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SOME RESULTS ON THE BIDISC ALGEBRA

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0. Introduction

The main purpose of this paper is to present some results on the existence of bounded bianalytic functions on the bidisc D x D, appearing as elements of A(DxD) or H^{∞}(DxD). The method is indirect and based on duality. This work appears in the same circle of ideas which enabled to show that the dual $A(D)^{\star}$ of the disc algebra A(D) has the cotype-2 property, a notion which will be explained later. It depends heavily on one-variable methods, such as the log modularity of A(D), as will be clear in what follows. In this context, several questions remain unsettled which seem natural.
In the last section of the paper, the structure of the dual of polydisc algebras , in particular $A(D^2)^{\star}$, is studied. As a corollary, the non-isomorphism of the Banach spaces $A(D^2)$ and $A(D^3)$ is deduced. The non-existence of a linear isomorphism between the spaces A(D) and $A(D^2)$ was already known for some time and previous results yield another contribution to the so-called dimension conjecture (cfr. [14], problem n. 1).
The reader will find in [3] full proofs of some results stated here. The material of the last section was not published earlier and appears here in detail.

1. Analytic projections

The main tool in the approach using duality and extrapolation arguments is a generalization of the classical Riesz-projection $\mathcal{R}$, which gives the orthogonal projection from $L^2(\Pi)$ onto the space H^2. Let m be the Lebesgue measure on the circle Π. If μ is a Radon probability measure on Π and $0 < p < \infty$, we denote by $H^p(\mu)$ the closure of the analytic trigonometric polynomials in the space $L^p(\mu)$. Let

$$d\mu = \Delta.dm + d\mu_s$$

be the Lebesgue decomposition of μ. As a consequence of peak-set theory, we obtain the splitting

$$H^p(\mu) = H^p(\Delta) \oplus L^p(\mu_s)$$

We are concerned with projections from $L^p(\mu)$ onto $H^p(\mu)$ for $1 < p < \infty$. In this problem, we may restrict ourselves to measures μ regular with respect to m, say $\frac{d\mu}{dm} = \Delta$.

It is well-known that if $1 < p < \infty$, the usual Riesz-operator is bounded on $L^p(\Delta)$ iff Δ satisfies Muckenhoupt's (A_p)-condition (see [12]), that is

$$\sup \left(\frac{1}{|I|}\int_I \Delta\right)\left(\frac{1}{|I|}\int_I \left(\frac{1}{\Delta}\right)^{\frac{1}{p-1}}\right)^{p-1} < \infty$$

where the supremum is taken over all intervals I in the circle. In particular, if Δ satisfies the A_1-condition, thus

$$\Delta^\star \leqslant \text{const.}\ \Delta,\quad \Delta^\star = \text{Hardy-Littlewood maximal function}$$

$\mathcal{R}$ is $L^p(\Delta)$-bounded for all $1 < p < \infty$ and $L^1(\Delta) - L^{1,\infty}(\Delta)$ bounded.

However, the measure μ appearing above follows in general from a Hahn-Banach separation argument (for instance the Grothendieck-Pietsch factorization procedure for summing operators) and is generally not a weight. This difficulty is overcome constructing ad-hoc projections as substitute for $\mathcal{R}$. In fact, for our purpose, Δ can be replaced by any $\Delta_1 \geqslant \Delta$, as long as

$$\int \Delta_1 \leqslant \text{const.} \int \Delta$$

Fix $1 < p < \infty$ and suppose $\log \Delta$ an $L^1(\Pi)$ function. Let Φ be an outer function with $|\phi| = \Delta^{1/p}$ on Π and define

$$P(\varphi) = \Phi^{-1}\mathcal{R}[\varphi\Phi] \qquad (1)$$

yielding a bounded projection from $L^p(\Delta)$ onto $H^p(\Delta)$. These projections, clearly also depending on p besides Δ, were considered by B. Mitjagin and A. Pelczynski in the context of absolutely summing and integral operators on the disc algebra (see [13] or [14]). The method was improved by S.V. Kisliakov (see [9]). Majorize Δ by Δ_1 such that $\Delta_1^{1/2}$ is an (A_1)-weight. Take for instance

$$\Delta_1^{1/2} = \sum_{j=0}^{\infty} \delta^j \, (\Delta^{1/2})^{\overbrace{\star\ldots\star}^{j}}$$

for a suitable constant $\delta > 0$ and where $\alpha^{\overbrace{\star\ldots\star}^{j}}$ denotes the j-fold maximal function. If $|\Phi| = \Delta_1^{1/2}$, the operator considered in (1) provides a bounded projection from $L^p(\Delta_1)$ onto $H^p(\Delta_1)$ in the interval $1 < p \leq 2$. There seems however to be no reason for P to be $L^1(\Delta_1) - L^1(\Delta_1)$, weak bounded, which is of importance for our purpose. The construction of such projections is done by a more sophisticated technique, which will be indicated later. One has the following theorem (see [3], Th. 1.1)

<u>THEOREM 1</u> : <u>Assume</u> $\Delta \in L^1_+(\Pi)$, $\int \Delta \, dm = 1$. <u>There exists</u> $\Delta_1 \in L^1_+(\Pi)$, $\Delta_1 \geq \Delta$ <u>and</u> $\int \Delta_1 \, dm \leq$ const. <u>and a projection</u> P <u>from</u> $L^2(\Delta_1)$ <u>onto</u> $H^2(\Delta_1)$ <u>which is bounded on</u> $L^p(\Delta_1)$ <u>for</u> $1 < p < \infty$ <u>and</u> $L^1(\Delta_1) - L^{1,\infty}(\Delta_1)$ <u>bounded</u>.

It may be an interesting question to decide for what functions Δ_1 (if there is any restriction) a projection P verifying these conditions can be found.
The proof of Th. 1 does not clarify this point.

