j\to\infty$. Thus the Schwartz support is contained in K. With this possible ambiguity removed we shall now prove

THEOREM 3.4. *If* $u\in\mathcal{E}'_L(\mathbf{R}^n)\cap\mathcal{A}'(K)$ *then* $u\in\mathcal{E}'_L(K)$.

Since on the other hand $u\in\mathcal{E}'_L(K)$ implies $u\in\mathcal{A}'(K)$, we obtain:

COROLLARY 3.5. *If* $u\in\mathcal{E}'_L$ *then there is a smallest compact set* $K\subset\mathbf{R}^n$ *such that* $u\in\mathcal{E}'_L(K)$; *it is equal to the support of* u *as an element of* $\mathcal{A}'(K)$.

Thus we may use the term support without specifying an L

such that $u\in\mathcal{E}'_L$. We shall say that $u=0$ in an open set X if $X\cap\operatorname{supp} u$ is empty.

PROOF OF THEOREM 3.4. Let $K\subset\subset Y\subset\subset X\subset\subset\mathbf{R}^n$, and let φ be a polynomial. We shall estimate $u(\varphi)$ in terms of the norm $M=|\varphi|_{L,r,X}$. By definition

$$(3.4)\qquad |D^\alpha\varphi(x)| \leqq M(L_{|\alpha|}/r)^{|\alpha|},\ x\in X.$$

The proof is a refinement of that of Proposition 3.2 where different regularizations are used in different frequency ranges.

1. Choose $\chi_\nu\in C_0^\infty(X)$ using Lemma 2.1 so that $\chi_\nu=1$ in Y and

$$|D^\alpha\chi_\nu| \leqq (C_1\nu)^{|\alpha|},\ |\alpha|\leqq\nu.$$

(Here C_1 is the constant C in (2.7) divided by the distance from Y to $\complement X$.) Since $\nu\leqq L_\nu$ we obtain using (3.4)

$$|D^\alpha(\chi_\nu\varphi)| \leqq M(L_\nu(C_1+1/r))^\nu,\ |\alpha|=\nu,$$

which implies for small r that

$$|\xi|^\nu|F(\chi_\nu\varphi)(\xi)| \leqq M(L_\nu C_2)^\nu m(X).$$

Here $C_2=2n/r$, and F is the Fourier transformation. Set

$$(3.5)\qquad \mathcal{L}(t) = \sup_{\nu\geqq 0}(t/L_\nu)^\nu,\ t>0.$$

Given $t>0$ we can choose $\nu=\nu(t)$ so that $\mathcal{L}(t)=(t/L_\nu)^\nu$, and then we obtain

$$|F(\chi_\nu\varphi)(\xi)| \leqq Mm(X)/\mathcal{L}(t)\ \text{if}\ |\xi|>C_2t.$$

Since $\mathcal{L}$ is increasing it follows that

$$(3.6)\qquad |F(\chi_\nu\varphi)(\xi)| \leqq Mm(X)/\mathcal{L}(|\xi|/4C_2)\ \text{if}\ \nu=\nu(t)\ \text{and}\ C_2<|\xi/t|<4C_2.$$

When $\nu=\nu(N,r)=\nu(2^{N-2}r/n)$ this estimate holds in the annulus where $2^{N-1}<|\xi|<2^{N+1}$. Choose $\psi_N\in C_0^\infty(\{\xi;\ 2^{N-1}<|\xi|<2^{N+1}\})$ when $N\neq 0$ and choose $\psi_0\in C_0^\infty(\{\xi;\ |\xi|<2\})$ so that $\psi_N\geqq 0$, $\sum\psi_N=1$, and set

$$(3.7)\qquad R\varphi(x) = \sum_0^\infty \psi_N(D)(\chi_{\nu(N,r)}\varphi)(x).$$

We claim that the sum converges in $C^L(\mathbf{R}^n)$ and that for some C_3 and $r'>0$ (depending on r)

(3.8) $$|R\varphi|_{L,r',\mathbf{R}^n} \leqq C_3|\varphi|_{L,r,X}.$$

From (3.6) it follows that

$$\sum_1^\infty |\xi^\alpha||\psi_N(\xi)||F(\chi_{\nu(N,r)}\varphi(\xi)| \leqq Mm(X)|\xi|^{|\alpha|}/L(|\xi|/4C_2) \leqq$$

$$\leqq Mm(X)|\xi|^{-n-1}(4C_2L_{|\alpha|+n+1})^{|\alpha|+n+1} \leqq C_4M(L_{|\alpha|}/r')^{|\alpha|}|\xi|^{-n-1}.$$

Here we have used (2.4). By Fourier's inversion formula we obtain

$$\sum_N |D^\alpha\psi_N(D)(\chi_{\nu(N,r)}\varphi)| \leqq C_3M(L_{|\alpha|}/r')^{|\alpha|}.$$

This proves (3.8) and also convergence in the norm $|\ |_{L,r'/2,\mathbf{R}^n}$. From (3.8) it follows that

$$u_1(\varphi) = u(R\varphi) = \sum_0^\infty u(\psi_N(D)(\chi_{\nu(N,r)}\varphi))$$

is continuous for the norm $|\ |_{L,r,X}$.

