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# Index theorem for constructible sheaves

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# INDEX THEOREM FOR CONSTRUCTIBLE SHEAVES

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## §0 - INTRODUCTION

0.1. Let X be a complex manifold of dimension n and let $\mathcal{M}$ be a holonomic module over the ring $\mathcal{D}_X$ of differential operators on X. Then the Rham complex $DR(\mathcal{M})$ of $\mathcal{M}$ has constructible sheaves as its cohomology groups, and its local index $\sum(-1)^i \dim \underline{H}^i(DR(\mathcal{M}))_x$ at a point x can be expressed in terms of the characteristic cycle $\underline{Ch}(\mathcal{M})$ of $\mathcal{M}$ (Kashiwara [3], Brylinski-Dubson-Kashiwara [1]). Recently Dubson [2] found a beautiful formula to describe this.

THEOREM - *If* X *is a compact complex manifold, we have*

$$\sum(-1)^i \dim H^i(X;DR(\mathcal{M})) = (-1)^n \underline{Ch}(\mathcal{M}).T^*_X X .$$

Here the last term means the intersection number of two n-cycles in $T^*X$.

0.2. The purpose of this lecture is to generalize his result to the real case.

Let X be a real analytic manifold of dimension n and F a constructible sheaf on X. First we shall define the characteristic cycle $\widetilde{SS}(F)$ of F as a $\pi^{-1}\omega_X$-valued n-cycle in $T^*X$. Here $\omega_X$ denotes the orientation sheaf of X and $\pi : T^*X \to X$ is the cotangent bundle to X. In order to define this, we use the micro-local theory of sheaves developped in Kashiwara-Schapira [4].

Secondly we prove the index theorem.

THEOREM - *Let* F *be a constructible sheaf, and* $\varphi : X \to \mathbb{R}$ *a* $C^2$*-function. Set* $Y_\varphi = \{d\varphi(x) \; ; \; x \in X\} \subset T^*X$. *We assume that* $\{x \in \operatorname{supp} F; \varphi(x) \leqq t\}$ *is compact for any* t *and that* $SSF \cap Y_\varphi$ *is compact. Then* $\dim H^j(X;F) < \infty$ *for any* j *and we have*

$$\sum(-1)^j \dim H^j(X;F) = (-1)^{n(n+1)/2} \widetilde{SS}(F).Y_\varphi .$$

The proof uses the micro-local version of Morse's theory. Similarly to the Morse function, we deform $\varphi$ a little in a generic position so that $Y_\varphi$ intersects SSF transversally. Then we consider $H^j(\{x \; ; \; \varphi(x) < t\} \; ; \; F)$ and vary t. Then the cohomology groups change at points $t \in \varphi(\pi(Y_\varphi \cap SSF))$, and the obstruction can be calculated locally and coincides with the intersection number of $Y_\varphi$

and $\widetilde{SS}(F)$ at $p \in SSF \cap Y_{\varphi}$ with $t = \varphi\pi(p)$.

## §1 - SUBANALYTIC CHAINS

1.1. For a topological manifold X, let us denote by $\omega_X$ the orientation sheaf of X. If X is oriented then $\omega_X \cong \mathbb{Z}_X$ and this isomorphism changes the signature when we take the opposite orientation of X.

1.2. If X is a differentiable manifold of dimension n and if $\theta$ is a nowhere vanishing n-form on X, then we shall denote by $\operatorname{sgn}\theta$ the section of $\omega_X$ given by the orientation that $\theta$ determines. Hence we have

(1.2.1) $$\operatorname{sgn}\varphi\theta = \operatorname{sgn}\varphi \ \operatorname{sgn}\theta$$

where $\operatorname{sgn}\varphi = \pm 1$ if $\pm\varphi > 0$.

1.3. From now on, we assume that X is a real analytic manifold. For an integer r, let us denote by $E_r(X)$ the set of pairs $(Y,s)$ of a subanalytic locally closed r-dimensional real analytic submanifold Y of X and a section s of $\omega_Y$. We define the equivalence relation $\sim$ on $E_r(X)$ as follows : $(Y_1,s_1) \sim (Y_2,s_2)$ if and only if there exists a subanalytic locally closed r-dimensional real analytic submanifold Y such that $Y \subset Y_1 \cap Y_2$, $s_1|_Y = s_2|_Y$ and $\overline{\operatorname{supp} s_1} = \overline{\operatorname{supp} s_2} = \overline{\operatorname{supp} s_1} \cap \bar{Y}$.