2. Results of Kisliakov on Fourier coefficients of functions in the bidisc algebra

Denote $A(D \times D)$ the algebra of bianalytic functions on $D \times D$, which are continuous on $\bar{D} \times \bar{D}$ and $H^\infty(D \times D)$ the algebra of bounded bianalytic functions on $D \times D$.
Passing to the radial limit, these spaces will be mostly seen as function spaces on $\Pi \times \Pi$. Define further

$$\hat{f}(m,n) = \int f(\theta,\psi)\ e^{-i(m\theta+n\psi)}\ m(d\theta)\ m(d\psi)$$

for $m,n \in \mathbb{Z}$.

The following theorem and its consequences were proved by S.V. Kisliakov a few years ago (see [10]).

PROPOSITION 2 : *Assume $(a_{mn})_{m,n\geq 0}$ a matrix of positive numbers such that*

$$\sum_{m,n=0}^{\infty} a_{mn}^2 \leq 1$$

For each $\delta > 0$, there exists a function $f \in A(D^2)$ satisfying

$$\left(\Sigma\ \left|a_{mn} - |\hat{f}(m,n)|\right|^2\right)^{1/2} < \delta \quad \text{and} \quad \|f\|_\infty < C(\delta)$$

where $C(\delta)$ is a function of δ.

The proof is indirect and based on an interpolation-inequality for the double Riesz-transform $K = \mathcal{R} \otimes \mathcal{R}$, which can be stated as follows

PROPOSITION 3 : *Assume $1 < p < q$ and $\frac{1}{p} = \theta + \frac{1-\theta}{q}$. Then following inequality holds*

$$\|K[\alpha]\|_p \leq C\ \|\alpha\|_1^\theta\ \|K[\alpha]\|_q^{1-\theta}$$

for $\alpha \in L^q(\Pi \times \Pi)$ and where $C = C(p,q)$ is a constant.

A proof will be outlined in section 4 of this paper.
Let us state 2 consequences of Prop. 2. The first is formal, the second follows from an abstract scheme developped by S.V. Hruščev (see [10] for details).

COROLLARY 4 : Denote $M(A(D^2),\ell^1)$ the space of multipliers from the bidisc algebra in the space $\ell^1(\mathbb{Z}_+ \times \mathbb{Z}_+)$. These are the positive systems $\lambda = (\lambda_{mn})_{m,n\geq 0}$ such that

$$\sum_{m,n} \lambda_{m,n} \ |\hat{f}(m,n)| < \infty \quad \text{for } f \in A(D^2)$$

Then $\lambda \in M(A(D^2),\ell^1)$ iff λ is square summable.

COROLLARY 5 : For any square-summable matrix $(a_{mn})_{m,n\geq 0}$ of positive numbers, there exists $f \in A(D^2)$ such that

$$a_{mn} \leq |\hat{f}(m,n)|, \text{ for all } m,n \geq 0.$$

Curiously, the characterization of ℓ^1-multipliers given in Cor. 4 seems unknown for the algebra $A(D^3)$. The validity of Prop. 3 for the 3-fold Riesz-projection is not settled. The extension of the result is not obvious, since again the argument depends on the log-modular property of the one-variable space.

3. Fourier coefficients of bianalytic functions with respect to one variable

For $f \in A(D^2)$ on $f \in H^\infty(D^2)$, we can define for each $m = 0,1,2,\ldots$ the function

$$f_m(\psi) = \int f(\theta,\psi)\ e^{-im\theta}\ d\theta.$$

In this way, a sequence of $A(D)$ (resp. $H^\infty(D)$) functions is obtained, which is clearly weakly 2-summable, thus

$$\sup_{z\in D} \left(\sum |f_m(z)|^2\right)^{1/2} < \infty$$

Combining Th. 1 and inequalities of the type stated in Prop. 3, several results concerning the 1-variable coefficients f_m can be proved. They refine Kisliakov's theorem. Let us state some of them.

PROPOSITION 6 : The weak*-closure of the convex hull of the set

$\{(\varepsilon_1 f_1, \varepsilon_2 f_2, \ldots) \mid \varepsilon_m = \pm 1$ <u>and</u> (f_m) <u>corresponding to</u> $f \in A(D^2), \|f\| \leq 1\}$ <u>in the space</u> $H^\infty_{\ell^2}(D)$ <u>has nonempty interior</u>.

<u>PROPOSITION 7</u> : <u>Assume</u> $\alpha_0, \alpha_1, \alpha_2, \ldots$ <u>a sequence in</u> $L^1(\Pi)$. <u>Then the map</u>

$$A(D^2) \to \ell^1 : f \leftrightarrow (\langle f_0, \alpha_0\rangle, \langle f_1, \alpha_1\rangle, \langle f_2, \alpha_2\rangle, \ldots)$$

<u>is bounded iff</u>

$$\inf_{h_m \in H^1_0} \int_\Pi (\Sigma\, |\alpha_m + \bar{h}_m|^2)^{1/2} < \infty$$

For convenience, we denote here $H^1_0 = \{h \in H^1 \mid h(0) = 0\}$.

Say that a subset S of $\mathbb{Z}$ is a Λ_2-set provided $L^1(\Pi)$ and $L^2(\Pi)$ norms are equivalent on the linear span of $\{e^{in\theta} \mid n \in S\}$.