2. To be able to estimate $u-u_1$ we must make an appropriate choice of ψ_N too. So far we have only used that the partition of unity is continuous. First we use Lemma 2.1 to choose h_N for $N=0$, 1, ... so that $0\leqq h_N\leqq 1$, $h_N(\xi)=1$ when $|\xi|<2^N$, $h_N(\xi)=0$ when $|\xi|>2^{N+1}$,

$$|D^\alpha h_N(\xi)| \leqq (C\delta)^{|\alpha|},\ |\alpha|\leqq 2^N\delta.$$

Here δ is a small positive number to be chosen later of the same order of magnitude as the distance from K to $\complement Y$. It is important that C does not depend on δ. The same will be true for the other constants below. Set $\psi_0 = h_0$ and

$$\psi_N = h_N - h_{N-1};\ N = 1, 2, \ldots$$

Since the derivatives of the terms have disjoint supports, we have

(3.9) $$|D^\alpha\psi_N(\xi)| \leqq (C\delta)^{|\alpha|},\ |\alpha|\leqq 2^{N-1}\delta,$$

and $2^{N-1}<|\xi|<2^{N+1}$ in supp ψ_N if $N\neq 0$.

The operator $\psi_N(D)$ consisting in multiplication of the Fourier transform by ψ_N is equal to convolution by Ψ_N where

$$\Psi_N(z) = (2\pi)^{-n}\int e^{i<z,\xi>}\ \psi_N(\xi)\ d\xi,\ z\in\mathbf{C}^n.$$

With $H = H_N = 2^N$ we have by (3.9)

$$|z^\alpha \Psi_N(z)| \leqq C'(C\delta)^{|\alpha|} H^n e^{2|\mathrm{Im}\, z|H}, \quad |\alpha| \leqq 2^{N-1}\delta.$$

Hence

$$|\Psi_N(z)| \leqq C'(C_5\delta/|z|)^{H\delta/2}\, H^n\, e^{2|\mathrm{Im}\, z|H} \leqq$$
$$\leqq C'H^n\, e^{2|\mathrm{Im}\, z|H-H\delta/2} \text{ if } |z|>C_5 e\delta,$$

so we have

(3.10) $$|\Psi_N(z)| \leqq C''\delta^{-n}\, e^{-H\delta/3} \text{ if } |z|>C_5 e\delta \text{ and } |\mathrm{Im}\, z|<\delta/13.$$

Let $\chi \in C_0^\infty(X)$ be equal to 1 in Y and set

$$T\varphi(x) = \sum_0^\infty \psi_N(D)((\chi-\chi_{\nu(N,r)})\varphi)(x).$$

Choose δ so small that $C_5 e\delta$ is smaller than the distance from K to $\complement Y$. Then there is a complex neighborhood Ω of K such that

$$|\Psi_N(z-y)| \leqq C_6 e^{-H_N\delta/3} \text{ if } z\in\Omega \text{ and } y\notin Y.$$

Hence we have for all $z\in\Omega$

$$|\psi_N(D)((\chi-\chi_{\nu(N,r)})\varphi)(z)| \leqq 2\int_{X\setminus Y} |\Psi_N(z-y)\varphi(y)|dy \leqq 2C_6 e^{-H_N\delta/3}\|\varphi\|$$

where $\|\varphi\|$ is the L^1 norm in X, so the series $T\varphi(z)$ converges for $z\in\Omega$, and

$$u_2(\varphi) = u(T\varphi) = \sum_0^\infty u(\psi_N(D)((\chi-\chi_{\nu(N,r)})\varphi))$$

is a well defined function in L^∞ with support in $\overline{X}$.

3. The proof of Theorem 3.4 will be completed if we show that $u=u_1+u_2$, for then we obtain an estimate $|u(\varphi)|\leqq C|\varphi|_{L,r,X}$ for any neighborhood X of K and any r>0. We have

$$u_1(\varphi)+u_2(\varphi) = \sum_0^\infty u(\psi_N(D)(\chi\varphi)).$$

To prove that $u_1+u_2=u$ it suffices to show that the sum of $\psi_N(D)(\chi\varphi)$ converges to φ in a complex neighborhood of K. It is clear that the sum converges to $\chi\varphi$ in $\mathcal{S}$, which implies that it converges to φ in $C^\infty(Y)$. Now consider the derivative of order α when $|\alpha|$ exceeds the degree of the polynomial φ. It is a finite sum of terms of the form

$$\Psi_N*((D^\beta\chi)D^\gamma\varphi)$$

where $|\beta|+|\gamma|=|\alpha|$ and $|\gamma|<|\alpha|$, hence $|\beta|\neq 0$. In view of (3.10) it follows that $\sum D^\alpha\psi_N(D)(\chi\varphi)$ is locally uniformly convergent in Ω. The sum must be equal to $D^\alpha\varphi$ since this is true in Y. Taylor's formula now shows that $\sum\psi_N(D)(\chi\varphi)$ converges locally uniformly to φ in Ω, which completes the proof.