We denote by $C_r(X)$ the set of equivalence classes in $E_r(X)$ and an equivalence class is called *subanalytic* r-*chain*. Remark that its support is not assumed to be compact.

We can define the boundary operator

$$\partial : C_r(X) \longrightarrow C_{r-1}(X),$$

so that $\partial\partial = 0$.

1.4. One can see easily that $C_r : U \longmapsto C_r(U)$ is a fine sheaf on X and we have the exact sequence

(1.4.1) $$0 \to \omega_X \to C_n \xrightarrow{\partial} C_{n-1} \to \cdots \to C_0 \to 0$$

This follows for example from the fact that any subanalytic set admits a subanalytic triangulation.

1.5. For a sheaf F on X, we set $C_r(F) = C_r \otimes F$. By (1.4.1), $\omega_X \otimes F$ is quasi-isomorphic to the complex of soft sheaves

(1.5.1) $$C_n(F) \longrightarrow C_{n-1}(F) \longrightarrow \cdots \longrightarrow C_0(F).$$

We set

(1.5.2) $$C_r(X;F) = \Gamma(X; C_r(F))$$

and call its elements *F-valued subanalytic* r-*chains*. We have isomorphisms

(1.5.3) $$H_r^{\inf}(X;F) \underset{\text{def}}{=} H_r(C.(X;F)) = H^{n-r}(X; F \otimes \omega_X).$$

(1.5.4) $$H_r(X;F) \underset{\text{def}}{=} H_r(\Gamma_c(X; C.(F))) = H_c^{n-r}(X; F \otimes \omega_X).$$

1.6. Assume further that F is locally constant. For a subanalytic r-dimensional real analytic submanifold Y of X and for a section s of $F \otimes \omega_Y$ over Y, the pair (Y,s) determines an F-valued subanalytic r-chain.

1.7. The following criterion for a chain to be a cycle is evident.

LEMMA 1.1 - *Let* $\alpha$ *be a subanalytic* r-*chain,* $\varphi : X \longrightarrow \mathbb{R}^r$ *be a real analytic map. We assume that*

(*i*) $\operatorname{Supp} \alpha \longrightarrow \mathbb{R}^r$ *is a finite map,*

(*ii*) $\operatorname{Supp} \partial\alpha \longrightarrow \mathbb{R}^r$ *is an immersion,*

(*iii*) *the intersection number of* $\alpha$ *and* $\varphi^{-1}(t)$ *is constant in* $t \in \mathbb{R}^r \setminus \varphi(\operatorname{Supp} \partial\alpha)$.

*Then* $\alpha$ *is a cycle, i.e.* $\partial\alpha = 0$.

## §2 - SYMPLECTIC GEOMETRY

2.1. Let X be an n-dimensional real analytic manifold of dimension n and $\pi : T^*X \longrightarrow X$ the cotangent bundle to X. Let $\theta_X$ denote the canonical 1-form on $T^*X$. Then $(d\theta_X)^n$ is nowhere vanishing and this gives the orientation of $T^*X$.

2.2. Now, let Y be a real analytic submanifold of X. Let $T_Y^*X$ be the conormal bundle to Y. Then we have the canonical isomorphism

$$\omega_{T^*_Y X} \otimes \pi^{-1}\omega_X \cong \mathbb{Z}_{T^*_Y X} \ . \tag{2.2.1}$$

Since the choice of signature is important in the future arguments, we shall write this explicitely. Let $(x_1, \ldots, x_n)$ be a local coordinate system of X such that Y is given by $x_1 = \ldots = x_r = 0$, and let $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ be the coordinates of $T^*X$ such that $\theta_X = \sum \xi_j \, dx_j$. Then the section $(-1)^r \operatorname{sgn}(d\xi_1 \ldots d\xi_r \, dx_{r+1} \ldots dx_n) \otimes \operatorname{sgn}(dx_1 \ldots dx_n)$ of $\omega_{T^*_Y X} \otimes \pi^{-1}\omega_X$ does not depend on the choice of coordinates and it determines the isomorphism (2.2.1).

