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JØRGEN HOFFMANN-JØRGENSEN

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THE LAW OF LARGE NUMBERS FOR NON-MEASURABLE AND NON-SEPARABLE RANDOM ELEMENTS

J. Hoffmann-Jørgensen

1. Introduction. The law of large numbers has been in the center of probability ever since it was discovered by James Bernoulli around 1695 (published in 1713 in "Ars Conjectandi"). Lately it has been generalized to random variables taking values in a Banach space, see [1], [4], [5] and [6]. However in these papers it is assumed, that the random variables are measurable and separably valued, two conditions which, weird as it may sound, are not fulfilled in the first and most natural example of an infinitely dimensional law of large number, viz. the Glivenko-Cantelli theorem, see [8, p.20] or [2, p.261].

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent identically distributed real random variable with distribution function $F(t) = P(\xi_n \leq t)$. Let F_n be the empirical distribution function based on $\xi_1, \ldots, \xi_n$, i.e.

$$F_n(t) = \frac{1}{n} \sum_{j=1}^{n} 1_{\{\xi_j \leq t\}} \tag{1.1}$$

Then the Glivenko-Cantelli theorem states that

$$\sup_t |F_n(t) - F(t)| \underset{n\to\infty}{\to} 0 \quad \text{a.s.} \tag{1.2}$$

Let $B(\mathbb{R})$ be the set of all bounded real valued function on $\mathbb{R}$ with its usual sup-norm:

$$\|f\|_\infty = \sup_t |f(t)|$$

Then $(B(\mathbb{R}), \|\cdot\|_\infty)$ is a Banach space, and if

$$X_n(\omega,t) = 1_{\{\xi_n \leq t\}}(\omega) \quad \text{and} \quad X_n(\omega) = X_n(\omega,\cdot)$$

Then X_n is a random variable with values in $B(\mathbb{R})$, and the Glivenko-Cantelli theorem just states, that the sequence $\{X_n\}$ satisfies the law of large numbers in $B(\mathbb{R})$, i.e. that we have

$$\frac{1}{n}\sum_{j=1}^{n} X_n \to F \quad \text{a.s. in} \quad (B(\mathbb{R}), \|\cdot\|_\infty)$$

However X_n is neither measurable with respect to the Borel σ-algebra on $B(\mathbb{R})$, nor is it separably valued.

This example shows, that the general Banach space versions of the strong law of large number are too special and too poor to cover the first and most natural example of an infinite dimensional strong law of large numbers. In this paper I shall prove an infinite dimensional version of the strong law of large numbers, which neither assumes measurability nor separability of the random vectors, and which covers the Glivenko-Cantelli theorem as well as many other uniform laws of large numbers for stochastic processes.

2. The general case.

In all of this section we let $(S,\mathcal{S},\mu)$ denote a probability space and $(B,\|\cdot\|)$ a Banach space with dual space $(B',\|\cdot\|)$ and second dual $(B'',\|\cdot\|)$. As usual we shall consider B as a closed subspace of B''.

Let $f: S \to B$ be a function. Then we say, that f is μ-measurable, if f is μ-measurable when B has its Borel σ-algebra. We say that f is weakly μ-measurable (respectively weakly μ-integrable), if $x'(f(\cdot))$ is μ-measurable (respectively μ-integrable) for all $x' \in B'$. If f is weakly μ-integrable then we define its mean:

$$Ef = \int_S f d\mu$$

to be the linear functional on B' defined by

$$(Ef)(x') = \int_S x'(f(s))\mu(ds) \quad \forall x' \in B'$$

It is wellknown that $Ef \in B''$. If f is weakly μ-integrable and $Ef \in B$, then we say that f is Gelfand μ-integrable. We say that f is Bochner μ-measurable, if f is μ-measurable and $f(S \setminus N)$ is separable in $(B, \|\cdot\|)$ for some μ-nullset $N \in \mathcal{S}$. Finally we say that f is Bochner μ-integrable, if f is Bochner μ-measurable and $\|f(\cdot)\|$ is μ-integrable. And we shall consider the following four function spaces

$$L^1_w(\mu,B) = \{f: S \to B \mid f \text{ is weakly } \mu\text{-integrable}\}$$

$$L^1_G(\mu,B) = \{f: S \to B \mid f \text{ is Gelfand } \mu\text{-integrable}\}$$

$$L^1(\mu,B) = \{f: S \to B \mid f \text{ is Bochner } \mu\text{-integrable}\}$$

$$L^1_*(\mu,B) = \{f: S \to B \mid \int^* \|f(s)\| \mu(ds) < \infty\}$$

It is wellknown that $L^1(\mu,B) \subseteq L^1_G(\mu,B) \subseteq L^1_w(\mu,B)$ and that the integral above coincides with the usual Bochner integral on $L^1(\mu,B)$ (see [3] p. 112 and p. 149).

As we shall work with non-measurable functions we shall use a few concepts concerning non-measurable sets and functions. Let $(\Omega,\mathcal{F},P)$ be a probability space, then P^* and P_* denotes <u>the outer</u> and <u>inner</u> P-<u>measure</u> and $\int^* f dP$ and $\int_* f dP$ denotes <u>the upper</u> and <u>lower</u> P-<u>integrals of</u> f, whenever f is an arbitrary map from Ω into $\overline{\mathbb{R}} = [-\infty,\infty]$. And if f is an arbitrary map from Ω in $\overline{\mathbb{R}}$, then f_* and f^* denotes the <u>lower</u> and <u>upper</u> P-<u>envelopes of</u> f, i.e.

(2.1) $\quad f_*$ and f^* are measurable: $(\Omega,\mathcal{F}) \to \overline{\mathbb{R}}$

(2.2) $\quad f_*(\omega) \leq f(\omega) \leq f^*(\omega) \quad \forall\, \omega \in \Omega$

(2.3) $\quad P_*(f_* < g \leq f) = P_*(f \leq g < f^*) = 0 \quad \forall$ P-measurable functions $g:\Omega \to \overline{\mathbb{R}}$

If ξ is a map from Ω into a measurable space $(M,\mathcal{B})$ we say that ξ is P-<u>perfect</u> if ξ is P-measurable and

(2.4) $\quad (P_\xi)_*(A) = P_*(\xi^{-1}(A)) \quad \forall\, A \subseteq M$

where P_ξ is <u>the distribution law</u> of ξ on $(M,\mathcal{B})$. It is easily checked that (2.4) is equivalent to either of the following three conditions

(2.5) $\quad \forall\, F \in \mathcal{F}\ \exists\, B \in \mathcal{B} : B \subseteq \xi(F)$ and $P(F \setminus \xi^{-1}(B)) = 0$

(2.6) $\quad P(F \mid \xi = x) = 0$ for P_ξ-a.a. $x \in M \setminus \xi(F)$, $\forall\, F \in \mathcal{F}$

(2.7) $\quad \int^* f \circ \xi \, dP = \int^* f \, dP_\xi \quad \forall\, f : M \to \overline{\mathbb{R}}$

Moreover the composition of perfect maps are perfect, i.e.

(2.8) If $\xi:\Omega \to (M,\mathcal{B})$ is P-perfect and $\eta:M \to (L,\mathcal{A})$ is P_ξ-perfect, then $\eta\circ\xi$ is P-perfect.

Let $LLN(\mu,B)$ denote the set of all function $f:S \to B$, which satisfies the following version of <u>the strong law of large numbers</u>:

$$\exists a \in B: \lim_{n\to\infty} \|a-\frac{1}{n}\sum_{j=1}^{n} f(s_j)\| = 0 \quad \text{for } \mu^\infty\text{-a.a. } (s_j) \in S^\infty \tag{2.9}$$

where $(S^\infty, \mathcal{S}^\infty, \mu^\infty)$ is the countable product of $(S,\mathcal{S},\mu)$ with itself. Notice, that we do not assume any measurability or separability of f, but of course a.s. convergence makes good sense no matter whether the functions are measurable or not.

Let $f \in LLN(\mu,B)$, then the vector $a \in B$ occuring in (2.9) is of course uniquely determined, and we shall call it <u>the mean</u> of f and it is denoted

$$a = Ef = \int_S f d\mu$$

Note that if $f \in LLN(\mu,B)$ and f is Gelfand integrable, then by the real valued law of large number we have that the vector a in (2.9) equals the Gelfand integral of f. So there is no ambiguity in our notation. Actually we shall see below that every function f in $LLN(\mu,B)$ is Gelfand integrable.

Clearly we have the following simple properties of $LLN(\mu,B)$:

(2.10) $LLN(\mu,B)$ is a linear space

(2.11) $E:LLN(\mu,B) \to B$ is a linear map

And if φ is a bounded linear map from $(B,\|\cdot\|)$ into a Banach space $(A,\|\cdot\|)$, then we have

(2.12) $\varphi(f(\cdot)) \in \mathrm{LLN}(\mu, A) \quad \forall f \in \mathrm{LLN}(\mu, B)$

(2.13) $\varphi(Ef) = E\varphi(f) \quad \forall f \in \mathrm{LLN}(\mu, B)$

Let (Ω, F, P) be a probability space, and let $\{\xi_n\}$ be a sequence of independent identically distributed random variables with values in $(S, \mathcal{S})$ and distribution law μ. Then evidently we have

$$(2.14) \qquad \|Ef - \frac{1}{n}\sum_{j=1}^{n} f(\xi_j)\| \underset{n\to\infty}{\to} 0 \quad \text{P-a.s.,} \quad \forall f \in \mathrm{LLN}(\mu, B)$$

However, even if $B = \mathbb{R}$, we may have functions f, such that the averages $n^{-1}(f(\xi_1) + \ldots + f(\xi_n))$ converges P-a.s., but $f \notin \mathrm{LLN}(\mu, B)$. However if the sequence $\{\xi_n\}$ is P-perfect, i.e. if the product map

$$\xi(\omega) = (\xi_n(\omega))_1^\infty$$

is P-perfect from Ω into $(S^\infty, \mathcal{S}^\infty)$, then we shall see in Theorem 2.3 below, that this cannot occur.

Note that we have not assumed any measurability or separability of functions in $\mathrm{LLN}(\mu, B)$. However it turns out (see Theorem 2.4) that any function f in $\mathrm{LLN}(\mu, B)$ is weakly measurable and Gelfand integrable. To see this we need a couple of lemmas.

Lemma 2.1. Let $S_n \subseteq S$ so that $\mu^*(S_n) = 1$ for all $n \geq 1$. Then we have

$$(2.1.1) \qquad (\mu^\infty)^*(\prod_{n=1}^{\infty} S_n) = 1$$

Moreover if $f_n : S \to B$ are maps, so that $f_n(s_n) \to 0$ for μ^∞-a.a.

$(s_j) \in S$, <u>then there exist a sequence</u> $\{g_n\}$ <u>of measurable maps</u> <u>from</u> $(S,\mathcal{S})$ <u>into</u> $\overline{\mathbb{R}}$, <u>so that</u>

$$(2.1.2) \qquad \|f_n(s)\| \le g_n(s) \quad \forall s \in S$$

$$(2.1.3) \qquad g_n(s_n) \to 0 \quad \underline{\text{for}} \quad \mu^\infty\text{-}\underline{\text{a.a.}} \ (s_n) \in S$$

<u>Actually we may take</u> g_n <u>to be the upper</u> μ-<u>envelope of</u> $\|f_n(\cdot)\|$.

<u>Proof</u>. Let $\mathcal{S}_n = \{F \cap S_n | F \in \mathcal{S}\}$ and $\mu_n(F) = \mu^*(F)$ for $F \in \mathcal{S}_n$. Then $\mathcal{S}_n$ is a σ-algebra on S_n and μ_n is a probability measure on $(S_n, \mathcal{S}_n)$. By Tulcea's theorem (see [2] p. 183), we know that the product probability space:

$$(S_\infty, \mathcal{S}_\infty, \mu_\infty) = (\prod_{j=1}^\infty S_j, \bigotimes_{j=1}^\infty \mathcal{S}_j, \bigotimes_{j=1}^\infty \mu_j)$$

is welldefined, and since

$$\mu_\infty(\prod_{j=1}^\infty F_j \cap S_j) = \prod_{j=1}^\infty \mu_j(F_j \cap S_j) = \prod_{j=1}^\infty \mu(F_j) = \mu^\infty(\prod_{j=1}^\infty F_j)$$

for all $\{F_j\} \subseteq \mathcal{S}$, we conclude that

$$\mu_\infty(F \cap S_\infty) = \mu^\infty(F) \quad \forall F \in \mathcal{S}^\infty$$

Hence if $F \supseteq S_\infty$ and $F \in \mathcal{S}^\infty$, then $\mu^\infty(F) = 1$. Thus (2.1.1) follows.

Let g_n be the upper μ-envelope of $\|f_n\|$ (see (2.1)-(2.3)) and let

$$S_n = \{s \in S | g_n(s) \le 2\|f_n(s)\| \quad \text{or} \quad \|f_n(s)\| \ge 1\}$$

Then I claim that $\mu^*(S_n) = 1$ for all $n \ge 1$. So let $n \ge 1$ and put

$$h_n = \min\{1, \tfrac{1}{2} g_n\}$$

then h_n is measurable and $S \setminus S_n \subseteq \{ \|f_n\| \le h_n < g_n \}$. Hence by (2.3) we have that $\mu^*(S_n) = 1$. Now let $L \in \mathcal{S}^\infty$, so that $f_n(s_n) \to 0$ for all $(s_n) \in L$ and $\mu^\infty(L) = 1$. Then by (2.1.1) we have that

$$\text{(i)} \qquad (\mu^\infty)^*(L_0) = 1 \quad \text{where} \quad L_0 = L \cap \prod_{n=1}^{\infty} S_n$$

Now let $(s_n) \in L_0$, then $f_n(s_n) \to 0$ so for some $p \ge 1$ we have that $\|f_n(s_n)\| < 1$ for all $n \ge p$, and since $s_n \in S_n$ we find that $g_n(s_n) \le 2\|f(s_n)\|$ for all $n \ge p$. Hence $g_n(s_n) \to 0$ for all $(s_n) \in L_0$, and since g_n is measurable for all $n \ge 1$ we conclude from (i) that $g_n(s_n) \to 0$ for μ^∞-a.a. $(s_n) \in S^\infty$. I.e. the sequence $\{g_n\}$ satisfies (2.1.2) and (2.1.3). □

Lemma 2.2. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and ξ a P-perfect map from Ω into a measurable space $(M, \mathcal{B})$. Let f be a P-measurable map from Ω into a measurable space $(L, \mathcal{A})$, and g an arbitrary map from M into L, such that*

$$\text{(2.2.1)} \qquad f(\omega) = g(\xi(\omega)) \quad \forall\, \omega \in \Omega_0$$

where Ω_0 is a subset of Ω. Then there exist a set $B_0 \in \mathcal{B}$ such that

$$\text{(2.2.2)} \qquad B_0 \subseteq \xi(\Omega_0) \quad \text{and} \quad P_*(\Omega_0 \setminus \xi^{-1}(B_0)) = 0$$

$$\text{(2.2.3)} \qquad g^{-1}(A) \cap B_0 \quad \text{is } P_\xi\text{-measurable} \quad \forall\, A \in \mathcal{A}$$

i.e. $g \restriction B_0$ is P_ξ-measurable, and $P_\xi(B_0) \ge P_*(\Omega_0)$.