As we shall see in section 4 the preceding fairly technical argument is superfluous in the non-quasianalytic case. In the quasianalytic case it will be used again in section 5.

4. The non-quasianalytic case. When $\sum 1/L_k<\infty$ the space C^L contains functions of x_1 vanishing for $x_1<0$ but not identically. Since C^L is a ring invariant under linear changes of variables it follows that the space C_0^L of elements in C^L with compact support contains non-negative functions with integral 1. Regularization by convolution with elements in C_0^L shows that C_0^L is dense in C_0^∞ and allows one to construct cutoff functions and partitions of unity in C_0^L just as in C_0^∞. The proof of Proposition 3.2 can be simplified for we may assume that $\varphi\in C_0^L$. Taking $\varphi_j=E_j*\varphi$ we get rid of all terms in the proof containing derivatives of χ. The proof of Theorem 3.4 for Schwartz distributions preceding the statement also gives $u(\varphi)=0$ if $\varphi\in C_0^L$ and $K\cap\operatorname{supp}\varphi=\emptyset$. The full result follows since $u(\varphi)=0$ if $\varphi\in C^L$ and $K'\cap\operatorname{supp}\varphi=\emptyset$ for some K' such that $u\in \mathcal{E}_L'(K')$. The following decomposition theorem is also proved just as for Schwartz distributions:

THEOREM 4.1. *If X_1 and X_2 are open sets in $\mathbf{R}^n$, C^L is non-quasianalytic, and $u\in\mathcal{E}_L'(X_1\cup X_2)$, then $u=u_1+u_2$ with $u_j\in\mathcal{E}_L'(X_j)$.*

PROOF. We can choose $\chi_j\in C_0^L(X_j)$ so that $\chi_1+\chi_2=1$ in a neighborhood of supp u and set

$$u_j(\varphi) = u(\chi_j\varphi).$$

Then $u_j\in\mathcal{E}_L'(X_j)$ and $u_1+u_2=u$.

It is not possible to replace the open sets X_j by compact sets in Theorem 4.1:

THEOREM 4.2. *For every non-quasianalytic class C^L one can find compact sets K_1, $K_2\subset\mathbf{R}^n$ and a Schwartz distribution $u\in \mathcal{E}_L'(K_1\cup K_2)$ of order 1 such that $u\neq u_1+u_2$ for all $u_j\in\mathcal{E}_L'(K_j)$.*

PROOF. Let K_1 be the closure of a sequence $x_j \in \mathbf{R}^n$ with $|x_1|>|x_2|> \dots \to 0$. Since $\sum 1/(L_j\delta_j) < \infty$ if $\delta_j \to 0$ sufficiently slowly, we can choose $\varphi_j \in C_0^L(\complement\{0\})$ so that

$$\varphi_j(x_j) = 1, \ \varphi_j(x_k) = 0 \text{ for } k \neq j, \quad a_j = |\varphi_j|_{L,1,\mathbf{R}^n} < \infty.$$

Next choose $y_j \neq x_k$ for every k so that $|x_j - y_j| a_j < j^{-3}$, and let K_2 consist of the points y_j and the limit 0. Then

$$u(\varphi) = \sum j a_j (\varphi(x_j) - \varphi(y_j))$$

is a Schwartz distribution of order 1. If $u = u_1 + u_2$ with $u_j \in \mathcal{E}'_L(K_j)$ and we write $\varphi_j = \psi_1 + \psi_2$ with $\psi_k \in C_0^L(\complement K_k)$, it follows that

$$u_1(\varphi_j) = u_1(\psi_2) = u(\psi_2) = j a_j \psi_2(x_j) = j a_j \varphi_j(x_j) = j a_j.$$

In view of the definition of a_j this contradicts that $u_1 \in \mathcal{E}'_L$.

5. <u>The quasianalytic case.</u> In this case we cannot find partitions of unity in C^L. Nevertheless there is a stronger version of Theorem 4.1 which by Theorem 4.2 is false in the non-quasianalytic case:

THEOREM 5.1. <u>Let</u> K_1 <u>and</u> K_2 <u>be compact sets in</u> $\mathbf{R}^n$ <u>and let</u> C^L <u>be quasianalytic. For every</u> $u \in \mathcal{E}'_L(K_1 \cup K_2)$ <u>one can then find</u> $u_j \in \mathcal{E}'_L(K_j)$, j = 1, 2, <u>such that</u> $u = u_1 + u_2$.