2.3. Let $\Lambda$ be a subanalytic conic locally closed Lagrangian subvariety of $T^*X$ such that the projection $\Lambda \to X$ has a constant rank. Then we have $\omega_\Lambda \otimes \pi^{-1}\omega_X \cong \mathbb{Z}_\Lambda$. In fact, locally, $\Lambda$ is an open subset of $T^*_Y X$ for a real analytic submanifold Y of X and we can apply 2.2. Therefore $\Lambda$ defines the $\pi^{-1}\omega_X$-valued n-chain in $T^*X$ (see 1.6), which we shall denote by $[\Lambda]$.

## §3 - CHARACTERISTIC CYCLE

3.1. Let us fix a commutative field k once for all, and vector spaces mean vector spaces over k. Let X be a real analytic manifold of dimension n. Let D(X) be the derived category of the abelian category of sheaves of vector spaces on X.

An object F of D(X) is called *constructible* if the following conditions are satisfied.

(3.1.1) $H^j(F) = 0$ except for finitely many j's.

(3.1.2) There exists a subanalytic locally finite decomposition $X = \bigcup X_\alpha$ of X such that $H^j(F)|_{X_\alpha}$ is a locally constant sheaf of finite rank for any j and any $\alpha$.

We denote by $D^b_c(X)$ the full subcategory of D(X) consisting of constructible complexes.

3.2. For the notion of micro-support and its properties, we refer to [4]. We just mention the following properties.

For $F \in \mathrm{Ob}(D^+(X))$, we can define the micro-support $SS(F)$ of $F$ as a closed conic subset of $T^*X$.

PROPOSITION 3.1 - *Let* $F \in \mathrm{Ob}(D^+(X))$, $\varphi$ *a* $C^1$- *function on* $X$ *and let* $t_1 \leqq t_2$ *be two real numbers. Assume that* $\varphi$ Supp $F \to \mathbb{R}$ *is proper and that* $d\varphi(x) \notin SSF$ *for any* $x \in X$ *with* $t_1 \leqq \varphi(x) < t_2$. *Then the restriction homomorphism*

$$H^j(\{x;\varphi(x) < t_2\};F) \to H^j(\{x;\varphi(x) < t_1\};F)$$

*is an isomorphism for any* j.

PROPOSITION 3.2. - *If* $F \in \mathrm{Ob}(D^b_c(X))$, *then* SSF *is a closed subanalytic Lagrangian subset of* $T^*X$.

3.3. A morphism $u : F \to F'$ in $D^+(X)$ is called an isomorphism at $p \in T^*X$, if, for a distinguished triangle $F \xrightarrow{u} F' \to F'' \to F[1]$, we have $p \notin SSF''$. We denote by $D^+(X;p)$ the category obtained by localizing $D^+(X)$ by the isomorphisms at p (see [4]).

In particular, if $\varphi$ is a $C^1$-function such that $d\varphi(\pi(p)) = p$ $\varphi(\pi(p)) = 0$, then $F \mapsto \mathbb{R}\Gamma_{\varphi^{-1}(\mathbb{R}^+)}(F)_{\pi(p)}$ is a functor from $D^+(X;p)$. Here $\mathbb{R}^+$ signifies the set of non-negative numbers.

PROPOSITION 3.3 - *Let* $F \in \mathrm{Ob}(D^b_c(X))$ *and* Y *a real analytic submanifold. If* $SSF \subset T^*_YX$ *on a neighborhood of* $p \in T^*_YX$, *then we have*

$$F \cong \underline{V}_Y \quad \text{in } D^+(X;p)$$

*where* V *is a bounded complex of finite-dimensional vector spaces and* $V_Y$ *is the constant sheaf on* Y *with* V *as fiber.*

3.4 Let F be an object of $D^b_c(X)$. Then $\Lambda = SSF$ is a subanalytic Lagrangian subvariety. