JEAN-LOUIS VERDIER

Extension of a perverse sheaf over a closed subspace

Astérisque, tome 130 (1985), p. 210-217

<http://www.numdam.org/item?id=AST_1985__130__210_0>

EXTENSION OF A PERVERSE SHEAF OVER A CLOSED SUBSPACE

by Jean-Louis VERDIER(*)

1.- REVIEW OF Sp

Let X be an analytic space, $Y \overset{i}{\hookrightarrow} X$ a closed subspace, $U = X - Y$ and $j : U \hookrightarrow X$ the inclusion map. Let C be the normal cone of Y in X. Denote again by $i : Y \hookrightarrow C$ the inclusion, set $U' = C - Y$, and let $j : U' \hookrightarrow C$ be the inclusion.

The specialization functor Sp goes from $D_{const}(X)$ to the category $D_{const,mon}(C)$ of constructible monodromic complexes [2]. It induces a functor again denoted by Sp from $D_{const}(U)$ to $D_{const,mon}(U')$ and by extension Sp will also denote the identity functor $D_{const}(Y) \longrightarrow D_{const}(Y)$. Let us list some properties of Sp : all those Sp's are exact.

1) Sp "commutes" with i_*, i^*, $i^!$, j_*, j^*, $\mathbb{D}$.

2) Sp transformes the fundamental octahedron

$$\begin{tikzcd}
& F \arrow[dl] \arrow[ddr] & \\
i_*i^*F \arrow[d] \arrow[ddr, dashed] & & i_*i^!F \arrow[ul] \arrow[ll, dashed] \\
j_!j^*F \arrow[uur] \arrow[rr] & & j_*j^*F \arrow[u] \arrow[dl] \\
& i_*i^*j_*j^*F \arrow[ul] \arrow[uur, dashed] &
\end{tikzcd}$$

into the corresponding one for $\mathrm{Sp}\,F$.

(*) Extrait d'une lettre adressée à R. Mac Pherson en novembre 1982. Cette lettre est un commentaire sur le résultat de R. Mac Pherson et K. Vilonen, présenté dans ce colloque.

3) Sp preserves perversity. Hence Sp is an exact functor $\mathrm{Per}(X) \longrightarrow \mathrm{Per\,Mon}(C)$, and Sp commutes with the perverse cohomology functor.

2.- THE EXTENSION PROBLEM : REDUCTION TO A CONE

Denote by $\mathrm{Gl}(U,C)$ the category whose objects are (G,S,α) where $G \in \mathrm{Per}(U)$, $S \in \mathrm{Per\,Mon}(C)$, $a : j^*S \xrightarrow{\sim} \mathrm{Sp}(G)$, and whose morphisms are the obvious one.

PROPOSITION.- *The functor* $\mathrm{gl} : \mathrm{Per}(X) \longrightarrow \mathrm{Gl}(U,C)$ *that sends* F *to* $\mathrm{gl}(F) = (j^*F,\ \mathrm{Sp}(F),\ \text{canonical})$ *is an equivalence of categories.*

gl is faithfull *:* Suppose that a map m satisfies $\mathrm{gl}(m) = 0$. Therefore $j_*j^*m = 0$ and by adjunction and distinguished triangle we see that there exists a factorization of m of the type $F \longrightarrow i_*i^*F \xrightarrow{m'} i_*i^!G \longrightarrow G$. Passing to the 0-th perverse cohomology we see that m factors through

$$F \longrightarrow i_*{}^Pi^*F \xrightarrow{n} i_*{}^Pi^!G \longrightarrow G \quad .$$

We know that $F \longrightarrow i_*{}^Pi^*F$ is surjective and $i_*{}^Pi^!G \longrightarrow G$ is injective. Hence n is unically defined by m but also by $\mathrm{Sp}(m)$. Hence $n = 0$.

gl is fully faithfull : Let $F, G \in \mathrm{Per}(X)$ and $(\alpha,\beta) : \mathrm{gl}(F) \longrightarrow \mathrm{gl}(G)$. We want to find $m : F \longrightarrow G$ such that $\mathrm{gl}(m) = (\alpha,\beta)$.

Consider the following commutative diagram with exact rows :

$$(*)\qquad \begin{tikzcd} 0 \arrow[r] & i_*{}^Pi^!F \arrow[r] \arrow[d] & \mathrm{Sp}(F) \arrow[r] \arrow[d, "\beta"] & {}^Pj_*j^*\mathrm{Sp}F \arrow[r] \arrow[d, "{}^Pj_*j^*\beta"] & i_*{}^PR^1i^!F \arrow[r] \arrow[d, "i_*{}^PR^1i^!\beta"] & 0 \\ 0 \arrow[r] & i_*{}^Pi^!G \arrow[r] & \cdots\cdots & \cdots\cdots & \cdots\cdots \arrow[r] & 0 \end{tikzcd}$$

The compatibility between α and β implies that

$$\mathrm{Sp}\,{}^Pj_*\alpha = {}^Pj_*j^*\beta \quad .$$

The first claim is that

$$(**)\qquad \begin{tikzcd} {}^Pj_*j^*F \arrow[r] \arrow[d, "{}^Pj_*\alpha"] & i_*{}^PR^1i^!F \arrow[r] \arrow[d, "i_*{}^PR^1i^!\beta"] & 0 \\ {}^Pj_*j^*G \arrow[r] & i_*{}^PR^1i^!G \arrow[r] & 0 \end{tikzcd}$$

is commutative. This is because

$$\mathrm{Hom}(F,G) \longrightarrow \mathrm{Hom}(\mathrm{Sp}\,F, \mathrm{Sp}\,G)$$

is bijective when G is supported on Y .

Call $I(F)$ the image of $F \longrightarrow {}^Pj_*j^*F$. We therefore have an exact sequence

$$0 \longrightarrow I(F) \longrightarrow {}^{P}j_*j^*F \longrightarrow i_*{}^{P}R^1i^!F \longrightarrow 0 \quad .$$

The commutativity of (**) gives an $\alpha' : I(F) \longrightarrow I(G)$ such that $j^*\alpha' = \alpha$. The second claim is that the diagramm of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & i_*{}^{P}i^!F & \longrightarrow & F & \longrightarrow & I(F) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i_*{}^{P}i^!\beta} & & & & \downarrow{\scriptstyle \alpha'} & & \\ 0 & \longrightarrow & i_*{}^{P}i^!G & \longrightarrow & G & \longrightarrow & I(G) & \longrightarrow & 0 \end{array}$$

is commutative : since $\mathrm{Ext}^1(F,G) \longrightarrow \mathrm{Ext}\,(\mathrm{Sp}(F),\mathrm{Sp}(G))$ is bijective when G is supported on Y , it is enough to check the commutativity after applying Sp , and this commutativity is then given by (*). Hence there is $\bar{\alpha} : F \longrightarrow G$ such that $j^*\bar{\alpha} = \alpha$. Modifying (α,β) by $gl(\bar{\alpha})$, we can assume now that $\alpha = 0$. But then ${}^{P}j_*j^*\beta = i_*{}^{P}R^1i^!\beta = 0$. Hence β factors through

$$\mathrm{Sp}(F) \longrightarrow i_*{}^{P}i^*\mathrm{Sp}F \xrightarrow{\beta'} i_*{}^{P}i^!\mathrm{Sp}\,G \longrightarrow \mathrm{Sp}\,G$$

since β' can be lifted, β can also be lifted.

gl <u>is essentially surjective</u> : Let $(G,S,\text{canonical})$ be an object of $Gl(U,C)$. The canonical map ${}^{P}j_*j^*S \longrightarrow i_*{}^{P}R^1i^!S \longrightarrow 0$ defines a map ${}^{P}j_*j^*G \xrightarrow{u} i_*{}^{P}R'i^!S \longrightarrow 0$. Set $G' = \ker u$. We have an exact sequence

$$0 \longrightarrow (0, i_*{}^{P}i^!S) \longrightarrow (G,S,C) \longrightarrow gl(G') \longrightarrow 0 \quad .$$

Since $\mathrm{Ext}^1(F,G) \longrightarrow \mathrm{Ext}^1(\mathrm{Sp}\,F\,,\,\mathrm{Sp}\,G)$ is bijective when G is supported in Y , this exact sequence lifts to $\mathrm{Per}(X)$.

3.- <u>REVIEW OF</u> ϕ <u>AND</u> Ψ .

Assume that Y is a principal divisor and let f be an equation of Y . Then $C(f) : C \longrightarrow \mathbf{A}$ and the projection $C \longrightarrow Y$ define an isomorphism $C \xrightarrow{\sim} Y \times \mathbf{A}$ ($\mathbf{A}$ the affine line). We have the usual functors Ψ_f , ϕ_f , $\Psi_{C(f)}$, $\phi_{C(f)}$ [1] and the commutative diagram

$$(*) \qquad \begin{tikzcd} & \phi_f \arrow[dl, "can"'] \arrow[dr, "var"] & \\ \Psi_f \arrow[rr, "セ = 1-\sigma"'] & & \Psi_f \end{tikzcd}$$

as well as a similar diagram for $C(f)$ (1).

1) We have $\phi_{C(f)} \circ \mathrm{Sp} = \phi_f$, $\Psi_{C(f)} \circ \mathrm{Sp} = \Psi_f$ can $\mathrm{Sp}\,F = \mathrm{can}\,f$, $\mathrm{var}\,\mathrm{Sp}\,F = \mathrm{var}\,F$
   セ$(\mathrm{Sp}\,F) =$ セ .

---

(1) セ est un caractère japonais de l'écriture Katakana qui se lit "se".

2) $\Psi_{C(f)}$ is simply the restriction at $Y \times \{1\}$.

Using these statements and what is known of Sp , we get that

3) $\Psi[-1]$ and $\phi[-1]$ commutes with duality

4) $\Psi[-1]$ and $\phi[-1]$ preserve perversity. They are exact on perverse sheaves. They commute with perverse cohomology.

5) $F \simeq j_* j^* F \Longleftrightarrow$ var is an isomorphism

$F \simeq j_! j^* F \Longleftrightarrow$ can is an isomorphism

6) Let $j_! j^* F \longrightarrow F \longrightarrow j_* j^* F$ be the canonical maps. Applying the functor Φ and using the isomorphism in 5), we get (*).

It is a nice exercise to look at the transform of the fundamental octahedron (section 1) by Φ .

4.- AN EQUIVALENCE OF CATEGORIES

Assume that $X = Y \times \mathbb{A}$ and $f = pr_2$. Denote by Ψ and ϕ the functors relative to pr_2 . All the complexes that we consider are monodromic with respect to pr_1 . Denote by $\mathrm{Per}(Y, t)$ the category of perverse object on Y equipped with an endomorphism t such that $1 - t$ is an automorphism.

PROPOSITION.- *The functor* $F \longmapsto (\Psi(F)[-1], t)$ *from* $\mathrm{Per\,Mon}(U)$ *to* $\mathrm{Per}(Y, t)$ *is an equivalence of categories.*

This comes from the fact that perverse objects are sheaves.

Denote by Per(Y,can,var) the category of object of the form

$$\Psi \xrightarrow{\text{can}} \phi \xrightarrow{\text{var}} \Psi$$

where Ψ , ϕ are perverse and $1 - \text{var can}$ is an automorphism (or equivalently $1 - \text{can var}$ is an automorphism).

PROPOSITION.- *The functor* $F \longmapsto (\Psi(F)[-1] \xrightarrow{\text{can}} \phi(F)[-1] \xrightarrow{\text{var}} \Psi(F)[-1])$ *is an equivalence of categories from* $\mathrm{Per\,Mon}(Y \times \mathbb{A})$ *to* Per(Y,can,var).

The proof is due to Deligne : the simple objects in $\mathrm{Per\,Mon}(Y \times \mathbb{A})$ are of three types

a) Simple perverse objects with support in Y

b) Simple perverse objects such that $S = \Psi(M)[-1]$ is simple on Y and $\sigma(M) =$ multiplication by $\lambda \in \mathbb{C}^*$ for $\lambda \neq 1$ (such objects will be denoted (S, λ))

c) Simple perverse M such that $\Psi(M)\ [-1]$ is simple on Y and $\sigma(M) = \mathrm{Id}$.

Notice that in case b) M is isomorphic to j_*j^*M and $j_!j^*M$ and in case c) M is isomorphic to $j_{!*}j^*M$ and $pr_1^*i^*M$ where $pr_1 : C \longrightarrow Y$ is the canonical projection.

Simple objects in $\mathrm{Per}(Y,\mathrm{can},\mathrm{var})$ are of three types

a) $0 \longrightarrow \phi \longrightarrow 0$ where ϕ is simple

b) $\Psi \xrightarrow{\mathrm{can}} \phi \xrightarrow{\mathrm{var}} \Psi$ where Ψ is simple and $\mathrm{var.can}$ is multiplication by $\lambda \neq 0$ (such object will be denoted by (Ψ,λ)).

c) $\Psi \longrightarrow 0 \longrightarrow \Psi$ where Ψ is simple (denoted by $(\Psi,0)$)

it is easy to check that the functor $F \longmapsto (\Psi(F)[-1] \longrightarrow \ldots)$ establishes a bijection between isomorphism classes. Then one has to check that this functor induces a bijection on Ext^i between simple objects. Now objects of type a) and b) don't mix, objects of type b) for different λ don't mix, objects of type b) and c) don't mix. We have $\mathrm{Ext}^i((S_1,\lambda), (S_2,\lambda)) = \mathrm{Ext}^i(S_1,S_2)$, $\mathrm{Ext}^i(M,N) = \mathrm{Ext}^i(\phi[-1]M, \phi[-1]N)$ when M and N have support in Y, $\mathrm{Ext}^i(M,N) = \mathrm{Ext}^i(\Psi[-1]M,\Psi[-1]N)$ if M and N are of type c). So it remains to study $\mathrm{Ext}^i(M,N)$ when M is of type a) and N of type c). In the topological case $N = pr_1^! N'[-1]$ where N' is simple on Y. So $\mathrm{Ext}^i(M,N) = \mathrm{Ext}^i(M,N')$ and we have a similar conclusion in $\mathrm{Per}(Y,\mathrm{can},\mathrm{var})$. The last case is $\mathrm{Ext}^i(M,N)$ when M is of type c) and N of type a), and we have

$$\mathrm{Ext}^i(M,N) = \mathrm{Ext}^i(\Psi(M),N) = \mathrm{Ext}^{i-1}(\Psi[-1](M),\phi[-1](N))$$

and this is the same as in $\mathrm{Per}(Y,\mathrm{can},\mathrm{var})$. Now the proof of the equivalence goes by induction on the length of the different objects.

<u>COROLLARY 1</u>.- *Let* $Y \subset X$ *be a principal divisor and* f *an equation of* Y. *The functor* $F \longmapsto (F/U, \Psi_f(F)[-1] \xrightarrow{\mathrm{can}} \phi_f(F)[-1] \xrightarrow{\mathrm{var}} \Psi_f(F)[-1]$ *is an equivalence between* $\mathrm{Per}(X)$ *and the category consisting of objects* $(G, \Psi \xrightarrow{\mathrm{can}} \phi \xrightarrow{\mathrm{var}} \Psi, \alpha)$ *where* $G \in \mathrm{Per}(U)$, $(\Psi \longrightarrow \phi \longrightarrow \Psi) \in \mathrm{Per}(Y,\mathrm{can},\mathrm{var})$ *and* $\alpha : \Psi(G) \xrightarrow{\sim} \Psi$ *is an isomorphism such that* $\mathfrak{t}(G) = \alpha^{-1} \circ \mathrm{var} \circ \mathrm{can} \circ \alpha$

<u>Remarks</u> : 1) It is interesting to look at simple objects of $\mathrm{Per\,Mon}(N)$ when N is a rank one vector bundle on Y. You see then the influence on the classification of the twisting of N (which provides a non trivial central extension of the π_1).
2) All of this goes over to the case of perverse étale sheaves. And in the corollary 1, one can use the tame ϕ and Ψ.

3) The involution $\Psi \circlearrowright \phi$ of the category $\mathrm{Per}(Y,\mathrm{can},\mathrm{var})$ is the Fourier transform of $\mathrm{Per\,Mon}(Y \times \mathbb{A})$.

Let $Y \subset X$ be closed subspace and $D : \{f = 0\}$ a principal divisor containing Y, and $F \in \mathrm{Per}(X - Y)$. Then $\Psi_f(F)\,[-1]$ is defined all over D and we have on D a morphism of perverse sheaves

$$(*) \quad \Psi_f(F)[-1] \xrightarrow{t} \Psi_f(F)[-1] \ .$$

Let $j : X - Y \hookrightarrow X$ be the inclusion. The diagram (*) induces on $D - Y$ the diagram

$$\Psi_{f\circ j}(F)[-1] \xrightarrow{t} \Psi_{f\circ j}(F)[-1]$$

and F being defined on $X - Y$ we have a commutative diagram

$$\begin{tikzcd}
 & \phi_{f\circ j}(F)[-1] \arrow[dr, "\mathrm{var}"] & \\
\Psi_{f\circ j}(F)[-1] \arrow[ur, "\mathrm{can}"] \arrow[rr, "t"] & & \Psi_{j\circ j}(F)[-1]
\end{tikzcd}$$

<u>COROLLARY 2</u>.- *The category* $\mathrm{Per}(X)$ *is equivalent to the category of objects of the type* (F,S,α) *where* $F \in \mathrm{Per}(X - Y)$, $S = \begin{tikzcd}\Psi_f(F)[-1] \arrow[r, bend left, "\phi"] \arrow[r, bend right, "t"'] & \Psi_f(F)[-1]\end{tikzcd}$ *and*

α *is an isomorphism* $S/D - Y \simeq \begin{tikzcd}\Psi_{f\circ j}(F)[-1] \arrow[r, bend left, "\phi_{f\circ j}(F)[-1]"] \arrow[r] & \Psi_{f\circ j}(F)[-1]\end{tikzcd}$ *canonical on* $\Psi_{f\circ j}(F)[-1]$.

This follows from the corollary 1.

Assume now that $F/X - Y$ is such that $\phi_{f\circ j}(F) = 0$. Hence $t/D - Y = 0$ and t factors through

$$\Psi_f(F)[-1] \longrightarrow {}^{P}i^{*}\Psi_f(F)[-1] \xrightarrow{t_{\{Y\}}} {}^{P}i^{!}\Psi_f(F)[-1] \longrightarrow \Psi_f(F)[-1] \ .$$

The factorization in the above cor. should be of the form

$$\begin{tikzcd}
\Psi_f[-1] \arrow[r] & {}^{P}i^{*}\Psi_f(F)[-1] \arrow[r, bend left, "\phi"] \arrow[r, bend right, "t_{\{Y\}}"'] & {}^{P}i^{!}\Psi_f(F)[-1] \arrow[r] & \Psi_f(F)[-1]
\end{tikzcd}$$

Denote by $\mathrm{Per}(X\,;D - Y)$ the category of perverse object on X such that $\phi_{f\circ j}(F) = 0$.

<u>COROLLARY 3</u>.- *The category* $\mathrm{Per}(X\,;D - Y)$ *is equivalent to the category of objects of the type* (F,S) *where* $F \in \mathrm{Per}(X - Y\,;D - Y)$, $S = \begin{tikzcd}{}^{P}i^{*}\Psi_f[-1] \arrow[r, bend left, "\phi"] \arrow[r, bend right, "t_{\{Y\}}"'] & {}^{P}i^{!}\Psi_f[-1]\end{tikzcd}$.

This follows from cor. 2.

As an example suppose Y is a point x and let F be an object of $\mathrm{Per}(X-\{x\})$. The generic divisors going through x will be transversal to F (i.e $\phi(F)=0$). To see this take a local embedding of (X,x) in $(\mathbb{A}^n,0)$ and look at $X\cap H$ where H is an hyperplan in $\mathbb{A}^n$.
Let $\tilde{X}$ be the blow-up of X with center 0 and $\tilde{X}_o\subset \mathbb{P}^{n-1}$ the exceptionnal divisor. Take a Whitney stratification $\mathcal{S}$ of $\tilde{X}$, that stratifies F (defined over $\tilde{X}-\tilde{X}_o$) and $\tilde{X}_o$. For generic H the proper transform $\tilde{H}$ will be transversal to the stratification $\mathcal{S}_o$ induced by $\mathcal{S}$ on $\tilde{X}_o$. But then by Whitney theory $\tilde{H}$ will be transversal to $\mathcal{S}$ in a neighbourhood of $\tilde{X}_o$. Hence for those H we will get $\phi(F)=0$. Pick such an H , set $X\cap H=D$, then $F\in \mathrm{Per}(X_i, X-D)$.
According to the corollary 3, an extension of F to all of X is given by a diagram

$$\begin{tikzcd} {}^{P}i^{*}\Psi_f(F)[-1] \arrow[r, bend left, "\phi"] \arrow[r, bend right, "t_{\{x\}}"'] & {}^{P}i^{!}\Psi_f(F)[-1] \end{tikzcd}$$

where the objects are perverse sheaves over $\{x\}$, i.e *vector spaces*. In fact we have

$$\begin{cases} {}^{P}i^{*}\Psi_f(F)[-1]=\mathcal{H}^{-1}(\Psi_f(F))_x & \text{(usual cohomology !) ,} \\ {}^{P}i^{!}\Psi_f(F)[-1]= H^{-1}_{\{x\}}(D,\Psi_f(F)) & . \end{cases}$$

In terms of balls and complex links we have

$$\begin{cases} {}^{P}i^{*}\Psi_f(F)[-1]= H^{-1}(D_\varepsilon\cap X_\delta,F) \, , \\ {}^{P}i^{!}\Psi_f(F)[-1]= H^{-1}_c(D_\varepsilon\cap X_\delta,F) \, . \end{cases}$$

5.- RELATION WITH FOURIER TRANSFORM : A PROBLEM

Assume now that $X=\mathbb{A}^n$, $Y=0$. Let F be an object of $\mathrm{Per\,Mon}(\mathbb{A}^n-\{0\})$, and $\bar{F}$ an extension to $\mathbb{A}^n$. We have a diagram

$$(*)\qquad \begin{tikzcd} & \bar{F} \arrow[dr] & \\ {}^{P}j_{!}F \arrow[ur] \arrow[rr] & & {}^{P}j_{*}F \end{tikzcd} \quad .$$

Let $D_\xi=\{\xi=0\}$ a hyperplane in general position with respect to F . Then it can be checked that

$$ {}^{P}i^{*}\Psi_\xi(F)[-1]= \mathcal{F}({}^{P}j_{!}F)_\xi $$

$$ \phi_\xi(F)[-1]= \mathcal{F}(\bar{F})_\xi $$

$$ {}^{P}i^{!}\Psi_\xi(F)[-1]= \mathcal{F}({}^{P}j_{*}F)_\xi $$

and that the map ${}^{P}i^{*}\Psi \underset{\mathfrak{t}_{\{0\}}}{\overset{\phi}{\longrightarrow}} {}^{P}i^{!}\Psi$ comes from (*) by applying the Fourier transform.
Hence those invariant are equipped with a natural action of $\pi_1(\hat{U})$ when $\hat{U}$ is an open subset of $\hat{\mathbb{A}}^n$, on which $\mathcal{F}({}^{P}j_{!}F)$ is non singular. But then comes the following problem :
To describe an extension, one can take any vector space V and a factorization of $\mathfrak{t}_{\{0\}}$. But interprating the construction in terms of Fourier transform, one sees that V is endowed with a structure of π_1-module ! I do not understand well what is happening. If all this is true this would certainly imply that the representations occuring in ${}^{P}i^{*}\Psi[-1]$ and ${}^{P}i^{!}\Psi[-1]$, well defined up to trivial representations, have an extremal property that I can not formulate for the moment (*)..

J.L VERDIER
Centre de Mathématiques de l'E.N.S
ERA 589

(*) Pour la réponse à cette question, voir la lettre de B. MALGRANGE p.

B I B L I O G R A P H I E

[1] P. DELIGNE - SGA II, exp XIV - Lectures Notes n° 340, Springer Verlag.

[2] J.L VERDIER - *Spécialisation de faisceaux et monodromie modérée.* Astérisque n°101-102, p. 332-384.