An essential point in the proof is that one can approximate distributions with support at one point with distributions having support at another. This can be derived from the following consequence of the proof of the Denjoy-Carleman theorem:

LEMMA 5.2. <u>Let</u> C^L <u>be quasianalytic, that is,</u> $\sum 1/L_k < \infty$, <u>and let</u> δ, r <u>be positive numbers. Then one can find an integer</u> N <u>and real numbers</u> $a_0, \dots, a_N$ <u>such that for</u> $\varphi \in C^L([0,1])$

$$|\varphi(1) - \sum_0^N a_j \varphi^{(j)}(0)| \leqq \delta |\varphi|_{L,r,[0,1]}. \tag{5.1}$$

This follows from the proof by Bang [1] of the Denjoy-Carleman theorem or the variant of the proof given in [4, section 1.3]. If the derivatives of φ up to some high order vanish at 0 we can just use the estimate (1.3.13)' there. Inspection of the proof shows that if we define $\varphi(t)=0$ for $t<0$ then Lemma 5.1 is obtained without any such restrictions.

Using Lemma 5.2 we can prove the following approximation lemma:

LEMMA 5.3. *Let* K *be a compact set* $\subset \mathbf{R}^n$, *and let* $K(t)$ *be the set of points at distance* $\leq t$ *from* K. *Assume that* C^L *is quasi-analytic. For arbitrary positive* δ, ρ, t *one can find* $r>0$ *independent of* δ *such that for every linear form* u *on* A *with*

$$|u(\varphi)| \leqq C|\varphi|_{L,r,K(t/2)}, \quad \varphi\in A,$$

for some C, *there is some* $v\in \mathcal{E}_L'(K)$ *with*

$$|\langle v-u,\varphi\rangle| \leqq \delta|\varphi|_{L,\rho,K(t)}, \quad \varphi\in A.$$

PROOF. Choose $\chi\in C_0^1(K(t))$ equal to 1 in $K(2t/3)$, and set with E_j defined as in the proof of Proposition 3.2

$$u_j = \chi(u*E_j); \quad u*E_j(z) = u(E_j(z-.)).$$

The proof of Proposition 3.2 gives for some $r>0$

$$|\varphi-E_j*(\chi\varphi)|_{L,r,K(t/2)} \leqq Cj^{-\frac{1}{2}}|\varphi|_{L,\rho,K(t)}, \quad \varphi\in A.$$

We fix j so that $C'j^{-\frac{1}{2}} < \delta/3$. It remains then to approximate the function $u_j\in C_0^\infty(K(t))$. Approximating $\langle u_j, \varphi\rangle$ by a Riemann sum we obtain a finite sum $\mu = \sum c_k\delta_{x_k}$ with $x_k\in K(t)$ such that

$$|\langle u_k, \varphi\rangle - \int d\mu| \leqq \delta|\varphi|_{L,\rho,K(t)}/3.$$

For every x_k we can find $y_k\in K$ with $|x_k-y_k|\leqq t$. If we note that the line segment between y_k and x_k belongs to $K(t)$ and apply Lemma 5.2 to the function $s\mapsto \varphi(y_k+s(x_k-y_k))$, it follows that we can find a finite sum v of derivatives of Dirac measures at the points y_k such that

$$|\int \varphi\, d\mu - \langle v,\varphi\rangle| \leqq \delta|\varphi|_{L,\rho,K(t)}/3.$$

Adding up these estimates, we have proved the lemma.

LEMMA 5.4. *Let the hypotheses of Theorem 5.1 be fulfilled, and let* X_j *be bounded open sets containing* K_j. *Then one can for* $r>0$ *find linear forms* u_j^r *on* A, $j=1,2$, *such that* $u=u_1^r+u_2^r$ *and*

(5.2) $$|u_j^r(\varphi)| \leqq C_r|\varphi|_{L,r,X_j};\ \varphi\in A;$$

(5.3) $$|\langle u_j^r - u_j^{r'},\ \varphi\rangle| \leqq C_{r,r'}|\varphi|_{L,r,\overline{X}_1\cap\overline{X}_2};\ \varphi\in A,\ 0<r'<r.$$

PROOF. It suffices to construct u_j^r for small $r>0$. Choose open sets Y_j and Z_j with $K_j\subset Y_j\subset\subset Z_j\subset\subset X_j$. We shall now follow the same steps as in the proof of Theorem 3.4.