<u>PROPOSITION 8</u> : <u>Suppose</u> $S \subset \mathbb{Z}_+$ <u>a</u> Λ_2-<u>set and</u> $(\varphi_m)_{m \in S}$ <u>a weakly 2-summable sequence in</u> H^∞. <u>Then there exists</u> $f \in H^\infty(D^2)$ <u>satisfying</u>

$$f_m = \int f(\theta, .)\, e^{-im\theta}\, d\theta = \varphi_m \text{ for each } m \in S$$

This fact is a 2-variable version of a well-known H^∞ property. In particular, one can take for the set S in Prop. 8 some lacunary set. Ignoring the analyticity of f with respect to the variable θ, the following fact appears

Denote (ε_m) the Rademacher sequence on the Cantor group $\{1,-1\}^{\mathbb{N}}$ and assume $(\varphi_m)_{m=1,2,\ldots}$ a sequence in $H^\infty_{\ell^2}(D)$. Then there exists f in $L^\infty(\{1,-1\}^{\mathbb{N}} \times D)$ such that

$$\hat{f}\{m\}(\psi) = \int f(\varepsilon, \psi)\, \varepsilon_m\, d\varepsilon = \varphi_m(\psi), \text{ for each } m$$

and

$$f(\varepsilon, .) \text{ is an } H^\infty\text{-function in the second variable.}$$

We say that a Banach space X has cotype q $(2 \leq q < \infty)$ provided the inequality

$$\int \|\Sigma\, \varepsilon_m x_m\| \, d\varepsilon \geq c_q(X) \, (\Sigma \|x_m\|^q)^{1/q}$$

holds for all finite sequences (x_m) in X.

The previous observation implies formally that the spaces $A(D)^*$ and hence L^1/H_0^1 have the cotype-2 property. This problem is open for $A(D^2)^*$. In fact, it seems even unknown whether or not $A(D^2)^*$ has a finite cotype, which means equivalently that the space c_0 is not finitely representable in $A(D^2)^*$.

The results mentioned before are derived from the following proposition

PROPOSITION 9 : *Assume μ a regular product measure on Π^2. Then there exist a regular product measure μ' on Π^2, $\mu < \mu'$ and a projection Q in $L^2(\mu')$, bounded on $L^r(\mu')$ for $1 < r < \infty$, such that*

(i) $\|\mu'\| \leq \text{const} \|\mu\|$

(ii) $\text{Spec}\,(I-Q) \cap \Lambda = \phi$ (I = identity)

(iii) *For* $1 < p < q$ *and* $p^{-1} = \theta + (1-\theta)q^{-1}$

$$\|Q[\alpha]\|_{L^p(\mu')} \leq \text{const.} \, (\inf_{\beta} \|\alpha+\beta\|_{L^1(\mu')})^{\theta} \, \|\alpha\|^{1-\theta}_{L^q(\mu')}$$

where Λ is the negative quadrant $\{(m,n) \in \mathbb{Z} \times \mathbb{Z} \mid m \leq 0,\ n \leq 0\}$ and the infimum in (iii) *is taken over functions β such that* $(\text{Spec}\,\beta) \cap \Lambda = \phi$.

The proof of Prop. 6 follows from the property

$$\lim_{C\to\infty} \inf_{\substack{f\in A(D^2),\, \|f\|\leq C \\ \varepsilon_m = \pm 1}} \int (\Sigma\, |\varphi_m(\psi) - \varepsilon_m f_m(\psi)|^2)^{1/2} \, \nu(d\psi) = 0$$

uniformly for regular probability measures ν on Π and finite sequences (φ_m) in $A(D)$ satisfying $\|\Sigma\, |\varphi_m|^2\|_\infty \leq 1$. This fact is deduced from Prop. 9 by a duality reasoning, taking $\mu = m \otimes \nu$.

Prop. 8 admits a generalization in a more Banach space theoretical language, which we state now.

PROPOSITION 10 : *Assume Y a reflexive subspace of* L^1/H_0^1 *and u a bounded linear operator from Y into* H^∞. *Then there exists a function* $f \in H^\infty(D \times D)$ *which "extends" the operator u, i.e. such that*

$$u(y)(\psi) = \int f(\theta,\psi)\, y(\theta)\, d\theta \quad \text{for each } y \in Y$$

This follows from Prop. 9.
The application is indeed clear. Take $Y = [e^{-im\theta} \mid m \in Y]$ and define u by $u(e^{-im\theta}) = \varphi_m$.

4. Outline of some proofs

The proof of Th. 1 uses the following "decomposition" lemma, depending crucially on the log modularity of A(D).

PROPOSITION 11 : *Assume* $f \in L^1(\Pi)$, $f \geq 0$. *There are sequences* (c_i) *of positive scalars and* (θ_i), (τ_i) *of* H^∞ *functions, such that following conditions are fulfilled*

1. $\|\theta_i\|_\infty \leq 1$ *for each i*
2. $\|\Sigma_i\, |\tau_i|\,\|_\infty \leq c$
3. $\Sigma_i\, \theta_i\, \tau_i^5 = 1$

Defining $F = \Sigma\, c_i\, |\tau_i|$

4. $f \leq F$
5. $|\tau_i|\, F \leq C\, c_i$ *for each i*
6. $\|F\|_1 \leq C\, \|f\|_1$

where C stands for a numerical constant.

The role of the power 5 in (3) will appear later. In fact, one could take any other power. The idea behind the H^∞-sequence (τ_i) is to generate something which looks like a partition of unity of Π with respect to the level sets of f.
The functions θ_i appear only for a technical reason. Details of the construction can be found in [3] (see Prop. 1.2).

