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NORMS OF CLASSICAL OPERATORS IN FUNCTION SPACES

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This is a survey of results on best constants in some classical inequalities like the Riesz's inequality for the Hilbert Transform, the Khinchine inequality for the Rademacher functions, the Marcinkiewicz-Paley inequality for the Haar system, etc ....

In the language of Functional Analysis that means evaluation of operator norms or ideal norms (in the sense of Banach ideals, cf. Pietsch [1]) of some classical operators acting in L^p-spaces.

Given a linear operator T acting between some spaces of measurable functions one considers the function

$$(p,q) \to \|T\|^{p,q} = \|T : L^p \to L^q\|$$

Often the following problems arise naturally

a) for what pairs (p,q) is the norm $\|T\|^{p,q}$ finite (the question of the boundedness of T) ;

b) if $\|T\|^{p_o,q_o} = +\infty$, determine the asymptotic behaviour of $\|T\|^{p,q}$ as $(p,q)\to(p_o,q_o)$;

c) identify the function $(p,q)\to\|T\|^{p,q}$ or at least describe the quantitative character of the function.

Similar problems arise if the operator norms are replaced by some ideal norms like nuclear or absolutely summing norms.

From the point of view of Banach spaces problems of type a) are connected with the isomorphic theory, problems of type b) - with the so called local theory, problems of type c) - with the isometric theory.

In this survey we are mainly interested in problems of type c).

The survey is far from being complete. In particular it does not cover the beautiful work of Beckner [1] (1975) on the best constant in the Hausdorff-Young inequality for the Fourier Transform as well as the recent progress on the best constants for the Sobolev embeddings, cf. Cartier [2].

1. THE HILBERT TRANSFORM

Recall that the classical Hilbert Transform on the real line is defined by

$$\mathcal{H}(f)(x) = \text{p.v.}\,\pi^{-1}\int_{-\infty}^{+\infty} f(x-t)t^{-1}\,dt = \lim_{\varepsilon\to 0}\pi^{-1}\int_{|t|>\varepsilon} f(x-t)t^{-1}\,dt,\quad x\in\mathbb{R}$$

(here p.v. means the principal value).

An important property of the Hilbert Transform says that if $F(z)$ is an analytic function in the upper half plane $\operatorname{Im} z>0$ and if $f+ig$ is the "radial" boundary value function of F defined on the real line $\operatorname{Im} z = 0$, then $g = \mathcal{H}(f)$ and $\mathcal{H}(g) = -f$.

M. Riesz discovered that $\|\mathcal{H}\|^{p,p} < \infty$ for $1 < p < \infty$ and unbounded for $p = 1,\infty$. Thus $\|\mathcal{H}\|^{p,q}$ is bounded at least for $1 < p < \infty$, $1 < q < \infty$. The best constant C_p in the Riesz inequality $\|\mathcal{H}(f)\|_p \leq C_p\|f\|_p$ i.e. the norm $\|\mathcal{H}\|^{p,p}$ has been identified by Pichorides [1] (1972) who based to some extent on an earlier observation of Gohberg and Krupnik [1] (1968). The result says :

<u>We have</u>

$$\|\mathcal{H}\|^{p,p} = \operatorname{tg}\frac{\pi}{2p}\quad \underline{\text{for}}\quad 1<p\leq 2$$

(1)

$$\|\mathcal{H}\|^{p,p} = \operatorname{ctg}\frac{\pi}{2p}\quad \underline{\text{for}}\quad 2\leq p<+\infty .$$

Let us present briefly the approach of Gohberg and Krupnik which naturally leads to the right conjecture. It bases on a proof due to the Cotlar [1] of the bouded-ness of the Hilbert Transform. Let us put $h_p = \|\mathcal{H}\|^{p,p}$. Observe that for nice real test functions $<\mathcal{H}(\varphi),\psi> = <\varphi,\mathcal{H}(\psi)>$. Thus $h_p = h_{p'}$ where $p' = p/(p-1)$ and it is enough to consider the case $p > 2$.

Observe that if $f+ig$ is the boundary value of an F analytic in the upper half plane, then $f^2-g^2+i.2fg$ is the boundary value of F^2. Thus remembering that $f = -\mathcal{H}(g)$ and $f^2-g^2 = -\mathcal{H}(2fg)$ we get the basic identity

$$(*) \qquad \mathcal{H}(g)^2 = g^2 + 2\mathcal{H}(g\,\mathcal{H}(g)) .$$

Now fix g with $\|g\|_{2p} = 1$. Using the identity $\|\varphi^2\|_p = \|\varphi\|_{2p}^2$ and the Schwarz inequality in the form $\|\varphi.\psi\|_p \leqslant \|\varphi\|_{2p}\|\psi\|_{2p}$, we get

$$\begin{aligned}\|\mathcal{H}(g)\|_{2p}^2 &= \|\mathcal{H}(g)^2\|_p \\ &\leqslant \|g^2\|_p + 2\|\mathcal{H}(g\mathcal{H}(g))\|_p \\ &\leqslant 1 + 2h_p\|g\mathcal{H}(g)\|_p \\ &\leqslant 1 + 2h_p\|g\|_{2p}\|\mathcal{H}(g)\|_{2p} \\ &\leqslant 1 + 2h_p h_{2p} .\end{aligned}$$

Hence, "suping" over all g with $\|g\|_{2p} = 1$,

$$h_{2p}^2 \leqslant 1+2h_p h_{2p} .$$

Thus

$$(2) \qquad h_{2p} \leqslant h_p + \sqrt{1+h_p^2} .$$

Next observe that the function $\mathrm{ctg}\,\frac{\pi}{2p}$ satisfies the functional equation

$$h_{2p} = h_p + \sqrt{1+h_p^2}.$$

On the other hand, a direct computation shows that for the one parameter family of functions

$$F_\gamma(z) = \frac{1}{z+1}\left(i\,\frac{z+1}{z-1}\right)^{2\gamma/\pi}, \quad \operatorname{Im} z > 0, \quad \frac{p'-1}{p'}\,\frac{\pi}{2} < \gamma < \frac{\pi}{2p'}$$

the functions $f_\gamma + i\,\mathcal{H}(f_\gamma)$ being the boundary value of F_γ satisfy

$$f_\gamma \in L^{p'} \quad \text{and} \quad \lim_{\gamma = \frac{\pi}{2p'}} \|\mathcal{H}(f_\gamma)\|_{p'} \cdot \|f_\gamma\|_{p'}^{-1} = \operatorname{tg}\frac{\pi}{2p'} .$$

Thus

$$h_p = \|\mathcal{H}\|^{p',p'} \geq \operatorname{ctg}\frac{\pi}{2p} \quad \text{for } p > 2 . \tag{3}$$

Combining (2) with (3) and taking into account that $h_2 = 1$ Gohberg and Krupnik [1] have shown that $h_{2^n} = \operatorname{ctg}\frac{\pi}{2.2^n}$ and they have conjectured (1). We believe that one could complete the proof of (1) using (2) and (3) and combining it with some missing a priori information on the behaviour of the function $p \to h_p$.

However, Pichorides [1] used a different argument. He worked with the Hilbert Transform on the circle

$$\mathcal{H}_{per}(f) = \text{p.v.}\ \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\operatorname{ctg}\frac{t}{2}\,dt .$$

A standard argument (cf. Zygmund [1], Chapter XVI, Theorem 3.8) shows that the Hilbert Transform on the line can be obtained as a limit case (by "blowing up the circle") of a Hilbert Transform on a circle. Thus

$$\|\mathcal{H}\|^{p,p} \leq \|\mathcal{H}_{per}\|^{p,p} . \tag{4}$$

Pichorides'argument is a refinement of the classical proof of M. Riesz inequality due to Calderon (1950) (cf. Zygmund [1], Chapt VII). Recall that in this proof the actual work is done for $1 < p < 2$. For one familiar with the proof we mention briefly main improvements due to Pichorides.

