MORIHIKO SAITO

Hodge structure via filtered $\mathcal{D}$-modules

Astérisque, tome 130 (1985), p. 342-351
<http://www.numdam.org/item?id=AST_1985__130__342_0>

HODGE STRUCTURE VIA FILTERED $\mathcal{D}$-MODULES

by Morihiko Saito *)

We explain a method which will give an analytic proof of the decomposition theorem [BBD] and a pure Hodge structure on the intersection cohomology.

In char. $p > 0$, Deligne defined the notion of a pure complex and proved its stability by proper morphisms [CW II]. Then, using the theory of t-structure, Beilinson-Bernstein and Deligne-Gabber proved the purity of the intersection complexes and the decomposition theorem for pure complexes, i.e. after a base change by $\otimes \bar{\mathbb{F}}_q$, (a) any pure complexes are isomorphic to the direct sum of their perverse cohomologies shifted by their degree [BBD, 5.4.5] and (b) any perverse pure complexes are semisimple (of finite length); the simple perverse complexes are given by the intersection complexes with twisted coefficients [loc.cit. 5.3.8] (cf. also [Br 2, p.131 and p. 147]). Gabber also proved that if K is a perverse pure complex, the weight filtration coincides with the monodromy filtration up to a shift (see [BBD, p. 17][Br 2, 3.2.9]).

In this note, we construct the category of "polarizable Hodge Modules" which might correspond to that of perverse pure complexes (cf. Remark in (3.1)).

Due to the dictionary of Deligne, the (mixed) Hodge structure corresponds to the action of Frobenius [TH,I] and the polarizable variation of Hodge structures to the smooth (= lisse) perverse pure complex. It is also known that the category of regular holonomic systems corresponds to that of perverse complexes by the Riemann-Hilbert correspondence, which enables us to consider the filtered regular holonomic $\mathcal{D}$-Modules with $\mathbb{Q}$-structure. Then, by induction on dimension, we can define the category of Hodge Modules as a full subcategory, so that it coincides with the variation of Hodge structures if the underlying perverse complex is a local system and the support is non singular (cf. (3.1)). Here it should be noted that we take the above result of Gabber as the definition and use the recent result of Kashiwara [K] on the description of ψ_f and ϕ_f via $\mathcal{D}$-Modules. Because these functors are compatible with direct images, this definition is convenient to inductive arguments. The details of the proof will be published elsewhere.

*)Supported by the Netherlands Foundation for Mathematics SMC with financial aid from the Netherlands Organization for the Advancement of Pure Research (Z.W.O.)

I would like to thank Prof. Deligne; the definition of the sign convention of the polarization is due to him [D3]. It should be noted that the sign is crucial in the positivity argument and very delicate in the derived category [D1,1.1][D2].

§1. VANISHING CYCLE FUNCTOR AND $\mathcal{D}$-MODULES.

(1.1) In this note, we use right $\mathcal{D}$-Modules; they are convenient to the operation of dual and direct image, and the left to inverse image. We use Deligne's convention of perverse complex (see [BBD]); in particular, $\mathbb{C}_X[\dim X] \in \mathrm{Perv}(\mathbb{C}_X)$ if X is smooth. The de Rham functor DR_X is given by $M \longrightarrow M \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathcal{O}_X$.

(1.2) Let X be a complex manifold and $f : X \to \mathbb{C}$ a holomorphic function. We define $i : X \longrightarrow X \times \mathbb{C}$ by $i(x) = (x,f(x))$. For $K^{\cdot} \in \mathrm{Perv}(\mathbb{Q}_X)$, set $K_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{Q}} K \in \mathrm{Perv}(\mathbb{C}_X)$, $\tilde{K} = i_* K \in \mathrm{Perv}(\mathbb{Q}_{X\times\mathbb{C}})$ and $\tilde{K}_{\mathbb{C}} = i_* K_{\mathbb{C}}$.

Let Ψ be the vanishing cycle functor of Deligne (see SGA 7 XIII,XIV). For $p = pr_2 : X \times \mathbb{C} \to \mathbb{C}$, $\Psi_p \tilde{K}$ coincides with the Verdier specialization $Sp_{X\times\{0\}}\tilde{K}$ ($= \nu_{X\times\{0\}}\tilde{K}$ in [K]), cf. [V]. Set $\Psi K = \Psi_p \tilde{K}$ and $\Psi K_{\mathbb{C}} = \Psi_p \tilde{K}_{\mathbb{C}}$.