Let us now pass to the proof of Th. 1. Apply Prop. 11 to $f = \Delta + 1$ and define

$$\Delta_1 = F \quad \text{and} \quad P(\varphi) = \Sigma_i \ \theta_i \ \tau_i^4 \ \mathcal{R}[\tau_i \varphi]$$

Clearly $P_0(\varphi) = \Sigma' \ \theta_i \ \tau_i^4 \ \mathcal{R}[\tau_i \varphi]$ makes sense (as $L^{1,\infty}$-function) for finite sums Σ' and $\varphi \in L^1(\Pi)$. For $1 < p < \infty$, the $L^p(\Pi)$-boundedness of $\mathcal{R}$ gives (using (2), (5) of Prop. 11)

$$\int |P_0\varphi|^p \ \Delta_1 \leqslant C \ \Sigma \int |\tau_i| \ |\mathcal{R}[\tau_i\varphi]|^p \ \Delta_1 \leqslant C \ \Sigma \ c_i \int |\mathcal{R}[\tau_i\varphi]|^p \leqslant$$

$$c_p \int |\varphi|^p \ \Sigma' \ c_i \ |\tau_i| \leqslant C \int |\varphi|^p \ \Delta_1.$$

Thus, for $\varphi \in L^p(\Delta_1)$, the series defining P converge in $L^p(\Delta_1)$ and

$$\|P\varphi\|_{L^p(\Delta_1)} \leqslant C \ \|\varphi\|_{L^p(\Delta_1)}$$

The above argument also easily shows that $P(\varphi) \in H^p(\Delta_1)$, using the inclusion $H^\infty \subset H^p(\Delta_1)$ and approximation of $\mathcal{R}$ by $\mathcal{R} \star P_\rho$ $(0 < \rho < 1)$, where P_ρ denotes the Poisson kernel. Also, if $\varphi \in H^p(\Delta_1)$, it follows from Prop. 11 (3) that $P\varphi = \varphi$.
It follows from the weak-type property of $\mathcal{R}$ (see [5]) that if $\alpha \in L^1(\Pi)$, $\beta \in L^\infty(\Pi)$

$$\int |\mathcal{R}[\alpha]|^{1/2} \ |\beta| \ dm \leqslant C \ \|\alpha\|_1^{1/2} \ \|\beta\|_\infty^{1/2} \ \|\beta\|_1^{1/2} \qquad (\star)$$

Hence, for any $\omega \in L_+^\infty(\Pi)$, $\|\omega\|_\infty = 1$

$$\int |P_0\varphi|^{1/2} \ \omega \ \Delta_1 \leqslant \Sigma \int |\tau_i|^2 \ |\mathcal{R}[\tau_i\varphi]|^{1/2} \ \omega \ \Delta_1$$

$$\leqslant C \ \Sigma \ c_i \int |\mathcal{R}[\tau_i\varphi]|^{1/2} \ |\tau_i| \ \omega$$

$$\leqslant C \ \Sigma \ c_i \ \|\tau_i\varphi\|_1^{1/2} \ \|\tau_i\omega\|_1^{1/2}$$

$$\leqslant C \ \{\int |\varphi| \ (\Sigma \ c_i \ |\tau_i|)\}^{1/2} \ \{\int \omega \ \Delta_1\}^{1/2}$$

using the Cauchy-Schwarz inequality.
Specifying ω as the indication function of the set $[\,|P_0\varphi| \geqslant \lambda]$, $\lambda>0$ a fixed number, it follows

$$\sqrt{\lambda}\ \mathbb{P}_{\Delta_2}[|P_0\varphi| \geq \lambda] \leq C\ \|\varphi\|^{1/2}_{L^1(\Delta_1)}\ \mathbb{P}_{\Delta_1}[|P_0\varphi| \geq \lambda]^{1/2}$$

and hence

$$\lambda\ \mathbb{P}_{\Delta_1}[|P_0\varphi| \geq \lambda] \leq C\ \|\varphi\|_{L^1(\Delta_1)}$$

If φ is in $L^1(\Delta_1)$, $P\varphi = \lim P_0\varphi$ in $L^{1,\infty}(\Delta_1)$ and previous estimation yields

$$\|P\varphi\|_{L^{1,\infty}(\Delta_1)} \leq C\ \|\varphi\|_{L^1(\Delta_1)}.$$

This completes the proof of Th. 1.

Prop. 3 is a consequence of the following 1-variable interpolation inequality for the Riesz transform $\mathcal{R}$.

<u>PROPOSITION 12</u> : <u>If</u> p,q,θ <u>are as in Prop.</u> 3, <u>the following inequality holds</u>

$$\|\mathcal{R}[\alpha]\|_p \leq C\ \|\alpha\|^{\theta}_{1,\infty}\|\mathcal{R}[\alpha]\|^{1-\theta}_q \tag{$\star$}$$

<u>for</u> $\alpha \in L^q(\Pi)$.

<u>Proof</u> : Take $\lambda = \|\alpha\|^{-q'/q}_{1,\infty}\|\mathcal{R}[\alpha]\|^{q'}_q$ and $\zeta = (1 \vee \lambda^{-1}|\alpha|)^{-1}$.
Consider the outer function τ defined by

$$\tau(z) = \int \log|\zeta(\theta)|\ \frac{e^{i\theta}+z}{e^{i\theta}-z}\ m(d\theta) \qquad (z \in D)$$

thus with boundary value

$\tau = \zeta\ e^{i\,\mathcal{H}[\log \zeta]}$, where $\mathcal{H}$ denotes the Hilbert-transform

Thus, by construction

$$|\alpha|\ |\tau| \leq \lambda \quad \text{on } \Pi$$

Next, write

$$\alpha = \bar{\tau}\ \alpha + (1-\bar{\tau})\ \alpha$$

$$\mathcal{R}[\alpha] = \mathcal{R}[\bar{\tau}\alpha] + \mathcal{R}[(1-\bar{\tau})\ \mathcal{R}[\alpha]]$$

implying

$$\|\mathcal{R}[\alpha]\|_p \leqslant c_p \|\tau\alpha\|_p + c_p \|(1-\tau)\mathcal{R}[\alpha]\|_p.$$

Estimate

$$\|\tau\alpha\|_p^p \leqslant c_p \lambda^{p-1} \|\alpha\|_{1,\infty}$$

and using Hölder's inequality

$$\|(1-\tau)\mathcal{R}[\alpha]\|_p^p \leqslant \left(\int |1-\tau|^{\frac{pq}{q-p}}\right)^{1-p/q} \|\mathcal{R}[\alpha]\|_q^p$$

where

$$\int |1-\tau|^{\frac{pq}{q-p}} \leqslant \int |1-\tau|^p \leqslant c_p \int_{[|\alpha|\geqslant\lambda]} \left(\log \frac{2|\alpha|}{\lambda}\right)^p \leqslant c_p \lambda^{-1} \|\alpha\|_{1,\infty}$$

Finally

$$\|\mathcal{R}[\alpha]\|_p \leqslant c_p \lambda^{1-1/p} \|\alpha\|_{1,\infty}^{1/p} + c_p \lambda^{1/q-1/p} \|\alpha\|_{1,\infty}^{1/p-1/q} \|\mathcal{R}[\alpha]\|_q$$

yielding (*) after substitution of λ.