1) A precise estimate for $|\sin x|^p$ given by the numerical inequality

$$C(p)(\cos^p x - \cos px) \leq |\sin x|^p \leq A(p,\gamma)\cos^p x - B(p,\gamma)\cos px,$$

where $1 < p \leq 2$, $0 < \gamma < \frac{\pi}{2}$ and

$$A(p,\gamma) = \frac{tg^{p-1}\gamma}{tg(p-1)\gamma}\ ,\ B(p,\gamma) = \frac{\sin^{p-1}\gamma}{\sin(p-1)\gamma}\ ,\ C(p) = -\frac{1}{\cos(p\pi/2)}\ .$$

2) Calderon's proof bases on the observation that if $f \geqslant 0$ and $f \not\equiv 0$ and if u and v are the Poisson integrals of f and $\mathcal{H}(f)$ respectively, $F = u+iv$, then $|F^p(z)| = |F(z)|^p \exp(pi \arg F(z))$ ($|\arg F(z)| \leqslant \frac{\pi}{2}$) is analytic in the open disc $|z| < 1$. Hence $F^p(0) = \|f\|_1^p = \left(\frac{1}{2\pi}\int_0^{2\pi} f\,dx\right)^p$. The usual device is now to replace an arbitrary real function by the difference of two positive functions. Pichorides has observed that in general F^p is a subharmonic function and this allows to obtain the same (sharp) estimates as for non-negative functions, namely

$$(5) \qquad \|\mathcal{H}_{per}(f)\|_p \leqslant tg\frac{\pi}{2p}\ \|f\|_p \qquad \text{for } f \in L^p_R\ .$$

Here and in the sequel L^p_R denotes the L^p-space of real functions and L^p is the L^p-space of complex functions.

Clearly, in view of (3), (4) and (5), one gets (1) and moreover

$$(6) \qquad \|\mathcal{H}\|^{p,p} = \|\mathcal{H}_{per}\|^{p,p}\ .$$

In this paper, Pichorides considered the Hilbert Transform as an operator acting between real spaces. However his result (i.e. the identities (1) and (6) extends to the complex case without difficulties in view of the following general fact .

<u>Proposition 1.</u> : <u>Let</u> $0 < p \leqslant r \leqslant \infty$. <u>Let</u> X <u>be a linear subspace of</u> L^p_R, $T: X \to L^r_R$ <u>a bounded (real) linear operator. Let</u> $T \otimes 1_{L^r} : X \otimes L^r_R \to L^r_R \otimes L^r_R$ <u>be the tensor of</u> T <u>with the identity on</u> L^r_R <u>(i.e. the operator induced by</u> T <u>with values in</u> L^r_R). <u>Then</u>

$$\|T \otimes 1_{L^r_R} : X \otimes L^r_R \to L^r_R \otimes L^r_R\| = \|T : X \to L^r_R\|\ .$$

(We consider here $X \otimes L^r_R$ as a subspace of $L^p_R(L^r_R)$ and $L^r_R \otimes L^r_R$ with the norm of $L^r_R(L^r_R)$; the latter is nothing else but the L^r_R space with respect to the product measure).

Corollary:(a) Let ℓ^2 denote the (real) Hilbert space.
Then $\| T : X \to L^r_R \| = \| T \otimes 1_{\ell^2} : X \otimes \ell^2 \to L^r_R \otimes L^2 \|$.

(b) Let $\tilde{T}$ denote the complexification of T i.e. the complex linear operator induced by T from $\tilde{X}$ into L^r (here $\tilde{X}$ denotes the subspace of L^p generated by X).
Then $\| T : X \to L^r_R \| = \| \tilde{T} : X \to L^r \|$.

To derive (a) from Proposition 1 observe that L^2_R contains a subspace isometric to ℓ^2 (for instance a subspace spanned by a sequence of independent normal Gaussian random variables). For (b) note that one can identify $\tilde{T}$ as a real operator with $T \otimes 1_{\ell^2_2}$ where ℓ^2_2 denotes the 2-dimensional real Hilbert space.

The proof of Proposition 1 in the full generality can be found in Figiel, Iwaniec and Pełczynski[1](cf. also Beckner [1] lemma 2 for a weaker statement). It uses only the Fubini theorem for $p = r$, and the integral form of the Minkowski inequality for $p < r$.

Remark. Proposition 1 and the Corollary fail for $p > r$.

The last remark leads to an open question concerning the weak (1-1) and the Kolmogorov inequalities for the Hilbert Transform.

Recall (Here m denotes the Lebesgue measure, $L^p_R = L^p_R(-\infty, +\infty)$).

The weak (1,1) inequality :

$$m(|\mathcal{H}(f)| > a) \leq \frac{C}{a} \|f\|_1 \quad (f \in L^1_R,\ a > 0)$$

C is a universal constant independent of f and a.

The Kolmogorov inequality

$$\| \mathcal{H}(f) \|_p = \left(\int_{-\infty}^{+\infty} |\mathcal{H}(f)(x)|^p \, dx\right)^{\frac{1}{p}} \leq C_p \|f\|_1 \quad (f \in L^1_R,\ 0<p<1).$$

C_p is a universal constant depending only on p but independent of f.

Denote by $\|\mathcal{H}\|^{weak(1,1)}$ the smallest possible C satisfying the weak (1,1) inequality and by $\|\mathcal{H}\|^{1,p}$ the smallest possible C_p satisfying the Kolmogorov inequality. These constants have been found by Burgess Davis [1][2] (1974),(1976) who used probabilistic methods. Later A. Baernstein [1][2](1978) gave analytic proofs of Burgess Davis' results which base on somewhat similar use of subharmonic functions as in Pichorides [1] and some properties of certain "counting functions". The constants are

$$\|\mathcal{H}\|^{weak(1,1)} = \pi^3\Big(\int_0^{\pi} \ln \operatorname{ctg}\frac{\vartheta}{2}\,d\vartheta\Big)^{-1} = \frac{1+3^{-2}+5^{-2}+\dots}{1-3^{-2}+5^{-2}-\dots}$$

$$\|\mathcal{H}\|^{1,p} = \Big(\frac{1}{2\pi}\int_0^{2\pi} |\sin\vartheta|^{-p}\,d\vartheta\Big)^{\frac{1}{p}} \qquad (0<p<1) .$$

We also have as in the case of Riesz inequality

$$\|\mathcal{H}\|^{weak(1,1)} = \|\mathcal{H}_{per}\|^{weak(1,1)}$$

$$\|\mathcal{H}\|^{1,p} = \|\mathcal{H}_{per}\|^{1,p} \qquad (0<p<1).$$

However as was pointed out to the author by A. Baernstein, the constants $\|\mathcal{H}\|^{weak(1,1)}$ and $\|\mathcal{H}\|^{1,p}$ are known only for real functions (!). To our best knowledge the following is open.

Problem 1 : Identify the best constants in the weak (1,1) and the Kolmogorov inequalities for the Hilbert Transform for f complex-valued.

In connection with the Pichorides result it might be interesting to solve

Problem 2 : Find $\|\mathcal{H}\|^{p,q}$ for $p > q$.