We have the natural action of the monodromy transformation T on ΨK, and T has the Jordan decomposition $T = T_s T_u$ in $\mathrm{Perv}(\mathbb{Q}_{X\times\mathbb{C}})$, because T has a minimal polynomial (at least locally). Set $N = (\log T_u)/2\pi i$ (it is independent of the choice of i). We denote by (n) the tensorization by $\mathbb{Z}_X(n)$ over $\mathbb{Z}_X$ for $n \in \mathbb{Z}$, where $\mathbb{Z}_X(n) = (2\pi i)^n \mathbb{Z}_X \subset \mathbb{C}_X$. Then N is a morphism of ΨK to $\Psi K(-1)$. For $\lambda \in \mathbb{C}$, set

$$\Psi_\lambda K_{\mathbb{C}} = \mathrm{Ker}(T_s - \lambda) \subset \Psi K_{\mathbb{C}} \ , \ \Psi_1 K = \mathrm{Ker}(T_s - 1) \subset \Psi K \ .$$

We say that K is quasi-unipotent along f, iff $T_s^m = id$ for some $m \in \mathbb{Z}$ and $m > 0$.

We denote by W the N-filtration on ΨK, i.e., W is a unique increasing filtration such that $NW_i \subset W_{i-2}(-1)$ and $N^j : Gr_j^W \xrightarrow{\sim} Gr_{-j}^W(-j)$ $(j > 0)$.

(1.3) Let $\pi' : \mathbb{C} \longrightarrow \mathbb{C}^*$ be a universal covering and set $\pi = id \times \pi' : X \times \mathbb{C} \to X \times \mathbb{C}^*$. Then there exists uniquely $\psi K \in \mathrm{Perv}(\mathbb{Q}_X)$ such that $\pi^* \Psi K \simeq \mathbb{Q}_{\mathbb{C}}[1] \boxtimes \psi K$ $(= pr_2^* \mathbb{Q}_{\mathbb{C}}[1] \otimes pr_1^* \psi K)$. (This definition is different from the usual one by the shift of complexes).

We have a canonical morphism $sp : \tilde{K}|_{X\times\{0\}} \to \psi K[1]$ and we define ϕK by the mapping cone, i.e. $\phi K[1] = \mathrm{Cone}(sp : \tilde{K}|_{X\times\{0\}} \to \psi K[1])$. Then it is known that $\phi K \in \mathrm{Perv}(\mathbb{Q}_X)$.

We can also define the action of T and the N-filtration W on $\psi K, \phi K$ and $\psi_\lambda K_{\mathbb{C}}$, $\psi_1 K, \phi_1 K$, etc. so that $\pi^*\Psi_\lambda K_{\mathbb{C}} = \mathbb{C}_{\mathbb{C}}[1] \boxtimes \psi_\lambda K_{\mathbb{C}}$ and $\pi^* \mathrm{Gr}_i^W \Psi K = \mathbb{Q}_{\mathbb{C}}[1] \boxtimes \mathrm{Gr}_i^W \psi K$.

(1.4) We now assume that $\tilde{K}$ has no nontrivial subobject nor quotient with support in $X\times\{0\}$. Then we have

$$\mathrm{Gr}_i^W \Psi_1 K = (\mathbb{Q}_{\mathbb{C}}[1] \boxtimes \mathrm{Gr}_i^W \psi_1 K) \oplus (\mathbb{Q}_{\{0\}} \boxtimes \mathrm{Gr}_i^W \phi_1 K)$$

for any i.

We denote by T_X the topological dualizing complex $\mathbb{Q}_X(\dim X)[2 \dim X]$ for a complex manifold X. We choose dualities c_1: $\mathbb{Q}_{\mathbb{C}}[1] \otimes \mathbb{Q}_{\mathbb{C}}[1] \to \mathbb{Q}_{\mathbb{C}}[2] = T_{\mathbb{C}}(-1)$ and c_0: $\mathbb{Q}_{\{0\}} \otimes \mathbb{Q}_{\{0\}} \to \mathbb{Q}_{\{0\}} = k_! T_{\{0\}} \to T_{\mathbb{C}}$ by $c_1(x\otimes y) = xy \in (\mathbb{Q}_{\mathbb{C}}[2])^{-2}$ and $c_0(x\otimes y) = xy \in \mathbb{Q}_{\{0\}}$ where $k : \{0\} \to \mathbb{C}$ and $k_! T_{\{0\}} \to T_{\mathbb{C}}$ is the trace morphism.

Let a be a duality $a: K \otimes K \to T_X(-n)$. Then it induces the dualities:

$$\pi^* a\circ(N^i \otimes \mathrm{id}) : \pi^*\mathrm{Gr}_i^W \Psi K \otimes \pi^*\mathrm{Gr}_i^W \Psi K \to T_{X\times\mathbb{C}}(-n-i)$$
$$a\circ(N^i \otimes \mathrm{id}) : \mathrm{Gr}_i^W \Psi_1 K \otimes \mathrm{Gr}_i^W \Psi_1 K \to T_{X\times\mathbb{C}}(-n-i)$$

hence there exist uniquely the dualities:

$$b_i' : \mathrm{Gr}_i^W \psi K \otimes \mathrm{Gr}_i^W \psi K \to T_X(-n-i+1),\ b_i'' : \mathrm{Gr}_i^W \phi_1 K \otimes \mathrm{Gr}_i^W \phi_1 K \to T_X(-n-i)$$

such that $\pi^* a\circ(N^i \otimes \mathrm{id}) = c_1 \boxtimes b_i'$ and $a\circ(N^i \otimes \mathrm{id}) = (c_1 \boxtimes b_{i,1}') \oplus (c_0 \boxtimes b_i'')$ where $b_{i,1}'$ is the restriction of b_i' to $\mathrm{Gr}_i^W \psi_1 K$. We say that b_i' and b_i'' are induced dualities by a and N (and c_0, c_1).

