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Circular Symmetry and Stationary-Phase Approximation

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§1. Introduction

Stationary-phase approximation applies to the computation of oscillatory integrals of the form $\int e^{itf(x)}dx$ and it asserts that for large t the dominant terms come from the critical points of the phase function $f(x)$. Many interesting examples are known where this approximation actually yields the exact answer. Recently a simple general symmetry principle has been found by Duistermaat and Heckman [6] which gives a geometrical explanation for such exactness. In the first part of this lecture I will explain the result of Duistermaat-Heckman. I will go on to outline a brilliant observation of the physicist E. Witten suggesting that an infinite-dimensional version of this result should lead rather directly to the index theorem for the Dirac operator. Such interactions between mathematics and physics played a prominent part in the work of Laurent Schwartz.

§2. The Duistermaat-Heckman Formula

We start with the following general situation. Let M be a compact symplectic $2n$-dimensional manifold with fundamental 2-form ω and its associated Liouville volume form $\frac{\omega^n}{n!}$. Assume that we have an action of the circle S^1 on M preserving ω with associated Hamiltonian function H. This means that the 1-form dH corresponds, under the duality defined by ω, to the vector field which generates the circle action on M. For simplicity let us also assume, for the moment, that the circle action has only isolated fixed points P. At each such point the circle action on the tangent space is described by n plane rotations, characterized by integers $m_j^P (j=1,\ldots,n)$. If we fix an orientation of M then we shall pick the signs of the m_j^P to be consistent with the orientation of M. We are now in a position to state the Duistermaat-Heckman exact integration formula:

$$(2.1) \qquad \int_M e^{-tH} \frac{\omega^n}{n!} = \sum_P \frac{e^{-tH(P)}}{t^n(\prod_j m_j^P)} .$$

In this formula t is a real or complex parameter. If it is taken purely imaginary the integral over M is oscillatory, the points P are the stationary points of H and the right-hand side is given by stationary-phase approximation. Thus (2.1) asserts that, when the phase function H is the Hamiltonian of a circular symmetry, stationary-phase approximation is exact.

By taking Fourier transforms it is easy to see that (2.1) is essentially equivalent to the assertion that the direct image of the Liouville measure under the map $H : M \to R$ is <u>piece-wise polynomial</u>. General considerations show that it must be piece-wise smooth with breaks at the critical values $H(P)$, but the exactness of (2.1) produces the piece-wise polynomial.

The proof of [6] consists in showing that, for non-critical values of $\lambda \in R$, the induced symplectic form ω_λ on the reduced manifold $M_\lambda = H^{-1}(\lambda)/S^1$, varies linearly in λ as a <u>cohomology class</u>. This shows that (2.1) is essentially a cohomological theorem - in the same way as Cauchy's residue theorem is cohomological In fact an alternative derivation of (2.1) is given in [2] which emphasizes more clearly this cohomological aspect.

Although we have, for simplicity, stated only the simplest form of the Duistermaat-Heckman result, the generalizations are important and not too difficult. In the first place the circle S^1 can be replaced by a torus. Thus the vector field dual to the Hamiltonian need not be periodic - it is enough if it is almost periodic (with finitely many 'periods'). Also the assumption that the fixed points are isolated can be dropped. In general the fixed points of a torus action consist of sub-manifolds. The right-hand side of (2.1) now becomes a sum over the components X of the fixed-point set, and the contribution of X is an integral of the form

$$(2.2) \qquad \int_X \frac{e^{-tH(X)} e^{\omega}}{\prod_{j=1}^{k} (tm_j - i\alpha_j)} .$$

Here $H(X)$ is the constant value taken by H on X. The m_j are the rotation numbers normal to X ($2k = \operatorname{codim} X$), and the α_j are symbolic differential 2-forms (or cohomology classes) so that the total Chern class of the normal bundle N to X is given by

$$\prod_{j=1}^{k} (1 + \alpha_j) .$$

Note that N has a complex structure once we orient the rotation planes of the action of S^1. The expression e^{ω} is to be expanded as a formal power series in ω, then the denominator is to be expanded by the binomial theorem in powers of the α_j, and finally we have to pick the differential forms of degree $2n - 2k = \dim X$ to evaluate on X. This form of the fixed point contribution is given correctly in [7] and also in [2].

Finally we can allow the symplectic form ω to become degenerate, although now we must be more careful about orientation. Typlically $\omega^n = o$ on a submanifold $Y \subset M$ of codimension 1 and the dual cohomology class $[Y] \in H^1(M, Z_2)$ represents the first Steifel-Whitney class of M and is the obstruction to orientability.

Thus we must now assume that this class is zero so that M is orientable. This can also be expressed analytically by picking a Riemannian metric ρ on M with Riemannian volume $d\rho$. This is related to the symplectic volume by the formula

$$\frac{\omega^n}{n!} = \mathrm{Pf}(\omega_\rho)d\rho \tag{2.3}$$

where ω_ρ is the skew-adjoint endomorphism of the tangent space associated to ω by the metric ρ, and Pf is the Pfaffian. Note that Pf is invariant under $SO(2n)$ but changes sign under $O(2n)$, so that $\mathrm{Pf}(\omega_\rho)$ is a function not on M but on its orientable double cover $\tilde{M}$. The orientability of M (or triviality of $\tilde{M} \to M$) is therefore equivalent to being able to define $\mathrm{Pf}(\omega_\rho)$ as a smooth function on M. Recalling that the Pfaffian is defined as the square root of the determinant (of a skew-adjoint endomorphism) we see that M is orientable if and only if $\det(\omega_\rho)$ has a smooth square-root. This remark will be useful when we come to the infinite-dimensional case considered by Witten.

§3. The loop space

Witten's idea was to apply the Duistermaat-Heckman theorem to the infinite-dimensional manifold

$$\mathcal{M} = \mathrm{Map}(S^1, X)$$

of smooth maps of the circle into a finite-dimensional compact orientable manifold X. It is well-known that Wiener integration on such a manifold of loops is related to the heat equation on X (once we have chosen a Riemannian metric on X), so one might hope to get interesting results about X by such a procedure.

Let us therefore consider the geometry of M. I shall discuss this quite heuristically so that analytical questions will be ignored.

A point of M is, by definition, a smooth map $\phi : S^1 \to X$, and the tangent space T_ϕ to M at ϕ can be identified with the space of sections of the vector bundle $\phi^*(TX)$, the tangent bundle TX of X pulled back to S^1 by ϕ. The metric on X defines a metric on $\phi^*(TX)$ and hence, by integration over S^1, we get an inner product on the space of sections. This defines a pre-Hilbert space structure on T_ϕ. Next we introduce the Levi-Civita connection ∇ on X. This induces a connection on the bundle $\phi^*(TX)$ and hence (evaluating along the tangent field $\frac{d}{dt}$ of S^1) a covariant derivative operator ∇_ϕ. This is a skew-adjoint operator on the space of sections T_ϕ and hence, using the inner product, it defines a skew bilinear form on T_ϕ. As we now vary the point $\phi \in M$ we get an exterior differential 2-form ω on M. One can now verify the following:

<u>Lemma 1</u>. ω <u>is closed</u>.

<u>Remark</u>. This lemma depends on the fact that we took the Levi-Civita connection. If we had taken an arbitrary orthogonal connection on X then the corresponding ω would not have been closed. In fact $d\omega$ is then the integral over S^1 of the skew part of the torsion of the connection. There is in fact another definition of ω which explains more directly why it is closed. We shall return to this later in §5.

Clearly ω is degenerate precisely at those ϕ for which ∇_ϕ has a zero-eigenvalue, i.e. a tangent vector to X which is covariant constant along the loop ϕ. For example ω is degenerate at any closed geodesic ϕ. As in the finite-dimensional case we may call M 'orientable' if $\mathrm{Det}\,\nabla_\phi$ has a smooth square-root. Of course, for this to make sense, we have first to define the determinant of the (ordinary) differential operator ∇_ϕ. However it is well-known how to make sense or 'regularize' such a determinant. A very general procedure (valid for elliptic partial differential operators) is to use Zeta-function methods as in [11]. However, in the present case, of an ordinary differential operator, more elementary methods comparing ∇_ϕ with ∇_o (for a constant loop o) also work.

If T_ϕ denotes the parallel transport round ϕ, one finds the following simple formula for computing the regularized determinant of ∇_ϕ:

<u>LEMMA 2</u>. $\mathrm{Det}\,\nabla_\phi = \det\,(1 - T_\phi)$.

<u>Remark</u>. We have written a capital "D" in the first determinant and a small 'd' in the second to emphasize that the first is that of an operator while the second is that of a finite-dimensional matrix.

The verification of Lemma 2 depends on the fact that we can explicitly compute all the eigenvalues of ∇_ϕ. When $\dim X$ is odd both sides of Lemma 2 vanish, so assume $\dim X = 2m$ and let the eigenvalues of T_ϕ be $\exp(\pm 2\pi i\alpha_j)$, $j=1,\ldots,m$. Then the eigenvalues of ∇_ϕ are

$$\pm in \pm i\alpha_j,\quad n=0,1,2,\ldots;\quad j=2,\ldots,m.$$

The determinant of ∇_ϕ is formally therefore the infinite product

$$\left\{\prod_{n=1}^{\infty} n^2\right\}^{2m} \prod_{j=1}^{m} \alpha_j^2 \prod_{n=1}^{\infty}\left(1 - \frac{\alpha_j^2}{n^2}\right)^2$$

$$= \left\{\prod_{n=1}^{\infty} n^2\right\}^{2m} \prod_{j=1}^{m}\left(\frac{\mathrm{Sin}\,\pi\alpha_j}{\pi}\right)^2 .$$

This is regularized by "renormalizing" the divergent first factor to be a finite constant C^{2m}. In fact the zeta-function approach of [11] to infinite determinants gives the value

$$C = \exp(-2\zeta'(0))$$

where $\zeta(s)$ is the Riemann zeta-function. On the other hand

$$\det(1 - T_\phi) = \prod_{j=1}^{m} (1 - e^{2\pi\alpha_j})(1 - e^{-2\pi i\alpha_j})$$

$$= \prod_{j=1}^{m} (2\mathrm{Sin}\,\pi\alpha_j)^2$$

which leads to Lemma 2 in virtue of the fact that

$$\zeta'(0) = -\tfrac{1}{2}\log 2\pi .$$

To understand whether M is 'orientable' we therefore have to see under what circumstances the function $\det(1 - T_\phi)$ on M has a smooth square-root. Now consider $\det(1 - T)$ as a function of $T \in SO(2m)$. This function does not have a smooth square-root: in fact one has to go to the double cover $\mathrm{Spin}(2m)$ to extract this root. Homologically this reflects the fact that the subspace $Y \subset SO(2m)$ consisting of matrices with 1 as an eigenvalue represents the generator of $H^1(SO(2m), Z_2)$. More explicitly in terms of characters we have the following identity:

LEMMA 3. $\sqrt{\det(1 - T)} = \chi(S_T^+) - \chi(S_T^-)$ *as characters of* $\mathrm{Spin}(2m)$,

<u>where</u> $S^{\pm}$ <u>are the two half-spin representations</u>.

Returning to our loop manifold $M = \mathrm{Map}(S^1, X)$ we see that if X is a <u>spin manifold</u> so that all $T_\phi \in \mathrm{Spin}\ (2m)$ then, from Lemmas 2 and 3, $\mathrm{Det}\ \nabla_\phi$ has a smooth square-root so that M is '<u>orientable</u>'.

<u>Remark</u>. In purely homological terms the 'orientability' of M can be understood as follows. We have the natural evaluation map

$$f: S^1 \times M \to X.$$

Pulling back by f^* and then integrating over S^1 induces a homomorphism

$$\alpha : H^2(X, Z_2) \to H^1(M, Z_2) .$$

The image of the second Stiefel-Whitney class ω_2 of X is then the obstruction to orientability of M. In particular

$$(*) \qquad X \text{ Spin} \Longrightarrow M \text{ orientable}$$

as we have just seen by a more explicit argument. Note also that the identification of the class $\alpha(\omega_2)$ with the class arising from trying to extract the square root of $\mathrm{Det}\ \nabla_\rho$, which we deduced from Lemmas 2 and 3, is essentially a special case of the index theorem for families of real elliptic operators [5]. Finally, if X is simply-connected, one can show that α is injective so that $(*)$ is an equivalence.

Although our closed 2-form ω on M is degenerate we can still define the Hamiltonian function H associated to the obvious action of the circle on M (rotating loops). Recall now that the <u>Energy</u> of a loop ϕ is defined by

$$E(\phi) = \tfrac{1}{2}\int_{S^1} |d\phi|^2 .$$

Computing the derivative of E in the direction of a tangent vector $\xi \in T_\phi$ we get

$$(dE,\xi) = \int_{S^1} \langle \frac{d\phi}{dt}, \nabla_\phi \xi \rangle$$

which establishes

LEMMA 4. $H = E$.

We have now found all the geometrical ingredients on M in order to formally investigate the infinite-dimensional version of the Duistermaat-Heckman theorem. In the next section we shall, following Witten, see how this is to be carried out.

§4. Relation to the index of the Dirac operator

We begin by recalling that the fundamental solution of the heat equation on the Riemannian manifold X can be described by a path integral. In particular, for the trace, one gets

$$(4.1)\qquad \mathrm{Tr}e^{2\pi\tau\Delta} = \int_M e^{-tE(\phi)}d\phi$$

where $d\phi$ denotes the (formal part of) Wiener measure on M, and Δ is the Laplace operator on X: the variable $\tau = t^{-1}$ arises by reparametrizing path length.

Suppose now we were to believe, naively, that we could apply the Duistermaat-Heckman theorem to the action of S^1 on M. The fixed-points of the action are just the constant point-loops and so are parametrized by $X \subset M$. We would then get a formula for the integral of e^{-tH} over M in terms of some explicit integral (as in (2.2)) over X. In view of Lemma 4 and (4.1) this would give a formula for the trace of the heat kernel as an explicit integral over X. Of course this trace is given by an integral over X but this integral is very global in nature while the formula of the type (2.2) is inherently local. Thus the Duistermaat-Heckman formula would appear to fail, since it would lead to a result which is much too strong and known to be false.

Witten points out however that this argument is based on the implicit assumption that the Wiener measure $d\phi$ is the same as the 'Liouville measure' of the symplectic manifold M. Now even in finite dimensions we have to be careful to distinguish these measures. Thus if M is a $2n$-dimensional symplectic manifold with a Riemannian metric ρ we have two different natural measures on M, the Liouville measure $\frac{\omega^n}{n!}$ and the Riemannian measure $d\rho$. These differ by the Pfaffian of the skew-adjoint operator ω_ρ as expressed by (2.3). This continues to hold when ω can degenerate so long as M is orientable so that $Pf(\omega_\rho)$ is a well-defined smooth function. Returning now to the infinite-dimensional case of our loop space we recognize that the Wiener measure $d\phi$ should be viewed as the 'Riemannian' measure, so that the 'Liouville' measure should be

$$P_f(\nabla_\phi)\; d\phi$$

where Pf is the regularized Pfaffian. Using Lemmas 2 and 3 this can be rewritten as

$$\{\operatorname{Tr} S^+(T_\phi) - \operatorname{Tr} S^-(T_\phi)\} d\phi$$

where $S^\pm$ denote the two half-spin representations of $\mathrm{Spin}\,(2m)$.

Thus, according to Witten, the correct analogue of the Duistermaat-Heckman integral over M is

$$(4.2)\qquad \int_M e^{-tE(\phi)}\{\mathrm{Tr}\ S^+(T_\phi) - \mathrm{Tr}\ S^-(T_\phi)\}d\phi\ ,$$

and it is this which should be given by the explicit formula, of the type (2.2), over X.

Before going on to examine this integral over X we shall make a couple of observations about (2.2). In the first place, the appearance of Pfaffian factors is well-known to physicists and is interpreted as the result of a "Fermionic" integration. Thus Witten interprets (4.2) as computing a "super-symmetric" trace. For an explanation of this idea the reader should consult [12], although that treats the de Rham complex rather than the Dirac operator.

The second observation about (4.2) is that formally it is what we would expect to replace (4.1) in computing

$$(4.3)\qquad \mathrm{Tr}\ e^{-2\pi\tau\Delta^+} - \mathrm{Tr}\ e^{-2\pi\tau\Delta^-}$$

where $\Delta^{\pm}$ are Laplace-type operators acting, not on scalars, but on the vector bundles S^+ and S^- of spinors on X. Next we recall that the Dirac operator acts on spinors, interchanging S^+ and S^-

$$D : S^+ \to S^-$$

and one can define the two Dirac Laplacians Δ^+ and Δ^- by

$$(4.4)\qquad \Delta^+ = D^*D \qquad \Delta^- = DD^*\ .$$

If we assume that these are the operators in (4.3) then, by an elementary and well-known argument all non-zero eigenvalues cancel, and (4.3) reduces to the spinor index:

(4.5) index D = dim (Ker D) - dim(Ker D*).

Thus (4.2) should give index D, for all t, and the integral on X which is supposed to generalize the Duistermaat-Heckman theorem will give us an explicit formula for this index.

We turn now to computing this integral formula on X. The first thing we need is to analyze the normal bundle of $X \subset M$. At any point $x \in X$ the tangent space to M consists of all functions on S^1 with values in T_x. The constant functions give the directions along X. Thus the normal vectors correspond to Fourier series (with values in T_x) having no constant term. Hence the normal bundle N to X in M can be decomposed as an infinite direct sum

$$N = T_1 \oplus T_2 \oplus \ldots$$

where each T_p is the complexified tangent bundle T of X on which the circle S^1 acts with rotation number p. Now the Chern class of the complexification of T can be factorized symbolically as

$$\prod_{j=1}^{m} (1 + \alpha_j)(1 - \alpha_j) \; .$$

Hence the denominator in (2.2), with $t = 1$, becomes

$$\prod_{j=1}^{m} \prod_{p=1}^{\infty} (p^2 + \alpha_j^2)$$

and this is formally

$$\left\{ \prod_{p=1}^{\infty} p^2 \right\}^m \prod_{j=1}^{m} \frac{\operatorname{Sinh} \pi\alpha_j}{\pi\alpha_j} \; .$$

Replacing the first divergent factor as before by its renormalized value $(2\pi)^m$ we get

$$(4.6) \qquad (2\pi)^m \prod_{j=1}^{m} \frac{\sinh \pi\alpha_j}{\pi\alpha_j}$$

On the other hand since point loops have zero energy, $H(X)=0$, and $\omega = 0$ on X because the constants are null-vectors for when ϕ is a constant loop. Hence the numerator in (2.2) is just 1. Thus we need to pick out the term from the expansion of the inverse of (4.6) of order m in the α_j, and this is the same as that of

$$(4.7) \qquad \prod_{j=1}^{m} \frac{\alpha_j/2}{\sinh \alpha_j/2} .$$

The integral of (4.5) over X is, by definition, denoted traditionally by $\hat{A}(X)$: it is a combination of Pontrjagin numbers. Finally therefore we see that the Duistermaat-Heckman formula, for the infinite-dimensional loop space M, ought to reduce to

$$\text{index } D = \hat{A}(X)$$

which is the index theorem [3] [4] for the Dirac operator.

Remark. We put $t = 1$ because the index was independent of t. This checks with the behaviour of the fixed-point contribution, provided we interpret $\prod_{p=1}^{\infty} t$ as $t^{\zeta(0)}$ where $\zeta(s)$ is the Riemann zeta function. Since $\zeta(0) = -\frac{1}{2}$ we get a total factor of t^m and this cancels with a factor t^{-m} arising from replacing α_j by α_j/t.

§5. Comments

Of course the discussion in §4 leading to the index formula of the Diract operator was purely heuristic. In particular it is not clear why we should use the Dirac Laplacians of (4.4) to interpret the path integral (4.2). There is always an ambiguity about lower order terms when one quantizes a classical system. For example one could envisage adding a constant c to the operators Δ^+ and Δ^-. Clearly this would multiply the difference of the traces in (4.3) by $\exp(-2\pi c\tau)$ so that the new expression would not be independent of τ. Thus the Dirac Laplacians appear to be distinguished in this respect: this is an aspect of the 'supersymmetry' expressed by (4.4).

More fundamentally the proof of [6] and also its homological variant given in [2] does not, as it stands, extend to infinite-dimensions. Naturally any such extension has to be compatible with the various renormalizations of infinite determinants involved.

Witten's supersymmetric treatment has been further elaborated by Alvarez-Gaumé [1] and while this claims to yield a proof of the index theorem the arguments are of a type more familiar in the physics literature. A more direct mathematical account has been given by Getzler [8] in which the presentation is purely classical but algebraic features of the supersymmetry have been exploited.

The proof of the index theorem along these lines is basically very close to the heat equation proof given in [3]. There is however one noteworthy difference. In the supersymmetric approach as developed in [8], the explicit local form of the $\hat{A}$-integrand appears directly from the algebra while in [3] an indirect invariance theory argument is given. It would be instructive to compare these two treatments more carefully. The direct approach in [8] is perhaps closer to the original direct argument of Patodi [9].

At this point it may perhaps be appropriate to add a few remarks concerning the (degenerate) symplectic structure we have used on the loop space M. The 2-form ω of §3 has an alternative description as the differential of a 1-form

(5.1) $\omega = d\theta$

where θ is the 1-form dual (via the metric) to the vector field given by the action of the circle. The simplest way to verify (5.1) (which of course implies Lemma 1) is to embed X isometrically in a Euclidean space R^N, then $M(X) \subset M(R^N)$ and it is sufficient to verify (5.1) for R^N which is elementary. Now the symplectic structure given by the 2-form in (5.1) can equivalently be described as that induced from the canonical symplectic structure on the cotangent bundle T^*M via the embedding given by the cross-section

(5.2) $\theta : M \to T^*M.$

Note that T^*M can also be described as the function space

$\mathrm{Map}(S^1, T^*X).$

Using the Liouville measure on M, rather than the Riemannian measure means essentially that we are integrating over the submanifold $\theta(M) \subset T^*M$. This fits in with the Fermionic point of view, since the Fermions should be thought of as in the fibre direction of T^*M.

Finally it should be pointed out that there are other infinite-dimensional examples in which the Duistermaat-Heckman principle appears to hold, but which are not supersymmetric. A notable example is given by the based loops space ΩG of a compact Lie group G. This is an infinite-dimensional Kähler manifold [10] so that in this case the Riemannian measure coincides with the Liouville measure. There is thus no Pfaffian and no Fermionic integration. The fixed points of the circle action now correspond to the closed geodesics or the closed one-parameter subgroups, and explicit calculations appear to check with the Duistermaat-Heckman formula.

It appears therefore that the task of fully understanding the infinite-dimensional case remains as a challenging problem.

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