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SOME ESTIMATES IN $\bar{\partial}_b$ NEUMANN BOUNDARY VALUE PROBLEM FOR STRONGLY PSEUDO-CONVEX CR STRUCTURES

by MASATAKE KURANISHI

Introduction. We consider a system of partial differential equations of the first order

$$(1) \qquad Xf = 0 \qquad \text{for all } X \in E ,$$

where f is an unknown complex valued function and E is a subbundle of the bundle $\underline{C}\,TM$ of the complex tangent vectors to a compact manifold M, possibly with boundary. We denote by $C^\infty(M,E)$ the vector space of smooth sections of E. We assume that, for any X and Y $\in C^\infty(M,E)$, their bracket $[X,Y]$ is also a section of E. We also assume that $E \cap \bar{E} = \{0\}$.

Let $M \to \underline{C}^n$ be a smooth embedding. Denote by E the set of all $X \in \underline{C}\,TM$ such that, when considered as a complex tangent vector to $\underline{C}^n$ via the embedding, it is of type (0,1). Then E satisfies the above conditions, provided E is a subbundle. This always happens when the codimension of M is 1. We say that E is embeddable when it is locally obtained by embedding in $\underline{C}^n$.

The nature of the equation (1) depends very much on its Levi-form. Namely, we consider

$$(2) \qquad \begin{aligned} &C^\infty(M,E) \times C^\infty(M,E) \ni (X,Y) \\ &\to \frac{1}{i}\,[X,\bar{Y}] \bmod E+\bar{E} \in C^\infty(M,\underline{C}\,TM/(E+\bar{E})) . \end{aligned}$$

We see easily that the map does not involve differentiation and actually comes from hermitian quadratic forms on the fibers of E valued in $\underline{C}\,TM/(E+\bar{E})$. Here, by hermitian we mean with respect to the bar operation induced on $\underline{C}\, T M/(E+\overline{E})$ by that of $\underline{C}\, T M$.

When the complex fiber dimension of $\underline{C}\, T M/(E+\overline{E})$ is 1 and the Levi-form is non-degenerate, we have another interesting example. Namely, we now consider the principal bundle on which its normal Cartan connection is defined. Then the vertical complex tangent vectors over E of the connection form a subbundle which satisfies our conditions.

The equation (1) is closely related to the complex it induces. Namely, denote by $\Lambda^q(E)$ the bundle of skew-symmetric multi-linear maps $E \times \ldots \times E$ (q factors) $\to \underline{C}$. Then we have the exterior derivative

$$D : C^\infty(M, \Lambda^q(E)) \to C^\infty(M, \Lambda^{q+1}(E)) \tag{3}$$

just as in the case of de Rham complex, i.e.

$$\begin{aligned} D u(X_o, \ldots, X_q) &= \textstyle\sum_{a=0}^q (-1)^a X_a u(X_o, \ldots, \hat{X}_a, \ldots, X_q) \\ &+ \textstyle\sum_{a<b} (-1)^{a+b} u([X_a, X_b], X_o, \ldots, \hat{X}_a, \ldots, \hat{X}_b, \ldots, X_q) . \end{aligned} \tag{4}$$

Introduce hermitian metrics on the fibers of E and a volume element of M. Then they induce a pre-Hilbert structure on $C^\infty(M, \Lambda^q(E))$. We wish to exhibit two formulas related to a semi-norm

$$\|\chi D u\|^2 + \|\chi D^* u\|^2 \tag{5}$$

where χ is an arbitrary real valued smooth function compactly supported in the interior of M. When the complex fiber dimension of $\underline{C}\, T M/(E+\overline{E})$ is 1, $\dim M = 2n-1$ with $q(n-2-q) > 0$, and the boundary of M satisfies rather strict conditions (cf. (31)) these formulas can be combined to find an estimate of (5). When we let χ converge to a function μ which may not be zero on the boundary, in the limit estimate we find terms which involve integrals on the boundary of M. Our main concern is to find an estimate of (5) such that, under D-Neumann boundary value condition (cf. (3) and 3) (48)), these boundary integrals are non-negative. The formulas are improved versions of those in (I) [4], where one finds the details which are omitted here. These formulas are not strong enough to solve the D-Neumann boundary value problem. In the last section we derive estimates from the above. Our hope is to find out eventually if the norm $||Du||^2 + ||D^* u||^2 + C||u||^2$ is compact with respect to L_2-norm, provided u satisfies conditions (cf. (48)) including the D-Neumann boundary value condition. However it seems that our estimates are not yet strong enough to show the compactness.