1. Choose $\chi_\nu^j\in C_0^\infty(Z_j)$ so that $\chi_\nu^j=1$ in Y_j and

$$|D^\alpha\chi_\nu^j| \leqq (C_1\nu)^{|\alpha|},\quad |\alpha|\leqq\nu.$$

Set $\tilde\chi_\nu^1 = \chi_\nu^1$ and $\tilde\chi_\nu^2 = (1-\chi_\nu^1)\chi_\nu^2$, which means that

$$1-\tilde\chi_\nu^1-\tilde\chi_\nu^2 = (1-\chi_\nu^1)(1-\chi_\nu^2) = 0 \text{ in } Y_1\cup Y_2.$$

We have the same estimates for $\tilde\chi_\nu^j$ (with C_1 replaced by $2C_1$). Thus (3.6) remains valid for small r when χ_ν is replaced by $\tilde\chi_\nu^j$ and M is replaced by $|\varphi|_{L,r,X_j}$. With ψ_N defined as before, with sufficiently small δ independent of r, we set for polynomials φ

$$R_j^r\varphi(x) = \sum_0^\infty \psi_N(D)(\tilde\chi^j_{\nu(N,r)}\varphi)(x).$$

Corresponding to (3.8) we obtain for some $r'>0$

(5.4) $$|R_j^r\varphi|_{L,r',\mathbf{R}^n} \leqq C_4|\varphi|_{L,r,Z_j}.$$

Thus

$$w_j^r(\varphi) = u(R_j^r\varphi) = \sum_0^\infty u(\psi_N(D)(\tilde\chi^j_{\nu(N,r)}\varphi))$$

defines a linear form on A which is continuous for $|\ \ |_{L,r,Z_j}$.

2. Choose $\chi\in C_0^\infty(Z_1\cup Z_2)$ equal to 1 in $Y_1\cup Y_2$. Since we have $\tilde\chi_\nu^1+\tilde\chi_\nu^2=1$ in $Y_1\cup Y_2$, it follows that

$$v^r(\varphi) = \sum_0^\infty u(\psi_N(D)((\chi-\tilde\chi^1_{\nu(N,r)}-\tilde\chi^2_{\nu(N,r)})\varphi))$$

is continuous for the L^1 norm in $Z_1\cup Z_2$ and is therefore defined by a function $v^r\in L^\infty$ with support in $\overline{Z}_1\cup\overline{Z}_2$.

3. Since

$$w_1^r(\varphi)+w_2^r(\varphi)+v^r(\varphi) = \sum_0^\infty u(\psi_N(D)(\chi\varphi))$$

the end of the proof of Theorem 3.4 gives that $w_1^r+w_2^r+v^r=u$. Now

$v^r \in L^\infty$ and supp $v^r \subset \overline{X}_1 \cup \overline{X}_2$, so $v_j{}^r \in \mathcal{E}'(\overline{X}_j)$ and $v_1{}^r + v_2{}^r = v^r$ if $v_1{}^r = v^r$ in $\overline{X}_1$, $v_1{}^r = 0$ in $\complement\overline{X}_1$, while $v_2{}^r = 0$ in $\overline{X}_1$ and $v_2 = v$ in $\complement\overline{X}_1$. Thus $u_j{}^r = v_j{}^r + w_j{}^r$ is a linear form on A which is continuous for $|\ |_{L,r,X_j}$, and $u = u_1{}^r + u_2{}^r$.

4. What remains is to prove (5.3). Since $u_1{}^r - u_1{}^{r'} = u_2{}^{r'} - u_2{}^r$ we may take j=1. By definition

$$w_1{}^r - w_1{}^{r'} = \sum_0^\infty u(\psi_N(D)((\tilde{\chi}^1_{\nu(N,r)} - \tilde{\chi}^1_{\nu(N,r')})\varphi)).$$

The support of $\tilde{\chi}^1_{\nu(N,r)} - \tilde{\chi}^1_{\nu(N,r')}$ is contained in $Z_1 \setminus Y_1$. Now choose a cutoff function $f_N \in C_0^\infty(X_2)$ such that $f_N = 1$ in Z_2 and

$$|D^\alpha f_N| \leqq (C_1\nu)^{|\alpha|}, \quad |\alpha| \leqq \nu, \text{ if } \nu = \nu(N,r) \text{ or } \nu = \nu(N,r').$$

It follows from the proof of Lemma 2.2 that this is possible with a constant C_1 depending only on X_2 and Z_2. Set

$$w^{r,r'}(\varphi) = \sum_0^\infty u(\psi_N(D)(f_N(\tilde{\chi}^1_{\nu(N,r)} - \tilde{\chi}^1_{\nu(N,r')})\varphi)).$$

Each of the terms is of the form already discussed, so $w^{r,r'}$ is continuous for the norm $|\ |_{L,r,X_1 \cap X_2}$. Next consider

$$W^{r,r'}(\varphi) = \sum_0^\infty u(\psi_N(D)((1-f_N)(\tilde{\chi}^1_{\nu(N,r)} - \tilde{\chi}^1_{\nu(N,r')})\varphi)).$$

Since $(1-f_N)(\tilde{\chi}^1_{\nu(N,r)} - \tilde{\chi}^1_{\nu(N,r')})\varphi$ has support in $(\complement Z_2) \cap (Z_1 \setminus Y_1) \subset Z_1 \setminus (Y_1 \cup Y_2)$, it follows as in step 2 that $W^{r,r'}$ is a function in L^∞ with support there.