Proof. It is no loss of generality to assume that P is complete. Let $F \in \mathcal{F}$ be chosen so that $F \subseteq \Omega_0$ and $P(F) = P_*(\Omega_0)$. And let $P(F|\xi=x)$ be a conditional expectation of 1_F given ξ. Then by (2.6) we may assume that $P(F|\xi=x) = 0$ for all $x \notin \xi(F)$. Now let

$$B_0 = \{x \mid P(F|\xi=x) > 0\}$$

Then $B_0 \in \mathcal{B}$, and

$$\text{(i)} \qquad P(F \cap \xi^{-1}(B)) = \int_{B \cap B_0} P(F|\xi=x) P_\xi(dx) \qquad \forall B \in \mathcal{B}$$

And since $P(F|\xi=x) = 0$ for $x \notin \xi(F)$ we have

$$\text{(ii)} \qquad B_0 \subseteq \xi(F) \subseteq \xi(\Omega_0)$$

From (i) we find that $P(F \setminus \xi^{-1}(B_0)) = 0$, and since $P_*(\Omega_0 \setminus F) = 0$, we see that B_0 satisfies (2.2.2).

Now let $F_0 = F \cap \xi^{-1}(B_0)$, then $F_0 \in \mathcal{F}$, and $\xi(F_0) = B_0$. Also since $F_0 \subseteq F \subseteq \Omega_0$, it follows easily from (2.2.1) that we have

$$\text{(iii)} \qquad g^{-1}(A) \cap B_0 = \xi(F_0 \cap f^{-1}(A)) \qquad \forall A \subseteq L$$

Now let $A \in \mathcal{A}$, then $F_0 \cap f^{-1}(A)$ and $F_0 \cap f^{-1}(A^c)$ belongs to $\mathcal{F}$, so by (2.5) there exist $B_1, B_1' \in \mathcal{B}$ so that

$$\text{(iv)} \qquad B_1 \subseteq B \subseteq B_0 \quad \text{and} \quad B_1' \subseteq B_0 \setminus B$$

$$\text{(v)} \qquad P(F_0 \cap f^{-1}(A) \setminus \xi^{-1}(B_1)) = P(F_0 \cap f^{-1}(A^c) \setminus \xi^{-1}(B_1')) = 0$$

where $B = g^{-1}(A) \cap B_0 = \xi(F_0 \cap f^{-1}(A))$. Now put $B_2 = B_0 \setminus B_1'$, then by (iv) we have that $B_1 \subseteq B \subseteq B_2$ and from (i) and (v) we find

$$0 = P(F_0 \cap \xi^{-1}(B_2 \setminus B_1)) = P(F \cap \xi^{-1}(B_2 \setminus B_1))$$

$$= \int_{B_2 \setminus B_1} P(F \mid \xi = x) \quad P_\xi(dx)$$

since $F_0 = F \cap \xi^{-1}(B_0)$ and $B_2 \subseteq B_0$. Now $B_2 \setminus B_1 \subseteq B_0$ and so $P(F \mid \xi = x) > 0$ for all $x \in B_2 \setminus B_1$, hence we have that $P_\xi(B_2 \setminus B_1) = 0$ and $B_1 \subseteq B \subseteq B_2$. Thus $B = g^{-1}(A) \cap B_0$ is P_ξ-measurable and so (2.2.3) holds. □

Theorem 2.3. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space and let* $\{\xi_n\}$ *be a P-perfect sequence of independent, identically distributed,* $(S, \mathcal{S})$*-valued random variables with* μ *as their common distribution law. Let* f *be a map from* S *into the Banach space* $(B, \|\cdot\|)$, *and suppose that there exist a Bochner measurable function* $a: \Omega \to B$, *and a set* $\Omega_0 \in \mathcal{F}$, *such that* $P(\Omega_0) > 0$ *and*

$$\text{(2.3.1)} \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(\xi_j(\omega)) = a(\omega) \qquad \forall\, \omega \in \Omega_0$$

Then $f \in \mathrm{LLN}(\mu, B)$ *and* $a(\omega) = Ef$ *for* P-a.a. $\omega \in \Omega_0$.

Proof. By removing a nullset from Ω_0, we may assume that there exist a separable subspace B_0 of B so that $a(\omega) \in B_0$ for all $\omega \in \Omega_0$. Let $\xi(\omega) = (\xi_1(\omega), \xi_2(\omega), \dots)$, then by assumption ξ is a P-perfect map from Ω into $(S^\infty, \mathcal{S}^\infty)$. Let us put

$$L = \{ (s_j) \in S^\infty \mid \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(s_j) \quad \text{exists in} \quad (B, \|\cdot\|) \}$$

$$\alpha(s) = \begin{cases} 0 & \text{if} \quad (s_j) \in S^\infty \setminus L \\ \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(s_j) & \text{if} \quad (s_j) \in L \end{cases}$$

Then by (2.3.1) we have that $\xi(\Omega_0) \subseteq L$ and $a(\omega) = \alpha(\xi(\omega))$ for all $\omega \in \Omega_0$. So by Lemma 2.2 there exist $L_0 \in \mathcal{S}^\infty$ so that $L_0 \subseteq \xi(\Omega_0) \subseteq L$, $\mu^\infty(L_0) \geq P(\Omega_0) > 0$, and $\alpha|L_0$ is μ^∞-measurable (note that $P_\xi = \mu^\infty$). Let

$$L_k = \{(s_j) \in S^\infty \mid (s_{k+1}, s_{k+2}, \dots) \in L_0\} \qquad \forall\, k \geq 0$$

$$L_\infty = \bigcup_{k=0}^{\infty} L_k$$

Then $L_k \in \mathcal{S}^\infty$ for all $0 \leq k \leq \infty$, and $\mu^\infty(L_k) = \mu^\infty(L_0) > 0$ for all $k \geq 0$. Hence

$$\mu^\infty(L_\infty) \geq \mu^\infty(\limsup L_k) \geq \limsup_{n\to\infty} \mu^\infty(L_k) > 0$$

and so by the zero-one law we see that $\mu^\infty(L_\infty) = 1$. Moreover if τ_k is the translation map:

$$\tau_k(s) = (s_{k+1}, s_{k+2}, \dots) \qquad \forall\, s = (s_j)$$

Then $\alpha(s) = \alpha(\tau_k(s))$ for all $s \in L$, and so $\alpha|L_k$ is μ^∞-measurable for all $0 \leq k < \infty$, and since $\mu^\infty(L_\infty) = 1$ we see that α is μ^∞-measurable on all of S^∞, and that $\mu^\infty(L) = 1$. Moreover since $\alpha(\xi(\omega)) = a(\omega) \in B_0$ for $\omega \in \Omega_0$, we have that $\alpha(s) \in B_0$ for all $s \in L_\infty$. Thus α is Bochner measurable from $(S^\infty, \mathcal{S}^\infty, \mu^\infty)$ into $(B, \|\cdot\|)$.

Now let C be a countable subset of B_0' which separates points in B_0. Since $\alpha(s) = \alpha(\tau_k(s))$ for all $s \in S^\infty$ and all $k \geq 1$, it follows from the zero-one law that $x_0'(\alpha(\cdot))$ is constant μ^∞-a.s. for all $x_0' \in C$. And since C separates points in B_0 and C is countable it follows that α is constant μ^∞-a.s., since $\alpha(s) \in B_0$ for μ^∞-a.a. $s \in S^\infty$. Thus there exist $a_0 \in B$ so that

$$\alpha(s) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(s_j) = a_0$$

for μ^∞-a.a. $s = (s_j) \in S^\infty$. Hence $f \in LLN(\mu,B)$, and clearly we have $a(\omega) = a_0 = Ef$ P-a.e. in Ω_0. □

Theorem 2.4. Let $(S,\mathcal{S},\mu)$ be a probability space and $(B,||\cdot||)$ a Banach space. Then we have

(2.4.1) $$L^1(\mu,B) \subseteq LLN(\mu,B) \subseteq L^1_G(\mu,B) \cap L^1_*(\mu,B)$$

(2.4.2) $$\int^* ||f||\,d\mu < \infty \qquad \forall\, f \in LLN(\mu,B)$$

Remark. The key point of the second inclusion in (2.4.1) is to prove that every f in $LLN(\mu,B)$ is weakly measurable. The proof below of this fact is due to M. Talagrand (private communication), who has also proved a very nice and surprising characterization of $LLN(\mu,B)$ (handwritten manuscript).

Proof. The first inclusion is a wellknown result due A. Beck (see [1] p. 26).

Let $f \in LLN(\mu,B)$ and let $x' \in B'$ and $g(s) = x'(f(s))$. Then g is real valued, $g \in LLN(\mu,\mathbb{R})$ and $Eg = x'(Ef)$. Now let g_* and g^* be the lower and upper μ-envelopes of g. We shall then show

(i) $$g_* = g^* \quad \mu\text{-a.s.}$$

To see this we choose two measurable functions h_0 and h_1 from S into $\overline{\mathbb{R}}$, such that

(ii) $\qquad g_*(s) = h_0(s) = h_1(s) = g^*(s) \qquad \forall\, s \in \{g_* = g^*\}$

(iii) $\qquad g_*(s) < h_0(s) < h_1(s) < g^*(s) \qquad \forall\, s \in \{g_* < g^*\}$

Note that $h_0(s) = h_1(s) = g(s)$ on $\{g_* = g^*\}$, so h_0 and h_1 are finite every where. Now by (2.3) we have

$$\mu_*(h_0 < g) \le \mu_*(g_* < h_0 \le g) = 0$$
$$\mu_*(g < h_1) \le \mu_*(g \le h_1 < g^*) = 0$$

Hence if $S_0 = \{g \le h_0\}$ and $S_1 = \{h_1 \le g\}$, then $\mu^*(S_j) = 1$ for $j = 0,1$. Now let $L \in \mathcal{S}^\infty$ so that $\mu^\infty(L) = 1$ and

$$\frac{1}{n}\sum_{j=1}^{\infty} g(s_j) \to Eg \qquad \forall\, (s_j) \in L$$

If we put

$$L_j = L \cap (S_j \times S_j \times \dots) \qquad \text{for} \quad j = 0,1$$

Then by Lemma 2.1 we have that $(\mu^\infty)^*(L_j) = 1$ for $j = 0,1$. And by definition of S_0 and S_1 we have

$$Eg \le \liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} h_0(s_j) \qquad \forall\, (s_j) \in L_0$$

$$Eg \ge \limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} h_1(s_j) \qquad \forall\, (s_j) \in L_1$$

Since h_0 and h_1 are measurable we consequently find that the two inequalities holds μ^∞-a.s., and since $h_0 \le h_1$ everywhere we have

$$Eg = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} h_0(s_j) = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} h_1(s_j)$$

for μ^∞-a.a. $(s_j) \in S^\infty$.

Now by the converse law of large numbers, we have that h_0 and h_1 are μ-integrable and

(iv) $$\int_S h_0 d\mu = \int_S h_1 d\mu = Eg$$

(see [2] p. 122). But $h_0 \leq h_1$ and $h_0 < h_1$ on the set $\{g_* < g^*\}$, hence by (iv) we have that $\mu(g_* < g^*) = 0$, and so $g_* = g^*$ μ-a.s..

Thus g is μ-measurable and μ-integrable and we have

$$\int_S g d\mu = \int_S x'(f(s))\mu(ds) = x'(Ef)$$

for all $x' \in B'$. Thus f is Gelfand integrable and Ef is the Gelfand integral of f.

Now let us show that $f \in L^1_*(\mu,B)$. First we note that if $f \in LLN(\mu,B)$, then

(v) $$\frac{1}{n} f(s_n) \to 0 \quad \text{for } \mu^\infty\text{-a.a. } (s_j) \in S^\infty$$

Let h be the upper μ-envelope of $||f(\cdot)||$. Then by Lemma 2.1 we have that $n^{-1}h(s_n) \to 0$ for μ^∞-a.a. $(s_n) \in S^\infty$, and so by Lemma 1.4 (p.53) in [5] we have that h is μ-integrable, and since $||f(s)|| \leq h(s)$ for all $s \in S$, we see that $f \in L^1_*(\mu,B)$. □

Theorem 2.4 gives a necessary condition for $f \in LLN(\mu,B)$ and we shall now seek sufficient conditions. To do this we shall introduce a topology on B^S (the set of all function from S into B), and show that $LLN(\mu,B)$ is closed in this topology. Since $L^1(\mu,B) \subseteq LLN(\mu,B)$ we will then know, that the closure of $L^1(\mu,B)$ is contained in $LLN(\mu,B)$, and in sections 3 and 4 we shall see that this fact implies the law of large number for a large class of stochastic processes.

Definition 2.5. (The π-topology). Let $(B,\|\cdot\|)$ be a Banach space, then a finite partition of the norm $\|\cdot\|$, is a finite set σ of functions from B into $\overline{\mathbb{R}}_+ = [0,\infty]$, such that

$$\text{(2.5.1)} \qquad \alpha(x+y) \le \alpha(x) + \alpha(y) \quad \forall\, x,y \in B \quad \text{and} \quad \alpha(0)=0 \quad \forall\, \alpha \in \sigma$$

$$\text{(2.5.2)} \qquad \|x\| \le \max_{\alpha\in\pi} \alpha(x) \qquad \forall\, x \in B$$

I.e. a finite partition of $\|\cdot\|$ is a finite set of subadditive $\overline{\mathbb{R}}_+$-valued functions on B whose maximum dominates the norm $\|\cdot\|$. We put

$$\Pi(\|\cdot\|) = \{\sigma \mid \sigma \text{ is a finite partition of } \|\cdot\|\}$$

If $(S,\mathcal{S},\mu)$ is a probability space and $(B,\|\cdot\|)$ is a Banach space we put:

$$\text{(2.5.3)} \qquad \sigma(f) = \max_{\alpha\in\sigma} \int_S^* \alpha(f(s))\,\mu(ds) \qquad \forall\, f \in B^S \;\forall\, \sigma \in \Pi(\|\cdot\|)$$

Note that $\sigma(f)$ is subadditive on B^S, but not necessatily homogeneous nor symmetric.

We can then define a convergence notion on B^S as follows. If $\{f_\lambda \mid \lambda \in \Lambda\}$ is a net in B^S and $f \in B^S$, we shall say that $\{f_\lambda\}$ is π-convergent to f, and we write $f_\lambda \overset{\pi}{\to} f$, if

$$\text{(2.5.4)} \qquad \forall\, \varepsilon > 0 \;\exists\, \lambda_0 \in \Lambda \;\exists\, \sigma \in \Pi(\|\cdot\|) : \sigma(f_\lambda - f) < \varepsilon \quad \forall\, \lambda \ge \lambda_0$$

Let Φ be a subset of B^S, then we say that Φ is π-closed, if for every π-convergent net $\{f_\lambda\} \subseteq \Phi$ with $f_\lambda \overset{\pi}{\to} f$, we have that $f \in \Phi$. Since a subnet of a π-convergent net clearly is π-convergent to the same limit, we have that the class of all π-closed sets is closed under finite unions and arbitrary intersections. Thus there exists a topology on B^S, which we shall call the π-<u>topology</u>, such that a set $\Phi \subseteq B^S$ is closed in the π-topology, if and only if Φ is π-closed. Clearly we have

$$(2.5.5) \qquad f_\lambda \overset{\pi}{\to} f \Rightarrow f_\lambda \to f \quad \text{in the } \pi\text{-topology.}$$

I do not know if the converse implication holds, i.e. if $\overset{\pi}{\to}$ is a topological convergence notion, but I strongly suspect that this is not so in general.

If $\Phi \subseteq B^S$ is π-closed, and Φ is a map from Φ into a topological space T, then it is easily checked, that φ is continuous in the restricted π-topology, if and only if φ satisfies:

$$(2.5.6) \qquad \varphi(f_\lambda) \to \varphi(f) \quad \forall f \in \Phi \; \forall \{f_\lambda\} \subseteq \Phi \quad \text{so that} \quad f_\lambda \overset{\pi}{\to} f$$

A function φ satisfying (2.5.6) is said to be π-<u>continuous</u> on Φ.

Finally we let $L^1_\pi(\mu,B)$ denote the π-closure of $L^1(\mu,B)$, i.e. $L^1_\pi(\mu,B)$ is the smallest π-closed set containing $L^1(\mu,B)$.

<u>Lemma 2.6</u>. Let $(S,\mathcal{S},\mu)$ <u>be a probability space, and</u> $(B,||\cdot||)$ <u>a Banach space. Then we have</u>

(2.6.1). <u>If</u> $f_\lambda \overset{\pi}{\to} f$ <u>and</u> $t \in \mathbb{R}$, <u>then</u> $tf_\lambda \overset{\pi}{\to} tf$

(2.6.2) <u>If</u> $f_\lambda \overset{\pi}{\to} f$ <u>and</u> $g \in B^S$, <u>then</u> $f_\lambda + g \overset{\pi}{\to} f+g$

Moreover if $\sigma \in \Pi(\|\cdot\|)$, and $x' \in B'$ with $\|x'\| \leq 1$, then we have

$$-\sigma(f) \leq \int_* x'(f(s))\mu(ds) \leq \int^* x'(f(s))\mu(ds) \leq \sigma(f) \tag{2.6.3}$$

$$\left|\int^* x'(f(s))\mu(ds)\right| \leq \sigma(f) \qquad \forall f \in B^S \tag{2.6.4}$$

$$\left|\int_* x'(f(s))\mu(ds)\right| \leq \sigma(f) \qquad \forall f \in B^S \tag{2.6.5}$$

$$\int^* |x'(f(s))|\mu(ds) \leq 2\sigma(f) \qquad \forall f \in B^S \tag{2.6.6}$$

$$\|Ef\| \leq \sigma(f) \qquad \forall f \in L^1_w(\mu,B) \tag{2.6.7}$$

Proof. (2.6.1): If $t = 0$ then (2.6.1) is obvious. So suppose that $t \neq 0$ and let $\varepsilon > 0$ be given. Then we choose $\lambda_0 \in \Lambda$ and $\sigma \in \Pi(\|\cdot\|)$ such that $\sigma(f_\lambda - f) < \varepsilon/|t|$ for $\lambda \geq \lambda_0$. If $\alpha \in \sigma$ we put

$$\tilde{\alpha}(x) = |t|\alpha(t^{-1}x) \qquad \forall x \in B$$

Then $\tilde{\sigma} = \{\tilde{\alpha} \mid \alpha \in \sigma\}$ belongs to $\Pi(\|\cdot\|)$, and

$$\tilde{\sigma}(tf_\lambda - tf) = |t|\sigma(f_\lambda - f) < \varepsilon$$

for all $\lambda \geq \lambda_0$. Thus $tf_\lambda \xrightarrow{\pi} tf$.

(2.6.2): Evident!

(2.6.3): Let $f \in B^S$, $\sigma \in \Pi(\|\cdot\|)$ and $x' \in B'$ with $\|x'\| \leq 1$. If $\sigma(f) = \infty$ then (2.6.3) is obvious. So suppose that $\sigma(f) < \infty$. Then

$$\int^* \|f\| d\mu \leq \sum_{\alpha \in \sigma} \int^* \alpha(f) d\mu \leq k\sigma(f) < \infty \tag{i}$$

where k is the number of elements in σ.

Let $g(s) = x'(f(s))$ and let g_* and g^* be the lower and upper μ-envelopes of g. Also let f_α be the upper μ-envelope of $\alpha(f(\cdot))$. Then by (i) we have that g_*, g^* and f_α are μ-integrable and

$$\text{(ii)} \qquad \int_S f_\alpha d\mu = \int^* \alpha(f(s))\mu(ds)$$

$$\text{(iii)} \qquad \int_S g^* d\mu = \int^* x'(f(s))\mu(ds)$$

$$\text{(iv)} \qquad \int_S g_* d\mu = \int_* x'(f(s))\mu(ds)$$

Now let m be any real number satisfying

$$\text{(v)} \qquad \int_S g_* d\mu < m < \int_S g^* d\mu \quad \text{or} \quad m = \int_S g_* d\mu = \int_S g^* d\mu$$

Then we can find a measurable function h, such that

$$\text{(vi)} \qquad h(s) = g_*(s) = g^*(s) \qquad \forall\, s \in \{g_* = g^*\}$$

$$\text{(vii)} \qquad g_*(s) < h(s) < g^*(s) \qquad \forall\, s \in \{g_* < g^*\}$$

$$\text{(viii)} \qquad \int_S h d\mu = m$$

As in the proof of Theorem 2.4 we find that

$$\mu^*(g \leq h) = \mu^*(g \geq h) = 1$$

Hence by Lemma 2.1 we have that the two sets:

$$M_0 = \{(s_j) \in S^\infty \mid g(s_j) \leq h(s_j) \quad \forall\, j \geq 1\}$$

$$M_1 = \{(s_j) \in S^\infty \mid g(s_j) \geq h(s_j) \quad \forall\, j \geq 1\}$$

have outer μ^{∞}-measure equal to 1. And by the real valued law of large numbers there exist $M \in S^{\infty}$ with $\mu^{\infty}(M) = 1$ and

$$\text{(ix)} \qquad \frac{1}{n}\sum_{j=1}^{n} h(s_j) \to m \qquad \forall\, (s_j) \in M$$

$$\text{(x)} \qquad \frac{1}{n}\sum_{j=1}^{n} f_\alpha(s_j) \to \int_S f_\alpha d\mu \qquad \forall\, (s_j) \in M$$

(see (viii)). Now note that we have

$$- \max_{\alpha\in\sigma} \frac{1}{n}\sum_{j=1}^{n} f_\alpha(s_j) \le - \max_{\alpha\in\sigma} \frac{1}{n}\sum_{j=1}^{n} \alpha(f(s_j))$$

$$\le - \max_{\alpha\in\sigma} \frac{1}{n}\alpha\Big(\sum_{j=1}^{n} f(s_j)\Big) \le -\frac{1}{n}\Big\|\sum_{j=1}^{n} f(s_j)\Big\|$$

$$\le x'\Big(\frac{1}{n}\sum_{j=1}^{n} f(s_j)\Big) = \frac{1}{n}\sum_{j=1}^{n} g(s_j)$$

$$\le \frac{1}{n}\Big\|\sum_{j=1}^{n} f(s_j)\Big\| \le \max_{\alpha\in\sigma} \frac{1}{n}\alpha\Big(\sum_{j=1}^{n} f(s_j)\Big)$$

$$\le \max_{\alpha\in\sigma} \frac{1}{n}\sum_{j=1}^{n} \alpha(f(s_j)) \le \max_{\alpha\in\sigma} \frac{1}{n}\sum_{j=1}^{n} f_\alpha(s_j)$$

since each α in σ is subadditive, $f_\alpha \ge \alpha(f(\cdot))$ and $\|x\| \le \max \alpha(x)$. Hence by (ii) and (x) we find

$$-\sigma(f) \le \liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} g(s_j)$$

$$\le \limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} g(s_j) \le \sigma(f)$$

for all $(s_j) \in M$. Now since $\mu^{\infty}(M) = (\mu^{\infty})^*(M_0) = 1$ we have that $M \cap M_0 \neq \phi$ and if $(s_j) \in M \cap M_0$, then

$$m = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} h(s_j) \ge \liminf_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} g(s_j) \ge -\sigma(f)$$

And similarly since $M \cap M_1 \neq \phi$ we have that $m \leq \sigma(f)$. I.e. we have shown that any real number m satisfying (v) also satisfies

$$-\sigma(f) \leq m \leq \sigma(f)$$

But then (2.6.3) follows from (iii) and (iv).

(2.6.4): By (2.6.3) we have

$$\int^* x'(f(s))\mu(ds) \leq \sigma(f)$$

$$-\int^* x'(f(s))\mu(ds) \leq -\int_* x'(f(s))\mu(ds) \leq \sigma(f)$$

and so (2.6.4) holds.

(2.6.5) follows from (2.6.3) as above.

(2.6.6): Let us put

$$S^+ = \{s \in S \mid x'(f(s)) \geq 0\} \qquad f^+ = 1_{S^+} f$$

$$S^- = \{s \in S \mid x'(f(s)) < 0\} \qquad f^- = 1_{S^-} f$$

Then by (2.6.3) we have

$$\int^* |x'(f)|\,d\mu \leq \int^* x'(f^+)\,d\mu - \int_* x'(f^-)\,d\mu$$
$$\leq \sigma(f^+) + \sigma(f^-)$$

If $\alpha \in \sigma$ then

$$\alpha(f^+) = 1_{S^+}\alpha(f) + 1_{S^-}\alpha(0) \leq \alpha(f)$$

$$\alpha(f^-) = 1_{S^+}\alpha(0) + 1_{S^-}\alpha(f) \leq \alpha(f)$$

Thus $\sigma(f^+) \leq \sigma(f)$ and $\sigma(f^-) \leq \sigma(f)$ and so (2.6.6) follows.

(2.6.7): Immediate consequence of (2.6.4). □

Theorem 2.7. Let $(S,\mathcal{S},\mu)$ be a probability space and $(B,\|\cdot\|)$ a Banach space, then we have

(2.7.1) $f \rightsquigarrow Ef$ is π-continuous: $L^1_W(\mu,B) \to (B'',\|\cdot\|)$

(2.7.2) $L^1_*(\mu,B)$ is a π-closed, and π-open linear space

(2.7.3) $L^1_\pi(\mu,B)$, $LLN(\mu,B)$, $L^1_G(\mu,B)$ and $L^1_W(\mu,B)$ are π-closed linear subspaces of B^S

(2.7.4) $L^1(\mu,B) \subseteq L^1_\pi(\mu,B) \subseteq LLN(\mu,B) \subseteq L^1_G(\mu,B) \cap L^1_*(\mu,B)$

Proof. (2.7.1) follows easily from (2.6.7).

(2.7.2): If $\{f_\lambda \mid \lambda \in \Lambda\}$ be a net in B^S, so that $f_\lambda \xrightarrow{\pi} f$ then there exist $\lambda \in \Lambda$ and $\sigma \in \Pi(\|\cdot\|)$, so that $\sigma(f_\lambda - f) \leq 1$. Now let k be the number of elements in σ, then

$$\begin{aligned}\int^* \|f\| d\mu &\leq \int^* \|f_\lambda\| d\mu + \int^* \|f_\lambda - f\| d\mu \\ &\leq \int^* \|f_\lambda\| d\mu + \sum_{\alpha\in\sigma} \int^* \alpha(f_\lambda - f) d\mu \\ &\leq k + \int^* \|f_\lambda\| d\mu\end{aligned}$$

Hence if $f_\lambda \in L^1_*(\mu,B)$ then so does f, and consequently we have that $L^1_*(\mu,B)$ is π-closed. Similarly we have

$$\int^* \|f_\lambda\| d\mu \le \int^* \|f_\lambda - f\| d\mu + \int^* \|f\| d\mu$$
$$\le k\sigma(f_\lambda - f) + \int^* \|f\| d\mu$$
$$\le k + \int^* \|f\| d\mu$$

so if $f_\lambda \in B^S \setminus L^1_*(\mu,B)$ then so does f, and consequently we have that $L^1_*(\mu,B)$ is π-open.

(2.7.3): The spaces $LLN(\mu,B)$, $L^1_G(\mu,B)$ and $L^1_w(\mu,B)$ are evidently linear spaces, however since the π-topology is not a linear topology in general (see (2.6. 1+2)), it is not evident that $L^1_\pi(\mu,B)$ is a linear space. To see that this is actually so we put

$$L_t = \{f \in B^S \mid tf \in L^1_\pi(\mu,B)\} \qquad \forall t \in \mathbb{R}$$
$$L(g) = \{f \in B^S \mid f + g \in L^1_\pi(\mu,B)\} \quad \forall g \in B^S$$

Then by (2.6.1) and (2.6.2) we have that L_t and $L(g)$ are π-closed for all $t \in \mathbb{R}$ and all $g \in B^S$. Clearly we have that $L^1(\mu,B) \subseteq L_t \cap L(g)$ for all $t \in \mathbb{R}$ and all $g \in L^1(\mu,B)$. Thus $L^1_\pi(\mu,B) \subseteq L_t \cap L(g)$ whenever $t \in \mathbb{R}$ and $g \in L^1(\mu,B)$. I.e. tf and $f+g$ belongs to $L^1_\pi(\mu,B)$ whenever $t \in \mathbb{R}$, $f \in L^1_\pi(\mu,B)$ and $g \in L^1(\mu,B)$. Hence $L^1(\mu,B) \subseteq L(g)$ for all $g \in L^1_\pi(\mu,B)$ and so as above we have that $f+g$ belongs to $L^1_\pi(\mu,B)$ whenever f and g belongs to $L^1_\pi(\mu,B)$. Thus $L^1_\pi(\mu,B)$ is a linear space.

By definition we have that $L^1_\pi(\mu,B)$ is π-closed.

Now let $\{f_\lambda \mid \lambda \in \Lambda\}$ be a net in $L^1_w(\mu,B)$, such that $f_\lambda \xrightarrow{\pi} f \in B^S$. Then there exist $\lambda_k \in \Lambda$ and $\sigma_k \in \Pi(\|\cdot\|)$ such that $\sigma_k(f_\lambda - f) \le 2^{-k}$ for all $\lambda \ge \lambda_k$. Now by (2.6.6) we have

$$\int^* |x'(f_\lambda(s)) - x'(f(s))| \mu(ds) \le 2^{-k+1} \|x'\|$$

for all $x' \in B'$, and all $\lambda \geq \lambda_k$. Now since $x'(f_\lambda(\cdot))$ is μ-integrable for all λ we find that $x'(f(\cdot))$ is μ-integrable, and so $L_W^1(\mu,B)$ is π-closed. Moreover if $f_\lambda \in L_G^1(\mu,B)$ for all $\lambda \in \Lambda$, then $||Ef_\lambda - Ef|| \to 0$ by (2.7.1), and so $Ef \in B$, since B is norm closed in B''. Thus $f \in L_G^1(\mu,B)$ and so $L_G^1(\mu,B)$ is π-closed.

Now suppose that $\{f_\lambda\} \subseteq LLN(\mu,B)$ so that $f_\lambda \overset{\pi}{\to} f$. Let $h_{\alpha\lambda}$ be the upper μ-envelope of $\alpha(f_\lambda(\cdot)-f(\cdot))$ whenever α is a map: $B \to \overline{\mathbb{R}}_+$ and $\lambda \in \Lambda$. By assumption there exist $\lambda_k \in \Lambda$ and $\sigma(k) \in \Pi(||\cdot||)$ so that

$$\text{(i)} \qquad \int^* \alpha(f_\lambda - f)\,d\mu = \int_S h_{\alpha\lambda}\,d\mu \leq 2^{-k} \quad \forall \lambda \geq \lambda_k \quad \forall \alpha \in \sigma(k) \quad \forall k \geq 1$$

Since $LLN(\mu,B) \subseteq L_G^1(\mu,B)$ by Theorem 2.4, we have that $f \in L_G^1(\mu,B)$ and $Ef_\lambda \to Ef$. Hence we may assume that λ_k is chosen so large that

$$\text{(ii)} \qquad ||Ef - Ef_\lambda|| \leq 2^{-k} \qquad \forall \lambda \geq \lambda_k$$

Let $k \geq 1$ and $\lambda \geq \lambda_k$ be fixed for a moment, then we have

$$||Ef - \frac{1}{n}\sum_{j=1}^n f(s_j)|| \leq || Ef - Ef_\lambda|| + || Ef_\lambda - \frac{1}{n}\sum_1^n f(s_j)||$$

$$\leq 2^{-k} + ||Ef_\lambda - \frac{1}{n}\sum_{j=1}^n f_\lambda(s_j)|| + \frac{1}{n}||\sum_{j=1}^n (f_\lambda(s_j) - f(s_j))||$$

$$\leq 2^{-k} + ||Ef_\lambda - \frac{1}{n}\sum_{j=1}^n f_\lambda(s_j)|| + \frac{1}{n}\max_{\alpha\in\sigma(k)} \alpha(\sum_{j=1}^n \{f_\lambda(s_j)-f(s_j)\})$$

$$\leq 2^{-k} + ||Ef_\lambda - \frac{1}{n}\sum_{j=1}^n f_\lambda(s_j)|| + \max_{\alpha\in\sigma(k)} \frac{1}{n}\sum_{j=1}^n h_{\alpha\lambda}(s_j)$$

for all $n \geq 1$. Now since $f_\lambda \in LLN(\mu,B)$ and $h_{\alpha\lambda}$ is μ-integrable we find by (i) that

$$\limsup_{n\to\infty} \|Ef - \frac{1}{n}\sum_{j=1}^{n} f(s_j)\| \leq 2^{-k} + \max_{\alpha\in\sigma(k)} \int_S h_{\alpha\lambda} d\mu \leq 2^{-k+1}$$

for μ^∞-a.a. $(s_j) \in S^\infty$ and all $k \geq 1$. Hence $f \in LLN(\mu,B)$ and so $LLN(\mu,B)$ is π-closed.

(2.7.4): The first inclusion is trivial, and since $LLN(\mu,B)$ is π-closed by (2.7.3) and contains $L^1(\mu,B)$ by Theorem 2.4, the second inclusion follows. The last inclusion was proved in Theorem 2.4. □

If B is separable then $L^1(\mu,B) = L^1_G(\mu,B) \cap L^1_*(\mu,B)$ and we have equality everywhere in (2.7.4). M. Talagrand has introduced a new measurability concept for B-valued functions, called <u>properly measurable</u>, and he has shown for an arbitrary Banach space B, that $f \in LLN(\mu,B)$, if and only if f is properly measurable and $f \in L^1_*(\mu,B)$ (to appear).

In the next two sections we shall see that in the non-separable case we may have, that $L^1(\mu,B) \neq L^1_\pi(\mu,B)$, and one may naturally pose the following problem

(2.15) Is: $L^1_\pi(\mu,B) = LLN(\mu,B)$?

In Definition 2.5 we introduced the π-topology on B^S, and in Lemma 2.6 we showed that $f \rightsquigarrow tf$ and $f \rightsquigarrow f+g$ are π-continuous for all $t \in \mathbb{R}$ and all $g \in B^S$. However if $f \notin L^1_*(\mu,B)$ then by (2.7.2) we have that $t \rightsquigarrow tf$ is discontinuous at $t = 0$ in the π-topology. Hence the π-topology is not a linear topology, if $L^1_*(\mu,B) \neq B^S$ (i.e. if $\dim L^1(\mu,\mathbb{R}) = \infty$ and $\dim B \geq 1$). However we may pose the following problem:

(2.16) Is the π-topology on $L^1_*(\mu,B)$ a linear topology?

In order to solve this problem one probably need to exhibit an explicitly defined neighborhood base at 0 for the π-topology, and I have not been able to do this. In connection with the π-topology one may pose several problems e.g.

(2.17) Does: $f_\lambda \to f$ in the π-topology imply $f_\lambda \overset{\pi}{\to} f$?

(2.18) Is: $cl_\pi(\Phi) = \{f \mid \exists \{f_\lambda\} \subseteq \Phi : f_\lambda \overset{\pi}{\to} f\}$?

where $cl_\pi(\Phi)$ is the π-closure of Φ.

3. Sample bounded stochastic processes

We shall now specialize the results of the previous section to the case where $B = B(T)$ is the set of all bounded real valued functions on a set T with its usual sup-norm (see [3] p.240).

Let T be a set and $\varphi: T \to \mathbb{R}$ a function. If $A \subseteq T$ and $t \in T$ we define

$$\|\varphi\|_A = \sup_{u \in A} |\varphi(u)|$$

$$w_A(\varphi, t) = \sup_{u \in A} |\varphi(u) - \varphi(t)|$$

$$W_A(\varphi) = \sup_{u,v \in A} |\varphi(u) - \varphi(v)|$$

with the convention: $\sup \emptyset = 0$. A finite cover of T is a set $\mathcal{A} = \{A_1, \ldots, A_n\}$ of non-empty subsets of T, such that $T = A_1 \cup \ldots \cup A_n$. We let

$$\Gamma(T) = \{\mathcal{A} \mid \mathcal{A} \text{ is a finite cover of } T\}$$

denote the set of all finite covers of T.

Let $(S, \mathcal{S}, \mu)$ be a probability space and let T be a set. A stochastic process g on $(S, \mathcal{S}, \mu)$ with time set T is a map $g: S \times T \to \mathbb{R}$, such that $g(\cdot, t)$ is μ-measurable for all $t \in T$. If $g: S \times T \to \mathbb{R}$ is a stochastic process we put

$$g(s) = g(s, \cdot) \in \mathbb{R}^T \qquad \forall s \in S$$

Then $s \leadsto g(s)$ is a μ-measurable map from S into $\mathbb{R}^T$ with its product σ-algebra. A first order stochastic process $g: S \times T \to \mathbb{R}$, is a stochastic process g, such that $g(\cdot, t) \in L^1(\mu)$ for all $t \in T$. The mean function of a first order stochastic process g is the function

$$M(t) = \int_S g(s,t)\, \mu(ds)$$

A stochastic process $g: S \times T \to \mathbb{R}$ is <u>uniformly bounded in</u> μ-<u>mean</u>, if we have

(3.1) $$\int^* \|g(s)\|_T \mu(ds) = \int^* \sup_{t \in T} |g(s,t)| \mu(ds) < \infty$$

(Note that $\|g(\cdot)\|_T$ need not be μ-measurable). Finally we say that a stochastic process $g: S \times T \to \mathbb{R}$ is <u>totally bounded in</u> μ-<u>mean</u>, if g is a first order stochastic process satisfying

(3.2) $$\forall \varepsilon > 0 \; \exists \mathcal{A} \in \Gamma(T): \int^* W_A(g(s)) \mu(ds) < \varepsilon \quad \forall A \in \mathcal{A}$$

Let $g: S \times T \to \mathbb{R}$ be a stochastic process, then clearly we have

(3.3) If g is uniformly bounded in μ-mean, then $g(s) \in B(T)$ for μ-a.a. $s \in S$ and $g \in L^1_*(\mu, B(T))$

(3.4) If g is totally bounded in μ-mean, then g is uniformly bounded in μ-mean.

<u>Proposition 3.1</u>. <u>Let</u> T <u>be a set and</u> $(S, \mathcal{S}, \mu)$ <u>be a probability space</u>. <u>Let</u> $g: S \times T \to \mathbb{R}$ <u>be a first order stochastic process, such that for all</u> $\varepsilon > 0$ <u>there exist sets</u> L_o <u>and</u> L_1 <u>and stochastic processes</u> $g_j: S \times L_j \to \mathbb{R}$ <u>for</u> $j = 0,1$ <u>satisfying</u>

(3.1.1) $\forall t \in T \; \exists (x_o, x_1) \in L_o \times L_1$ <u>so that</u> $g_o(s, x_o) \le g(s,t) \le g_1(s, x_1)$ <u>for all</u> $s \in S$ <u>and</u> $\int_S \{g_1(s,x_1) - g_o(s,x_o)\} \mu(ds) \le \varepsilon$

(3.1.2) $\exists \mathcal{A}_j \in \Gamma(L_j): \int^* W_A(g_j(s)) \mu(ds) \le \varepsilon \quad \forall A \in \mathcal{A}_j \quad \forall j = 0,1$

<u>Then</u> g <u>is totally bounded in</u> μ-<u>mean</u>.

<u>Proof</u>. Let $\varepsilon > 0$ be given, and let $\mathcal{A}_o = \{A_1, \ldots, A_k\}$ and $\mathcal{A}_1 = \{B_1, \ldots, B_m\}$ be the finite covers of L_o and L_1 from (3.1.2).

Now put

$$C_{ij} = \left\{ t \in T \;\middle|\; \begin{array}{l} \exists (x_o, x_1) \in A_i \times B_j : g_o(s,x_o) \leq g(s,t) \leq g_1(s,x_1) \\ \text{for all} \quad s \in S \quad \text{and} \quad \int_S \{g_1(s,x_1) - g_o(s,x_o)\}\, \mu(ds) < \varepsilon \end{array} \right\}$$

$$\Lambda = \{(i,j) \mid 1 \leq i \leq k, \quad 1 \leq j \leq m, \quad C_{ij} \neq \emptyset\}$$

$$\theta_{ij} = \inf\{|g_1(s,x_1) - g_o(s,x_o)| \mid (x_o, x_1) \in A_i \times B_j\}$$

Then by (3.1.1) we have that $\{C_{ij} \mid (i,j) \in \Lambda\}$ is a finite cover of T and

$$\text{(i)} \qquad \int^* \theta_{ij}\, d\mu \leq \varepsilon \quad \forall (i,j) \in \Lambda$$

Now let $(i,j) \in \Lambda$ and $t', t'' \in C_{ij}$, then we choose (x_o', x_1') and (x_o'', x_1'') in $A_i \times B_j$ according to the defining property of C_{ij}. Then we have

$$g(s,t') - g(s,t'') \leq g_1(s,x_1') - g_o(s,x_o'')$$

$$= g_1(s,x_1') - g_1(s,v) + g_1(s,v) - g_o(s,u) + g_o(s,u) - g_o(s,x_o'')$$

$$\leq W_{B_j}(g_1(s)) + \{g_1(s,v) - g_o(s,u)\} + W_{A_i}(g_o(s))$$

for all $(u,v) \in A_i \times B_j$. Taking infimum over all $(u,v) \in A_i \times B_j$ we find

$$g(s,t) - g(s,t'') \leq W_{B_j}(g_1(s)) + \theta_{ij}(s) + W_{A_i}(g_o(s))$$

So interchanging t' and t'' gives

$$w_{C_{ij}}(g(s)) \leq w_{B_j}(g_i(s)) + \theta_{ij}(s) + w_{A_i}(g_o(s))$$

for all $s \in S$ and all $(i,j) \in \Lambda$. And by (3.1.2) and (i) we have

$$\int^* w_{C_{ij}}(g(s))\mu(ds) \leq 3\varepsilon \qquad \forall\, (i,j) \in \Lambda$$

Thus g is totally bounded in μ-mean, since $\{C_{ij} \mid (i,j) \in \Lambda\}$ is a finite cover of T. □

Theorem 3.2. Let T be a set, and $(S,\mathcal{S},\mu)$ a probability space. Let $g\colon S \times T \to \mathbb{R}$ be a first order stochastic process, then the following four statements are equivalent:

(3.2.1) g is totally bounded in μ-mean

(3.2.2) $\forall\ \varepsilon > 0\ \ \exists\, \mathcal{A} \in \Gamma(T)\ \ \exists\, f_o \in L^1(\mu, B(T))$ such that

$$\int^* \|g(s) - f_o(s)\|_A \mu(ds) \leq \varepsilon \quad \forall\, A \in \mathcal{A}$$

(3.2.3) There exists a totally bounded, ultra pseudo-metric ρ on T, satisfying: $\forall \varepsilon > 0\ \ \exists\, \delta > 0$, such that

$$\int^* w_{B_\rho(z,\delta)}(g(s),t)\mu(ds) \leq \varepsilon \quad \forall\, t \in T$$

(3.2.4) There exists a totally bounded, uniformity $\mathcal{U}$ on T, satisfying: $\forall\, \varepsilon > 0\ \exists\, U \in \mathcal{U}$, such that

$$\int^* w_{u(t)}(g(s),t)\mu(ds) \leq \varepsilon \quad \forall\, t \in T$$

where $B_\rho(t,\delta) = \{u \in T \mid \rho(u,t) < \delta\}$ and $U(t) = \{u \in T \mid (u,t) \in U\}$ whenever ρ is a pseudo-metric on T and $U \subseteq T \times T$.

Remark (a): A totally bounded pseudo-metric on T is a pseudo-metric ρ on T, such that T may be covered by finitely many ρ-balls of radius ε for all $\varepsilon > 0$.

(b): A totally bounded uniformity is defined similarly, i.e. the uniformity $\mathcal{U}$ is totally bounded if the cover: $\{U(t) \mid t \in T\}$, of T admits a finite subcover for all $U \in \mathcal{U}$.

(c): An ultra pseudo-metric on T is a pseudo-metric ρ on T satisfying the following strong triangle inequality:

$$\rho(u,v) \leq \max\{\rho(u,t),\rho(t,v)\}$$

for all $u,v,t \in T$.

Proof. (3.2.1) ⇒ (3.2.2): Let $\varepsilon > 0$ be given and choose a finite cover $\mathcal{A}$ of T, so that (3.2) holds. By replacing $\mathcal{A}$ with a suitable refinement we may assume that the sets in $\mathcal{A}$ are mutually disjoint. For each $A \in \mathcal{A}$ we choose a point $t_A \in A$ and we define

$$f_o(s) = \sum_{A \in \mathcal{A}} g(s,t_A) 1_A$$

Then f_o maps S into $B(T)$ and since $g(\cdot,t_A) \in L^1(\mu)$ we have that $f_o \in L^1(\mu,B(T))$. If $A \in \mathcal{A}$, then

$$\|g(s) - f_o(s)\|_A = w_A(g(s),t_A) \leq W_A(g(s))$$

and so by (3.2) we have that

$$\int^* \|g(s) - f_o(s)\|_A \mu(ds) \leq \int^* W_A(g(s))\mu(ds) \leq \varepsilon$$

for all $A \in \mathcal{A}$. Thus (3.2.2) holds.

(3.2.2) $\Rightarrow$ (3.2.3): By (3.2.2) there exist $\mathcal{A}_k \in \Gamma(T)$ and $f_k \in L^1(\mu, B(T))$, so that

$$\text{(i)} \qquad \int^* \|g(s) - f_k(s)\|_A \mu(ds) \leq 2^{-k-1} \quad \forall A \in \mathcal{A}_k \quad \forall k \geq 1$$

Now notice that the set of functions of the form

$$h(s) = \sum_{B \in \mathcal{B}} h_B(s) \, 1_B(\cdot)$$

where $\mathcal{B} \in \Gamma(T)$ and $h_B \in L^1(\mu)$ $\forall B \in \mathcal{B}$, is $\|\cdot\|_1$-dense in $L^1(\mu, B(T))$. Hence we may assume that f_k is of this form for all k, i.e.

$$\text{(ii)} \qquad f_k(s) = \sum_{B \in \mathcal{B}_k} h_{kB}(s) \, 1_B(\cdot)$$

where $\mathcal{B}_k \in \Gamma(T)$ and $h_{kB} \in L^1(\mu)$ $\forall B \in \mathcal{B}_k$ $\forall k \geq 1$. Now let $\mathcal{F}_k = \sigma(\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k \cup \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_k)$, then $\mathcal{F}_k$ is a finite σ-algebra on T. If $t \in T$ we let $\mathcal{F}_k(t)$ denote $\mathcal{F}_k$-atom containing t. Since $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$ we have

$$\text{(iii)} \qquad \mathcal{F}_1(t) \supseteq \mathcal{F}_2(t) \supseteq \ldots \qquad \forall t \in T$$

And we define

$$\rho(u,v) = \sup\{2^{-n} \mid n \in \mathbb{N}, \ \mathcal{F}_n(u) \cap \mathcal{F}_n(v) = \emptyset\}$$

for $u, v \in T$, with the convention: $\sup \emptyset = 0$. Then I claim that we have

(iv) $\qquad \rho$ is an ultra pseudo-metric on T

(v) $\quad B_\rho(t,2^{-n}) = F_n(t) \quad \forall t \in T \quad \forall n \geq 1$

(vi) $\quad \rho$ is totally bounded.

(iv): Clearly $\rho(t,t) = 0$ and $\rho(u,v) = \rho(v,u)$. Now let $u,v,t \in T$, and let $\rho(u,v) = 2^{-p}$. Then $F_k(u) = F_k(v)$ for $1 \leq k < p$ and $F_p(u) \cap F_p(v) = \emptyset$, and so either $t \notin F_p(u)$ or $t \notin F_p(v)$. In the first case we have that $\rho(u,t) \geq 2^{-p}$, and in the second case we have that $\rho(v,t) \geq 2^{-p}$. Thus in any case we have

$$\rho(u,v) \leq \max\{\rho(u,t),\rho(t,v)\}$$

Thus ρ is an ultra pseudo-metric.

(v): If $u \in B_\rho(t,2^{-n})$, then $\rho(u,t) < 2^{-n}$, and so $u \in F_n(u) = F_n(t)$. If $u \in F_n(t)$, then $u \in F_j(t)$ for all $1 \leq j \leq n$ by (iii). Hence $F_j(t) = F_j(u)$ for all $1 \leq j \leq n$ and so $\rho(u,t) < 2^{-n}$, and $u \in B_\rho(t,2^{-n})$.

(vi): Since F_n is a finite σ-algebra, we have that the set of F_n-atoms is finite. But then it follows from (v) that ρ is totally bounded.

Now let us show that the pseudo-metric ρ satisfies the condition in (3.2.3).

So let $\varepsilon > 0$ be given and choose $k \geq 1$, such that $2^{-k} < \varepsilon$. Now put $\delta = 2^{-k}$ and (see (ii))

$$f_k(s,t) = f_k(s)(t) = \sum_{B \in B_k} h_{kB}(s) 1_B(t)$$

Then $f_k(s,\cdot)$ is F_k-measurable and so $f_k(s,\cdot)$ is constant on all F_k-atoms. Now let $t \in T$ and $u \in F_k(t)$, since $A_k \subseteq F_k$ is a covering of T, there exists an $A \in A_k$ such that $u,t \in F_k(t) \subseteq A$. Moreover since $f_k(s,t) = f_k(s,u)$ we have

$$|g(s,t) - g(s,u)| \leq |g(s,t) - f_k(s,t)| + |f_k(s,u) - g(s,u)|$$

$$\leq 2 \, \|g(s) - f_k(s)\|_A$$

And since $B_\rho(t,\delta) = B_\rho(t,2^{-k}) = F_k(t)$ by (v) we have

$$w_{B_\rho(t,\delta)}(g(s),t) \leq 2 \, \|g(s) - f_k(s)\|_A$$

Thus by (i) we conclude that

$$\int^* w_{B_\rho(t,\delta)}(g(s),t)\mu(ds) \leq 2^{-k} < \varepsilon \qquad \forall t \in T$$

and so (3.2.3) holds.

(3.2.3) $\Rightarrow$ (3.2.4): Evident!

(3.2.4) $\Rightarrow$ (3.2.1): Let $\varepsilon > 0$ be given, then by (3.2.4) there exist $U \in \mathcal{U}$ so that

$$\text{(vii)} \qquad \int^* w_{U(t)}(g(s),t)\mu(ds) \leq \varepsilon \qquad \forall t \in T$$

And since the uniformity is totally bounded there exist $t_1, t_2, \ldots, t_n \in T$ such that

$$T = \bigcup_{j=1}^{n} U(t_j)$$

Hence $\mathcal{A} = \{U(t_1), \ldots, U(t_n)\} \in \Gamma(T)$, and by (vii) we see that (3.2) holds. Thus g is totally bounded in μ-mean, and the theorem is proved. □

Theorem 3.3. *Let* T *be a set and* $(S,\mathcal{S},\mu)$ *a probability space. Let* $g: S \times T \to \mathbb{R}$ *be a stochastic process, which is totally bounded in* μ*-mean, then we have*

(3.3.1) $$g \in L^1_\pi(\mu, B(T)) \subseteq LLN(\mu, B(T))$$

Hence if $\{\xi_n\}$ is a sequence of independent, identically distributed $(S,\mathcal{S})$-valued random variables defined on a probability space $(\Omega,\mathcal{F},P)$, and with μ as their common distribution law, then

(3.3.2) $$\sup_{t\in T}\left|m(t) - \frac{1}{n}\sum_{j=1}^{n} g(\xi_j,t)\right| \xrightarrow[n\to\infty]{} 0 \qquad P\text{-a.s.}$$

where

$$m(t) = \int_S g(s,t)\mu(ds) = Eg(\xi_j,t)$$

is the mean function of g.

Proof. By (3.3) and (3.4) we have that $g \in L^1_*(\mu,B(T))$. By (3.2.2) there exists $f_k \in L^1(\mu,B(T))$ and $A_k \in \Gamma(T)$, such that

(i) $$\int^* \|g(s) - f_k(s)\|_A \mu(ds) \le 2^{-k} \qquad \forall A \in A_k \quad \forall k \ge 1$$

Now put $\sigma_k = \{\|\cdot\|_A \mid A \in A_k\}$, since A_k is a finite cover of T we have that σ_k is a finite partition of the norm $\|\cdot\|_T$, and by (i) we have that $\sigma_k(f_k - g) \le 2^{-k}$. Hence $f_k \xrightarrow{\pi} g$ (see Definition 2.5) and so (3.3.1) holds by Theorem 2.7. Now it is easily checked that $Eg = m$ (see Example 3.5 (in particular (3.5.8)), and so (3.3.2) follows from (3.3.1) and (2.14). □

Theorem 3.4. Let T be a set and $(\Omega,\mathcal{F},P)$ a probability space. Let $X_n\colon \Omega\times T \to \mathbb{R}$ be a sequence of independent identically distributed stochastic processes, such that for some $k \ge 1$ we have

(3.4.1) X_k is P-perfect: $\Omega \to (\mathbb{R}^T, \mathcal{B}^T)$

(3.4.2) $\quad X_k$ <u>is totally bounded in</u> μ-<u>mean</u>

<u>Then we have</u>

$$(3.4.3) \qquad \sup_{t\in T}\left|m(t) - \frac{1}{n}\sum_{j=1}^{n} X_j(\omega,t)\right| \xrightarrow[n\to\infty]{} 0 \qquad P\text{-}\underline{a.s.}$$

<u>where</u> $m(t) = EX_n(t)$ <u>is the common mean function of the processes</u> X_n.

<u>Proof</u>. Let $(S,\mathcal{S}) = (\mathbb{R}^T,\mathcal{B}^T)$ and let μ be the common distribution law on $(S,\mathcal{S})$ of the processes X_n. Put

$$g(s,t) = s(t) \qquad \forall\, s \in S = \mathbb{R}^T \qquad \forall\, t \in T$$

Then g is a stochastic process on $(S,\mathcal{S},\mu)$ and $X_n(\omega) = g(X_n(\omega))$ for all $\omega \in \Omega$ and all $n \geq 1$. Now let $\varepsilon > 0$ be given, then by (3.4.2) there exists a finite cover $\mathcal{A}$ of T, such that

$$\int^* W_A(X_k(\omega))P(d\omega) \leq \varepsilon \qquad \forall\, A \in \mathcal{A}$$

And since X_k is P-perfect and $X_k(\omega) = g(X_k(\omega))$ it follows from (2.7) that

$$\int^* W_A(g(s))\mu(ds) = \int^* W_A(X_k(\omega))P(d\omega) \leq \varepsilon \;\forall A \in \mathcal{A}$$

Hence g is totally bounded in μ-mean, and so the theorem follows from Theorem 3.3. □

<u>Example 3.5</u>. Let $(S,\mathcal{S},\mu)$ be a probability space, T a set and $g\colon S\times T \to \mathbb{R}$ a map such that

$$(3.5.1) \qquad g(s) = g(s,\cdot) \in B(T) \qquad \forall\, s \in S$$

From Corollary IV.5.2 in [3] we have that $B(T)' = ba(T)$ is the set of all finitely additive real valued set functions on $(T, 2^T)$ which are of bounded variation, and the total variation of $\lambda \in ba(T)$ equals its norm as an element of $B(T)'$.

Thus g is weakly μ-measurable (μ-integrable), if and only if the function

$$(3.5.1) \qquad s \rightsquigarrow \lambda(g(s)) = \int_T g(s,t)\lambda(dt)$$

is μ-measurable (μ-integrable) for all $\lambda \in ba(T)$. In view of the result in [10] (see also [9] p.364-366) we have that non-σ-additive functions $\lambda \in ba(T)$ are highly non-measurable, so weak measurability is a severe restriction, which in general is difficult to verify.

Now let $g \in L^1_w(\mu, B)$, then $Eg \in B(T)''$, and

$$(3.5.2) \qquad (Eg)(\lambda) = \int_S \mu(ds) \int_T g(s,t)\lambda(dt)$$

$$(3.5.3) \qquad |(Eg)(\lambda)| \leq \|Ef\| \; \|\lambda\|_1$$

where $\|\lambda\|_1$ is the total variation of λ over T. In particular we see (put λ equal to the Dirac measure at t), that

$$(3.5.4) \qquad m(t) = \int_S g(s,t)\mu(ds)$$

exists for all $t \in T$ and m is bounded, i.e. $m \in B(T)$.

Hence we see that g is Gelfand integrable, if and only if g satisfies the following 3 conditions:

(3.5.5) $\quad s \sim \int_T g(s,t)\lambda(dt)$ is μ-integrable $\forall \lambda \in ba(T)$

(3.5.6) $\quad m(t) = \int_S g(s,t)\mu(ds)$ exists and is bounded on T

(3.5.7) $\quad \int_S \mu(ds) \int_T g(s,t)\lambda(dt) = \int_T \lambda(dt) \int_S g(s,t)\mu(ds) \ \forall \lambda \in ba(T)$

And if so, then

(3.5.8) $$Eg = m$$

Note that (3.5.7) states that g, μ and λ satisfies the Fubini Theorem for all $\lambda \in ba(T)$. Now the Fubini Theorem is only rarely true for finitely additive set functions, so condition (3.5.7) is indeed a severe restriction. □

<u>Example 3.6</u>. Let $(S,\mathcal{S},\mu)$ be a probability space and let $\eta_1,\eta_2,\dots$ be a sequence of real valued random variables on $(S,\mathcal{S},\mu)$. Now put

$$g(s,j) = \eta_j(s), \quad g(s) = g(s,\cdot) = (\eta_j(s))_{j=1}^{\infty}$$

Then by the Borel-Cantelli lemmas we have

(3.5.1) $\quad \sum_{j=1}^{\infty} \mu(|\eta_j|>a)<\infty$ for some $a \in \mathbb{R} \Rightarrow g(s) \in \ell^{\infty}$ μ-a.s.

and the converse implication holds, if $\eta_1,\eta_2,\dots$ are independent.

Similarly it is easily checked that if $\{\eta_n\}$ satisfies the following condition

(3.6.2) $\quad \forall \varepsilon > 0 \ \exists m \geq 1: \sum_{j=1}^{\infty} \sum_{n=m}^{\infty} \mu^{\infty}(s \mid |\frac{1}{n} \sum_{i=1}^{n} \eta_j(s_i)|>\varepsilon) < \infty$

then $g(s) \in \ell^{\infty}$ μ-a.s. and

(3.6.3) $\quad \sup_j |\frac{1}{n} \sum_{i=1}^{n} \eta_j(s_i)| \xrightarrow[n\to\infty]{} 0$ $\quad \mu^{\infty}$-a.s.

i.e. $g \in \mathrm{LLN}(\mu, \ell^\infty)$ and $Eg = 0$.

Now suppose that $\eta_1, \eta_2, \ldots$ are independent gaussian random variables with $E\eta_j = 0$ and $E\eta_j^2 = \sigma_j^2$, then a straight forward argument using the remarks above shows that the following 3 statements are equivalent

(3.6.4) $\quad g(s) \in \ell^\infty \quad \mu\text{-a.s.}$

(3.6.5) $\quad g \in \mathrm{LLN}(\mu, \ell^\infty)$

(3.6.6) $\quad \exists\, a > 0 : \sum_{j=1}^{\infty} \exp(-a/\sigma_j^2) < \infty$

And similarly that the following 3 statements are equivalent

(3.6.7) $\quad g(s) \in c_o \ \mu\text{-a.s.}$ (see [3], p.239)

(3.6.8) $\quad g$ is totally bounded in μ-mean

(3.6.9) $\quad \sum_{j=1}^{\infty} \exp(-a/\sigma_j^2) < \infty \quad \forall\, a > 0$

Putting $\sigma_j^2 = \frac{1}{\log j}$ for $j \geq 2$, we thus obtain an example of a gaussian sequence, which satisfies the uniform law of large numbers, but which is not totally bounded in μ-mean.

Finally suppose that $\eta_1, \eta_2, \ldots$ is a Bernoulli sequence, i.e. $\eta_1, \eta_2, \ldots$ are independent and

$$\mu(\eta_j = 1) = \mu(\eta_j = -1) = \frac{1}{2}$$

Then $g(s) \in \ell^\infty$ and $\|g(s)\|_\infty = 1$ for all $s \in S$. However, if (η_j) is a perfect sequence, then by [10] we have that the map

$$s \rightsquigarrow \int_{\mathbb{N}} \eta_j(s) \ \lambda(dj)$$

where $\lambda \in ba(\mathbb{N})$ (see Example 3.5) is μ-measurable, if and only if λ is σ-additive (i.e. if and only if $\lambda \in \ell^1$). Hence g is <u>not</u> weakly μ-measurable and so $g \notin LLN(\mu, \ell^\infty)$. □

<u>Example 3.7</u>. Let $(S, \mathcal{S}, \mu)$ be an atomfree probability space, such that $\{s\} \in \mathcal{S}$ for all $s \in S$. Let T be a subset of S and define

$$g(s,t) = \begin{cases} 1 & \text{if } (s,t) \in S \times T \text{ and } s = t \\ 0 & \text{if } (s,t) \in S \times T \text{ and } s \neq t \end{cases}$$

$$g(s) = g(s,\cdot) = 1_{\{s\}}$$

Then g is a map from S into $B(T)$, and by Example 3.4 we have

$$\lambda(g(s)) = \int_T g(s,t)\, \lambda(dt) = \begin{cases} 0 & \text{if } s \in S \setminus T \\ \lambda(\{s\}) & \text{if } s \in T \end{cases}$$

for all $\lambda \in ba(T) = B(T)'$. Hence $\lambda(g(s))$ is only non-zero for countably many $s \in S$, and since μ is atomfree we find

(3.7.1) $\quad s \rightsquigarrow \lambda(g(s))$ is $\mathcal{S}$-measurable $\quad \forall \lambda \in ba(T)$

(3.7.2) $\quad \lambda(g(s)) = 0 \quad \mu\text{-a.s.} \quad \forall \lambda \in ba(T)$

(3.7.3) $\quad g \in L^1_G(\mu, B(T)) \cap L^1_*(\mu, B(T))$ and $Eg = 0$

(3.7.4) $\quad \|g(s)\|_T = 1_T(s) \quad \forall s \in S$

Moreover if $\varepsilon > 0$ is given, then since μ is atomfree there exist $S_1, \dots, S_n \in \mathcal{S}$ so that $S = S_1 \cup \dots \cup S_n$ and $\mu(S_j) \le \varepsilon$ for all $j = 1, \dots, n$. Then $\mathcal{A} = \{T \cap S_j \mid 1 \le j \le n\}$ belongs to $\Gamma(T)$, and if $A = T \cap S_j \in \mathcal{A}$ then

$$\int^* W_A(g(s))\,\mu(ds) = \int^* \sup_{u,v \in A} |1_{\{u\}}(s) - 1_{\{v\}}(s)|\,\mu(ds) \le \mu^*(A) \le \varepsilon$$

Hence we have

(3.7.5) $\quad g$ is totally bounded in μ-mean

(3.7.6) $\quad g \in LLN(\mu, B(T))$

Note that even though $\lambda(g(\cdot)) = 0$ μ-a.s. for all $\lambda \in B(T)'$, then $\|g(\cdot)\|_T$ need not vanish μ-a.s. (take $T \subseteq S$ with $\mu^*(T) > 0$). Also note that even though g is Gelfand integrable, then $\|g(\cdot)\|_T$ need not be μ-measurable (take T to be a non-μ-measurable subset of S). $\square$

4. Sample continuous stochastic processes

We shall now see that if a first order stochastic process has sufficiently continuous sample paths, then it is totally bounded in mean.

Let L be a topological space, $\varphi: L \to \mathbb{R}$ a function and A a subset of L; then we define the boundary function by

$$\partial_A(\varphi,x) = \inf\{W_{A\cap U}(\varphi) \mid U \in N(x)\}$$

where $N(x)$ is the set of all neighbourhoods of x. Clearly, if x belongs to the closure of A, then we have

$$\partial_A(\varphi,x) = 0 \Leftrightarrow \lim_{\substack{y\to x\\ y\in A}} \varphi(y) \quad \text{exists and is finite.} \tag{4.1}$$

And so φ is continuous at x, if and only if $\partial_L(\varphi,x) = 0$. If $\alpha: 2^L \to \overline{\mathbb{R}}$ is a set function and $a \in \mathbb{R}$, then we write $a = \lim_{U\to x} \alpha(U)$ if

$$\forall \varepsilon > 0 \ \exists U \in N(x): \ |\alpha(U\cap V)-a| \leqq \varepsilon, \qquad \forall V \in N(x). \tag{4.2}$$

Note that since $U \rightsquigarrow W_{A\cap U}(\varphi)$ is increasing in U, we have

$$\partial_A(\varphi,x) = \lim_{U\to x} W_{A\cap U}(\varphi). \tag{4.3}$$

Theorem 4.1. *Let T be a set and $(S,\mathcal{S},\mu)$ a probability space. Let $g: S\times T \to \mathbb{R}$ be a first order stochastic process and suppose that for all $\varepsilon > 0$ there exist compact topological spaces L_j and stochastic processes $g_j: S\times L_j \to \mathbb{R}$ for $j = 0,1$, satisfying*

(4.1.1) $\forall t \in T \ \exists (x_0,x_1) \in L_0 \times L_1: \ g_0(s,x_0) \leqq g(s,t) \leqq g_1(s,x_1)$

for all $s \in S$ *and* $\int_S \{g_1(s,x_1) - g_0(s,x_0)\}\mu(ds) \leqq \varepsilon$

(4.1.2) $\qquad \forall j = 0,1 \quad \forall x \in L_j \quad \exists \mathcal{A}$ a finite cover of L_j, so that

$$\lim_{U\to x} \int^* W_{A\cap U}(g_j(s))\mu(ds) < \varepsilon \qquad \forall A \in \mathcal{A}.$$

Then g is totally bounded in μ-mean and so $g \in \mathrm{LLN}(\mu, B(T))$.

Proof. We shall apply Proposition 3.1. So let $\varepsilon > 0$ be given and choose L_0, L_1, g_0 and g_1 according to (4.1.1) and (4.1.2). Then evidently (3.1.1) holds. Now let $x \in L_j$ and choose $\mathcal{A}_x \in \Gamma(L_j)$ so that

$$\lim_{U\to x} \int^* W_{A\cap U}(g_j(s))\mu(ds) < \varepsilon \qquad \forall A \in \mathcal{A}_x.$$

Then we may choose a neighbourhood U_x of x so that

(i) $$\int^* W_{A\cap U_x}(g_j(s))\mu(ds) < \varepsilon \qquad \forall A \in \mathcal{A}_x$$

since $\mathcal{A}_x$ is finite. By compactness of L_j we can find a finite set $F \subseteqq L_j$ so that $L_j = \cup_{x\in F} U_x$. Now put

$$\mathcal{A} = \{A\cap U_x \mid A \in \mathcal{A}_x,\ x \in F\}.$$

Then $\mathcal{A}$ is a finite cover of L_j, and by (i) we have

$$\int^* W_A(g_j(s)) \ (ds) < \varepsilon \qquad \forall A \in \mathcal{A}.$$

Thus (3.1.2) holds and the Theorem follows from Proposition 3.1. □

Theorem 4.1 turns out to be a very useful interpolation principle. But in order to make its application more convenient, we shall study Condition (4.1.2) a bit closer. First notice that if L is a topological space and $f: S \times L \to \mathbb{R}$ is a stochastic process, then we have

$$\int^* \partial_A(f(s),x)\mu(ds) \leqq \lim_{U \to x} \int^* W_{A \cap U}(f(s))\mu(ds) \tag{4.4}$$

for all $A \subseteq L$ and all $x \in L$. Our next lemma gives some sufficient conditions for equality in (4.4).

Lemma 4.2. Let L be a topological space $(S,\mathcal{S},\mu)$ a probability space and $f: S \times L \to \mathbb{R}$ a stochastic process. Let $x_0 \in L$, let $A \subseteq L$, and let F be a μ-measurable subset of S satisfying

(4.2.1) $\exists U \in \mathcal{N}(x_0)$ so that $\int_F^* W_{A \cap U}(f(s))\mu(ds) < \infty$

(4.2.2) L has a countable neighbourhood base at x_0

(4.2.3) $\forall U \in \mathcal{N}(x_0)\ \exists V \in \mathcal{N}(x_0)$ such that $V \subseteq U$ and the map: $s \leadsto 1_F(s) W_{A \cap V}(f(s))$ is μ-measurable

where $\mathcal{N}(x_0)$ is the set of all neighbourhoods of x_0. Then we have

(4.2.4) $s \leadsto 1_F(s)\partial_A(f(s),x_0)$ is μ-integrable

$$\lim_{U \to x_0} \int_S^* W_{A \cap U}(f(s))\mu(ds) = \lim_{U \to x_0} \int_{S \setminus F}^* W_{A \cap U}(f(s))\mu(ds) + \int_F \partial_A(f(s),x_0)\mu(ds). \tag{4.2.5}$$

Moreover, if $f: S\times L \to \mathbb{R}$ is an arbitrary stochastic process, and $F \subseteq S$ is μ-measurable, then (4.2.3) holds in either of the following two cases:

Case 1°. There exists a countable set $D \subseteq A$ such that $f(s,\cdot)$ is D-separable on A for all $s \in F$.

Case 2°. There exists a Blackwell σ-algebra $\mathcal{G}$ on L containing a neighbourhood base at x_0 such that $A \in \mathcal{G}$ and f is $\mathcal{S} \otimes \mathcal{G}$-measurable.

Remarks. (a): Let L and M be topological spaces and $\varphi: L \to M$ a function. If D and A are subsets of L, we say that φ is D-separable on A if

$$(4.2.6) \qquad \varphi(G\cap A) \subseteq \mathrm{cl}\,\varphi(G\cap D) \qquad \forall G \text{ open} \subseteq L$$

where $\mathrm{cl}\,B$ denotes the closure of B. Note that if φ restricted to A is continuous, then φ is D-separable on A whenever D is a dense subset of A.

(b): A σ-algebra $\mathcal{G}$ on a set L is called a Blackwell σ-algebra if $\varphi(L)$ is an analytic subset of $\mathbb{R}$ in the sense of [9], whenever φ is a real $\mathcal{G}$-measurable function on L.

(c): Let L be a topological space; then the Baire σ-algebra, denoted $\mathcal{Ba}(L)$, is the smallest σ-algebra on L making all real valued continuous functions measurable. It is well-known that if L is K-analytic in the sense of [9] (e.g. if L is a compact Hausdorff space), then $\mathcal{B}a(L)$ is a Blackwell σ-algebra.

(d): The main feature about Blackwell σ-algebras, which we shall use here, is the so-called Projection Theorem. Suppose that $\mathcal{G}$ is a Blackwell σ-algebra on L, and $\mathcal{S}$ is an arbitrary σ-algebra on S; if $A \subseteqq S\times L$ belongs to the product σ-algebra $\mathcal{S}\otimes\mathcal{G}$, then we have

(4.2.7) $\pi_S(A)$ is μ-measurable, $\forall\mu$ a probability on $(S,\mathcal{S})$, where $\pi_S(s,x) = s$ is the natural projection of $S\times L$ onto S.

Proof. By (4.2.1) - (4.2.3) there exists a countable neighbourhood base $\{V_n \mid n \geqq 1\}$ at x_0 such that

(i) $V_1 \supseteqq V_2 \supseteqq \cdots \supseteqq V_n \supseteqq, \ldots$

(ii) $\int_F^* W_{A\cap V_1}(f(s))\mu(ds) < \infty$

(iii) $\varphi_n(s) = 1_F(s)W_{A\cap V_n}(f(s))$ is μ-measurable $\forall n \geqq 1$.

By (i) we have that $\{\varphi_n\}$ decreases, and since $\{V_n\}$ is a base at x_0, we have that

$$\varphi_n(s) \downarrow 1_F(s)\partial_A(f(s),x_0).$$

By (ii) we have that φ_1 is μ-integrable, and so by the monotone convergence theorem we have that (4.2.4) holds and

$$\lim_{n\to\infty}\int\varphi_n d\mu = \lim_{n\to\infty}\int_F W_{A\cap V_n}(f(s))\mu(ds) = \int_F \partial_A(f(s),x_0)\mu(ds).$$

Now since $\{V_n\}$ is a neighbourhood base at x_0, we find

$$\text{(iv)} \qquad \lim_{U\to x_0} \int_F^* W_{A\cap U}(f(s))\mu(ds) = \int_F \partial_A(f(s),x_0)\mu(ds).$$

Finally, since F is μ-measurable, we have

$$\lim_{U\to x_0} \int^* W_{A\cap U}(f(s))\mu(ds) = \lim_{U\to x_0} \int_F^* W_{A\cap U}(f(s))\mu(ds)$$
$$+ \lim_{U\to x_0} \int_{S\setminus F}^* W_{A\cap U}(f(s))\mu(ds)$$

and so (4.2.4) follows from (iv).

<u>Case 1°</u>. Let f be a stochastic process such that $f(s,\cdot)$ is D-separable on A for all $s\in F$, where D is a countable subset of A. Then it is easily verified that

$$W_{A\cap U}(f(s)) = W_{D\cap U}(f(s)) = \sup_{x,y\in D\cap U} |f(s,x) - f(s,y)|$$

for all $s\in F$, and all open sets U in L. Since $D\cap U$ is at most countable and $f(\cdot,x)$ is μ-measurable for all $x\in L$, it follows that $W_{A\cap U}(f(\cdot))1_F$ is μ-measurable whenever U is open. Thus (4.2.3) holds.

<u>Case 2°</u>. Now suppose that $A\in \mathcal{G}$ and f is $\mathcal{S}\otimes\mathcal{G}$-measurable where $\mathcal{G}$ is some Blackwell σ-algebra on L containig a neighbourhood base at x_0. Now let $G\in\mathcal{G}$ and put

$$\varphi(s,x,y) = |f(s,x) - f(s,y)|$$

$$\varphi_0(s) = W_G(f(s)) = \sup_{x,y\in G} \varphi(s,x,y).$$

Then φ is $\mathcal{S}\otimes\mathcal{G}\otimes\mathcal{G}$-measurable, and $\mathcal{G}\otimes\mathcal{G}$ is a Blackwell σ-algebra.

Hence if $a \in \mathbb{R}$, then the set

$$A = \{(s,x,y) \in S\times L\times L \mid \varphi(s,x,y) > a,\ x \in G,\ y \in G\}$$

belongs to $\mathcal{S} \otimes \mathcal{G} \otimes \mathcal{G}$. Let $\pi_S(s,x,y) = s$ be the natural projection of $S\times L\times L$ onto S. Then by the Projection Theorem (4.2.7) we have that the set

$$\pi_S(A) = \{s \mid \exists x,y \in G\colon \varphi(s,x,y) > a\} = \{s \mid \varphi_0(s) > a\}$$

is μ-measurable. Thus $W_G(f(\cdot))$ is μ-measurable for all $G \in \mathcal{G}$, and since $A \in \mathcal{G}$ and $\mathcal{G}$ contains a neighbourhood base at x_0, we conclude that (4.2.3) holds whenever $F \subseteqq S$ is μ-measurable. □

Recall that a topological space L is called <u>first countable</u> if every point in L has a countable neighbourhood base. And a subset A of L is called <u>separable</u> if there exists a countable set D such that $D \subseteqq A \subseteqq \operatorname{cl} D$, where $\operatorname{cl} D$ denotes the closure of D.

<u>Theorem 4.3.</u> *Let T be a compact, first countable, topological space, and $(S,\mathcal{S},\mu)$ a probability space. Let $g\colon S\times T \to \mathbb{R}$ be a stochastic process which is uniformly bounded in μ-mean, and suppose that for all $\delta > 0$ there exists a finite cover $\mathcal{A}$ of T satisfying*

(4.3.1) $\quad A$ *is separable* $\quad \forall A \in \mathcal{A}$

(4.3.2) $\quad \mu_*(s \in S \mid \partial_A(g(s),t) = 0 \quad \forall t \in T) \geqq 1-\delta \qquad \forall A \in \mathcal{A}.$

Then g is totally bounded in μ-mean, and $g \in LLN(\mu,B(T))$.

Remarks. (a): It follows from the proof below that we may replace Condition (4.3.1) by the following condition:

$$(4.3.3)\quad \begin{cases} \forall A \in \mathcal{A}\ \exists D \subseteqq A \quad \exists F \in \mathcal{S} \quad \text{such that } D \text{ is countable,} \\ \mu(F) \geqq 1-\delta \quad \text{and} \quad g(s,\cdot) \text{ is } D\text{-separable on } A \quad \forall s \in F. \end{cases}$$

(b): Let $F_A = \{s \mid \partial_A(g(s),t) = 0 \quad \forall t \in T\}$ be the set occurring in (4.3.2). It is then easily checked that we have

$$(4.3.4)\qquad F_A = \left\{ s \in S \;\middle|\; \begin{array}{l} \text{the restriction of } g(s,\cdot) \text{ to } A \text{ has a continuous} \\ \text{real valued extension to the closure of } A \end{array} \right\}$$

i.e. $s \in F_A$ if and only if there exists a continuous function ψ from $\mathrm{cl}\,A$ into $\mathbb{R}$ such that $g(s,t) = \psi(t) \quad \forall t \in A$.

Proof. We shall apply Theorem 4.1, so let $\varepsilon > 0$ be given. Let ψ be the upper μ-envelope of $\|g(\cdot)\|_T$; then by assumption we have that $\psi \in L^1(\mu)$ and hence there exists $\delta > 0$ such that

$$\text{(i)}\qquad \int_B \psi d\mu \leqq \tfrac{1}{2}\varepsilon \qquad \forall B \in \mathcal{S} \text{ with } \mu(B) \leqq \delta.$$

Now put $L_0 = L_1 = T$ and $g_0 = g_1 = g$. Then evidently (4.1.1) holds. Now let $t_0 \in T$ and choose a finite cover $\mathcal{A}$ of T satisfying (4.3.1) - (4.3.2). Let $A \in \mathcal{A}$; then by (4.3.2) there exist $F \in \mathcal{S}$ so that $\mu(F) \geqq 1-\delta$ and $\partial_A(g(s),t) = 0$ for all $t \in T$ and all $s \in F$. Then the restriction of $g(s,\cdot)$ to A is continuous, so by (4.3.1) there exists a countable set $D \subseteqq A$ such that $g(s,\cdot)$ is D-separable on A for all $s \in F$. Hence by Lemma 4.2 (see Case 1°) we have

$$\lim_{U\to t_0}\int^* W_{A\cap U}(g(s)) \leqq \int_{S\smallsetminus F} 2\psi d\mu + \int_F \partial_A(g(s),t_0)\mu(ds)$$

$$= \int_{S\smallsetminus F} 2\psi d\mu$$

since $W_{A\cap U}(g(s)) \leqq 2\psi(s)$ for all $s\in S$ and $\partial_A(g(s),t_0) = 0$ for all $s\in F$. Now since $\mu(S\smallsetminus F) \leqq \delta$, it follows from (i) that (4.1.2) holds. Thus the Theorem follows from Theorem 4.1. □

Corollary 4.4. *Let T be a compact, separable, first countable, topological space and $(S,\mathcal{S},\mu)$ a probability space. If $C(T)$ is the Banach space of all real valued continuous functions on T with its usual sup-norm: $\|\cdot\|_T$, then we have*

$$(4.4.1)\qquad L^1_\pi(\mu,C(T)) = LLN(\mu,C(T)) = L^1_*(\mu,C(T)) \cap L^1_w(\mu,C(T)).$$

Remark. If T is a compact metric space, then $C(T)$ is $\|\cdot\|_T$-separable, and so the equalities in (4.4.1) follows from Beck's theorem (see [1] p. 26). However, if T is not metrizable, then $C(T)$ is not $\|\cdot\|_T$-separable. The split interval, i.e. $[0,1]\times\{-1,1\}$ with its lexicographic order topology, provides an example of a compact, first countable, separable, Hausdorff space which is not metrizable. □

Theorem 4.5. *Let T be a compact, first countable topological space, $\mathcal{G}$ a Blackwell σ-algebra on T containing a base for the topology on T, and $(S,\mathcal{S},\mu)$ a probability space. Let $g: S\times T \to \mathbb{R}$ be an $\mathcal{S}\otimes\mathcal{G}$-measurable process which is uniformly bounded in μ-mean, and* suppose that for all $\delta > 0$ and all $t \in T$ there exists a finite cover $\mathcal{A}_{t,\delta}$ of T satisfying

$$(4.5.1) \qquad \mathcal{A}_{t,\delta} \subseteq \mathcal{G} \qquad \forall t \in T \quad \forall \delta > 0$$

$$(4.5.2) \qquad \mu^*(s \in S \mid \partial_A(g(s),t) = 0) \geq 1-\delta \qquad \forall A \in \mathcal{A}_{t\delta} \ \forall t \in T \quad \forall \delta > 0.$$

Then g is totally bounded in μ-mean, and $g \in LLN(\mu, B(T))$.

Remark. Note that $\partial_A(g(s),t) = 0$ for all $t \in T \setminus \mathrm{cl}\, A$ and all $s \in S$, and that if $t \in \mathrm{cl}\, A$, then we have

$$(4.5.3) \qquad \{s \mid \partial_A(g(s),t) = 0\} = \{s \mid \lim_{\substack{u \to t \\ u \in A}} g(s,u) \text{ exists and is finite}\}.$$

Also notice that Condition (4.5.2) is much weaker than Condition (4.3.2).

Proof. We shall apply Theorem 4.1 in much the same way as in the proof of Theorem 4.3. So let $\varepsilon > 0$ be given. Let ψ be the upper μ-envelope of $\|g(\cdot)\|_T$, then by assumption we have that $\psi \in L^1(\mu)$, and so there exists $\delta > 0$ such that

$$(\mathrm{i}) \qquad \int_B \psi \, d\mu \leq \tfrac{1}{2}\varepsilon \qquad \text{if} \quad B \in \mathcal{S} \quad \text{and} \quad \mu(B) \leq \delta.$$

Now put $L_0 = L_1 = T$ and $g_0 = g_1 = g$. Then evidently (4.1.1) holds. Now let $t \in T$ and choose a finite cover $\mathcal{A} = \mathcal{A}_{t\delta}$ of T satisfying (4.5.1) and (4.5.2). By Lemma 4.2 (see Case 2° and put $F = S$) we have that $\partial_A(g(\cdot),t)$ is μ-integrable and

$$\text{(ii)} \qquad \lim_{U \to t} \int^* W_{A \cap U}(g(s))\mu(ds) = \int_S \partial_A(g(s),t)\mu(ds)$$

for all $A \in \mathcal{G}$, in particular for all $A \in \mathcal{A}_{t\delta}$ (see (4.5.1)). Let $A \in \mathcal{A}_{t\delta}$ and put

$$B = \{s \mid \partial_A(g(s),t) > 0\};$$

then B is μ-measurable and $\mu(B) \leqq \delta$ by (4.5.2). Hence by (i) and (ii) we have

$$\lim_{U \to t} \int^* W_{A \cap U}(g(s),t)\mu(ds) = 2\int_B \psi d\mu \leqq \varepsilon$$

since $\partial_A(g(s),t) \leqq 2\, 1_B(s)\psi(s)$. Thus (4.2.1) holds, and so the theorem follows from Theorem 4.1. □

Corollary 4.6. *Let $\hat{T}$ be a compact, first countable, topological space, $\hat{\mathcal{G}}$ a Blackwell σ-algebra on $\hat{T}$ containing a base for the topology on $\hat{T}$ and $T \in \hat{\mathcal{G}}$ a subset of $\hat{T}$. Let $(S,\mathcal{S},\mu)$ be a probability space, and $g\colon S \times T \to \mathbb{R}$ a stochastic process satisfying*

$$\text{(4.6.1)} \qquad g \text{ is } \mathcal{S} \otimes \mathcal{G}\text{-measurable}$$

$$\text{(4.6.2)} \qquad \int^* \sup_{t \in T} |g(s,t)|\mu(ds) < \infty$$

where $\mathcal{G} = \{G \in \hat{\mathcal{G}} \mid G \subseteqq T\}$ is the trace of $\hat{\mathcal{G}}$ on T. Suppose that for every $\hat{t} \in \hat{T}$ there exists a finite cover $\hat{\mathcal{A}}_{\hat{t}}$ of $\hat{T}$ such that $\hat{\mathcal{A}}_{\hat{t}} \subseteqq \hat{\mathcal{G}}$ and

$$\text{(4.6.3)} \qquad \mu^*(s \mid \lim_{\substack{t \to \hat{t} \\ t \in A \cap T}} g(s,t) \text{ exists}) = 1$$

for all $A \in \hat{\mathcal{A}}_{\hat{t}}$ with $\hat{t} \in \mathrm{cl}(A \cap T)$. Then g is totally bounded in μ-mean, and $g \in LLN(\mu, B(T))$.

Remark. Since $T \in \hat{G}$, it follows easily that G is a Blackwell σ-algebra on T, and from (4.6.1) and the Projection Theorem (4.2.7) it then follows that

(4.6.4) $\qquad \|g(s)\|_T$ is μ-measurable

(4.6.5) $\qquad \{s \mid \lim_{\substack{t \to \hat{t} \\ t \in A \cap T}} g(s,t) \text{ exists}\}$ is μ-measurable

for all $A \in \hat{G}$ and all $\hat{t} \in \mathrm{cl}(A \cap T)$.

Proof. We extend g to $\hat{g}: S \times \hat{T} \to \mathbb{R}$ by putting $\hat{g}(s,\hat{t}) = 0$ for $s \in S$ and $\hat{t} \in \hat{T} \setminus T$. Since $T \in \hat{G}$, we have that $\hat{g}$ is $S \otimes \hat{G}$-measurable, and by (4.6.2) we have that $\hat{g}$ is uniformly bounded in μ-mean. If $\hat{t} \in \hat{T}$, we put

$$A_{\hat{t}} = \{A \cap T \mid A \in \hat{A}_{\hat{t}}\} \cup \{\hat{T} \setminus T\}.$$

Then $A_{\hat{t}}$ is a finite cover of $\hat{T}$, and if $A \subseteq \hat{T}$ and $\hat{t} \in \hat{T}$, then

$$\partial_{A \cap T}(\hat{g}(s),\hat{t}) = \partial_{A \cap T}(g(s),\hat{t})$$

$$\partial_{\hat{T} \setminus T}(\hat{g}(s),\hat{t}) = 0.$$

Since $\|\hat{g}(s)\|_T$ is finite μ-a.s., it follows from (4.6.3) and (4.5.3) that (4.5.2) holds. And since $T \in \hat{G}$ and $\hat{A}_{\hat{t}} \subseteq \hat{G}$, we see that $A_{\hat{t}} \subseteq G$, and so (4.5.1) holds.

Thus by Theorem 4.5 we have that $\hat{g}$ is totally bounded in μ-mean, but this clearly implies that g is totally bounded in μ-mean. $\square$

Putting $\hat{T} = \overline{\mathbb{R}}$, $\hat{G} = \mathcal{B}(\overline{\mathbb{R}})$ and

$$\hat{A}_{\hat{t}} = \{[-\infty,\hat{t}[, \{\hat{t}\},]\hat{t},\infty]\},$$

we get the following corollary

Corollary 4.7. *Let* T *be a Borel subset of* $\bar{\mathbb{R}}$, $(S,\mathcal{S},\mu)$ *a probability space, and* $g: S\times T \to \mathbb{R}$ *a stochastic process satisfying*

(4.7.1) $\quad g$ *is* $\mathcal{S}\otimes\mathcal{B}(T)$*-measurable*

(4.7.2) $\quad \int^* \sup_{t\in T}|g(s,t)|\,\mu(ds) < \infty$

(4.7.3) $\quad \mu^*(s \mid \lim_{\substack{t\to u\\ t<u}} g(s,t) \text{ exists}) = 1 \qquad \forall u \in T^-$

(4.7.4) $\quad \mu^*(s \mid \lim_{\substack{t\to u\\ t>u}} g(s,t) \text{ exists}) = 1 \qquad \forall u \in T^+$

where T^- *and* T^+ *are the set of all left, respectively, right limits points of* T, *i.e.*

$$T^- = \{u \in \bar{\mathbb{R}} \mid \exists\{t_n\} \subseteqq T:\ t_n \to u \text{ and } t_n < u\ \forall n\}$$

$$T^+ = \{u \in \bar{\mathbb{R}} \mid \exists\{t_n\} \subseteqq T:\ t_n \to u \text{ and } t_n > u\ \forall n\}.$$

Then g *is totally bounded in* μ*-mean, and* $g \in LLN(\mu,\mathcal{B}(T))$. □

Clearly we have a similar result in more dimensions for any given family $\{\hat{A}_{\hat{t}} \mid \hat{t} \in \bar{\mathbb{R}}^q\}$ of finite Borel covers of $\bar{\mathbb{R}}^q$.

In the Glivenko-Cantelli case (see (1.1) and (1.2)) we have $S = T = R$ and

$$g(s,t) = \begin{cases} 1 & \text{if } s \leqq t \\ 0 & \text{if } s > t \end{cases}$$

and so the Glivenko-Cantelli theorem (1.2) is a direct consequence of Corollary 4.7.

In the q-dimensional Glivenko-Cantelli case we have $S = T = \mathbb{R}^q$ and

$$g(s,t) = \begin{cases} 1 & \text{if } s_1 \leqq t_1, \ldots, s_q \leqq t_q \\ 0 & \text{otherwise.} \end{cases}$$

Let $\hat{T} = \bar{\mathbb{R}}^q$ and

$$A_{\alpha\beta\gamma}(\hat{t}) = \{\hat{u} \in \bar{\mathbb{R}}^q \mid \hat{u}_j = \hat{t}_j \ \forall j \in \alpha,\ \hat{u}_j > \hat{t}_j \ \forall j \in \beta, \hat{u}_j < \hat{t} \ \forall j \in \gamma\}$$

whenever $\hat{t} = (\hat{t}_1, \ldots, \hat{t}_q) \in \bar{\mathbb{R}}^q$ and $\{\alpha, \beta, \gamma\}$ is a disjoint partition of $\{1, \ldots, q\}$. Then putting

$$\hat{A}_{\hat{t}} = \{A_{\alpha\beta\gamma}(\hat{t}) \mid \{\alpha,\beta,\gamma\} \text{ a disjoint partition of } \{1,\ldots,q\}\},$$

we see that the multi dimensional version of the Glivenko-Cantelli theorem follows from Corollary 4.6.

<u>Theorem 4.8.</u> <u>Let</u> (Ω, F, P) <u>be a probability space, and</u> $\{\xi_n\}$ <u>a sequence of independent, identically distributed</u>, q-<u>dimensional</u> $(1 \leqq q < \infty)$, <u>random variables on</u> (Ω, F, P) <u>such that</u>

(4.8.1) $\qquad \mu(\partial K) = 0$ <u>for all convex sets</u> $K \subseteqq \mathbb{R}^q$

<u>where</u> μ <u>is the common distribution law of the</u> ξ_n <u>on</u> $\mathbb{R}^q$. <u>Let</u> $M\colon \mathbb{R}^q \to [0,\infty]$ <u>be an upper semi-continuous function such that</u> M <u>is</u> μ-<u>integrable. If</u> Φ <u>is the set of all functions,</u> $\varphi\colon \mathbb{R}^q \to \bar{\mathbb{R}}$ <u>satisfying:</u>

(4.8.2) $\qquad \{s \in \mathbb{R}^q \mid \varphi(s) \geqq a\}$ <u>is convex for all</u> $a \in \mathbb{R}$

(4.8.3) $\qquad |\varphi(s)| \leqq M(s) \qquad \forall s \in \mathbb{R}^q,$

<u>then</u> $\Phi \subseteqq L^1(\mu)$ <u>and we have</u>

(4.8.4) $$\sup_{\varphi \in \Phi} \left| \int \varphi\, d\mu - \frac{1}{n} \sum_{j=1}^{n} \varphi(\xi_j) \right| \underset{n\to\infty}{\to} 0 \qquad P\text{-a.s.}$$

Remarks. (a): If A is a subset of a topological space, then ∂A denotes the boundary of A, i.e. $\partial A = \mathrm{cl}\, A \setminus \mathrm{int}\, A$.

(b): Notice that (4.8.1) holds in particular if μ is absolutely continuous with respect to a product of atom-free one-dimensional probability measures.

(c): A function φ satisfying (4.8.2) is usually called unimodal or quasi-concave.

Proof. First notice that by (4.8.1) we have that every convex set and every unimodal function is μ-measurable. Hence by integrability of M and (4.8.3) we find that $\Phi \subseteqq L^1(\mu)$.

We shall apply Proposition 3.1 with $S = \mathbb{R}^q$, $T = \Phi$ and $g(s,\varphi) = \varphi(s)$ for $(s,\varphi) \in \mathbb{R}^q \times \Phi$. To do this, we need the so-called upper and lower Fell topologies.

Let Usc denote the set of all upper semi-continuous functions from $\mathbb{R}^q$ into $\overline{\mathbb{R}}$. Then the upper Fell topology $\bar{\eta}$ on Usc is the weakest topology on Usc satisfying

(i) $\qquad \varphi \rightsquigarrow \sup_{s \in G} \varphi(s)$ is lower semi-continuous $\forall G$ open

(ii) $\qquad \varphi \rightsquigarrow \sup_{s \in K} \varphi(s)$ is upper semi-continuous $\forall K$ compact.

Then $(\mathrm{Usc}, \bar{\eta})$ is a compact metric space and

(iii) $$\varphi_n \to \varphi \text{ in } \bar{\eta} \iff \begin{cases} \varphi(s) \leqq \limsup\limits_{n\to\infty} \varphi_n(s_n) & \forall s \quad \forall s_n \to s \\ \forall s \quad \exists \{s_n\}: s_n \to s & \text{and} \quad \varphi_n(s_n) \to \varphi(s). \end{cases}$$

Now let $L_1 = \Phi \cap \mathrm{Usc}$ and $g_1(s,\varphi) = \varphi(s)$ for (s,φ) $\mathbb{R}^q \times L_1$. It is then a routine matter to verify the following propositions:

(iv) L_1 is a closed subset of $(\mathrm{Usc}, \bar{\eta})$

(v) g_1 is upper semi-continuous on $\mathbb{R}^q \times L_1$

(vi) $\{\varphi_n\} \subseteq L_1$, $\varphi_n \to \varphi$ in $\bar{\eta} \Rightarrow \varphi_n(s) \to \varphi(s) \quad \forall s \in C(\varphi)$

where $C(\varphi)$ is the set of continuity points of φ. If $D(\varphi)$ denotes the set of discontinuity points of φ and Q is a dense subset of $\mathbb{R}$, then the reader easily verifies the following inclusion:

(vii) $D(\varphi) \subseteq \bigcup_{q \in Q} \partial\{\varphi \geq q\} \qquad \forall \varphi: \mathbb{R}^q \to \bar{\mathbb{R}}$

So by (4.8.1) and (vi) - (vii) we have

(viii) $\mu(C(\varphi)) = 1 \qquad \forall \varphi \in \Phi$

(ix) $\mu(s \mid \partial_S(g_1(s), \varphi) = 0) = 1 \qquad \forall \varphi \in L_1$

Thus by Theorem 4.5 we have

(x) g_1 is totally bounded in μ-mean.

Now let Lsc denote the set of all lower semi-continuous functions $\psi: \mathbb{R}^q \to \bar{\mathbb{R}}$ with its <u>lower Fell topology</u> η^0, i.e. the weakest topology Lsc satisfying

(i)* $\psi \leadsto \inf_{s \in G} \psi(s)$ is upper semi-continuous $\forall G$ open

(ii)* $\psi \leadsto \inf_{s \in K} \psi(s)$ is lower semi-continuous $\forall K$ compact.

Then exactly as above we have

(x)* g_0 is totally bounded in μ-mean

where $L_0 = \Phi \cap \mathrm{Lsc}$ and $g_0(s, \varphi) = \varphi(s)$ for $(s, \varphi) \in \mathbb{R}^q \times L_0$.

Now let $\varphi \in \Phi$, and φ^0 and $\bar{\varphi}$ be the lower respectively upper semi-continuous envelopes of φ. Since

$$\{\varphi^0 > a\} = \text{int}\{\varphi > a\} \qquad \text{and} \qquad \{\bar{\varphi} \geqq a\} = \text{cl}\{\varphi \geqq a\},$$

we see that φ^0 and $\bar{\varphi}$ are unimodal, and since M is upper semi-continuous and $(-M)$ is lower semi-continuous and $-M \leqq \varphi \leqq M$, we have

$$-M(s) \leqq \varphi^0(s) \leqq \varphi(s) \leqq \bar{\varphi}(s) \leqq M(s).$$

Thus $\varphi^0 \in L_0$ and $\bar{\varphi} \in L_1$ and moreover we have

$$g_0(s,\varphi^0) = \varphi^0(s) \leqq \varphi(s) \leqq \bar{\varphi}(s) = g_1(s,\bar{\varphi})$$

$$g_1(s,\bar{\varphi}) - g_0(s,\varphi^0) = 0 \qquad \forall s \in C(\varphi).$$

Hence conditions (3.1.1) and (3.1.2) hold by (4.8.1), (viii), (x) and (x)*. Thus by Theorem 3.1 we have that g is totally bounded in μ-mean and $g \in \text{LLN}(\mu,B(\Phi))$, and so the theorem follows from (2.14). □

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J.HOFFMANN-JØRGENSEN
Mathematisk Institut
Åarhus Universitet
Ny Munkegade 06-12 71 88-
8000 Åarhus C
Danemark