<u>Preliminary</u>. We first fix a complex vector subbundle F of $\underline{C}\,T\,M$ (with $F = \overline{F}$) supplementary to $E + \overline{E}$. Write for $X, Y \in C^\infty(M,E)$

$$[X,\overline{Y}] = i\, C_F(X,Y) + [X,\overline{Y}]_E + [X,\overline{Y}]_{\overline{E}}\ , \qquad \text{where}$$

$$(6) \qquad C_F(X,Y) \in C^\infty(M,F), [X,\overline{Y}]_E \in C^\infty(M,E)\ , \quad \text{and}$$

$$[X,\overline{Y}]_{\overline{E}} \in C^\infty(M,\overline{E})\ .$$

We define E-hessian of a function f by

$$(7) \qquad H^f(X,Y) = X\,\overline{Y}\,f - [X,\overline{Y}]_{\overline{E}}\, f$$

for any $X, Y \in C^\infty(M,E)$. We find easily that it does not involve differentiation. When f is real valued

$$(8) \qquad H^f(X,Y) = \overline{H^f(Y,X)} + i\, C_F(X,Y)\, f\ .$$

The exterior product induces an algebra structure on $\Lambda(E) = \sum \Lambda^q(E)$, and

$$D(u_\Lambda v) = (D\,u)_\Lambda\, v + (-1)^p\, u_\Lambda\, D\,v\ , \quad u_\Lambda v = (-1)^{pq}\, v_\Lambda\, u$$

for $u \in C^\infty(M,\Lambda^p(E))$ and $v \in C^\infty(M,\Lambda^q(E))$. In terms of the metric we introduce the interior product $\llcorner v : \Lambda^p(E) \to \Lambda^{p-1}(E)$ by

(9) $$\langle u \llcorner v , w \rangle = \langle u , v_\wedge w \rangle .$$

If $\alpha \in \Lambda^1(E)$ and $u \in \Lambda^p(E)$

(10) $$(u_\wedge v) \llcorner \alpha = (u \llcorner \alpha)_\wedge v + (-1)^p u_\wedge (v \llcorner \alpha) .$$

This formula plays a crucial role in the proof of (23). We assume that the support of X is so small that we can pick an orthonormal base $e_1,\ldots,e_m$ of $\Lambda^1(E)$ defined on a neighborhood U of the support of X. Let $g : \Lambda^1(E) \to \Lambda^1(E)$ be a homomorphism of vector bundles over the identity map of M. Then we let g also operate on $\Lambda^q(E)$ by

(11) $$g u = \sum_k (g e_k)_\wedge (u \llcorner e_k) .$$

Thus $g f = 0$ for a scalar valued function f. We see easily that the above g coincide with the given g when $q = 1$. Since the right-hand side of the above is clearly independent of the choice of orthonormal $e_1,\ldots,e_m$, (11) is defined globally. We also see easily that the adjoint of the above g is equal to the map induced by $g^* : \Lambda^1(E) \to \Lambda^1(E)$, where $*$ is in terms of the metric. Moreover

(12) $$g(u_\wedge v) = (g u)_\wedge v + u_\wedge g v$$

(13) $$g(u \llcorner v) = (g u) \llcorner v - u \llcorner g^* v .$$

We denote by $Y_1,\ldots,Y_m$ a base of E dual to $e_1,\ldots,e_m$. We set

(14) $$[Y_j , Y_k] = \sum_\ell r_{jk\ell} Y_\ell$$

and define $r_{(k)} : \Lambda^1(E) \to \Lambda^1(E)$ by

(15) $$r_{(k)} e_\ell = \sum_j r_{kj\ell} e_j .$$

For $K = (k_1,\ldots,k_q)$ and $u \in C^\infty(U,\Lambda^q(E))$, set

$$u_K = u(Y_{k_1}, \ldots, Y_{k_q}) \ . \tag{16}$$

If G is a vector field on U, we let G operate on $C^\infty(U,\Lambda^q(E))$ by

$$(Gu)_K = Gu_K \ . \tag{17}$$

By definition we see that the operation of G commutes with the exterior and the interior product by e_k. We define $\tilde{Y}_k : C^\infty(U,\Lambda^q(E)) \to C^\infty(U,\Lambda^q(E))$ by

$$\tilde{Y}_k u = Y_k u - \frac{1}{2} r_{(k)} u \ . \tag{18}$$

It then follows (cf. §.1 I [4])

$$Du = \sum_k e_{k_\Lambda} \tilde{Y}_k u \ . \tag{19}$$

Hence by duality

$$D^* u = \sum_k \tilde{Y}_k^* (u \,\llcorner\, e_k) \ . \tag{20}$$

Finally, set

$$[Y_j, Y_k^*] = \frac{1}{i} C_F(Y_j, Y_k) + \sum_\ell \overline{q_{kj\ell}} \tilde{Y}_\ell + \sum_\ell \tilde{Y}_\ell^* q_{jk\ell} + \tilde{q}_{jk} \tag{21}$$

(cf. (6)). We define $q_{(k)} : \Lambda^1(E) \to \Lambda^1(E)$ by

$$q_{(k)} e_\ell = \sum q_{k\ell j} e_j \ . \tag{22}$$

<u>A priori estimate</u>. Let X be as indicated in (5). Pick a section u of $\Lambda^q(E)$ which is smooth on a neighborhood of the support of X. To find a formula for the semi-norm (5), it is enough to consider $D^* X^2 D + D X^2 D^*$. For simplicity we consider X instead of X^2 for a while.

(23) *PROPOSITION 1.*

$$(D^* \chi D + D \chi D^*)u = A u + B u + C u, \text{ where } A = A_1 + A_2 \text{ with}$$

$$A_1 u = \sum_k \tilde{Y}_k^* \chi \tilde{Y}_k u$$

$$A_2 u = \sum_{k,j} \Big((q^*_{(k)} + \tfrac{1}{2} r^*_{(k)}) e_j \wedge \chi \tilde{Y}_j (u \llcorner e_k)$$

$$+ e_k \wedge \tilde{Y}_j^* \chi (u \llcorner (q^*_{(k)} + \tfrac{1}{2} r^*_{(k)}) e_j) \Big) - \sum_{k,j} r^*_{(k)} e_j \wedge (u \llcorner r^*_{(j)} e_k) ,$$

$$B u = - \sum_{j,k} e_j \wedge (\chi i c_F (Y_j, Y_k) + H^\chi (Y_j, Y_k)) (u \llcorner e_k) ,$$

$$C u = D (u \llcorner D \chi) + D \chi \wedge D^* u .$$

<u>Outline of the proof</u>. First work out the commutator relation between $\tilde{Y}_k$ and the exterior product by e_j (cf. (12),(13)). Similarly for the interior product by e_j. Write down $(D^* \chi D + D \chi D^*)u$ using (19) and (20). Apply (10) and rewrite it as the sum of $\sum_k \tilde{Y}_k^* \chi \tilde{Y}_k u$ and terms containing $u \llcorner e_k$. Then our formula follows by (21) and (7).

We note that the above formula is a precise version of the one given by J.J. Kohn in [2].

Note that $< A_1 u , u > \geqq 0$. In view of A_1, we do not have to worry too much about the term $< A_2 u , u >$. When we let χ converge to a function which may not be zero on the boundary of M, $D \chi$ will blow up on the boundary. Note that $D \chi$ appears in $C u$. The Neumann boundary condition we consider later is exactly the one which makes $< C u , u >$ go to zero. When we want to obtain an a priori estimate for $\| D u \|^2 + \| D^* u \|^2 + C \| u \|^2$, we see then that the main difficulty comes through $< B u , u >$. We try to eliminate this term by taking advantage of the term $\sum_j (\tilde{Y}_j)^* \chi \tilde{Y}_j$ in A.

We assume that $\overline{M}$ is in a manifold $\tilde{M}$ and E extends to $\tilde{M}$. Let t be a real valued function on $\tilde{M}$. For later application we consider the case when the boundary of M is defined by $t = 0$. However, there is no need to do so now. Set

$$(24)\qquad Y_j t = \sigma_j \quad , \quad \omega_j = \sigma_j / b \quad , \quad b = (\sum \sigma_j \bar{\sigma}_j)^{1/2} \ .$$

We also define (where $b \neq 0$)

$$(25)\qquad \tilde{Y}^o = \sum_k \bar{\omega}_k \tilde{Y}_k \quad , \quad \tilde{W}_j = \tilde{Y}_j - \omega_j \tilde{Y}^o = \sum_k Q_{kj} \tilde{Y}_k \quad ,$$

$$Q_{kj} = \delta_{kj} - \bar{\omega}_k \omega_j \quad , \quad W_j = \sum_k Q_{kj} Y_k$$

so that

$$(26)\qquad W_j t = 0 \quad , \quad \sum \bar{\omega}_j \tilde{W}_j = 0 \ .$$

Then we see that

$$(27)\qquad \sum_k \tilde{Y}_k^* X \tilde{Y}_k = (\tilde{Y}^o)^* X \tilde{Y}^o + \sum_k \tilde{W}_k^* X \tilde{W}_k \ .$$

We set

$$(28)\qquad \tilde{W}_j^{\wedge} = \sum Q_{jk} \tilde{Y}_k^* \ .$$

When X is a vector field and f is a function, we often write $[X,f]$ instead of $X f$, i.e. we regard a function as a multiplication operator.

(29) PROPOSITION 2. *Assume that the support of X is contained in $M' = \{p \in M \, ; \, b(p) \neq 0\}$. Then $\sum \tilde{W}_j^* X \tilde{W}_j = A' + B' + C'$, where $A' = A_1' + A_2' + G$ with*

$$A_1' = \sum (\tilde{W}_j\hat{\ })^* \chi (\tilde{W}_j\hat{\ }) + \sum_j (\chi\, b^{-1}\, \overline{H^t(W_j, W_j)}\, \tilde{Y}^o + (\tilde{Y}^o)^* \chi\, b^{-1} H^t(W_j, W_j)),$$

$$A_2' = -\sum_{j,\ell} (\chi([\overline{Y}_\ell, Q_{j\ell}] + \sum_k Q_{\ell k} \overline{q_{k\ell j}})\tilde{W}_j - (\tilde{W}_j)^* \chi([Y_\ell, Q_{\ell j}] + \sum_k Q_{k\ell}\, q_{k\ell j}) \;,$$

$$G = -\chi \sum_{j,k} ([\overline{Y}_j\,,\,[Y_k, Q_{kj}]] + \sum_\ell q_{jk\ell}\, [\overline{Y}_\ell, Q_{jk}] + Q_{jk}\, \tilde{q}_{jk}) \;,$$

$$B' = \sum_j i\, \chi\, C_F\, (W_j, W_j) + \sum_j H^\chi\, (W_j, W_j) \;,$$

$$C' = -\sum_j 2[\overline{Y}_j\,,\,[W_j, \chi]] - \sum_j (\tilde{W}_j)^* [W_j, \chi] - 2\, i\, \mathfrak{Im}\, \alpha \;,$$

$$\alpha = \sum_{j,k} ([Y_j, \chi][\overline{Y}_k, Q_{jk}] + [\overline{Y}_j\,,\,[Y_k, \chi]]Q_{kj}) \;.$$

Outline of the proof. Since the matrix (Q_{jk}) defines a projection operator

$$\sum_j (\tilde{W}_j)^* \;\chi \tilde{W}_j = \sum_{j,k} \tilde{Y}_k^* \chi\, Q_{jk}\, \tilde{Y}_j$$

$$= \sum_j (\tilde{W}_j\hat{\ })^* \chi \tilde{W}_j\hat{\ } - \sum_{j,k} (\chi\, Q_{jk}\, [\tilde{Y}_j\,,\,\tilde{Y}_k^*] + [\overline{Y}_k, \chi\, Q_{jk}]\, \tilde{Y}_j$$

$$+ \tilde{Y}_k^*[Y_j, \chi\, Q_{jk}]) + R \;, \qquad \text{where}$$

$$R = -\sum_{j,k} [\overline{Y}_k, [Y_j, \chi\, Q_{jk}]]$$

$$= -2\sum_k [\overline{Y}_k, [W_k, \chi]] + \sum_{j,k} ([\overline{Y}_k, [Y_j, \chi]Q_{jk}] - [\overline{Y}_k, \chi[Y_j, Q_{jk}]])$$

$$= -2\sum_k [\overline{Y}_k, [W_k, \chi]] + (\text{a purely imaginary number})$$

$$+ \sum_{j,k} [\overline{Y}_k, [Y_j, \chi]]Q_{jk} - \sum_{j,k} \chi\, [\overline{Y}_k, [Y_j, Q_{jk}]] \;.$$

The last term of the above goes into **G**, and the second from the last (modulo i $\mathbb{C}$) is $\sum_{j,k}[Y_k[\bar{Y}_j,X]]Q_{kj}$ which goes into B'.

<u>The case of codimension 1 with definite Levi-form</u>. We fix a generator $S \in C^\infty(M,TM)$ of F. Write

$$C_F(X,Y) = C_S(X,Y)S \quad . \tag{30}$$

To get rid of the B term in Proposition 1, we put a condition on the boundary of M. Pick a real valued function t on $\tilde{M}$ without any critical point on the boundary of M and such that M is defined by $t \leqslant 0$ and the boundary of M is defined by $t = 0$.

(31) DEFINITION 1. Assume that M is of dimension $2n-1$ with $n \geqslant 3$. We say that the boundary of M is admissible when

1) There is a smooth function γ on $\tilde{M}$ such that

$$H^t = \gamma\, C_S$$

at each point on the boundary of M, provided $n \geqslant 4$. If $n = 3$, we assume further that all the first order partial derivatives of $C_S - \gamma H^t$ also vanish at each boundary point of M.

2) At each boundary point $p \in M$ such that $b(p) = 0$ (cf. (24))

$$\gamma(p) \neq 0 \quad ,$$

3) For any $X, Y \in C^\infty(M,E)$ and for any p as above

$$X\,Y\,t(p) = 0 \quad .$$

We see easily that the above definition is independent of the choice of t as well as of the choice of a supplementary real vector field S.

To give an exemple of M with admissible boundary, consider a real hypersurface $\tilde{M} \subseteq \underline{C}^n$ of codimension 1. We write $(z,w) \in \underline{C}^n$, where $z \in \underline{C}^{n-1}$ and $w \in \underline{C}$. We assume that the origin is in $\tilde{M}$ and $\tilde{M}$ is given by an equation :

$$(32) \qquad y = h(z,\overline{z},x)$$

where $x = \mathcal{R}w$ and $y = \mathcal{I}m\, w$. We assume further that

$$(33) \qquad h(z,\overline{z},x) = \sum_{j,k} h_{j\overline{k}} z^j \overline{z}^k \mod (z,\overline{z},x)^3$$

where $(h_{j\overline{k}})$ is a positive definite hermitian matrix. As is shown by Chern and Moser in [1] we can always find a holomorphic chart (z,w) so that the above is valid locally, provided M is strongly pseudoconvex.

(34) PROPOSITION 3. For a sufficiently small $r > 0$ set

$$t = h(z,\overline{z},x) + \mathcal{R}w^2 - r \ .$$

Then the equation $t \leq 0$ defines a submanifold M with admissible boundary.

In the following we always assume that the boundary of M is admissible. We pick a smooth real valued function $\varphi(t)$ on $\underline{R}$ supported in $\{t \in \underline{R} \, ; x < 0\}$ such that $\varphi(t) = 1$ for $t < -c_1$ for a positive number c_1. We also pick $\mu \in C^\infty(\overline{M},\underline{R})$ with compact support. We assume that its support is small enough so that we can find an orthonormal base $Y_1,\ldots,Y_{n-1}$ of E on a neighborhood of its support. We set

$$(35) \qquad \chi = \mu\, \varphi(t) \ .$$

In the following we outline how to get rid of the B term in (23), which is the main obstruction to obtain an a priori estimate.

Note that for functions f and g

$$H^{fg}(X,Y) = f\,H^{g}(X,Y) + g\,H^{f}(X,Y) + (Xf)(\overline{Y}g) + (\overline{Y}f)(Xg) \quad . \tag{36}$$

Also we see that

$$H^{\varphi(t)}(X,Y) = \varphi'(t)\,H^{t}(X,Y) + \varphi''(t)\,(Xt)(\overline{Y}t) \quad . \tag{37}$$

Write

$$H^{t}(Y_j,Y_k) = \gamma\, C_S\,(Y_j,Y_k) + r_{jk} \;,\; r_{jk} = 0 \quad \text{on bd}\, M\,. \tag{38}$$

We pick our hermitian metric to be the one defined by C_S (which we can always assume to be positive definite by replacing S by $-$ S if necessary). Then we see that

$$< Bu,u > = < B_1 u,u > + < B_2 u,u > \quad , \quad \text{where}$$

$$\begin{aligned}
&< B_1 u,u > = - \textstyle\sum_k < (i\,\chi\,S + \mu\,\gamma\,\varphi')(u\,L\,e_k)\;,u\,L\,e_k > \\
&< B_2 u,u > = - \textstyle\sum_{j,k} < \mu\,\varphi'\;(t)\,r_{jk}(u\,L\,e_k),u\,L\,e_j > \\
&\qquad - 2\,\Re < \varphi'\,u\,L\,Dt,u\,L\,D\mu > - < \varphi''\,u\,L\,Dt\;,u\,L\,Dt > \\
&\qquad - \textstyle\sum_{j,k} < \varphi\,H^{\mu}\,(Y_j,Y_k)(u\,L\,e_k),u\,L\,e_j > \quad .
\end{aligned} \tag{39}$$

Note that

$$\begin{aligned}
&\textstyle\sum_j H^{t}\,(W_j,W_j) = \gamma(n-2)(1 + r_o) \quad , \quad r_o = 0 \text{ on bd}\, M \quad , \\
&\textstyle\sum_j C_S(W_j,W_j) = (n-2) \quad .
\end{aligned} \tag{40}$$

and

$$q < v,w > = \textstyle\sum_k < v\,L\,e_k,w\,L\,e_k > \tag{41}$$

where q is the degree of v and w. In view of (27) we can always take $\sum_j (q/(n-2))\tilde{W}_j^{*}\,\chi\,\tilde{W}_j u$ out of the A term, and

$$(1/(n-2))\;\textstyle\sum_k < B'\;(u\,L\,e_k),u\,L\,e_k > = < B_1' u,u > + < B_2' u,u > \;,\; \text{where}$$

$$< B_1'u,u > = \sum_k < (i\chi S + \mu\gamma\varphi')(u \mathbin{\llcorner} e_k), u \mathbin{\llcorner} e_k >$$

(42) $$< B_2'u,u > = < \mu\varphi' r_o\, u \mathbin{\llcorner} e_k, u \mathbin{\llcorner} e_k >$$

$$+ \ (1/(n-2)) < \sum_j \varphi H^\mu(W_j,W_j)(u \mathbin{\llcorner} e_k), u \mathbin{\llcorner} e_k > .$$

(cf. (26)). Hence

$$< Bu,u > + \sum_k (1/(n-2)) < B'(u \mathbin{\llcorner} e_k), u \mathbin{\llcorner} e_k >$$

$$= < B_2 u,u > + < B_2'u,u > \quad .$$

Note that the right-hand side of the above does not cause any trouble at the boundary under the D-Neumann boundary condition. By (23) and (29) we find then that for χ as in (35)

(43)
$$\begin{aligned}
< (D^*\chi D + D\chi D^*)u,u > &= < \tilde{Y}_o^* \chi \tilde{Y}_o\, u, u > \\
&+ \frac{q}{n-2}\,(2\,\mathcal{R} < \chi\tilde{Y}_o\, u, \sum_j b^{-1} H^t(W_j,W_j)u > + < Gu,u >) \\
&+ \frac{n-2-q}{n-2} \sum_j < \tilde{W}_j^* \chi \tilde{W}_j\, u, u > + \frac{q}{n-2} \sum_j < (\tilde{W}_j\hat{\ })^* \chi W_j\hat{\ }\, u, u > \\
&+ < B_2 u,u > + < B_2'u,u > + < (A_2 + C + (q/(n-2))(A_2' + C')u,u > .
\end{aligned}$$

Now when we calculate G more explicitly, we find that

$$G = \chi b^{-2} |\sum_j H^t (W_j,W_j)|^2 + G_1 , \quad \text{with}$$

(44) $$G_1 = - \chi b^{-4} \sum_{\ell,k} \sigma_\ell \sigma_k [\overline{Y}_\ell, \overline{\sigma}_k](\sum_j H^t (W_j,W_j) + \sum_{j,i,s} Q_{ji} q_{jis} \overline{\sigma}_s)$$

$$+ \chi b^{-1} R$$

where R is bounded. Therefore we obtain :

$$\|X^{\frac{1}{2}} Du\|^2 + \|X^{\frac{1}{2}} D^* u\|^2 = \|X^{\frac{1}{2}}(\tilde{Y}_o + \frac{q}{n-2} b^{-1} \sum_j H^t (W_j, W_j))u\|^2$$

$$+ \frac{(n-2-q)q}{(n-2)^2} \|X^{\frac{1}{2}} b^{-1} \sum_j H^t (W_j, W_j)u\|^2$$

$$+ \frac{n-2-q}{n-2} \sum_j \|X^{\frac{1}{2}} \tilde{W}_j u\|^2 + \frac{q}{n-2} \sum_j \|X^{\frac{1}{2}} \tilde{W}_j^{\wedge} u\|^2 \tag{45}$$

$$+ < G_1 u,u > + < B_2 u,u > + < B_2' u,u >$$

$$+ < (A_2 + C + (q/(n-2))(A_2' + C')u,u > .$$

Note by (40), (44), and 3) (31) that $< b^2 G_1 u,u >$ vanishes where $b = 0$, provided the support of X is contained in a sufficiently small neighborhood of a boundary point.

<u>D-Neumann boundary value problem</u>. The classical method of Kohn and Nirenberg (cf. [3]) to solve the problem is to find a norm $\|u\|'$ on $C^2(M,\Lambda^q(E))$ such that

1) $\|u\|'$ is compact with respect to L_2-norm $\|u\|$,

2) $\|Du\|^2 + \|D^* u\|^2 + C\|u\|^2 \geqslant c(\|u\|')^2$

for all u satisfying the boundary condition : $u \mathbin{\llcorner} Dt = 0$.

We apply the same method in our case. However, the nature of the formulas in PROPOSITION 1 and 2 forces us to modify it. Firstly, since b^{-1} comes in our picture which is not smooth, it is more natural to enlarge the space $C^2(\overline{M},\Lambda^q(E))$. Secondly, since we localize and use different methods to prove our estimate depending where we are, we replace a single norm $\|u\|'$ by a pre-fréchet space structure.

We first study neighborhoods of boundary points p with $b(p) = 0$. They are the characteristic boundary points. By the non-degeneracy of

C_S and 1), 2) in (31) we see easily that :

(46) PROPOSITION 4. *Let p be a characteristic boundary point. Then, on a sufficiently small neighborhood of p, (t,σ) is a chart.*

(47) COROLLARY. *The set of characteristic boundary points is isolated.*

(48) DEFINITION. *We denote by $C'(M,\Lambda^q(E))$ the vector space of sections u of $\Lambda^q(E)$ on M satisfying the following conditions :*

1) u is C^1 in the interior of M.

2) $D\,u$ and D^*u are in L_2, $b^{-1}u$ is in L_2 on a neighborhood of each characteristic boundary point, and $W_j u$ is in L_2 in a neighborhood of each boundary point.

3) For each C^∞ function f on $\overline{M}$ whose support is compact and disjoint from the set of characteristic boundary points, $f\,t^{-1}\,u \llcorner D\,t$ is in L_2.

We prove a priori estimate on $C'(M,\Lambda^q(E))$. The above condition 3) is the D-Neumann boundary condition. We work separately on neighborhoods of interior points of M, of characteristic boundary points, and of non characteristic boundary points.

(49) PROPOSITION 5. *Let χ be a C^∞ function with compact support on $\overline{M}$ which is zero on the boundary of M. Then there are constants $C>0$ and $c>0$ depending on χ such that for any $u \in C'(M,\Lambda^q(E))$*

$$\|D\,u\|^2 + \|D^*u\|^2 + C\,\|u\|^2 \geqslant c\Big(\sum_j \|Y_j\,\chi\,u\|^2 + \sum_j \|Y_j^*\,\chi\,u\|^2 + |\langle S\,\chi\,u\,,\,\chi\,u\rangle|\Big)\,.$$

This follows by (23) because we can get rid of B term (without introducing b^{-1}) by the well-known method of Kohn. Note in the above that the term $\sum_j \|Y_j Xu\|^2$ is independent of a choice of a local orthonormal base $Y_1,\dots,Y_{n-1}$ and hence has a global meaning. Similarly $\sum_j \|Y_j^* Xu\|^2$ has a global meaning modulo a term $\leqslant C'\|u\|^2$.

We next consider a small neighborhood U of a boundary point p_o. In (35) we take $\mu \in C^\infty(\bar{M},\underline{R})$ with support in $U\cap\{b \neq 0\}$. We also replace $\varphi(t)$ by $\varphi_\varepsilon = \varphi(t/\varepsilon)$ and let $\varepsilon \to 0$. In (45) the term which contains the derivatives of $\varphi(t/\varepsilon)$ in t and the derivatives of μ is

$$E = \langle B_2u,u\rangle + \langle B_2'u,u\rangle + \langle (C + (q/(n-2))C')u,u\rangle .$$

Because $r_{jk} = r_o = 0$ on the boundary of M we see by 3) (48) that E converges to

$$E_\mu' = 2\,\mathcal{R}\langle D^*u, u \mathbin{\llcorner} D\mu\rangle - (2q/(n-2))\,(\langle \textstyle\sum_k [\bar{Y}_k,[W_k,\mu]]u,u\rangle$$

$$+ \textstyle\sum_j \langle [W_j,\mu]u\,,\tilde{W}_j u\rangle - \sum_{j,k}\langle H^\mu(Y_j,Y_k)\,(u \mathbin{\llcorner} e_k), u \mathbin{\llcorner} e_j\rangle)$$

$$+ (q/(n-2))\textstyle\sum_j \langle H^\mu\,(W_j,W_j)u,u\rangle$$

because (cf. (26))

$$\mu[W_j,\varphi_\varepsilon] = 0 \quad , \quad [\bar{Y}_j,\varphi_\varepsilon][W_j,\mu] = 0 \ .$$

Assume now that p_o is a characteristic boundary point. We assume that U is sufficiently small so that (t,σ) is a chart on U (cf. (46)). We pick $\mu_1 \in C^\infty(\bar{M},\underline{R})$ with support in U and set

$$\mu = \mu_1\,\varphi\,(\tfrac{1}{\varepsilon}\,b)$$

and let $\varepsilon \to 0$. Because $b^{-1}u$ is in L_2 (cf. 2) (48)), we find that E_μ' converges to E_{μ_1}'. In view of (40) and 3) (31) we then find by (45) the following :

(50) *PROPOSITION 6. Let* p_o *be a characteric boundary point of M. Assume that* $q(n-2-q) > 0$. *Then there is a neighborhood U of* p_o *such that, for any* $\mu \in C^\infty(\overline{M}, \underline{R})$ *with support in U, there are constants* $C, c > 0$ *such that*

$$\|Du\|^2 + \|D^*u\|^2 + C\|u\|^2 \geqq c\|b^{-1}\mu u\|^2$$

for any $u \in C'(\overline{M}, \underline{R})$.

We next consider a non-characteristic boundary point p_o and pick a sufficiently small U which does not contain any characteristic boundary point. Let $\mu \in C^\infty(\overline{M}, \underline{R})$ with support in U. Then the above argument proves that for any $u \in C'(\overline{M}, \underline{R})$

$$(51) \quad \|Du\|^2 + \|D^*u\|^2 + C\|u\|^2 \geqq c\sum_k (\|Y_k \mu u\|^2 + \|\overline{W}_k \mu u\|^2) \ .$$

Looking at terms B and B' in (23) and (29) we also find that

$$(52) \quad \|Du\|^2 + \|D^*u\|^2 + C\|u\|^2 \geqq c|\langle iS\mu u, \mu u\rangle - \langle \gamma \mu u, \mu u\rangle_{bd}|$$

where $\langle u, u\rangle_{bd}$ denotes the square of the L_2-norm of the restriction of u to the boundary of M. With φ as in (35), $[Y_o, \varphi] = \varphi'(t)b$. Hence we see easily that

$$-\langle \gamma \mu u, \mu u\rangle_{bd} = -\langle b^{-1}\overline{\gamma}Y^o \mu u, \mu u\rangle + \langle \mu u, (Y^o)^* \mu \gamma b^{-1} u\rangle .$$

Therefore by (51) and (52)

$$(53) \qquad \|Du\|^2 + \|D^*u\|^2 + C\|u\|^2 \geqslant c|\langle ib^{-1}X_o \mu u, \mu u\rangle|$$

where

$$(54) \qquad X_o = ibS + \overline{\gamma}Y^o - \gamma\overline{Y}_o \ .$$

Note that $X_o, W_j, \overline{W}_j$ are tangential to the boundary of M. We denote by (bd)'M the set of non-characteristic boundary point.