We split $v^r - v^{r'}$ in the same way, noting that

$$\langle v^r - v^{r'}, \varphi\rangle = \sum_0^\infty u(\psi_N(D)((\tilde{\chi}^1_{\nu(N,r')} + \tilde{\chi}^2_{\nu(N,r')} - \tilde{\chi}^1_{\nu(N,r)} - \tilde{\chi}^2_{\nu(N,r)})\varphi)).$$

Since $1-f_N$ vanishes in Z_2 the terms involving $\tilde{\chi}^2_\nu$ drop out in the term where we insert a factor $1-f_N$, so $v^r - v^{r'} = v^{r,r'} - W^{r,r'}$ where $v^{r,r'}$ has support in $\overline{X}_2$. Hence $W^{r,r'} + v_1{}^r - v_1{}^{r'} = 0$ outside $\overline{X}_2$. The support is therefore in $\overline{X}_1 \cap \overline{X}_2$, so

$$u_1{}^r - u_1{}^{r'} = w^{r,r'} + W^{r,r'} + v_1{}^r - v_1{}^{r'}$$

satisfies (5.3).

PROOF OF THEOREM 5.1. Define $K_j(t)$ as in Lemma 5.3. We shall first prove that for fixed $t>0$ one can find $u_j \in \mathcal{E}'_L(K_j(t))$ so that $u_1+u_2=u$. Set $K=K_1\cap K_2$ and choose t' so that $K_1(t')\cap K_2(t')\subset K(t/2)$, thus $t'\leqq t/2$. We choose a decreasing positive sequence $r_\nu \leqq 1/\nu$ so that Lemma 5.3 holds with $r=r_\nu$ and $\rho=1/\nu$. With X_j equal to the interior of $K_j(t')$ we define u_j^r by Lemma 5.4 and set $U_j^\nu = u_j^{r_\nu}$. Then

$$|U_j^\nu(\varphi)| \leqq C_\nu |\varphi|_{L,r_\nu,X_j}$$

and

$$|\langle U_1^{\nu+1} - U_1^\nu, \varphi\rangle| \leqq C_\nu |\varphi|_{L,r_\nu,K(t/2)}.$$

Hence it follows from Lemma 5.3 that we can find $v_\nu \in \mathcal{E}'_L(K)$ so that

$$|\langle U_1^{\nu+1} - U_1^\nu - v^\nu, \varphi\rangle| \leqq 2^{-\nu} |\varphi|_{L,1/\nu,K(t)}.$$

This implies that

$$u_1(\varphi) = U_1^1(\varphi) + \sum_1^\infty \langle U_1^{\nu+1} - U_1^\nu - v^\nu, \varphi\rangle =$$

$$= U_1^k(\varphi) + \sum_k^\infty \langle U_1^{\nu+1} - U_1^\nu - v^\nu, \varphi\rangle - \sum_1^{k-1} \langle v^\nu, \varphi\rangle$$

exists and is bounded with respect to $|\varphi|_{L,1/k,K_1(t)}$ for every k. Hence $u_1 \in \mathcal{E}'_L(K_1(t))$. Set $u_2 = u - u_1 = U_1^k + U_2^k - u_1$. Then

$$u_2(\varphi) = U_2^k(\varphi) - \sum_k^\infty \langle U_1^{\nu+1} - U_1^\nu - v^\nu, \varphi\rangle + \sum_1^{k-1} \langle v^\nu, \varphi\rangle$$

so we obtain in the same way that $u_2 \in \mathcal{E}'_L(K_2(t))$.

Changing notation we have for every $t>0$ found $u_j^t \in \mathcal{E}'_L(K_j(t))$ so that $u = u_1^t + u_2^t$. Thus

$$u_1^t - u_1^T = u_2^T - u_2^t \in \mathcal{E}'_L(K_1(T)\cap K_2(T)),\ t \leqq T.$$

When t and T are small we can use Lemma 5.3 to approximate this difference by elements in $\mathcal{E}'_L(K_1\cap K_2)$. The same argument as above then shows that $u=u_1+u_2$ for some $u_j \in \mathcal{E}'_L(K_j)$. (See also the proof of Theorem 5.6 below.) This ends the proof of Theorem 5.1.

The following reformulation of Theorem 5.1 will be useful in section 6.

COROLLARY 5.5. *Let* $u_j \in \mathcal{E}'_L(\mathbf{R}^n)$ *and let* X_1, X_2 *be open sets such that* $u_1 - u_2 = 0$ in $X_1 \cap X_2$. *If* C^L *is quasianalytic it follows* *that one can find* $u\in\mathcal{E}'_L(\mathbf{R}^n)$ *so that* $u-u_j=0$ *in* X_j *for* $j=1,2$ *and* $\operatorname{supp} u \subset \operatorname{supp} u_1 \cup \operatorname{supp} u_2$.

PROOF. By hypothesis $\operatorname{supp}(u_1-u_2)\subset K\cap(\complement X_1\cup\complement X_2)$ if $K=\operatorname{supp} u_1 \cup\operatorname{supp} u_2$. Hence Theorem 5.1 shows that one can find $v_j\in\mathcal{E}'_L(K\cap\complement X_j)$ so that $u_1-u_2=v_1-v_2$. Thus $u=u_1-v_1=u_2-v_2$ has the required properties.