(1.5) Let M be a regular holonomic system such that $DR_X(M) = K_{\mathbb{C}}$. Set $\tilde{M} = \int_i M$ so that $DR_{X\times\mathbb{C}}(\tilde{M}) = \tilde{K}_{\mathbb{C}}$. Let $V_0\mathcal{D}_{X\times\mathbb{C}}$ be the $\mathcal{O}_{X\times\mathbb{C}}$-subAlgebra of $\mathcal{D}_{X\times\mathbb{C}}$ generated by $pr_1^*\mathcal{D}_X \otimes pr_2^* \mathbb{C}[t\partial_t]$ where t is the coordinate function of $\mathbb{C}$. We define the increasing filtration V on $\mathcal{D}_{X\times\mathbb{C}}$ by $V_p\mathcal{D}_{X\times\mathbb{C}} = \sum_{0\le i\le p} (V_0\mathcal{D}_{X\times\mathbb{C}})\partial_t^i$ $(p \ge 0)$ and $(V_0\mathcal{D}_{X\times\mathbb{C}})t^{-p}$ $(p \le 0)$. We have the following result due to Kashiwara [K]:

If K is quasi-unipotent along f, there is a unique increasing filtration $\{V_\alpha\}_{\alpha\in\mathbb{Q}}$ on $\tilde{M}$ such that:

0) $\bigcup_{\alpha\in\mathbb{Q}} V_\alpha\tilde{M} = \tilde{M}$,

i) $V_\alpha\tilde{M}$ are coherent $V_0\mathcal{D}_{X\times\mathbb{C}}$-subModules of $\tilde{M}$

ii) $(V_\alpha\tilde{M})t \subset V_{\alpha-1}\tilde{M}$, $(V_\alpha\tilde{M})\partial_t \subset V_{\alpha+1}\tilde{M}$ for any α and $(V_\alpha\tilde{M})t = V_{\alpha-1}\tilde{M}$ for $\alpha << 0$.

iii) $t\partial_t - \alpha$ is nilpotent on $Gr^V_\alpha \tilde{M}$ $(= V_\alpha \tilde{M} / \bigcup_{\beta<\alpha} V_\beta \tilde{M})$

iv) $DR_{X\times\mathbb{C}}\,(Gr^V \tilde{M} \otimes_{Gr^V \mathcal{D}} \mathcal{D}_{X\times\mathbb{C}}) = \Psi K_{\mathbb{C}}$

$$DR_X\,(Gr^V_\alpha \tilde{M}) = \begin{cases} \psi_{e(\alpha)} K_{\mathbb{C}} & (\alpha \neq 0,1,2,\dots) \\ \phi_1 \quad K_{\mathbb{C}} & (\alpha = 0,1,2,\dots) \end{cases}$$

Moreover the action of $t\partial_t - \alpha$ is identified with N by the second equality of iv). Here $e(\alpha) = \exp\,(2\pi i\alpha)$.

§2 FILTERED $\mathcal{D}$-MODULES WITH $\mathbb{Q}$-STRUCTURE.

(2.1) Let $MF_{rh}(\mathcal{D}_X)$ be the category of regular holonomic $\mathcal{D}_X$-Modules with good filtration. (We assume that $Gr^F M$ is coherent over $Gr^F\mathcal{D}$, but not that Ann $Gr^F M$ is reduced). Here the filtration F on $\mathcal{D}$ is given by the order of operator. We have the functors DR_X: $MF_{rh}(\mathcal{D}_X) \to Perv(\mathbb{C}_X)$ and $\mathbb{C}\otimes$: $Perv(\mathbb{Q}_X) \to Perv(\mathbb{C}_X)$ given by $DR_X(M,F) = DR_X(M)$ and $\mathbb{C}\otimes K^{\cdot}_{\mathbb{Q}} = \mathbb{C}\otimes(K^{\cdot}_{\mathbb{Q}})$. We define the category $MF(\mathcal{D}_X,\mathbb{Q})$ to be the fiber product of $MF_{rh}(\mathcal{D}_X)$ and $Perv(\mathbb{Q}_X)$ over $Perv(\mathbb{C}_X)$: the objects consist of $((M,F),K_{\mathbb{Q}},\alpha)$ where $(M,F) \in MF_{rh}(\mathcal{D}_X), K_{\mathbb{Q}} \in Perv(\mathbb{Q}_X)$ and α is an isomorphim $DR_X(M,F) \simeq \mathbb{C} \otimes (K_{\mathbb{Q}})$ in $Perv(\mathbb{C}_X)$ and the morphisms are pairs of morphisms in $MF_{rh}(\mathcal{D}_X)$ and $Perv(\mathbb{Q}_X)$ compatible with α. For simplicity we shall denote by (M,F,K) an object in $MF(\mathcal{D}_X,\mathbb{Q})$.