(55) PROPOSITION 7. (bd)'M has a foliation of codimension 1 such that X_o, W_j, $\overline{W}_j$ generate the complex tangent vector space of each leaf.

<u>Outline of the proof</u>. It is enough to show that the equation $X_o = W_j = \overline{W}_j = 0$, when restricted to (bd)'M, is completely integrable. This follows by the same calculation made in §.2 II [4]. As long as we do not differentiate $H^t(Y_j, Y_k)$, the calculation there is still valid for our more general t. In view of 1) (31), no modification is needed when $n = 3$. For $n \geqslant 4$ we have to take a little more care for terms containing $[Y_j, Y]$. Set $P_j = [Y_j, Y] - \sum_k i\, r_{j\bar{k}}\, \bar{\sigma}_k - Y\, r_j$ and $H^t(Y_j, Y_k) = Y\,\delta_{jk} + \alpha_{jk}$ with $\alpha_{jk} = 0$ on the boundary. Then instead of the formula $P_j\,\delta_{k\ell} = P_k\,\delta_{j\ell}$ (cf. the middle of the proof of (2.23) II [4]), we have

$$P_j\,\delta_{k\ell} + [Y_j, \alpha_{k\ell}] = P_k\,\delta_{j\ell} + [Y_k, \alpha_{j\ell}] \ .$$

Apply $\sum_j Q_{ji} \sum_{\ell,k} Q_{k\ell}$. We then find $\sum_j Q_{ji}\, P_j = 0$ on (bd)'M, provided $n \geqslant 4$. This is what we need. The term containing $[Y_j, \overline{Y}]$ can be also handled similarly.

In view of (51) and (53) we find by the above the following :

(56) COROLLARY. Let V be any complex tangent vector field on U which is tangential at the boundary to the leaves of the foliation in (55). Then

$$\|D\,u\|^2 + \|D^* u\|^2 + C\|u\|^2 \geqslant c\,|\langle V\,\mu\, u, \mu\, u\rangle| \ .$$

Note that $Y^o - \overline{Y}^o$ is tangential to the boundary and its restriction together with the restrictions of X_o, W_j, $\overline{W}_j$ generate $\underline{C}\,T$ (bd M).

(57) PROPOSITION 8. The flow generated by $i\, b^{-1}(Y^o - \overline{Y}^o)$ preserves the foliation of (55).

Outline of the proof. By the same calculation as in §.2 II [4] we find that $[b^{-1}(Y^o - \overline{Y}^o), W_j] \equiv 0 \mod W_k, \overline{W}_k$. Hence $[b^{-1}(Y^o - \overline{Y}^o), \overline{W}_j] \equiv 0 \mod W_k, \overline{W}_k$. Since $W_j, \overline{W}_j$ with bracket generate X_o, our contention follows.

We are now going to prove the following :

(58) PROPOSITION 9. There is a neighborhood $\tilde{U}$ of p_o in $\tilde{M}$ with a chart (x, y_1, y'), $y' = (y_2, \ldots, y_{2n-3})$, centered at p_o satisfying the following : 1) $U = \tilde{U} \cap M$ is given by $x \leqslant 0$, 2) the equations $x = 0$ and y_1 = constants define the local fibering of (52), 3) $Y^o = \frac{1}{2} b(\partial/\partial x + i\,\partial/\partial y^1) + B$ with $B = 0$ at each boundary point in U, and 4) for any $u \in C'(M, \Lambda^q(E))$ with $q(n-2-q) > 0$

$$\|Du\|^2 + \|D^*u\|^2 + C\|u\|^2 \geqslant c(\|\mu u\|'_{1/2})^2$$

where $(\| \quad \|'_{1/2})^2$ denotes the integral in (x, y_1) of the square of the Sobolev norm with respect to the variable y'.

PROOF. Pick a chart y' centered at p_o of the local fiber F_o of the foliation in (55). Consider the flow generated by $i(\overline{Y}^o - Y^o)/b$. Let $y = (y_1, \ldots, y')$ be the point on the boundary with the parameter y_1 originating from y' in F_o. This gives a chart of bd M. We now use the flow generated by $(\overline{Y}^o + Y^o)/b$ to define a chart (x, y). By the construction

$$Y^o - \overline{Y}^o = i\, b\, \partial/\partial y_1 + 2B$$
$$Y^o + \overline{Y}^o = b\, \partial/\partial x$$

with $B = 0$ at each boundary point. Note also that $(\overline{Y}^o + Y^o)/b \equiv 2\,\partial/\partial t$ modulo a vector field tangential to the boundary. Hence the inequality $x \leqslant 0$ defines U. Now our contention follows by (56) and (57), q.e.d.

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