A close relative of the complex Hilbert Transform is the so called Riesz Projection from L^p onto the Hardy space H^p. The Riesz Projection R can be formally defined by

$$R = \frac{1}{2}(I-i\mathcal{H}) \quad \text{for the real line}$$

$$R_{per} = \frac{1}{2}(I-i\mathcal{H}_{per}) + \hat{f}(0).1 \quad \text{for the circle.}$$

Here I denotes the identity operator and

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,e^{-ikt}\,dt \quad \text{for } k = 0,\pm1,\pm2,\ \dots$$

It is annoying and of some interest for the metric theory of Hardy spaces that the following is open.

Problem 3 : Find $\|R\|^{p,p}$ and $\|R_{per}\|^{p,p}$.

Also we do not know the best constants in the weak (1,1) and the Kolmogorov inequalities for the Riesz projection.

The Hilbert Transform is a special case of a Calderon-Zygmund singular integral operator. The question of identifying best constants for inequalities involving such operators seems to be too general. However it might be possible to attack successfully the case of operators with some additional algebraic properties like operators satisfying the identity (*). P. Cartier has suggested to consider the operators satisfying the Baxter identity $V(fg) + V(f)\,V(g) = V(f\,V(g)) + V(V(g)f)$ which is somewhat similar to (*). The reader is referred to Cartier [1] for a discussion of algebraic consequences of the Baxter identity.

Another interesting example is the operator S of convolution with z^{-2} defined for nice test functions on the plane by

$$S(f) = -\text{ p.v. } \frac{1}{\pi} \iint_{R^2} \frac{f(\xi,\eta)}{[(x-\xi)^2+i(y-\eta)]^2}\, d\xi\, d\eta .$$

An important property of S is that it changes the $D_{\bar z}$ derivative into D_z, in symbols

$$S\, D_{\bar z}\, f = D_z\, f \quad \text{for nice } f .$$

T. Iwaniec [1] (1982) discovered a relation of the asymptotic behaviour of the norm $\|S\|^{p,p}$ with the theory of quasi-conformal mappings. For this purpose it would be interesting to solve

Problem 4 (T. Iwaniec) : Find $\|S\|^{p,p}$ for $1<p<\infty$ or at least decide whether $\lim_{p=\infty} p^{-1}\, \|S\|^{p,p} = 1$.

2. THE MARCINKIEWICZ-PALEY INEQUALITY FOR THE HAAR SYSTEM.

In this section we discuss briefly the recent result due to Burkholder on the best constant in the Marcinkiewicz-Paley inequality for the Haar system.

Recall that the Haar system $(h_j)_{j=0}^{\infty}$ is the orthonormal system in $L^2[0,1]$ obtained by the Schmidt orthogonalization process from the sequence of characteristic functions of the intervals

$[0,1]$, $[2m\,2^{-k},(2m+1)2^{-k}]$ for $m = 0,1,\dots,2^{k-1}-1$; $k = 1,2,\dots$

The Marcinkiewicz-Paley inequality says :

Given p with $1 < p < \infty$, there is a constant C_p such that

$$\sup_{\varepsilon_j=\pm 1} \| \Sigma\, \varepsilon_j t_j h_j \|_p \leqslant C_p \| \Sigma\, t_j h_j \|_p \tag{7}$$

for all sequences of scalars (t_j).

Note that in view of corollary to Proposition 1, one can take the same constant both for real anf for complex sequence (t_j).

D. Burkholder [3] (1982) has shown that the best constant $b_p = \inf\{ C_p : C_p \text{ satisfies (7)} \}$ is :

$$b_p = p-1 \quad \text{for } p > 2$$

$$b_p = (p-1)^{-1} \quad \text{for } 1 < p < 2 .$$

This constant is to some extent important both for the theory of vector-valued martingales and for the Banach spaces because of the following facts.

a) The Haar system is the worst possible sequence of martingale differences with values in L^p (the worst possible in the sense that for any sequence in L^p of martingale differences the analogue of the Marcinkiewicz-Paley inequality holds with a constant not greater than b_p (Maurey [1] (1975) .

b) The Haar system is the best possible unconditional Schauder basis in L^p (i.e. if (e_j) is a sequence of functions in L^p such that for each $f \in L^p$ there is unique sequence of scalars (c_j) such that $f = \Sigma\, c_j e_j$ and if the inequality (7) holds with (h_j) replaced by (e_j) with a constant C_p, then $C_p \geqslant b_p$ (Olevskiĭ [1] (1967), [2]; Lindenstrauss and Pełczynski [2] (1971).

Burkholder's proof is highly ingeneous but complicated. It combines probabilistic ideas with geometric ones. A crucial role is played by the existence of some biconvex real function on $L^p \times L^p$ satisfying certain minimal properties.

In the final stage the problem reduces to solving the non linear partial differential equation which we write here for curiosity

$$(p-1)[yF_y - xF_x]F_{yy} - [(p-1)F_y - xF_{xy}]^2 + x^2 F_{xx}F_{yy} = 0$$

for F non constant and subjected to some other constraints on a suitable domain in the plane. The reader interested in the subject is referred to Burkholder [1],[2] [3],[4].

The measuring of unconditionality of a Schauder basis in complex Banach spaces is slightly different than that in the real case. From this point of view, it seems to be interesting to solve

Problem 5 : Identify the best constant $\tilde{b}_p = \inf\{\tilde{C}_p : \tilde{C}_p$ satisfying $(\tilde{7})\}$ where

$(\tilde{7})$ $\sup_{|\alpha_j|=1} \|\Sigma\, \alpha_j t_j h_j\|_p \leq \tilde{C}_p \|\Sigma\, t_j h_j\|_p$ for all sequences of scalars (t_j)

and the supremum is taken over all complex sequences (α_j) with $|\alpha_j| = 1$ for $j=0,1,..$

Clearly $\tilde{b}_p \geq b_p$ we conjecture that $\tilde{b}_p = b_p$ for $1 < p < \infty$.

3. CRITICAL EXPONENTS

From the examples which we have discussed one might get a wrong idea that the function $p \to K_p$ (where K_p is the best constant in some classical inequality) is always a nice one (for $p \neq 2$) like $\operatorname{ctg} \frac{\pi}{2p}$ or $p-1$.

However the answer may be more complicated even in very simple cases.

All results discussed in this section will appear in the paper Figiel, Iwaniec, and Pełczynski [1] .

We begin with the simple 2-dimensional example.

Let $L^p = L^p(-\pi,\pi)$ with respect to the normalized Lebesgue measure. Let $\mathcal{S}_2$ be the 2-dimensional orthogonal projection defined by

$$\mathcal{S}_2(f) = \hat{f}(0) + \hat{f}(1)\, e^{it} .$$

Let us consider the function $p \to \|\mathcal{S}_2\|^{\infty,p}$ for $p \geq 2$ where as usual $\|\mathcal{S}_2\|^{\infty,p} = \|\mathcal{S}_2 : L^\infty \to L^p\|$.

Then we have

Fact. $\|\mathcal{S}_2\|^{\infty,p} = 1$ for $2 \leq p \leq 4$; $\|\mathcal{S}_2\|^{\infty,p}$ is a strictly increasing function for $4 < p \leq \infty$; $\|\mathcal{S}_2\|^{\infty,\infty} = \frac{4}{\pi}$.

Sketch of the proof : First, we describe potential extremals using the following general

Lemma 1 : *Let* $E \subset L^\infty$ *be a finite dimensional linear subspace such that for every* $0 \neq e \in E$, $m\{t : e(t) = 0\} = 0$. *Let* P_E *denote the orthogonal projection onto* E .

Then every $f \in L^\infty$ *with* $|f| \equiv 1$ *satisfying* $\|P_E f\|_p = \|P_E\|^{\infty,p}$ *for some* p *with* $2 \leqslant p \leqslant \infty$ *is of the form* $f = e.|e|^{-1}$ *for some* $e \in E$.

Proof by variational method.

Corollary : *Given* p *with* $2 \leqslant p \leqslant \infty$ *there is an* $r = r(p)$ *with* $0 \leqslant r(p) \leqslant 1$ *such that*

$$\|\mathcal{P}_2\|^{\infty,p} = \mathcal{P}_2(f_r)$$

where

$$f_r = \frac{1+r e^{it}}{|1+r e^{it}|} .$$

Now we are ready to outline the proof of the Fact.

We have $\mathcal{P}_2(f_r) = A(r) + B(r) e^{it}$ where

$$A(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_r(t)dt = \hat{f}_r(0) , \quad B(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_r(t) e^{-it} dt = \hat{f}_r(1) .$$

Expanding the integrands into powers of r one can identify $A(r)$ and $B(r)$ with some Gauss hypergeometric series. Using the expansions it is not difficult to show that

$$\left(\frac{d}{dr} \|A(r) + B(r) e^{it}\|_p^p \right)_{r=0} > 0 \quad \text{for } p > 4 .$$

Since $A(0) = 1$, $B(0) = 0$, we infer that $\|\mathcal{P}_2(f_r)\|_p > 1 = \|\mathcal{P}_2(f_o)\|_p$ for $p > 4$ and r sufficiently close to 0 .

To prove that $\|\mathcal{P}_2\|^{\infty,p} = 1$ for $2 \leqslant p \leqslant 4$ it is enough to show that

$$\|\mathcal{P}_2\|^{\infty,4} = \sup\{\|\mathcal{P}_2(f_r)\|_4 : 0 \leqslant r \leqslant 1 \} = 1 .$$

For the latter identity it suffices to prove that

$$\phi(r) = \| A(r) + B(r)\, e^{it} \|_4^4 = A(r)^4 + B(r)^4 + 4\, A^2(r)\, B^2(r)$$

is a monotonely decreasing function of r.

This is the most difficult part of the argument. It requires a careful analysis of the derivative $\phi'(r)$ which among other things bases on a few non obvious formulae on Gauss hypergeometric series due to Gauss.

In fact the function ϕ is "almost constant" ; we have

$$\Phi(0) = 1 \quad \text{and} \quad \phi(1) = 6.\left(\frac{2}{\pi}\right)^4 \approx 0.98 .$$

The qualitative phenomenon described in the Fact holds in general. One has (cf. Figiel, Iwaniec and Pełczynski [1]).

Proposition 2 : Let E be a finite dimensional subspace of $L^\infty(\mu)$, μ-a probability atomless measure. Let P_E denote the orthogonal (with respect to $L^2(\mu)$) projection onto E.

Then

(8) $$cr(P_E) = \sup\{p : \|P_E\|^{\infty,p} = 1\} > 2 .$$

Precisely, if $K_E = \sup\{\|e\|_\infty . \|e\|_1^{-1} : e \in E \setminus \{0\}\}$ then $cr(P_E) = \infty$ for $K_E = 1$, and $cr(P_E) \geq p_o(K_E)$ where $p_o(K_E)$ is the unique root of the equation $K_E^x - 1 = x(K_E^2 - K_E)$ in the interval $2 < x < 3$.

The inequality (8) is sharp in the following sense :

Given $K > 1$ there is an $E \subset L^\infty[0,1]$ with dim E = 1 such that $K_E = K$ and $cr(P_E) = p_o(K)$.

The proof of proposition 2 reduces to the variational problem of finding $\sup a_p(g)$ where

$$a_p(g) = \|g\|_p\, K^{-1} \|g\|_2^{-2} ,$$

with $p \in (2,\infty)$ and $K \in (1,\infty)$ fixed : the supremum is extended over all functions g on [0,1] such that

$$1 \geq g \geq 0 , \quad \int_o^1 g\, dx = K^{-1} , \quad g \text{ non-increasing}.$$

To solve this problem we use a long argument which involves variational method and elementary Lagrange multipliers technique.

The symbol $cr(P_E)$- stands for "critical exponent". To justify the name observe that if the subspace E contains a unimodular function (i.e. there is an $e \in E$ with $|e| \equiv 1$) then it follows immediately from Proposition 2 that the funtion $p \to \|P_E\|^{\infty,p}$ is constant in the interval $[2, cr(P_E)]$.

The sketch of the proof of "Fact" indicates that the evaluation of $cr(P_E)$ for a fixed E may be difficult. In particular we have

Problem 6 : Evaluate $cr(\mathcal{S}_n)$ for $n \geq 3$ where $\mathcal{S}_n(f) = \sum_{j=0}^{n-1} \hat{f}(j)\, e^{ijt}$ is the orthogonal projection from $L^2(-\pi,\pi)$ onto the span $[1, e^{it}, \ldots, e^{i(n-1)t}]$.

4. THE KHINCHINE INEQUALITY AND ITS RELATIVES

Somewhat similar effect to the critical exponent can be observed in the behaviour of the best constant function in the Khinchine inequality.

Recall that the j-th Rademacher function r_j is defined by $r_j(t) = \text{sign} \sin 2^j \pi t$ for $0 \leq t \leq 1$ $(j=1,2,\ldots)$. The sequence (r_j) is a realization of a Bernoulli sequence, i.e. a sequence (δ_j) of independent random variables each distributed according to the law $P\{\delta_j = 1\} = P\{\delta_j = -1\} = \frac{1}{2}$.

The Khinchine inequality says

For each p with $0 < p < \infty$ there are constants A_p and B_p such that

$$A_p \|\sum_j a_j r_j\|_2 \leq \|\sum_j a_j r_j\|_p \leq B_p \|\sum_j a_j r_j\|_2 \tag{9}$$

for all sequences of scalars (a_j).

In the language of Banach spaces the best possible constants appearing in (9) or their inverses can be interpreted as the norm of embedding of ℓ^2 into L^p which takes the unit vectors of ℓ^2 into the Rademacher functions or the norm of the inverse operator.

Note : 1) From this interpretation and from Proposition 1 and its Corollary, it follows that the best constants in (9) are the same for real sequences (a_j) as well as for the complex ones.

2) (r_j) is an orthonormal sequence. Thus $\|\sum_j a_j r_j\|_2 = (\sum |a_j|^2)^{\frac{1}{2}}$.

Several mathematicians contributed to the problem of finding the best constants in the Khinchine inequality : Stečkin [1] (1961), Young [1] (1976), Szarek [1] (1976) evaluated A_p and B_p for some values of p. Finally in 1978 Haagerup [1][2] solved the problem completely using techniques of analytic probability and theory of special functions.

To formulate the result let from now A_p and B_p denote the best constants in the Khinchine inequality. Let γ_p denote the p-th moment of the standard Gaussian variable ,

$$\gamma_p = \left((2\pi)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} |t|^p e^{-\frac{t^2}{2}} dt\right)^{\frac{1}{p}} = 2^{\frac{1}{2}}\left(\pi^{-\frac{1}{2}} . \Gamma(\frac{p+1}{2})\right)^{\frac{1}{p}} .$$

Then $A_p = 1$ for $p \geq 2$ and $B_p = 1$ for $p \leq 2$, and

$$A_p = \begin{cases} 2^{\frac{1}{2}-\frac{1}{p}} & \text{for } < 0 \leq p_o \\ \gamma_p & \text{for } p_o < p < 2 , \end{cases}$$

$$B_p = \gamma_p \qquad \text{for } p > 2 ,$$

p_o is the root of the equation $\gamma_p = 2^{\frac{1}{2}-\frac{1}{p}}$ or equivalently the equation $2\Gamma(\frac{p+1}{2}) = \sqrt{\pi}$ for $0 < p < 2$. The p_o is the critical point at which the behaviour of the function $p \to A_p$ changes !

The Khinchine type inequality for Steinhaus variables

One may also think of the Rademacher functions as the coordinate functions of the infinite product of two elements group Z_2^∞ It is natural to consider the coordinate functions of the infinite torus $\mathbb{T}^\infty$ i.e. the infinite product of the circle groups.

The j-th Steinhaus function s_j is the j-th coordinate function of the infinite torus $\mathbb{T}^\infty$. The sequence (s_j) of the Steinhaus functions consists of mutually independent equally distributed variables, each distributed as the function $t \to e^{i2\pi t}$ on $[0,1]$.

The following problem is well known and still open.

Problem 7 : Find the best constants in the Khinchine type inequality (10) for Steinhaus functions, where

(10) $\qquad \tilde{A}_p \| \sum_j a_j s_j \|_2 \leqslant \| \sum_j a_j s_j \|_p \leqslant \tilde{B}_p \| \sum_j a_j s_j \|_2 \qquad (0<p<\infty)$

for all sequences of scalars (a_j).

Note that in the metric theory of complex Banach spaces, the Steinhaus functions play the same role as the Rademacher functions for the real spaces. This makes Problem 7 interesting from the point of view of geometry of Banach spaces and the theory of Banach ideals. However the problem is also interesting just to see whether the behaviour of the best constant functions in the inequality (10), $p \to \tilde{A}_p$ and $p \to \tilde{B}_p$, has the same qualitative character as in the case of the Khinchine inequality.

In a discussion during the Leipzig Conference in 1977, Haagerup has conjectured that for the function $p \to \tilde{A}_p$ there is a critical point, say $\tilde{p}_o$ with $0<\tilde{p}_o<1(!)$; for $\tilde{p}_o \leqslant p < 2$ one should have $\tilde{A}_p = \tilde{\gamma}_p$ while for $p<\tilde{p}_o$ the character of the function $p \to \tilde{A}_p$ changes ; we should also have $\tilde{B}_p = \tilde{\gamma}_p$ for $p \geqslant 2$.*

Here $\tilde{\gamma}_p$ is the p-th moment of the standard complex Gaussian variable,

$$\tilde{\gamma}_p = \left(\frac{1}{\pi} \iint_{R^2} e^{-x^2-y^2} (x^2+y^2)^{\frac{p}{2}} \, dx\, dy \right)^{\frac{1}{p}} = \left[\Gamma\left(\frac{p+2}{2} \right) \right]^{\frac{1}{p}} .$$

Observe that the identity

$$\| \sum_j |a_j| s_j \|_p = \| \sum_j a_j r_j(t) s_j \|_p \qquad \text{for a.e. } t \in [0,1]$$

combined with the Fubini Theorem yields

$$A_p \leqslant \tilde{A}_p \quad \text{and} \quad \tilde{B}_p \leqslant B_p \quad \text{for} \quad 0<p<\infty .$$

There is an unexpected connection between moments of linear combinations of Steinhaus functions and the Bessel functions. We shall discuss it now.

Already Kluyver [1] (1906) solving the problem of random flight in the plane has shown that if $r>0$, $a_1, a_2, \ldots, a_n$ are complex numbers and $p_n(a_1,a_2,\ldots a_n;r)$ denotes the probability that $|\sum a_j s_j| < r$ (i.e. $p_n(a_1,a_2,\ldots a_n;r)$ is the normalized Haar measure of the subset $\{ w \in \mathbb{T}^n : |\sum a_j s_j(w)| < r \}$ of the n-dimensional torus $\mathbb{T}^n$) then

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*) It is not hard to prove that $\tilde{B}_m = \tilde{\gamma}_m$ for $m = 2,4,6, \ldots$

$$P_n(a_1,a_2,\dots,a_n;r) = \begin{cases} r\int_0^\infty \mathcal{J}_1(rt)\prod_{j=1}^n \mathcal{J}_o(|a_j|t)\,dt & \text{for } \max_{1\le j\le n}|a_j| < r \\ 1-r\int_0^\infty \mathcal{J}_1(rt)\prod_{j=1}^n \mathcal{J}_o(|a_j|t)\,dt & \text{for } \max_{1\le j\le n}|a_j| \ge r \end{cases}$$

where

$$\mathcal{J}_n(t) = \frac{1}{2\pi}\int_0^{2\pi} \cos(t\sin\vartheta - n\vartheta)\,d\vartheta \qquad n = 0, 1$$

are the zero and the first Bessel function respectively (cf. Watson [1] p.419 for details). Thus

(11) $$\|\sum_{j=1}^n a_j s_j\|_p^p = \int_0^\infty r^p \frac{\partial}{\partial r} P_n(a_1,a_2,\dots a_n;r)\,dr \qquad 0 < p < \infty .$$

It is likely that from (11), one can derive Kwapień's Formula

(12) $$\|\sum_{j=1}^n a_j s_j\|_1 = \int_0^\infty \frac{1-\prod_{j=1}^n \mathcal{J}_o(|a_j|t)}{t^2}\,dt$$

We present here a direct proof of (12) due to Kwapień. It is a modification of an argument of Haagerup [2] p.235 for the Rademacher functions.

(i) One has $|a| = \frac{2}{\pi}\int_0^\infty \frac{1-\cos at}{t^2}\,dt$, $a \in R$ (For, substitute $u = at$, use the integration by parts and the formula for the Dirichlet integral $\frac{\pi}{2} = \int_0^\infty \frac{\sin t}{t}\,dt$.

(ii) The map $U : L^1(P) \to L^1(P \times \frac{d}{2\pi})$ (P- a probability measure) defined by

$$Uf(w,\vartheta) = \frac{\pi}{2}(\mathrm{Re} f\cos\vartheta + \mathrm{Im} f\sin\vartheta) = \frac{\pi}{2}|f|\cos(\vartheta - \arg f) , \ 0 \le \vartheta < 2$$

is the isometric imbedding of the complex space $L^1(P)$ into the real space $L^1_R(P \times \frac{d\vartheta}{2\pi})$; it is a real linear map. (For the proof use the Fubini theorem and note that $\frac{1}{2\pi}\int_0^{2\pi}|\cos(\vartheta - \arg f)|d\vartheta = \frac{1}{2\pi}\int_0^\infty|\cos\vartheta|\,d\vartheta = \frac{2}{\pi}$).

(iii) Let $c_1, c_2, \dots c_n$, be mutually independent random variables each distributed as the function $t \to \cos 2\pi t$ $0 \le t \le 1$. Then for arbitrary complex scalars $a_1, a_2, \dots, a_n$

$$\|\sum a_j s_j\|_1 = \frac{\pi}{2}\|\sum |a_j| c_j\|_1 .$$

Clearly $\| \Sigma a_j s_j \|_1 = \| \Sigma |a_j| s_j \|_1$. Next observe that if s_j is realized as e^{itj} - the j-th coordinate on the torus Π^n, then $U(s_j) = \cos(\vartheta - t_j)$. Now (iii) follows from (ii) and the properties of the Haar measure on $\mathbb{T}^n$.