Let us point out here that Prop. 12 clearly does not hold any more if $\mathcal{R}$ is replaced by the double Riesz-projection $\mathcal{R} \otimes \mathcal{R}$.

The deduction of Prop. 10 from Prop. 8 uses a "lifting" property for reflexive subspaces of $L^1/_{H^1_0}$, namely (see [2], Cor. 2.13)

PROPOSITION 13 : *Let Y be a reflexive subspace of $L^1/_{H^1_0}$ and $q : L^1(\Pi) \to L^1/_{H^1_0}$ the quotient map. Then there exists a subspace $\bar{Y}$ of $L^1(\Pi)$ such that the restriction map $q|\bar{Y}$ is an isomorphism from $\bar{Y}$ onto Y, more precisely*

$$\|\bar{y}\|_1 \leqslant C \|q(\bar{y})\| \quad \text{for } \bar{y} \in \bar{Y}$$

where C depends on the so-called type $p > 1$ and type-constant $T_p(Y)$ of Y.

The reader is referred to [11] for definition of type and to [3] for details. To $\bar{Y}$, H. Rosenthal's theorem (see [15]) is applied, yielding an embedding of $\bar{Y}$ in an L^r-space $(1 < r < p)$ by a change of density. The full proof of Prop. 13 is given in [3], section V.

5. On the dual of polydisc algebra's

For $d = 1,2,\dots$, let $A(D^d)$ be the algebra of functions which are analytic on D^d and continuous on $\bar{D}^d$. The problem whether or not the spaces $A(D^d)$ are mutually non-isomorphic (sometimes called "dimension conjecture") is unsolved in general. It is known that $A(D)$ and $A(D^2)$ are not linearly isomorphic (cfr. [13], [14]) and the purpose of this section is to show

THEOREM 14 : The Banach spaces $A(D^2)$ and $A(D^3)$ are not isomorphic.

The non-isomorphism of $A(D)$ and $A(D^d)$ for $d \geq 2$ is already a consequence of local structure. The invariant is the so-called $(i_p - \pi_p)$-ratio k_p $(1 \leq p < \infty)$ defined as follows for a Banach space X

$$k_p(X) = \sup i_p(u)$$

where the supremum is taken over all p-summing operators u from X into an arbitrary Banach space such that $\pi_p(u) = 1$ (i_p denotes the strictly p-integral norm).
For $X = A(D)$ and $1 \leq p < \infty$, it is known that $k_p(X) < \infty$ and more precisely $b_p(X) \sim \frac{p^2}{p-1}$. Now if $X = A(D^d)$, it is not difficult to see that $k_p(X) \geq c\, (\frac{p^2}{p-1})^d$, considering the natural identity map $u : A(D^2) \to H^p(\Pi^d)$.
Details and proofs of these results can be found in [14]. The previous distinction between dimension 1 and $d \geq 2$ will be used in proving Th. 2, which relies also on infinite dimensional considerations.

In fact, it will shown that the dual spaces $A(D^2)^*$ and $A(D^3)^*$ are not isomorphic. The first step consists in a description of $A(D^2)^*$ as a direct sum of a separable Banach space and a space related to the one-variable Riesz-projection. This part of the proof generalizes to arbitrary dimension. Denote C the space of continuous functions on Π.

PROPOSITION 15 : Let $i : A(D^d) \to C(\Pi^d)$ be the embedding. The spaces

$$A(D^d)^* \quad \text{and} \quad i^*(L^1(\Pi^d)) \oplus (A(D^{d-1}) \check{\otimes} C)^*$$

are isomorphic to the d-fold projective tensorproduct of $A(D)^*$

COROLLARY 16 : There is a linear isomorphism between the spaces $A(D^2)^\star$ and $i^\star(L^1(\Pi^2)) \oplus (A \overset{\vee}{\otimes} C)^\star$.

Postponing the proof of Prop. 15, let us derive Th. 14 from Cor. 16.

LEMMA 17 : The dual space $A(D^3)^\star$ contains a complemented copy of the space $\ell^1_\Gamma(\Sigma\ A(D^2)^\star)$, where Γ denotes the continuum.

Proof : The argument is essentially contained in [14], section 11. For each $a \in \Pi$, consider the subset $F_a = \{a\} \times \Pi^2$ of Π^3 and the subspace D_a of those elements of $A(D^3)^\star$ which are induced by a measure, supported by F_a. Since the F_a are disjoint peak-sets for $A(D^3)$, the subspace $\bigoplus_{a\in\Pi} D_a$ of $A(D^3)^\star$ satisfies the lemma.

Notice that if we write Prop. 15 for $d = 3$, the ℓ^1-sum appears immediately as complemented subspace of the second component.

The following elementary result from general Banach space theory will be needed.

LEMMA 18 : Let X,Y,Z be Banach spaces, Z being separable. If $\ell^1_\Gamma(\Sigma\ X)$, with Γ uncountable, embeds complementably in $Y \oplus Z$, then X embeds complementably in Y.

Proof : Denote $\mathfrak{X} = \ell^1_\Gamma(\Sigma\ X)$, $i : \mathfrak{X} \to Y\oplus Z$ the embedding and Q the projection onto $i(\mathfrak{X})$. Let $P_Y = P_Z$ be the Y,Z projections and $i_\gamma : X \to \mathfrak{X}$, $P_\gamma : \mathfrak{X} \to X$ $(\gamma \in \Gamma)$ have the obvious meaning. For each $\gamma \in \Gamma$, consider following operators

$$j_\gamma : P_Y\, i\, i_\gamma : X \to Y \quad \text{and} \quad Q_\gamma = P_\gamma\, i^{-1}\, Q : Y \to X$$

We show that $Q_\gamma\, j_\gamma$ is the identity for some $\gamma \in \Gamma$, which will complete the proof.

Notice that the operator $i^{-1}\, Q\, P_Z : Y \oplus Z \to \mathfrak{X}$ has separable range by hypothesis and hence there is a countable subset Γ_0 of Γ such that $P_\gamma\, i^{-1}\, Q\, P_Z = 0$ for $\gamma \in \Gamma\setminus\Gamma_0$. For $\gamma \in \Gamma\setminus\Gamma_0$, we obtain indeed

$$\begin{aligned} Id_X &= P_\gamma\, i^{-1}\, Q\, i\, i_\gamma = P_\gamma\, i^{-1}\, Q\, P_Y\, i\, i_\gamma + P_\gamma\, i^{-1}\, Q\, P_Z\, i\, i_\gamma \\ &= Q_\gamma\, j_\gamma . \end{aligned}$$

Proof of Th. 14 : If $A(D^2)^\star$ and $A(D^3)^\star$ are assumed isomorphic, then $A(D^2)^\star$ contains a complemented copy of $\ell^1_\Gamma(\Sigma\, A(D^2)^\star)$ and hence, by Cor. 16 and Lemma 18, $A(D^2)^\star$ embeds complementably in $(A \overset{\vee}{\otimes} C)^\star$. Taking the local structure of these spaces in account, this is however impossible.

Indeed, if $A(D^2)^{\star\star}$ was isomorphic to a complemented subspace of $(A \overset{\vee}{\otimes} C)^{\star\star}$, it should follow for $1 < p < \infty$

$$k_p(A(D^2)) \leqslant \text{const } k_p(A \overset{\vee}{\otimes} C) = \text{const } k_p(A)$$

using the general facts $k_p(X) = k_p(X^{\star\star})$ (see [8]) and $k_p(X) = k_p(X \overset{\vee}{\otimes} C)$.

Since the constant does not depend on p, the previous discussion yields a contradiction for $p \to \infty$.

It remains to prove Prop. 15. We will make use of Bishop's generalized Rudin-Carleson theorem [2] as stated in [14].

THEOREM 19 : Let X be a subspace of a C(K)-space. Let F be a closed subset of K such that

$$\mu(F_1) = 0 \text{ for every closed } F_1 \subset F \text{ and every } \mu \in X^\perp$$

(where $X^\perp$ denotes the annihilator of X).

Then for every $u \in C(F)$, $\varepsilon > 0$ and every open set $G \supset F$, there is an $f_u \in X$ such that

(i) $f_u(s) = u(s)$ for $s \in F$

(ii) $|f_u(s)| \leqslant \varepsilon \, \|u\|$ for $s \notin G$

(iii) $\|f_u\| = \|u\|$.

Moreover, if X is separable, the map $u \mapsto f_u$ from C(F) into X can be chosen to be a linear isometry.

This result will be used to prove the following fact. Compact spaces are assumed separable and metrizable (they will appear as k-dimensional torus).

COROLLARY 20 : Let X,Y be subspaces of C(K), C(L) respectively. Let $B \subset M(K)$ be the band of measures on K singular with $X^\perp$, thus

$$B = \{\mu \in M(K) \mid \mu \perp \nu \text{ \underline{for} \underline{each} } \nu \in X^{\perp}\}.$$

Denote $i_1 : X \to C(K)$, $i_2 : Y \to C(L)$ the embedding operators and let $i = i_1 \otimes i_2 : X \overset{\vee}{\otimes} Y \to C(KxL)$ and $j = Id \otimes i_2 : C(K) \overset{\vee}{\otimes} Y \to C(KxL)$.

(a) For $\xi \in B \hat{\otimes} M(L)$, one has

$$\|i^{\star}(\xi)\| = \|j^{\star}(\xi)\|$$

Consequently, if E is a subspace of $M(L)$, then $i^{\star}(B \hat{\otimes} E)$ and $B \hat{\otimes} i_2^{\star}(E)$ are isometric.

(b) Suppose $K = K' \times K''$, R a separable band in $M(K')$ and $B \perp R \hat{\otimes} M(K'')$. Then with previous notations, also

$$\|i^{\star}(\xi+\xi_1)\| \geqslant \|j^{\star}(\xi)\|$$

for $\xi \in B \hat{\otimes} M(L)$ and $\xi_1 \in R \hat{\otimes} M(K''xL)$.

LEMMA 21 : If B is as in Cor. 20, every element $\mu \in B$ is localized on an F_{σ}-subset of K, of $|\nu|$-measure 0 for each $\nu \in X^{\perp}$.

Proof : Clearly we may assume μ a probability measure. Consider a countable dense subset $\mathcal{D}$ of $C(K)$ of non-vanishing functions τ. Fix $\tau \in \mathcal{D}$ and $\varepsilon > 0$. Since by hypothesis $d(\mu, \tau X^{\perp}) = \|\mu\|$ and $\tau X^{\perp}$ is w-closed in $M(K)$, there is a function $\varphi = \varphi_{\tau}$ in $C(K)$, $\|\varphi\| = 1$ satisfying $\operatorname{Re} \int \varphi \, d\mu > 1 - \varepsilon^3$ and $\int \varphi \, \tau \, d\nu = 0$ for each $\nu \in X^{\perp}$.

Thus $\varphi \, \tau \in X$. Defining $F_{\tau} = [\,|1-\varphi| \leqslant \varepsilon]$, we get

$$\int_{K \setminus F_{\tau}} d\mu \leqslant \varepsilon^{-2} \int |1-\varphi|^2 \, d\mu \leqslant 2\, \varepsilon^{-2} \int (1 - \operatorname{Re} \varphi) \, d\mu < 2\varepsilon.$$

Hence, fixing any number $\delta > 0$ and choosing a suitable sequence $\varepsilon = \varepsilon_{\tau}$ for $\tau \in \mathcal{D}$, the set $F = \bigcap_{\tau \in \mathcal{D}} F$ will satisfy $\mu(K \setminus F) < \delta$.

It remains to show that $|\nu|(F) = 0$ if $\nu \in X^{\perp}$. Choose $\kappa > 0$ and a neighborhood G of F with $|\nu|(G \setminus F) < \kappa$. Let α be any $C(K)$-function vanishing outside G and $\tau \in \mathcal{D}$ with $\|\alpha - \tau\| < \kappa$ and $\varepsilon_{\tau} < \kappa$. Write

$$\alpha \, \chi_F = \tau \, \varphi_{\tau} + (\alpha - \tau)\varphi_{\tau} - \alpha \, \varphi_{\tau} \, \chi_{K \setminus F} + \alpha \, (1 - \varphi_{\tau}) \, \chi_F$$

from where, since $F \subset F_{\tau}$ and $\tau \, \varphi_{\tau} \in X$

$$\left| \int_F \alpha \, d\nu \right| \leqslant \kappa \, \|\nu\| + \kappa \, \|\alpha\| + \kappa \, \|\alpha\| \|\nu\|.$$

Consequently $\int_F \alpha \, d\nu = 0$, completing the proof.

Proof of Cor. 20 : It clearly suffices to prove (b). A density argument allows to assume $\xi = \Sigma_\lambda' \mu_\lambda \otimes \nu_\lambda$ and $\xi_1 = \Sigma_\lambda' \rho_\lambda \otimes \eta_\lambda$, where $\mu_\lambda \in B$, $\nu_\lambda \in M(L)$, $\rho_\lambda \in R$ and $\eta_\lambda \in M(K''xL)$. Fix $\varepsilon > 0$ and apply Lemma 21 to find a closed subset F of K satisfying

(i) $|\mu| \, (F) = 0$ for each $\mu \in X$

(ii) $|\mu_\lambda| \, (K\backslash F) < \varepsilon$ for each λ.

Moreover, the hypothesis on R and a similar construction as given in the proof of Lemma 21, permits to ensure the condition on $F \subset K' \times K''$.

(iii) $\rho \, (F_{x''}) = 0$ for each $x'' \in K''$ and $\rho \in R$.

Let G be some open neighborhood of F and $e : C(F) \to X$ the linear extension operator given in Th. 3, which applies again by Lemma 21. Take

$$a = \Sigma' \, \alpha(x) \, \psi(y)$$

such that

(iv) $\alpha \in C(K)$, $\psi \in Y$

(v) $\|a\| \leq 1$

(vi) $\mathrm{Re} \, \langle \xi, a\rangle > \|j^\star(\xi)\| - \varepsilon$

Define

$$b = \Sigma' \, e(\alpha|_F)(x) \, \psi(y)$$

Then clearly by the properties of e

(vii) $b \in X \check{\otimes} Y$

(viii) $\|b\| \leq \|a\|$

(ix) $a = b$ on $F \times L$

(x) $|b| \leq \varepsilon$ on $(K\backslash G) \times L$

Write, using (vii), (ix), (ii), (iii)

$$\begin{aligned}\|i^\star(\xi+\xi_1)\| &\geq \mathrm{Re}\,\langle \xi, b\rangle - |\langle \xi_1, b\rangle| \\ &\geq \mathrm{Re}\,\langle \xi, a\rangle - 2|\xi|((K\backslash F)xL) - |\xi_1|(GxL) - \varepsilon \, \|\xi_1\| \\ &> \|j^\star(\xi)\| - (1+\|\xi_1\|) \quad - \Sigma' |\mu_\lambda| (K\backslash F) \|\nu_\lambda\| - |\xi_1| ((G\backslash F)xL) \\ &> \|j^\star(\xi)\| - (1+\|\xi_1\| + \Sigma' \|\nu_\lambda\|)\varepsilon - |\xi_1| \; ((G\backslash E)xL)\end{aligned}$$

Since ε was arbitrary and G any neighborhood of F, we may conclude.

Use the symbol $[\![\ ]\!]$ to denote the generated band of measures.

LEMMA 22 : $[\![A(D^d)^\perp]\!] = A(D^d)^\perp + \sum_{j=1}^{d} L^1(\theta_j) \hat{\otimes} M(\theta_1,\dots,\tilde{\theta}_j,\dots,\theta_d)$ ($\sim$ meaning omission).

Proof : Clearly the left member contains the right member. It remains to prove that the right member is closed and contains all measures of the form $\alpha.\nu$ for $\nu \in A(D^d)^\perp$ and α a trigonometric polynomial in $\theta_1,\dots,\theta_d$.
The idea consists in considering a 1-variable polynomial F whose transform $\hat{F}$ equals 1 on the set $\{-N,-N+1,\dots,N\}$ and with $\|F\|_1 \leqslant 2$. Denote F_j the convolution operation on Π^d with the function $F(\theta_j)$. We may then use the decomposition

$$Id = F_1 + (I-F_1)F_2 + \dots + (I-F_1)\ \dots\ (I-F_d)$$

The method is standard and we leave details to the reader.

REMARK : In case of the ball-algebras $A = A(B_m)$, the set $[\![A^\perp]\!]$ are the Henkin measures and Lemma 22 corresponds to the theorems of Valskii and Henkin (cfr. [16], ch. 9).

Let $B = \{\mu \in M(\Pi^j)\ |\ \mu \perp \nu$ for each $\nu \in A(D^j)^\perp\}$, i.e. the band singular with respect to $[\![A(D^j)^\perp]\!]$. Fixing $S \subset \{1,\dots,d\}$ and identifying Π^j with the variables θ_i $(i \in S)$, $|S| = j$, the corresponding band B will be denoted B_S.

LEMMA 23 : $M(\Pi^d) = \sum_{S\subset\{1,\dots,d\}} X_S + A(D^d)^\perp$, where

$$X_S = L^1(\theta_i\ |\ i \in S)\ \hat{\otimes}\ B_{S^c}.$$

Proof : We argue by induction on the dimension d. One has

$$M(\Pi^d) = [\![A(D^d)^\perp]\!] \oplus B_{\{1,\dots,d\}}$$

and apply next Lemma 20 to the first component. Then the induction hypothesis permits to decompose each of the spaces $M(\theta_1,\dots,\tilde{\theta}_j,\dots,\theta_d)$. This gives the result, taking into account that

$$L^1(\theta_j)\ \hat{\otimes}\ A(\theta_1,\dots,\tilde{\theta}_j,\dots,\theta_d)^\perp \subset A(D^d)^\perp .$$

<u>LEMMA 24</u> : $A(D^d)^\star$ <u>identifies with the direct sum</u>

$$\bigoplus_{S\subset\{1,\dots,d\}} (Q_S\ \hat{\otimes}\ B_{S^c})$$

<u>where</u> $Q_S = i_S^\star(L^1(\theta_i \mid i\in S))$ <u>and</u> $i_S : A(\theta_i \mid i\in S) \to C(\theta_i \mid i\in S)$ <u>the embedding</u>.

<u>Proof</u> : Denoting $i = i_{\{1,\dots,d\}}$, one clearly has

$A(D^d)^\star = \sum_{S\subset\{1,\dots,d\}} i^\star(X_S)$ by Lemma 23. For fixed $S \subset \{1,\dots,d\}$, apply Cor. 20 in the situation $K \leftrightarrow (\theta_i \mid i \notin S)$, $L \leftrightarrow (\theta_i \mid i \in S)$ and $X = A(\theta_i \mid i \notin S)$, $Y = A(\theta_i \mid i \in S)$.

Thus B_{S^c} corresponds to B, and (a) applied with $E = L^1(\theta_i \mid i \in S)$ yields the identification of $i^\star(X_S)$ with $Q_S\ \hat{\otimes}\ B_{S^c}$.

If S is fixed, then

$\|i^\star(\xi+\xi_1)\| \geq \|i^\star(\xi)\|$ for $\xi\in L^1(\theta_i \mid i\in S)\hat{\otimes}B_{S^c}$ and $\xi_1\in \sum_{j\notin S} L^1(\theta_j)\hat{\otimes}M(\theta_i \mid i\neq j)$.

Previous inequality follows from (b) of Cor. 20 if we fix some $j \notin S$ and consider more particularly $\xi_1 \in L^1(\theta_j)\ \hat{\otimes}\ M(\theta_1,\dots,\tilde{\theta}_j,\dots,\theta_d)$. Write indeed $K = K' \times K''$, $K' \leftrightarrow (\theta_j)$ and $K'' \leftrightarrow (\theta_i \mid i \notin S,\ i \neq j)$, and use the fact that B_{S^c} is singular with respect to $L^1(\theta_j)\ \hat{\otimes}\ M(\theta_i \mid i \notin S,\ i \neq j)$, contained in $[\![A(\theta_i \mid i \notin S)^\perp]\!]$ by Lemma 22. But the argument to obtain Cor. 20 (b) yields also the inequality

$$\|i^\star(\xi+\xi_1)\| \geq \|j^\star(\xi)\|$$

for ξ_1 an element of the sum $\sum_{j\notin S} L^1(\theta_j)\ \hat{\otimes}\ M(\theta_i \mid i \neq j)$.

It is now routine to verify that the sum in Lemma 24 is a direct sum.

<u>Proof of Prop. 15</u> : Notice that if $S \neq \phi$, the band B_S is isomorphic to the space $M = M(\Pi)$, as a consequence of the decomposition method (see [11]). If $i : A(D) \to C(\Pi)$ is the 1-variable embedding, the

space Q_S identifies with the projective tensorproduct $\hat{\otimes}_{i\in S}\, i^\star(L^1(\theta_i))$.
Lemma 24 therefore gives the isomorphism of

$$A(D^d)^\star \text{ and } \bigoplus_{S\subset\{1,\dots,d\}} \{ \hat{\otimes}_{i\in S}\, i^\star(L^1(\theta_i)) \,\hat{\otimes}\, \mathcal{B}_{S^c}\}$$

Since $A(D)^\star$ identifies with $i^\star(L^1(\Pi)) \oplus M_S(\Pi)$, latter space is further isomorphic to the d-fold tensorproduct $A(D)^\star \hat{\otimes} \dots \hat{\otimes} A(D)^\star$.
These considerations may be applied to the space $Z = C(\Pi) \otimes A(D^{d-1})$ as well, yielding the formula

$$Z^\star \sim M \,\hat{\otimes}\, \underbrace{A(D)^\star \,\hat{\otimes} \dots \hat{\otimes}\, A(D)^\star}_{(d-1)\text{-fold}}$$

By P. Wojtaszczyk theorem $A(D^j) \sim c_0(\Sigma\, A(D^j))$ (see [17]) and relying again on the decomposition method, Prop. 15 is obtained.

In [4], it is shown that the dual spaces $A(D^d)^\star$ are weakly complete among other linear topological properties.

Added in proof : The author proved more recently the non-isomorphism of $A(D^m)$ and $A(D^n)$ for $m \neq n$, by showing that $A(D^n)^\star$ is not isomorphic to a subspace of $A(D^m)^\star$ for $m < n$. Details will appear elsewhere.

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