SIGERU MIZOHATA

On the meromorphic propagation of singularities and the Levi condition

Astérisque, tome 131 (1985), p. 127-135
<http://www.numdam.org/item?id=AST_1985__131__127_0>

On the meromorphic propagation of singularities and the Levi condition

by Sigeru MIZOHATA

§1. Levi condition

We are concerned with the partial differential operator $P(x,t;\partial_x,\partial_t)$ of order m with analytic coefficients, defined in a neighborhood of the origin, whose principal symbol P_m is of constant multiplicity, namely,

$$\sigma(P_m) = (\tau-\lambda_1(x,t;\xi))^{m_1}(\tau-\lambda_2(x,t;\xi))^{m_2} \ldots (\tau-\lambda_s(x,t;\xi))^{m_s},$$

where, for all i,j $(i\neq j)$, $\lambda_i(x,t;\xi) \neq \lambda_j(x,t;\xi)$, for $\xi \in \mathbb{R}^n\backslash 0$.

First we explain the Levi condition. In view of the hypothesis of constant multiplicity, using the pseudo-differential operators, P can be factorized in the form:

$$(1.1) \qquad P = P_s \circ P_{s-1} \circ \ldots \circ P_1 + R,$$

where each P_j has its principal part $(\partial_t - i\lambda_j(x,t;D_x))^{m_j}$, and R is an analytic regularizing operator. This is called perfect factorization in the analytic class. More precisely P_j has the form:

$$(1.2) \qquad \begin{aligned} p_j = {} & (\partial_t - i\lambda_j(x,t;D_x))^{m_j} + a_{1,j}(x,t;D_x)(\partial_t - i\lambda_j(x,t;D_x))^{m_j-1} \\ & + \ldots + a_{k,j}(x,t;D_x)(\partial_t - i\lambda_j)^{m_j-k} + \ldots + a_{m_j,j}(x,t;D_x), \end{aligned}$$

where

$$(1.3) \qquad \text{order } a_{k,j}(x,t;\xi) \leq k-1 \quad (1 \leq k \leq m_j).$$

Here the total symbol of $a_{k,j}$ is determined uniquely as an analytic formal symbol; see for example [12]. Then, by a suitable modification in the ξ-space, we associate to it a true symbol, which Treves calls pseudo-analytic symbol [14]. R is represented by

$$R = \sum_{j=1}^{m} r_j(x,t;D_x)\partial_t^{m-j},$$

and each r_j is an analytic regularizing operator, namely

$$u(x) \in \mathcal{E}'_x \longrightarrow r_j(x,t;D_x)u(x) \in A(\Omega_x)$$

is continuous, depending smoothly on t, where $A(\Omega_x)$ is the space of analytic functions in Ω_x. $\Omega_x(\subset \mathbb{R}^n)$ is an open neighborhood of the origin.

<u>Levi condition.</u> We say that P_j satisfies the Levi condition if for all k,

$$\text{(1.4)} \qquad \underset{\xi}{\text{order}}\; a_{k,j}(x,t;\xi) \le 0.$$

We say that P satisfies the Levi condition if all P_j satisfy the Levi condition.

Levi condition has been introduced to characterize the hyperbolicity, assuming the characteristic roots are real.

Now we are concerned with the following local Cauchy problem

$$\text{(1.5)} \quad \begin{cases} Pu = 0 \\ \partial_t^j u|_{t=0} = u_j(x) \in C_0^\infty(\Omega_x), \quad 0 \le j \le m-1 , \end{cases}$$

where Ω_x is a neighborhood of the origin. If for any C^∞-Cauchy data $\psi = (u_0(x), \ldots, u_{m-1}(x))$ there exists a solution $u(x,t) \in C^\infty$ in a neighborhood of the origin, we say that the homogeneous Cauchy problem is locally solvable at the origin.

<u>Theorem 1.</u> <u>The above Cauchy problem is locally solvable if and only if all P_j are locally solvable. Each P_j is locally solvable if and only if $\lambda_j(x.t;\xi)$ is real and that it satisfies the Levi condition.</u>

Let us remark that the above homogeneous Cauchy problem is merely concerned with the problem arround the origin. The sufficiency is almost evident. However the proof of the necessity is far from trivial. We assumed that the coefficients are analytic. This assumption enables us to prove the necessity by using the techniques developed in [10].