Lemma 5.3 also gives an important completeness property:

THEOREM 5.6. *Let* K *be a compact set in* $\mathbf{R}^n$ *and let* C^L *be quasianalytic. If* $u_j\in\mathcal{E}'_L(\mathbf{R}^n)$, $j=1,2,\ldots$ *and for every neighborhood* X *of* K *we have*

$$u_j\in\mathcal{E}'_L(X),\quad j>J(X), \tag{5.5}$$

then one can choose $u\in\mathcal{E}'_L(\mathbf{R}^n)$ *so that for every such* X

$$u-\sum_{j\leqq J(X)} u_j\in\mathcal{E}'_L(X). \tag{5.6}$$

(5.6) *determines* u *uniquely modulo* $\mathcal{E}'_L(K)$.

PROOF. Let $u_j\in\mathcal{E}'_L(K(t(j)))$ where $K(t)$ is defined as in Lemma 5.3 and $t(j)\downarrow 0$. By Lemma 5.3 we can choose $v_j\in\mathcal{E}'_L(K)$ so that for all polynomials φ

$$|\langle u_j-v_j,\varphi\rangle| \leqq 2^{-j}|\varphi|_{L,1/j,K(t(j))}.$$

Hence

$$\langle u,\varphi\rangle = \sum_1^\infty \langle u_j-v_j,\varphi\rangle$$

is well defined, and

$$|\langle u - \sum_{j<k} \langle u_j-v_j,\varphi\rangle| \leqq 2^{1-k}|\varphi|_{L,1/k,K(t(k))}$$

for every k. Since $\sum_{j<k}(u_j-v_j)\in\mathcal{E}'_L(K(t(1)))$ we conclude that $u\in\mathcal{E}'_L(K(t(1)))$. Hence

$$u-\sum_{j<k}(u_j-v_j)\in\mathcal{E}'_L(K(t(k)))$$

for every k, which proves (5.6). The last statement is obvious.

6. *The spaces* $\mathcal{D}'_L(X)$. We define a presheaf on $\mathbf{R}^n$ by assigning to each open set $X\subset\mathbf{R}^n$ the quotient space $\mathcal{E}'_L(\mathbf{R}^n)/\mathcal{E}'_L(\complement X)$. The stalk at x of the corresponding sheaf $\mathcal{D}'_L$ is the quotient space

$$\mathcal{E}'_L(\mathbf{R}^n)/\{u\in\mathcal{E}'_L(\mathbf{R}^n),\ x\notin \operatorname{supp} u\}.$$

If $u\in\mathcal{E}'_L(\mathbf{R}^n)$ and X is any open neighborhood of x then we can by Theorem 4.1 or Theorem 5.1 find $u_1\in\mathcal{E}'_L(X)$ and $u_2\in\mathcal{E}'_L(\complement\{0\})$ such that $u=u_1+u_2$. Thus $x\notin \operatorname{supp} u_2$ which proves that the stalk of $\mathcal{D}'_L$ at x is also equal to

$$\mathcal{E}'_L(X)/\{u\in\mathcal{E}'_L(X),\ x\notin \operatorname{supp} u\}.$$

Now let $u\in\mathcal{D}'_L(X)$ be a section of the sheaf over an open set $X\subset\mathbf{R}^n$. This means that $X=\cup X_j$ where X_j are open and that for every j we have some $u_j\in\mathcal{E}'_L(X)$ defining u in X_j. Thus $u_j-u_k=0$ in $X_j\cap X_k$. We claim that for every open $Y\subset\subset X$ one can find $u_Y\in\mathcal{E}'_L(X)$ such that

$$X_j\cap Y\cap \operatorname{supp}\,(u_Y-u_j)=\emptyset \text{ for all } j. \tag{6.1}$$

In the non-quasianalytic case this follows if we take $u=\sum\varphi_j u_j$ where $\varphi_j\in C_0^L(X_j)$, only finitely many terms are non-zero, and $\sum\varphi_j=1$ in a neighborhood of $\overline{Y}$. In the quasianalytic case the statement follows by repeated use of Corollary 5.5. Thus we obtain the following description of $\mathcal{D}'_L(X)$:

THEOREM 6.1. *Let X be an open set in $\mathbf{R}^n$ and let $u\in\mathcal{D}'_L(X)$. Then one can find $v_j\in\mathcal{E}'_L(X)$ such that the supports are locally finite and for any $Y\subset\subset X$ we have $u=\sum v_j$ in Y, the sum taken over the terms with support intersecting Y. Conversely, every such sum defines an element in $\mathcal{D}'_L(X)$.*

PROOF. Choose an increasing sequence of relatively compact open sets Y_1, Y_2, ... with union X, and for every j choose $u_j\in\mathcal{E}'_L(X)$ with $u_j=u$ in Y_j. Then the statement is valid with $v_1=u_1$ and $v_j=u_j-u_{j-1}$ for $j\neq 1$.

If the class C^L is non-quasianalytic and K is a compact subset of X, we can define

$$\langle u,\ \varphi\rangle = \langle v,\ \varphi\rangle;\ \varphi\in C_0^L(K);$$

where $v\in\mathcal{E}'_L(X)$ defines u in a neighborhood of K. The definition is clearly independent of the choice of v. From (3.1) we obtain

$$|\langle u,\ \varphi\rangle| \leqq C_{r,K}|\varphi|_{L,r,K},\ \varphi\in C_0^L(K). \tag{6.2}$$

Conversely, assume that we have a linear form u on $C_0^L(X)$ satisfying (6.2). If $\chi \in C_0^L(X)$ then

$$\langle \chi u, \varphi \rangle = \langle u, \chi\varphi \rangle$$

defines $\chi u \in \mathcal{E}'_L$ with support in supp χ. If $\chi_j \in C_0^L(X)$, $\sum \chi_j = 1$, and the supports are locally finite in X, then $\sum (\chi_j u)$ defines a distribution $U \in \mathcal{D}'_L(X)$, and it is clear that U gives rise to the linear form u on $C_0^L(X)$ which we started from. Thus we can identify $\mathcal{D}'_L(X)$ with the space of linear forms on $C_0^L(X)$ satisfying (6.2) for every compact set $K \subset X$ and every $r>0$. This is just as in the case of Schwartz distributions.

In the quasianalytic case we get another simple description of $\mathcal{D}'_L(X)$:

THEOREM 6.2. *If X is a bounded open set in $\mathbf{R}^n$ and C^L is quasianalytic, then $\mathcal{D}'_L(X)$ is isomorphic to $\mathcal{E}'_L(\overline{X})/\mathcal{E}'_L(\partial X)$.*

PROOF. This follows if we apply Theorem 5.6 to the series in Theorem 6.1.

The meaning of the theorem is that the distribution which is equal to u in X and 0 in $\complement\overline{X}$ can be extended to a distribution in the whole space. This remains true for any open set:

THEOREM 6.2'. *If X is any open set in $\mathbf{R}^n$ and C^L is quasianalytic, then every $u \in \mathcal{D}'_L(X)$ is the restriction of some $U \in \mathcal{D}'_L(\mathbf{R}^n)$.*

PROOF. Using Theorem 6.1 we can write $u = \sum v_j$ with $v_j \in \mathcal{E}'_L(X \setminus K_j)$ for a sequence of compact sets $K_j \subset X$ containing every compact subset of X for large j. Repeated use of Theorem 5.1 gives

$$v_j = \sum_{k=0}^{\infty} u_{jk}; \text{ supp } u_{jk} \subset \{x \in X \setminus K_j;\ k \leqq |x| \leqq k+1\};$$

where the sum is actually finite. If we apply Theorem 5.6 to $\sum_j u_{jk}$ we obtain $u_k \in \mathcal{E}'_L$ with support in $\{x \in \overline{X};\ k \leqq |x| \leqq k+1\}$ such that the support of $u_k - \sum_{j<J} u_{jk}$ does not meet K_J for any J. Hence $U = \sum u_k$ is an element in $\mathcal{D}'_L(\mathbf{R}^n)$ with support in $\overline{X}$ which is equal to u in X. The proof is complete.

Theorem 6.2' means that the distribution sheaf is flabby. Summing up, we have proved:

COROLLARY 6.3. *The following properties are equivalent:*

(i) $\mathcal{D}'_L$ *is flabby.*

(ii) <u>If</u> $u \in \mathcal{E}'_L(K_1 \cup K_2)$ <u>where</u> K_1 <u>and</u> K_2 <u>are compact subsets of</u> $\mathbf{R}^n$, <u>then</u> $u=u_1+u_2$ <u>for some</u> $u_j \in \mathcal{E}'_L(K_j)$.

(iii) C^L <u>is quasianalytic.</u>

PROOF. (iii) $\Rightarrow$ (i) by Theorem 6.2', and (ii) $\Rightarrow$ (iii) by Theorem 4.2. To prove that (i) $\Rightarrow$ (ii) we must for given $u \in \mathcal{E}'_L(K_1 \cup K_2)$ find $u_1 \in \mathcal{E}'_L(K_1)$ so that $u-u_1 \in \mathcal{E}'_L(K_2)$. This means that $u_1=0$ in $\complement K_1$ and that $u_1=u$ in $\complement K_2$. Now $\complement K_1 \cap \complement K_2 = \complement(K_1 \cup K_2)$, and by hypothesis $u=0$ there. Thus we have a well defined distribution $u_1 \in \mathcal{D}'_L(\complement K_1 \cup \complement K_2)$ and by condition (i) it can be extended to $\mathbf{R}^n$. The proof is complete.

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