Write $E(f) = \int f\, dP$. Recall that if $\psi_1, \psi_2, \ldots, \psi_n$ are mutually independent symmetric random variables, then $E(\cos \sum_{j=1}^n \psi_j) = \prod_{j=1}^n E(\cos \psi_j)$. We also have $E(\cos a c_j) = \frac{1}{2\pi} \int_0^{2\pi} \cos (a \cos t)\, dt = \frac{1}{2\pi}\int_0^{2\pi} \cos(a \sin t) dt = \mathcal{J}_0(at)$ for $j = 1,2,\ldots n$.
Now using (i) and (iii) and the Fubini Theorem, we complete the proof of (12) as follows

$$\| \Sigma a_j s_j \| = \frac{\pi}{2} E | \sum_{j=1}^n |a_j| c_j |$$

$$= E \int_0^\infty \frac{1 - \cos (t \sum_{j=1}^n |a_j| c_j)}{t^2} dt$$

$$= \int_0^\infty \frac{1 - E \cos (t \sum_{j=1}^n |a_j| c_j)}{t^2} dt$$

$$= \int_0^\infty \frac{1 - \prod_{j=1}^n E \cos(t |a_j| c_j)}{t^2} dt$$

$$= \int_0^\infty \frac{1 - \prod_{j=1}^n \mathcal{J}_0(t|a_j|)}{t^2} dt .$$

Using (12), Kwapien and Sawa (cf. Sawa [1]) obtain :

$$\widetilde{A}_1 = \widetilde{\gamma}_1 \quad . \tag{13}$$

Their argument is similar to that of Haagerup [2], Theorem 1.1. Put

$$\phi(s) = \int_0^\infty \frac{1 - |\mathcal{J}_0(t/\sqrt{s})|^s}{t^2} dt.$$

Now fix $a_j > 0$ $(j = 1,2,\ldots,n)$ with $\Sigma a_j^2 = 1$. Combining (12) with the elementary inequality

$$\prod_{j=1}^n |\mathcal{J}_0 (t a_j)| \leq \sum_{j=1}^n a_j^2 |\mathcal{J}_0 (t a_j)|^{\frac{1}{a_j^2}}$$

one gets

$$\|\sum_{j=1}^{n} a_j s_j\|_1 \geq \int_0^\infty \frac{1 - \sum_{j=1}^{n} a_j^2 |\gamma_o(ta_j)|^{\frac{1}{a_j^2}}}{t^2} dt$$

$$= \int_0^\infty \sum_{j=1}^{n} a_j^2 \frac{1 - |\gamma_o(t a_j)|^{\frac{1}{a_j^2}}}{t^2} dt$$

$$= \sum_{j=1}^{n} a_j^2 \; \phi(a_j^2) \quad .$$

The difficult part of the proof of (13) is a detailed analysis of the behaviour of the function Φ ; in particular to show that $\phi(s) \geq \tilde{\gamma}_1$ for large s.

The vector-valued Khinchine inequality

The Khinchine inequality in the form (9) stated in this paper remains valid for the a_j's being elements of any Banach space. This fact has been discovered by J.P. Kahane [1] (1964). As an easy consequence one gets analogous inequalities for the Steinhaus functions and for standard Gaussian variables. Of course, the constants appearing in the vector valued inequalities need not be the same as in the scalar ones. However, the new constants can be taken to be independent of a choice of a Banach space. The result can be stated as follows.

Let $\alpha = (\alpha_j)$ be a sequence of equally distributed mutually independent random variables having all moments. Then for every pair p,q with $0 < p \leq q < \infty$ there is a constant $A_{p,q}(\alpha) \in (0,\infty)$ such that for every Banach space X and every eventually zero sequence $(a_j) \subset X$ with not all $a_j = 0$,

$$(14) \qquad \left[E\left(\|\Sigma a_j \alpha_j\|^q\right)\right]^{\frac{1}{q}} \leq A_{p,q}(\alpha) \left[E\left(\|\Sigma a_j \alpha_j\|^p\right)\right]^{\frac{1}{p}}$$

Observe that in the scalar case the second moment is distinguished by the identity $E|\Sigma a_j \alpha_j|^2 = E|\alpha_1|^2 (\sum_j |a_j|^2)$ while for every Banach space X which is not isometric to a Hilbert space the analogous identity fails for some sequence $(a_j) \subset X$. Moreover, by Kwapień's Theorem (cf. Kwapień [1]) if X is not isomorphic to a Hilbert space, then the integrals $E\|\Sigma a_j \alpha_j\|^2$ are not uniformly (with respect of all eventually finite sequences $(a_j) \subset X$) of the same order as

$\Sigma \|a_j\|^2$. This is a reason that we prefer to state the vector valued Khinchine type inequality in a different form than scalar ones.

The most interesting cases of the inequality (14) are : for real Banach spaces $\alpha = r$ - the sequence of Rademacher functions and $\alpha = g$ - the sequence of mutually independent standard real Gaussian variables; for complex Banach spaces $\alpha = s$ - the sequence of Steinhaus functions and $\alpha = \tilde{g}$ - the sequence of mutually independent standard complex Gaussian variables.

The best constants $A_{p,q}^{(r)}$, $A_{p,q}^{(g)}$, $A_{p,q}^{(s)}$, $A_{p,q}^{(\tilde{g})}$ are unknown. Some effort was made to identify $A_{1,2}^{(r)}$. Clearly $A_{1,2}^{(r)} \geqslant A_1^{-1} = \sqrt{2}$ (where A_1 is the best constant in the Khinchine inequality). B. Tomaszewski [1] has shown that $A_{1,2}^{(r)} \leqslant \sqrt{3}$. He conjectured that $A_{1,2}^{(r)} = \sqrt{2}$. As was observed by C. Borell [1] using a numerical inequality due to Beckner [1], Lemma 1, one can get good estimates from above for $A_{p,q}^{(r)}$. The inequality (14) for $\alpha = g$ has been discovered by Landau and Shepp [1], and independently by Fernique [1].

5. GROTHENDIECK'S CONSTANTS

In 1956 in his remarkable São Paulo paper, Grothendieck has proved " The Fundamental Theorem in the Metric Theory of Tensor Products". The theorem has several equivalent formulations. Grothendieck [1] has stated it in the language of bilinear forms on C(S) - spaces (cf. also Cartier [3] and the excellent survey Pisier [2]). We prefer to formulate it in the language of operator ideals.

(I) Every operator from any $L^1(\mu)$ space into a Hilbert space H is 1-summing.

Recall that a linear operator $u : X \to Y$ is p-summing $(0 < p < \infty)$ if $\pi_p(u) < \infty$ where $\pi_p(u) = \sup(\sum_j \|u(x_j)\|^p)^{\frac{1}{p}}$; the supremum extends on all finite sequences (x_j) in X such that for every linear functional $x^* \in X^*$, $\Sigma |x^*(x_j)|^p \leqslant \|x^*\|^p$.

By a standard Baire category argument (I) is equivalent to

(I') There is a constant G such that

(15) $$\pi_1(u) \leqslant G \|u\| \quad \text{for every linear operator } u : L^1(\mu) \to H.$$

The best constant in the inequality (15) depends on the field of scalars. We shall write G_R to denote the best constant for real spaces and $G_{\mathbb{C}}$ for complex spaces. For statements concerning both constants we shall write G.

It is not hard to see that (I') is equivalent to the following elementary statement about matrices (cf. Lindenstrauss and Pełczynski [1]. The constant appearing below is the same as that in (15).

(I") <u>The Grothendieck inequality</u> : <u>There is a constant</u> G <u>such that for every</u> $n = 1,2, \ldots$ <u>and for every scalar</u> $n \times n$ - <u>matrix</u> $(a_{j,k})$ <u>the condition</u>

$$\Big|\sum_{j,k} a_{j,k}\, \varepsilon_j \eta_k\Big| \leq 1 \quad \text{for all scalars } \varepsilon_j \text{ and } \eta_k \text{ with } |\varepsilon_j| = |\eta_k| = 1$$

$(j,k = 1,2,\ldots,n)$.

<u>implies</u>

$$(16) \qquad \Big|\sum_{j,k} a_{j,k} \langle x_j, y_k\rangle\Big| \leq G \quad \text{for all vectors } x_j \text{ and } y_k \text{ in } \ell^2$$

$$\text{with } \|x_j\| = \|y_k\| = 1 \quad (j,k = 1,2,\ldots,n).$$

Now one of Grothendieck's problems (cf. Grothendieck [1] Problem 3, cf. also Pisier [2]) can be restated as follows.

<u>Problem 8</u> : Find the best constants G_R and G_C for the real and complex inequalities (16).

There are many beautiful proofs of the Grothendieck inequality which yield better or worse estimates from above for G (cf. Kaijser [1], Krivine [1], [2], Maurey [2], Pełczynski and Wojtaszczyk (see Pełczynski [1], Pisier [1], Rietz [1]). We recommend to the reader the papers Pisier [1] and Krivine [2] where the best estimates from above for $G_{\mathbb{R}}$ and $G_{\mathbb{C}}$ respectively are obtained.

We have

$$G_R \leq \frac{\pi}{2\ln(1+\sqrt{2})} \approx 1{,}782 \ldots \text{ (Krivine (1977))}$$

$$G_{\mathbb{C}} \leq e^{1-c} \approx 1{,}527 \ldots < \frac{\pi}{2} \text{ (Pisier (1978))},$$

here c is the Euler constant.

Very little is known on the estimates from below for G . The only published estimates are that of Grothendieck [1] .

We have

(18) $$\frac{\pi}{2} \leqslant G_R \quad ; \quad \frac{4}{\pi} \leqslant G_{\mathbb{C}} \ .$$

Consequently $$G_{\mathbb{C}} < G_R \ .$$

Krivine [2] anounced that $\frac{\pi}{2} < G_R$; he also conjectured that $G_R = \frac{\pi}{2 \ln(1+\sqrt{2})}$.

We present here a proof of (18) similar to that of Grothendieck [1].

Consider the map $U_n : \ell_n^2 \xrightarrow{\mathfrak{J}} L^\infty(S^{(n)},\lambda^{(n)}) \xrightarrow{Q} \ell_n^2$ where $S^{(n)}$ is the unit sphere of ℓ_n^2 ; $\lambda^{(n)}$- the normalized Haar measure on $S^{(n)}$; $\mathfrak{J}(t)(x) = \langle t,x \rangle$ for $t \in \ell_n^2$, $x \in S^{(n)}$; $Q(f)(x) = c^{(n)} \int_{S^{(n)}} f(y) \langle x,y \rangle \lambda^{(n)}(dy)$
$c(n) = \int_{S^{(n)}} |\langle y,e_1 \rangle| \ \lambda^{(n)}(dy)$, $e_1 = (1,0,0\ldots0)$, $\langle .,. \rangle$
- denotes the scalar product on ℓ_n^2. Clearly $\mathfrak{J}$ is an isometric embedding, and $\| Q \| \leqslant 1$.

Next consider the map
$$U_n^* U_n : \ \ell_n^2 \xrightarrow{\mathfrak{J}} L^\infty \xrightarrow{Q^*Q} L^1 \xrightarrow{\mathfrak{J}^*} \ell_n^2 \ .$$

Use the facts : (a) the composition wv of an one-summing operator w with an operator v from L^∞ is nuclear and the nuclear norm $\underline{n}(wv) \leqslant \pi_1(w) \|v\|$, (b) the trace tr(u) of a nuclear operator $u = \ell^2 \to \ell^2$ admits the estimate $|tr(u)| \leqslant \underline{n}(u)$.

Now assuming that $\pi_1(\mathfrak{J}^*) \leqslant \|\mathfrak{J}^*\| G = G$ and observing that $tr(U^*U) = tr((c^{(n)})^2 I_{\ell_n^2}) = n(c^{(n)})^2$ we get

(19) $$n(c^{(n)})^2 \leqslant \underline{n}(\mathfrak{J}^*(Q^*Q)\mathfrak{J}) \leqslant G \|\mathfrak{J}^*\| \|QQ^*\| \|\mathfrak{J}\| = G \ .$$

Note that in the real case
$$c_R^{(n)} = c_R^{(n)} = \left(\int_{S_R^{(n)}} |x_1| \ d\lambda_R^{(n)} \right)^{-1}$$

where x_1 denotes the first coordinate of the vector $x = (x_1,x_2,\ldots,x_n) \in S_R^{(n)}$;

While in the complex case

$$c^{(n)} = c_{\mathbb{C}}^{(n)} = \left(\int_{S_R^{(2n)}} \sqrt{x_1^2 + y_1^2}\, d\lambda_R^{(2n)} \right)^{-1}$$

where $z_1 = x_1 + iy_1$ denotes the first coordinate of the vector $z = (z_1, z_2, \ldots, z_n) \in S_{\mathbb{C}}^{(n)}$.

A straightforward computation shows that

$$\lim_n \sqrt{n}\; c_R^{(n)} = \sqrt{\frac{\pi}{2}} \;;\; \lim_n \sqrt{n}\; c_{\mathbb{C}}^{(n)} = \frac{2}{\sqrt{\pi}}$$

which in view of (19) completes the proof of (18).

Specifying classes of operators from L^1 into L^2 one can consider questions related to Problem 8; one of them arose from a discussion with S. Hartman a few years ago.

Consider the operator $u_f : L^1(\mathbf{T}) \to L^2(\mathbf{T})$ of convolution with a function f. Then $\|u_f\| = \|f\|_2$.

Problem 9 : Compute : $D_{\mathbb{C}} = \sup\{\pi_1(u_f) : f \in L^2(\mathbb{T})\}$ and $D_R = \sup\{\pi_1(u_f) : f \in L^2(\mathbb{T})\}$.

Clearly, if D denotes either $D_{\mathbb{R}}$ or $D_{\mathbb{C}}$, then $D \leqslant G$.

One can consider similarly the constant D(A) for any compact Abelian group A.

There are other constants related to G like the best constant (also unknown) of factorization of every operator from an L^∞-space into an L^1-space through a Hilbert space as well as the constant relating 2-summing norm of such operators to their operator norms. The reader is referred to Pisier [2] for further information. These constants do not seem to be objects of intensive study.

Before formulating the last problem, let us remark that in all previous inequalities we have considered in fact one parameter families of inequalities. There are several ways of including the Grothendieck inequality into a one parameter family. The most natural seems to be the following.

Observe that every one-summing operator is p-summing for $p \geqslant 1$. However, by a result of Maurey [2], Theorem 9, (1979) every operator from an L^1-space into a Hilbert space is p-summing for every $p > 0$. Thus, for each $p \in (0,\infty)$ there is a constant C_p such that $\pi_p(u) \leqslant C_p\|u\|$ for every operator u from an L^1-space into a Hilbert space. Put $G(p) = \inf C_p$. Again we have two functions $G_R(.)$

and $G_{\mathbb{C}}(.)$ depending on the field of scalars. Clearly $G_R(1) = G_R$ and $G_{\mathbb{C}}(1) = G_{\mathbb{C}}$. The inequality $\pi_p(u) \leqslant G(p)\|u\|$ can be also restated in elementary language as follows :

For every 1,2,... and for every scalar $n \times n$-matrix $(a_{j,k})$ the condition

$$\left(\sum_j \left| \sum_k a_{jk} \eta_k \right|^p \right)^{\frac{1}{p}} \leqslant 1 \quad \text{for scalars } \eta_j \text{ with } |\eta_j| \leqslant 1 \quad (j=1,2,\dots,n).$$

implies

$$\left(\sum_k \left| \sum_j a_{jk} < x_j, y_k > \right|^p \right)^{\frac{1}{p}} \leqslant G(p) \quad \text{for all vectors } x_j \text{ and } y_k \text{ in } \ell^2$$

$$\text{with } \|x_j\| \, \|y_k\| \leqslant 1 \quad (j,k = 1,2,\dots,n).$$

Problem 10 : Describe the functions $p \to G_R(p)$ and $p \to G_{\mathbb{C}}(p)$.

Let us observe that using the operator U defined in the proof of (18) Grothendieck [1] has shown that

$$G_R(2) = \sqrt{\frac{\pi}{2}} \quad \text{and} \quad G_{\mathbb{C}}(2) = \frac{2}{\sqrt{\pi}} .$$

R E F E R E N C E S

A. BAERNSTEIN [1] Some sharp Inequalities for Conjugate Funtions - Indiana Univ. Math. Journal 27 (1978), 833-852.

[2] Some Sharp Inequalities for Conjugate Functions - Proc. Symp. Pure Math. 35 (1979), 409-416.

W. BECKNER [1] Inequalities in Fourier Analysis, Annals of Math. 102 (1975), 159-182.

C. BOREL [1] On the integrability of Banach space valued Walsh polynomials, Séminaire de Probabilités XIII, Lecture Notes in Mat. 721, Springer Verlag 1979.

P.L. BURKHOLDER [1] Martingale transforms, Ann. Math. Statis. 37 (1966), 1494-1504.

[2] A geometric characterization of Banach spaces in which martingale difference sequences are unconditional, Ann. Probab. 9 (1981), 997-1011.

[3] A nonlinear partial differential equation and the unconditional constant of the Haar system in L^p, Bull. Amer. Math. Soc. (New Series) 7 (1982), 591-595.

[4] Boundary value problems and sharp inequalities for martingale transforms, Ann. Probab. (to appear).

P. CARTIER [1] On the structure of Free Baxter Algebras, Advances in Mathematics 9 (1972), 253-265.

[2] Spectre de l'équation de Schrödinger, Application à la Stabilité de Matière (d'après J. Lebowitz, E. Lieb, B. Simon et W. Thinning), Séminaire Bourbaki 29e année, 1976/77, n°496.

[3] Classes de formes bilinéaires sur les espaces de Banach (d'après Grothendieck), Séminaire Bourbaki, 13e année, 1960/61, exp. 211, 14 pp. Benjamin New York, 1966.

M. COTLAR [1] A unified theory of Hilbert transforms and ergodic theory, Rev. Mat. Cuyana I, (1955), 105-167.

B. DAVIS [1] On the week type (1,1) inequality for conjugate functions and circular symmetrization, Proc. Amer. Math. Soc. 44 (1974), 307-311.

[2] On Kolmogorov's Inequality $\|\tilde{f}\|_p \leqslant C_p \|f\|_1$, $0<p<1$, Amer. Math. Soc. 222 (1976), 179-192.

X. FERNIQUE [1] Intégrabilité des vecteurs Gaussiens, Compte Rendus Acad. Sci. Paris 270 (1970), 1698-1699.

T. FIGIEL, T. IWANIEC and A. PEŁCZYNSKI :

[1] Computing norms and critical exponents of some operators in L^p-spaces, Studia Math. to appear.

I.C. GOHBERG and N. KRUPNIK

[1] Norm of the Hilbert transformation in the L^p-space, Functional Analysis and its application 2.2 (1968), 91-92 (Russian).

A. GROTHENDIECK :

[1] Résumé de la théorie métrique des produits tensoriels topologiques, Bull. Soc. Mat. São Paulo (1956), 1-79.

U. HAAGERUP [1] Les meilleures constantes de l'inégalité de Khinchine, C.R. Acad. Sci. Paris 286.série A (1978), 259-262.

[2] The best constants in the Khinchine inequality, Studia Math. 52 (1982), 231-283.

T. IWANIEC [1] Extremal inequalities in Sobolev spaces and quasiconformal mappings, Zeitschrift f. Analysis und ihre Anwendungen, Bd 1(6) (1982).

J.P. KAHANE [1] Sur les sommes vectorielles $\Sigma \pm u_n$, C.R. Acad. Sci. Paris 259 (1964), 2577-2580.

S. KAIJSER [1] A note on the Grothendieck constant with an application to Harmonic Analysis, UUDM Report N° 1973 ; 10, Uppsala University.

J.C. KLUYVER [1] A local probability problem, Proc. Section of Sci. K. Akad. van Wet. te Amsterdam. VIII (1906), 341-350.

J.L. KRIVINE [1] Sur la constante de Grothendieck, C.R.Acad. Sci. Paris, 284 (1977) Série A, 445-446.

[2] Constantes de Grothendieck et fonctions de type positif sur les sphères, Advances in Math. 31 (1979), 16-30.

S. KWAPIEŃ [1] Isomorphic characterizations of inner product spaces by othogonal series with vector valued coefficients, Studia Math. 44 (1972) 583-595.

H.J. LANDAU and L.A. SHEPP :

[1] On the supremum of a Gaussian process, Sankhya 32 (1970), 369-378.

J. LINDENSTRAUSS and A. PEŁCZYNSKI :

[1] Absolutely summing operators in $\mathcal{L}_p$ spaces and their applications, Studia Math. 29 (1968), 275-326.

[2] Contributions to the theory of the classical Banach spaces, J. Funct. Anal. 8 (1971), 225-249.

B. MAUREY [1] Système de Haar, Séminaire Maurey-Schwartz (1974-1975), Ecole Polytechnique, Paris 1975.

[2] Théorèmes de factorization pour les opérateurs linéaires à valeurs dans un espace $L^p(\Omega,\mu)$, Astérisque N° 11, Soc. Math. France 1974.

A. OLEVSKIĬ [1] Fourier series and Lebesque functions, Uspehi Mat. Nauke 22 (1967), 237-239 (Russian).

[2] Fourier series with respect to general orthogonal systems, Springer-Verlag, New York, Heidelberg-Berlin 1975.

A. PEŁCZYNSKI [1] Banach spaces of Analytic Functions and absolutely summing operators, Conf. Board of Math. Regional Conf. Ser. Math. 30 (1977).

S.K. PICHORIDES :

[1] On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov, Studia Math. 44 (1972), 165-179.

A. PIETSCH [1] Operator Ideals VEB Deutscher Verlag der Wissenschaften, Berlin 1978.

G. PISIER [1] Grothendieck's theorem for non-commutativ C^*-algebras with an appendix on Grothendieck constants, Journal of Funct. Anal. 29 (1978), 397-415.

[2] Les produits tensoriels d'espaces de Banach depuis Grothendieck, Séminaire Initiation à l'analyse, G. Choquet, M. Rogalski, J. Saint Raymond, 21e année, 1981/1982 N° 10, Université Paris VI.

A. RIETZ [1] A proof of the Grothendieck inequality, Israël J. Math. 19 (1974), 271-276.

J. SAWA [1] Master thesis, Warsaw University 1983.

S.B. STEČKIN [1] On the best lacunary system of functions, Izv. Acad. Nauk SSSR, Ser. Math. 25 (1961), 357-366 (Russian).

S.J. SZAREK [1] On the best constants in the Khinchine inequality, Studia Math. 58 (1978), 197-208.

B. TOMASZEWSKI [1] Two remarks on the Khinchine-Kahane inequality, Coll. Math. 46 (1982), 283-288.

G.N. WATSON [1] A Treatise on the theory of Bessel Functions, Cambridge Univ. Press. 1944.

R.M.G. YOUNG [1] On the best constants in the Khinchine inequality, J. London Math. Soc. 14 (1976), 496-504.

A. ZYGMUND [1] Trigonometric series, Vols I and II, Cambridge 1968.

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