<u>§2. Cauchy-Kowalewski Theorem with meromorphic initial data</u>

For simplicity, first we consider the case $n = 1$. We are concerned with the following Cauchy problem.

$$\text{(2.1)} \quad \begin{cases} Pu = 0 \\ \partial_t^i u|_{t=0} = \dfrac{w_i(x)}{(x-y)^{p_i}} \qquad (0 \le i \le m-1), \end{cases}$$

$w_i(x)$ being analytic; p_i are non-negative integers.

The solution $u(x,t)$ exists in a neighborhood of the origin, and it is represented in the form.

$$(2.2)\qquad u(x,t) = \sum_{\mu=1}^{s} \sum_{-\infty<j<\infty} f_j(\phi_\mu(x,t)-y)a_{j,\mu}(x,t),$$

where

$$(2.3)\quad \begin{cases} f_0(s) = \log s, \quad f_j'(s) = f_{j-1}(s) \\ f_{-j}(s) = (-1)^{j-1} \dfrac{(j-1)!}{s^j}, \quad (j \geq 1) \\ f_j(s) = \dfrac{1}{j!} \{s^j \log s - s^j (1+\frac{1}{2}+\dots+\frac{1}{j})\}, \quad (j\geq 1)\ ; \end{cases}$$

$\phi_\mu(x,t)$ is the phase function corresponding to $\lambda_\mu(x,t)$:

$$(2.4)\quad \begin{cases} \partial_t\phi_\mu(x,t) = \lambda_\mu(x,t)\partial_x\phi_\mu(x,t) \\ \phi_\mu(x,0) = x. \end{cases}$$

Let us remark that all $a_{j,\mu}(x,t)$ are analytic in (x,t), more precisely they can be continued analytically in a common complex domain, and they have the following form of majorations:

$$(2.5)\quad \begin{cases} |a_{j,\mu}(x,t)| \leq j!\ C^{j+1} \quad \text{for } j \geq 0, \\ |a_{j,\mu}(x,t)| \leq \dfrac{\varepsilon^{-j}}{(-j)!} C_\varepsilon, \ \text{for any } \varepsilon > 0, \ \text{for } j < 0. \end{cases}$$

This result is usually called Hamada's theorem.

Historical Note.

In 1969, Hamada first proved the above theorem under the assumption that all λ_i are simple, and in 1970 he proved the same result when the characteristic roots are at most double, assuming the Levi condition on P [5]. Next De Paris introduced the notion "bien décomposable", and showed that the above result is also true for these operators [4]. Chazarain showed in [3] that the notion of bien décomposable is equivalent to the Levi condition. In all these works, the number of the terms with negative j which appear in (2.2) is finite. In 1973, Hamada showed that, in the case when the multiplicity is at most double, the result is even true without assuming the Levi condition [6]. Let us note that, in this case, in general, the essential singularities appear, namely the number of the terms with negative j may become infinite. Finally, in 1976 Hamada-Leray-Wagschal [7] proved the above theorem in general case (even for general systems).

We say that the operator P has the property of meromorphic propagation of singularities, or shortly meromorphic propagation property, if for any meromorphic initial data, the solution $u(x,t)$ has only a finite number of the terms with negative j in the expression (2.2). Thus, the result of De Paris can be stated as follows.

Theorem 2. (De Paris). When P satisfies the Levi condition, P has the meromorphic propagation property.

The purpose of this article is to prove the converse. Namely,

Theorem 3. When the operator P has the meromorphic propagation property, then P satisfies the Levi condition.

The plan of the proof is the following. First we suppose that all the characteristic roots $\lambda_\mu(x,t;\xi)$ are real. Then our proof is carried out in the manner,

meromorphic propagation $\Longrightarrow$ P is C^∞-wellposed $\Rightarrow$ Levi condition
(Theorem 1)

In the case when λ_μ are not real, we reduce the problem to the case of λ_μ real by some artifice in the calculus of Fourier integral operators (canonical transformation). Let us mention that our proof is not direct. It is desirable to prove the above theorem by direct method.

§3.

In this section we give the proof of Theorem 3. First we consider the case $n=1$, and suppose further the characteristic roots $\lambda_\mu(x,t)$ are all real. The proof is almost evident. In this case, in the definition of $f_j(s)$, we can take $f_0(s) = \log|s|$, and

$$f_{-j}(\phi_\mu(x,t)-y) = (-1)^{j-1}(j-1)!\ \mathrm{Pf}\,\frac{1}{(\phi_\mu(x,t)-y)^j}, \quad j > 0,$$

where y is real and Pf denotes the finite part.