(2.2) For $(M,F,K) \in MF(\mathcal{D}_X,\mathbb{Q})$, we define the Tate twist by $(M,F,K)(n)=(M\otimes_{\mathbb{Z}}\mathbb{Z}(n)$, $F[n],K \otimes_{\mathbb{Z}} \mathbb{Z}(n))$ where $\mathbb{Z}(n) = (2\pi i)^n \mathbb{Z}_X \subset \mathbb{C}_X$ and $(F[n])_p = F_{p-n}$.

If $Gr^F M$ is Cohen-Macaulay, we can define the dual $(M,F,K)^* \in MF(\mathcal{D}_X,\mathbb{Q})$ by $(M,F,K)^*=((M,F)^*,K^*)$ where $K^* = \mathbb{R}\underline{Hom}(K,T_X)$ is the Verdier dual and $(M,F)^* = \mathbb{R}\underline{Hom}_{\mathcal{D}_X}((M,F),$ $(\Omega_X^{\dim X} \otimes_{\mathbb{Z}} (\dim X)) \otimes_{\mathcal{O}_X} (\mathcal{D}_X,F)[\dim X])$ can be calculated by taking a filtered resolution of (M,F).

(2.3) Let $f : X \to Y$ be a proper morphism of complex manifolds.For $(M,F) \in MF_{rh}(\mathcal{D}_X)$, we can define the direct image $\int_f (M,F) \in DF(\mathcal{D}_Y)$ by taking a semi-free resolution of (M,F) (at least locally on Y; we have to use a Cech covering to define globally). Here (M,F) is called semi-free iff $(M,F) \simeq \bigoplus_p L_p \otimes_{\mathcal{O}_X} (\mathcal{D}_X,F[p])$ with L_p coherent $\mathcal{O}_X$-Modules, and for $(M,F) = L \otimes (\mathcal{D}_X,F[p])$ we set $\int_f (M,F) = f_* L \otimes_{\mathcal{O}_Y} (\mathcal{D}_Y,F[p])$. To induce the morphisms between the direct image of semi-free Modules, we use the relation with the usual definition:

$$\int_f L \otimes \mathcal{D}_X = f_*((L \otimes \mathcal{D}_X) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) = f_* L \otimes_{\mathcal{O}_Y} \mathcal{D}_Y$$

If $(\int_f M,F) = \int_f (M,F)$ is strict, i.e. $\underline{H}^i(F_p \int_f M) \to \underline{H}^i(\int_f M)$ is injective for any i,p, we can define the cohomologies $\underline{H}^i \int_f (M,F) = (\underline{H}^i \int_f M,F)$ by $F_p(\underline{H}^i \int_f M) = \underline{H}^i(F_p \int_f M)$

For $(M,F,K) \in MF(\mathcal{D}_X,\mathbb{Q})$ such that $\int_f (M,F)$ is strict, we define $\underline{H}^i f_*(M,F,K) = (\underline{H}^i \int_f (M,F), {}^p\underline{H}^i f_* K) \in MF(\mathcal{D}_Y,\mathbb{Q})$, where ${}^p\underline{H}^i$ is the perverse cohomology (see [BBD]) and the isomorphism $DR_Y \underline{H}^i \int_f M \simeq \mathbb{C} \otimes ({}^p\underline{H}^i f_* K)$ is induced by $DR_Y \int_f M \simeq f_* DR_X M$.

If f is a closed immersion, then $\int_f (M,F)$ is strict, $\int_f (M,F) \in MF_{rh}(\mathcal{D}_Y)$ and $f_*(M,F,K) \in MF(\mathcal{D}_Y,\mathbb{Q})$.

(2.4) Let $f : X \to \mathbb{C}$ be a holomorphic function. For $(M,F,K) \in MF(\mathcal{D}_X,\mathbb{Q})$ such that K is quasi-unipotent along f, let $\tilde{M} = \int M$ and V be as in (1.5). Because $i:X \to X\times\mathbb{C}$ is a closed immersion, we have $(\tilde{M},F) = \int_i (M,F) \in MF_{rh}(\mathcal{D}_{X\times\mathbb{C}})$ so that $F_p\tilde{M} = i_*(\sum_{j\geq 0} F_{p-j}M \otimes \partial_t^j)$ by the isomorphism $\tilde{M} = i_*(M\otimes\mathbb{C}[\partial_t])$.

We say that (M,F) is compatible with the V-filtration along f, iff

$$(V_\alpha F_p\tilde{M})t = V_{\alpha-1}F_p\tilde{M} \qquad \text{for } \alpha < 0$$

$$(Gr_\alpha^V F_p\tilde{M})\partial_t = Gr_{\alpha+1}^V F_{p+1}\tilde{M} \cap (Gr_\alpha^V\tilde{M})\partial_t \qquad \text{for } \alpha \geq -1$$

If $\underline{H}^0_{X\times\{0\}}(\tilde{M}^*)=0$ (i.e. $(Gr_{-1}^V\tilde{M})\partial_t = Gr_0^V\tilde{M}$), the two conditions are equivalent to:

$$F_p\tilde{M} = \sum_{i\geq 0} (V_{<0}\tilde{M} \cap j_*j^{-1}F_{p-i}\tilde{M})\partial_t^i$$

where $j:X\times\mathbb{C}^* \hookrightarrow X\times\mathbb{C}$ and $V_{<0}\tilde{M} = \bigcup_{\alpha<0} V_\alpha\tilde{M}$.

Set
$$\psi(M,F,K) = (\bigoplus_{-1\leq\alpha<0} Gr_\alpha^V\tilde{M},F[1],\psi K)$$
$$\phi_1(M,F,K) = ((Gr_0^V\tilde{M},F),\phi_1 K)$$

Because the action of $t\partial_t-\alpha$ is identified with N, $\psi(M,F,K)$ and $\phi_1(M,F,K)$ have the N-filtration W, i.e. $NW_i \subset W_{i-2}(-1)$, $N^j:Gr_j^W \xrightarrow{\sim} Gr_{-j}^W(-j)$ $(j > 0)$ if we forget the filtration F. Taking the induced filtration by F, we get

$$Gr_i^W\psi(M,F,K),\ Gr_i^W\phi_1(M,F,K) \in MF(\mathcal{D}_X,\mathbb{Q}),\ \text{if } Gr^F\, Gr_i^W \text{ cohérent.}$$

We set $P_N Gr_i^W = \mathrm{Ker}\, N^{i+1}: Gr_i^W \to Gr_{-i-2}^W$ with the induced filtration by F on Gr_i^W.

§ 3. POLARIZABLE HODGE MODULES.

(3.1) Let X be a complex manifold, $x \in X$, and Z a germ of closed (locally) irreductible analytic subvariety of X at x. Set $MF(\mathcal{D}_X,\mathbb{Q})_x = \varinjlim_{U \ni x} MF(\mathcal{D}_U,\mathbb{Q})$ where U are open neighborhoods of x.

We define a full subcategory $MH_Z(X,n)'_x$ of $MF(\mathcal{D}_X,\mathbb{Q})_x$ by induction on dim Z :

a) If X is a point, $MF_{rh}(\mathcal{D}_X)$ is the category of finite dimensional vector spaces over $\mathbb{C}$ with finite filtration. Then $(H_{\mathbb{C}},F,H_{\mathbb{Q}}) \in MH_X(X,n)'_x$ iff $H_{\mathbb{C}} = \bigoplus_{p+q=n} (F^p \cap \bar{F}^q)$ where $F^p = F_{-p} H_{\mathbb{C}}$ and $\bar{F}^q$ is the complex conjugate by the $\mathbb{Q}$-structure $H_{\mathbb{C}} = \mathbb{C} \otimes H_{\mathbb{Q}}$ (cf. [TH])

b) If $Z = \{x\}$, then $(M,F,K) \in MH_Z(X,n)'_x$, iff $(M,F,K) \simeq i_*(M',F,K')$ for $(M',F,K') \in MH_Z(Z,n)'_x$, where $i : Z \hookrightarrow X$.

c) If dim $Z > 0$, then $(M,F,K) \in MH_Z(X,n)'_x$, iff supp $M = Z$ or ϕ and, for any $f \in \mathcal{O}_{X,x}$, $\psi_f K$ is quasi-unipotent along f (see (1.2)), (M,F) is compatible with the filtration V along f (see (2.4)), and, if dim $(f^{-1}(0) \cap Z) <$ dim Z, we have :

$$(Gr^V_{-1} \tilde{M})\partial_t = Gr^V_0 \tilde{M},\ \mathrm{Ker}(\ t : Gr^V_0 \tilde{M} \to Gr^V_{-1} \tilde{M}) = 0$$

$$Gr^W_k \psi_f(M,F,K) \in \bigoplus_{\dim Z < \dim Z} MH_{Z'}(X,n+k-1)'_x$$

$$Gr^W_k \phi_{f,1}(M,F,K) \in \bigoplus_{\dim Z' < \dim Z} MH_{Z'}(X,n+k)'_x. \quad \text{cf. (2.4).}$$

Set $MH(X,n)'_x = \bigoplus_Z MH_Z(X,n)'_x$. Let Z be a closed (globally) irreductible analytic subvariety of X. We define a full subcategory $MH(X,n)$ (resp. $MH_Z(X,n)$) of $MF(\mathcal{D}_X,\mathbb{Q})$ by :

$(M,F,K) \in MH(X,n)$(resp. $MH_Z(X,n)$) iff $(M,F,K)_x \in MH(X,n)'_x$ (resp. $\bigoplus_i MH_{Z_i}(X,n)'_x$) for any $x \in X$,

where $(M,F,K)_x$ is the image of (M,F,K) in $MF(\mathcal{D}_X,\mathbb{Q})_x$ and $(Z,x) = \cup Z_i$ is the irreducible decomposition at $x \in X$. Then we have $MH(X,n) = \bigoplus_Z MH_Z(X,n)$ (locally finite direct sum) and, if $(M,F,K) \in MH_Z(X,n)$, we have $K = IC_Z(L)$ for some local system L on a Zariski open subset of Z. (But I don't know whether $MH(X,n)'_x = \varinjlim_{U \ni x} MH(U,n)$)

We say that $(M,F,K) \in MH(X,Z,n)$ is a Hodge Module of weight n with support Z. We can verify that if $K[-\dim X]$ is a local system $H_{\mathbb{Q}}$, there is a variation of Hodge structures $(H_{\mathbb{Q}} \otimes \mathcal{O}_X, F)$ of weight n-dim X such that $M = H_{\mathbb{Q}} \otimes \Omega_X^{\dim X}$, $F_pM = \Omega_X^{\dim X} \otimes_{\mathcal{O}_X} (F_{p+\dim X}(H_{\mathbb{Q}} \otimes \mathcal{O}_X))$ i.e. $(M,F) = (\Omega_X^{\dim X}(\dim X) \otimes_{\mathcal{O}_X} (H_{\mathbb{Q}} \otimes \mathcal{O}_X, F))(-\dim X)$.

REMARK. If we consider mixed Hodge Modules, it is natural to assume the compatibility of the weight filtration with the monodromy (cf [CWII,1.8.7]) (and some condition on F). It is not clear whether $\psi(M,F,K)$ and $\phi_1(M,F,K)$ satisfy these conditions, hence we might get a stronger definition of Hodge Modules by assuming these conditions on ψ and ϕ_1.

(3.2) We also define the notion of polarization by induction on dim Z:

$(M,F,K) \in MH_Z(X,n)'_x$ is polarizable, iff $Gr^F M$ is Cohen-Macaulay and there is a duality $a : K \otimes K \to T_X(-n)$ (called a polarization of (M,F,K)) which satisfies:

a) a is compatible with the Hodge filtration, i.e. a induces a duality $(M,F,K)^* \simeq (M,F,K)(n)$.

b) If X is a point, $a : H_{\mathbb{Q}} \times H_{\mathbb{Q}} \to \mathbb{Q}(-n)$ satisfies $a(x,y) = (-1)^n a(y,x)$ and $(2\pi i)^n a(x,C\bar{x}) > 0$ for any $x,y \in H_{\mathbb{C}}$, where C is the Weil operator (cf.[TH]) (Here $a(F^p,F^{n-p+1}) = 0$ follows from a).) $(x \neq 0)$.

c) If $Z = \{x\}$, there is a polorization a' on (M',F,K') (where $i_*(M',F,K')=(M,F,K)$, cf.(3.1.b)) such that $i_*a' : i_*K' \otimes i_*K' \longrightarrow i_*T_Z(-n) \to T_X(-n)$ coincides with a.

d) If dim > 0, then for any $f \in \mathcal{O}_{X,x}$ such that $\dim(f^{-1}(0) \cap Z) < \dim Z$, we have the following. Let b'_i and b''_i be the induced dualities on $Gr_i^W \psi K$ and on $Gr_i^W \phi_1 K$ by a,N (and c_0,c_1)(cf. (1.4)). Then, for i=0, $b'_{0,1}$ and b''_0 give a polarization on $P_N Gr_0^W \psi_1$ and on $P_N Gr_0^W \phi_1$, and for any i, b'_i and b''_i give a polarization up to sign on $P_N Gr_i^W \psi$ and on $P_N Gr_0^W \phi_1$ (cf. (2.4)).

A duality of $(M,F,K) \in MH_Z(X,n)$ is a polorization, if its restriction to any $(M,F,K)_x \in MH(X,n)'_x$ is.

REMARK. If $H_{\mathbb{Q}} = K[-\dim X]$ is a local system, then the conditions imply that

$$(-1)^{m(m-1)/2}\ a: H_{\mathbb{Q}} \otimes H_{\mathbb{Q}}\ (= (K \otimes K)^{-2m}) \to \mathbb{Q}_X(-n+m)\ (=(T_X(-n))^{-2m})$$

gives a polarization on the variation of Hodge structures, where m = dim X. We note that this is the sign convention of polarization given by Deligne [D3].

(3.3) The main results will be:

(A) Let $f : X \to Y$ be a projective morphism of complex manifolds and ℓ the first chern class of a relatively ample line bundle for f. If $(M,F,K) \in MH_Z(X,n)$ has a polarization a, then:

i) $\int_f (M,F)$ is strict and $\underline{H}^i f_*(M,F,K) \in MH(Y,n+i)$, cf (2.3),

ii) $\ell^i : \underline{H}^{-i} f_*(M,F,K) \xrightarrow{\sim} \underline{H}^i f_*(M,F,K)(i)$ for $i > 0$,

iii) let a^i be the induced duality on ${}^p\underline{H}^{-i} f_*K$ by $f_*a : {}^p\underline{H}^i f_*K \otimes {}^p\underline{H}^{-i} f_*K \to T_Y(-n)$ and ℓ^i, then, for $i=0$, a^0 gives a polarization on $P_\ell\underline{H}^0 f_*(M,F,K)$, and for $i > 0$, a^i gives a polarization up to sign on $P_\ell\underline{H}^{-i} f_*(M,F,K)$. Here $P_\ell\underline{H}^{-i} f_*(M,F,K) = \operatorname{Ker} \ell^{i+1} \subset \underline{H}^{-i} f_*(M,F,K)$. (The sign is given by $(-1)^{i(i-1)/2}$).

(B) We define the Hodge filtration on Ω_X^m by $F_{-m}\Omega_X^m = \Omega_X^m$ and $F_{-m-1}\Omega_X^m = 0$, where $m = \dim X$. Then $(\Omega_X^m, F, \mathbb{Q}_X[m]) \in MH(X,m)$ and it is polarizable.

Combined with [D4], (A-ii) implies:

(C) $f_*K \cong \bigoplus_{i \in \mathbb{Z}} {}^p\underline{H}^i f_*K[-i]$ in $D_c^b(\mathbb{Q}_Y)$

By the definition of $MH(Y,n+i)$, we get:

(D) ${}^p\underline{H}^i f_*K$ is the direct sum of intersection complexes with twisted coefficients.

Using Hironaka's desingularization theorem, (A) and (B) imply:

(E) Let Z be an irreducible projective variety, then $H^i(Z,\underline{IC}(Z))$ has a Hodge structure of weight $\dim Z + i$.

§4. REMARKS ON ISOLATED SINGULARITIES.

(4.1.) If $f : X \to \mathbb{C}$ has an isolated singularity at $x \in Y := f^{-1}(0)$ and X is non-singular, then $\operatorname{supp} \phi_f(\mathbb{Q}_X[n+1]) = \{x\}$, where $n = \dim Y$. Here we define ψ and ϕ so that $\psi K, \phi K \in \operatorname{Perv}(\mathbb{Q}_Y)$ if $K \in \operatorname{Perv}(\mathbb{Q}_X)$, cf. (1.3), hence $\phi_f(\mathbb{Q}_X[n+1])$ can be regarded as a $\mathbb{Q}$-module. Let X_∞ be a Milnor fiber of f, then we have canonically $H^n(X_\infty) \simeq \phi_f(\mathbb{Q}_X[n+1])$ so that the decomposition $H^n(X_\infty) = H^n(X_\infty)_1 \oplus H^n(X_\infty)_{\neq 1}$ corresponds to $\phi = \phi_1 \oplus \phi_{\neq 1}$, where $\mathbb{C} \otimes \left(\phi_{\neq 1}K\right) = \bigoplus_{\lambda \neq 1} \phi_\lambda K_{\mathbb{C}}$. Because $\psi_{\neq 1} \simeq \phi_{\neq 1}$, we get a mixed Hodge structure on the vanishing cohomology such that the weight filtration is given by the monodromy filtration. Then its coincidence with Steenbrink's mixed Hodge structure implies a result of Varchenko, Scherk-Steenbrink:

If we also denote by $f : X \to S$ a Milnor fibration, the Gauss-Manin system

$\int_f \mathcal{O}_X$ $(= \int_f \Omega_X^{n+1} \otimes (\Omega_S^1)^{\otimes -1})$ calculates $\mathbb{R}f_* \mathbb{C}_X[n+1]$. The functors ψ,ϕ commute with $\mathbb{R}f_*$ and the filtration V with $\int_f$, i.e. V is strict on $\int_f \Omega_X^{n+1}$. But the Hodge filtration F is not strict, it is strict on $Gr_\alpha^V \int_f \Omega_X^{n+1}$ for $\alpha \neq -1,-2,\ldots$. But the Gauss-Manin system $\underline{H}^0 \int_f \Omega_X^{n+1}$ coincides with the micro local Gauss-Manin system (which can be regarded as a $\mathcal{D}_{S,0}$ - module) on which F is strict, because the process of microlocalization changes only ψ_1 $(= Gr_{-1}^V)$ to $\phi_1 (= Gr_0^V)$ so that ∂_t acts bijectively (and $\phi = \phi_1 \oplus \psi_{\neq 1}$ remains invariant). Moreover the induced filtration $\mathrm{Im}(\underline{H}^0 F_p \int_f \Omega_X^{n+1} \to \underline{H}^0 \int_f \Omega_X^{n+1})$ coincides with the Hodge filtration on the micro-local Gauss-Manin system. Thus we see that our Hodge filtration on ϕ coincides with the induced filtration on $\bigoplus_{-1\le\alpha<0} Gr_\alpha^V \underline{H}^0 \int_f \Omega_X^{n+1}$ by $\partial_t^{n-p}\left(\Omega_{X,x}^{n+1}/df \wedge d\Omega_{X,x}^{n-1}\right) \subset \underline{H}^0 \int_f \Omega_X^{n+1} (p \in \mathbb{Z})$. (We suppose that $n > 0$).

(4.2) If X has an isolated singularity, the weight filtration on $H^n(X_\infty)$ is not the monodromy filtration, but it is so for $\psi\, \mathbb{Q}_X[n+1] \simeq \psi \underline{IC}(X)$. The difference can be analyzed by the weight spectral sequence and the local invariant cycle theorem, and we find a formula:

$$\sum_{pq} h_1^{pq} t^p s^q = \sum_{pq} a^{pq} \left(\sum_{i=1}^{n-p-q} (ts)^i\right) t^p s^q + \sum_{pq} b^{pq} \left(\sum_{i=0}^{n-p-q} (ts)^i\right) t^p s^q \text{ in } \mathbb{Z}[t,s], \text{where}$$

h_1^{pq}, a^{pq}, b^{pq} are the Hodge numbers (i.e. $\dim Gr_F^p Gr_{p+q}^W$) of $H_n(X_\infty)_1$, $H_{\{x\}}^n(Y)/H_{\{x\}}^n(X)$ and $H_{\{x\}}^{n+1}(X)$ respectively and $Y = f^{-1}(0)$. In fact, using a theory of Steenbrink, (in his Arcata paper), we can show the following:

There is a direct sum decomposition $\bigoplus_i Gr_i^W H^n(X_\infty)_1 = A \oplus B$ as a graded module (compatible with Hodge structures) such that $N^i: A_{n+1+i} \xrightarrow{\sim} A_{n+1-i}(-i)$, $N^i: B_{n+i} \xrightarrow{\sim} B_{n-i}(-i)$ for $i > 0$,

$$\mathrm{Ker}\, N : A(1) \to A \simeq \bigoplus_i Gr_i^W H_{\{x\}}^n(Y)/H_{\{x\}}^n(X)$$

$$\mathrm{Ker}\, N : B \to B(-1) \simeq \bigoplus_i Gr_i^W H_{\{x\}}^{n+1}(Y).$$

We note that this is compatible with the exact sequence

$$0 \to H_{\{x\}}^{n+1}(Y) \to H^n(X_\infty)_1 \to \phi_1 \underline{IC}(X) \to 0$$

where the weight filtration on ϕ_1 is given by the monodromy filtration (i.e., $N^i\colon \mathrm{Gr}^W_{n+1+i}\ \phi_1 \xrightarrow{\sim} \mathrm{Gr}^W_{n+1-i}\ \phi_1(-i)$, here W is the weight filtration).

REFERENCES

[BBD] A.A. BEILINSON, J. BERNSTEIN, P. DELIGNE: *Faisceaux pervers* . Astérisque 100 (1982).

[Br1] J-L. BRYLINSKI: *Modules holonomes à singularités régulières et filtration de Hodge*. I, Springer Lecture Note 961 (1982), II, Astérisque 101-102 (1983).

[Br2] J-L. BRYLINSKI: *Cohomologie d'intersection et faisceaux pervers*, Séminaire Bourbaki 585, Astérisque 92-93 (1982).

[CW] P. DELIGNE: *La conjecture de Weil* II, Publ.Math. I.H.E.S. 52(1980).

[TH] P. DELIGNE: *Théorie de Hodge*, I, Actes du Congrès international des Mathématiciens (Nice, 1970), II and III, Publ.Math.I.H.E.S. 40(1971) and 44(1974).

[D1] P. DELIGNE: *Cohomologie à supports propres*, SGA4, XVII,Springer Lecture Note 305(1973).

[D2] P. DELIGNE: *Signes* (Appendix to "*Intégration sur un cycle évanescent*"), Inv.Math. 76(1984)

[D3] P. DELIGNE: *Positivité: signes* (manuscript, 16-2-84).

[D4] P. DELIGNE: *Théorème de Lefschetz et critères de dégéneresecence de suites spectrales*, Publ.Math. I.H.E.S., 35(1968).

[K] M. KASHIWARA: *Vanishing cycle sheaves and holonomic systems of differential equations*. Springer Lecture Note 1016(1983).

[V] J-L. VERDIER: *Spécialisation de faisceaux et monodromie modérée*. Astérisque 101-102(1983)

Mathematisch Instituut
Rijksuniversiteit Leiden
Wassenaarseweg 80
2333 AL Leiden - The Netherlands.