We represent (2.2) by Fourier integral operators, Let us recall

$$F[f_j(s)] = \begin{cases} -i^{-j}\ \pi(\mathrm{Pf}\ \dfrac{Y(\xi)}{\xi^{j+1}} - \mathrm{Pf}\ \dfrac{Y(-\xi)}{\xi^{j+1}}) + i^j\ \dfrac{2\pi c}{j!}\ \delta^{(j)}, & j \geq 0 \\ -i^{-j}\ \pi(Y(\xi) - Y(-\xi))\ \xi^{-j-1}, & i \leq -1. \end{cases}$$

In particular, $\frac{i}{\pi}$ p.v. $\frac{1}{s} \longrightarrow Y(\xi) - Y(-\xi)$

Let the Cauchy problem be

$$(3.1)\begin{cases} Pu = 0 \\ \partial_t^k u|_{t=0} = g(x) \in C_0^\infty \\ \partial_t^i u|_{t=0} = 0, \quad (k\neq i,\ 0 \leq i \leq m-1). \end{cases}$$

We consider the solution $u(x,t)$ of the corresponding Cauchy problem :

$$(3.2)\begin{cases} Pu = 0 \\ \partial_t^k u|_{t=0} = \frac{i}{\pi}\ \frac{1}{x-y} \\ \partial_t^i u|_{t=0} = 0 \quad (i\neq k,\ 0 \leq i \leq m-1) \end{cases}$$

We decompose

$$\hat{g}(\xi) = \hat{g}_+(\xi) + \hat{g}_0(\xi) + \hat{g}_-(\xi),$$

where $\hat{g}_0(\xi)$ has compact support, and $\hat{g}_\pm(\xi)$ has its support strictly in $\xi > 0$ (resp. $\xi<0$). Then we consider.

$$(3.3) \quad u_+(x,t) = \sum_{\mu=1}^{s} (2\pi)^{-1} \int_0^\infty e^{i\phi_\mu(x,t)\xi}\ [\sum_j c_j a_{j,\mu}(x,t)\chi_{j,+}(\xi)\xi^{-j-1}]\ \hat{g}_+(\xi)d\xi,$$

$$(c_j = -i^{-j}\pi).$$

where $\{\chi_{j,+}(\xi)\}$ is a sequence of suitable cutoff functions to make analytic formal symbol $\sum_j c_j a_{j,\mu}(x,t)\xi^{-j-1}$ a true symbol which Treves calls pseudo-analytic symbol; see [14].

Now it is easy to see that

$$\begin{cases} Pu_+ = \text{analytic function}, \\ \partial_t^k u_+ |_{t=0} = g_+(x) + \text{analytic function}, \\ \partial_t^i u_+ |_{t=0} = \text{analytic function}. \end{cases}$$

The same procedure can be applied to $g_-(x)$. Since $g_0(x)$ (= Fourier inverse transform of $\hat{g}_0(\xi)$) is analytic, by applying the Cauchy-Kowalewski Theorem, we conclude that problem (3.1) is C^∞-wellposed.

In general case, we extend the above result as follows. (3.2) is replaced by

$$(3.4)\quad \begin{cases} Pu = 0 \\ \partial_t^k u|_{t=0} = \dfrac{\alpha_n}{\langle x-y,\omega\rangle^n}, \quad (\alpha_n = (-1)^n i^{-n}(n-1)!/\pi), \\ \partial_t^i u|_{t=0} = 0 \quad (k\neq i,\ 0 \le i \le m-1), \end{cases}$$

where $\omega \in \mathbb{R}^n$, $|\omega| = 1$. Then (2.2) becomes

$$(3.5)\quad u(x,t) = \sum_{\mu=1}^{s} \sum_j f_j(\phi_\mu(x,t;\omega) - \langle y,\omega\rangle) a_{j,\mu}(x,t;\omega).$$

Here $\phi_\mu(x,t;\xi)$ is the phase function satisfying

$$\begin{cases} \partial_t\phi_\mu = \lambda_\mu(x,t;\partial_x\phi) \\ \phi_\mu(x,0;\omega) = \langle x, \omega\rangle , \end{cases}$$

$\omega(\in S^{n-1})$ being a parameter in a neighborhood of ω_0.

Now, by partition of unity in the dual space, we can suppose that $\hat{g}(\xi)$ has its support in a conical neighborhood of ω_0. Put

$$a_{j,\mu}(x,t;\xi) = c_j a_{j,\mu}(x,t;\xi')/|\xi|^{j+1}, \quad (\xi' = \xi/|\xi|),$$

Then $\sum_j a_{j,\mu}(x,t;\xi)$ is an analytic formal symbol. Thus, by suitably chosen cutoff functions $\chi_j(|\xi|)$, we form a true symbol

$$a_\mu(x,t;\xi) = \sum_j a_{j,\mu}(x,t;\xi)\chi_j(|\xi|).$$

It follows easily that

$$(3.6)\quad u(x,t) = \sum_{\mu=1}^{s} (2\pi)^{-n} \int e^{i\phi_\mu(x,t;\xi)} a_\mu(x,t;\xi)\hat{g}(\xi)d\xi$$

solves the Cauchy problem (3.1), except analytic functions.

§4. Canonical transformations

When the phase function is complex-valued, we introduce a positive small parameter ε. Instead of (3.4), we consider modified Cauchy data.

$$(4.1)\begin{cases} Pu = 0 \\ \partial_t^i u\big|_{t=0} = \dfrac{w_i(x)}{(\langle x-y,\omega\rangle+i\varepsilon)^{p_i}}, \qquad (0 \le i \le m-1). \end{cases}$$

The solution takes the form

$$v_\varepsilon(x,t) = \sum_{\mu=1}^{s} \sum_j f_j(\phi_\mu(x,t;\omega)+i\varepsilon-\langle y,\omega\rangle)a_{j,\mu}(x,t;\omega).$$

This implies that

$$(4.2)\qquad u_\varepsilon(x,t) = \sum_{\mu=1}^{s} (2\pi)^{-n}\int e^{i\phi_\mu(x,t;\xi)-\varepsilon|\xi|} a_\mu(x,t;\xi)\hat{g}_\mu(\xi)d\xi,$$

satisfies $Pu = 0$ modulo analytic function.

To introduce a small parameter ε is used by many authors in the case of complex phase functions. We are inspired by Baouendi-Treves [1]. In (4.2), $\mathrm{Im}\ \Phi_\mu(x,t;\omega)+\varepsilon > 0$ is required.

Hereafter, we consider (t,ε) under the restriction,

$$(4.3)\qquad c|t| < \varepsilon\ ,$$

where c is a suitable (large) constant.

We say that an operator $A_\varepsilon(x,t)$, from $\mathcal{E}'_x$ into $A(\Omega_x)$, is <u>uniformly</u> (with respect to ε and t) <u>analytic regularizing</u>, if we restrict (t,ε) by (4.3), it is equi-continuous mapping, and that all derivative $\partial_t^k A(x,t)$ has the same property (where the constant c in (4.3) may depend on k).

Now in view of the form $P = P_s \circ P_{s-1} \circ \dots \circ P_1 + R$, and since λ_μ are distinct, we see that

$$(4.4)\qquad P_1 \circ (2\pi)^{-n}\int e^{i\phi_1(x,t;\xi)-\varepsilon|\xi|} a_1(x,t;\xi)\hat{u}(\xi)d\xi \sim 0,$$

where " $\sim$ " means that the left-hand side operator is uniformly analytic regularizing.

Next, put

$$I_{\phi_1+i\varepsilon}u = (2\pi)^{-n}\int e^{i\phi_1-\varepsilon|\xi|}\hat{u}(\xi)d\xi\ .$$

There exists an analytic symbol $a_{1,\varepsilon}{}^{\#}(x,t;\xi)$ such that

$$(2\pi)^{-n}\int e^{i\phi_1-\varepsilon|\xi|} a_1(x,t;\xi)\hat{u}(\xi)d\xi \sim I_{\phi_1+i\varepsilon}\circ a_{1,\varepsilon}{}^{\#}(x,t;D)u\ .$$

In the same way, there exists another analytic symbol $p_{1,\varepsilon}{}^{\#}(x,t;\partial_t,D_x)$ such that

$$P_1 \circ I_{\phi_1+i\varepsilon} u(\cdot,t) \sim I_{\phi_1+i\varepsilon} \circ P_{1,\varepsilon}^{\#}\, u(\cdot,t).$$

Thus,

$$(4.5) \qquad I_{\phi_1+i\varepsilon} \circ P_{1,\varepsilon}^{\#} \circ a_{1,\varepsilon}^{\#} \sim 0.$$

Let us remark that, $P_{1,\varepsilon}^{\#}$ and $a_{1,\varepsilon}^{\#}$ are analytic symbols depending analytically on ε (even for $\varepsilon = 0$). Further, the symbols $P_{1,0}^{\#}$, $a_{1,0}^{\#}$ corresponding to $\varepsilon = 0$ are nothing but those ones when the phase function ϕ_1 is supposed real (canonical transformation). By fairly delicate argument, we deduce from (4.5) that

$$(4.6) \qquad P_{1,0}^{\#} \circ a_{1,0}^{\#} \sim 0 \quad \text{(modulo analytic regularizing operators)},$$

By hypothesis, a_1 is analytic symbol (more precisely pseudo-analytic symbol) thus $a_{1,0}^{\#}$ is also analytic symbol. Now, $P_{1,0}^{\#}$ has its principal part $\partial_t^{m_1}$ So we can apply Theorem 1, because the existence of $a_{1,0}^{\#}$ implies that $P_{1,0}^{\#}$ is a C^∞-wellposed operator. Thus $P_{1,0}^{\#}$ should satisfy the Levi condition, which implies further P_1 itself satisfies the Levi condition. This proves Theorem 3.

Finally, the author would like to call attention to the work of Pallu de la Barrière and Schapira [15], which proves Hamada's theorem using the canonical transformations in the complex spaces.

References

[1] M.S. Baouendi and F. Treves, Approximation of solutions of linear PDE with analytic coefficients, Duke Math. J., 50(1983), 285-301.

[2] L. Boutet de Monvel and P. Krée, Pseudo-differential operators and Gevrey classes, Ann. Inst. Fourier, 17(1967), 295-323.

[3] J. Chazarain, Opérateurs hyperboliques à caractéristiques de constante multiplicité, Ann. Inst. Fourier, 24(1974), 173-202.

[4] De Paris, Problème de Cauchy analytique à données singulières pour un opérateur différentiel bien décomposable, J. Math. pure et appl., 51(1972), 465-488.

[5] Y. Hamada, The singularities of the solutions of the Cauchy problem, R.I.M.S. Kyoto Univ., 5(1969), 21-40; ibid., 6(1970), 357-384.

[6] _________, Problème analytique de Cauchy à caractéristiques multiples dont les données de Cauchy ont des singularités polaires, C.R. Acad. Sc. 276, Sér.A (1973), 1681-1684.

[7] Y. Hamada, J. Leray et C. Wagschal, Systèmes d'équations aux dérivées partielles à caractéristiques multiples; problème de Cauchy ramifié; hyperbolicité partielle, J. Math. pure et appl., 55(1976), 297-352.

[8] L. Hörmander, Uniqueness theorems and wave front sets for solutions of differential equations with analytic coefficients, Comm. Pure Appl. Math., 24(1971), 671-704.

[9] H. Komatsu, Irregularities of characteristic elements and construction of null solutions, J. Fac. Sc. Univ. Tokyo, Ser.IA 23(1976), 297-312.

[10] S. Mizohata, On the Cauchy-Kowalewski Theorem, Mathematical Analysis and Applications, Part B, 617-652, Academic Press, 1981.

[11] L. Nirenberg and F. Treves, On the local solvability of linear partial differential equations, II : Sufficient condition, Comm. Pure Appl. Math., 25(1970), 459-510.

[12] T. Nishitani, On the Lax-Mizohata theorem in the analytic and Gevrey classes, J. Math. Kyoto Univ., 18(1979), 501-512.

[13] M. Sato, T. Kawai, M. Kashiwara, Microfunctions and pseudo-differential equations, Lecture Notes in Math., Springer-verlag, 287(1972), 265-529.

[14] F. Treves, Introduction to pseudodifferential and Fourier integral operators, I, Plenum Press (1980).

[15] P. Pallu de la Barrière et P. Schapira, Application de la théorie des microfonctions holomorphes au problème de Cauchy à données singulières, Séminaire Goulaouic-Schwartz, 1975-76, Exposé n° 23.

Sigeru MIZOHATA
Department of Mathematics
Faculty of Science
Kyoto University
60 Kyoto, Japan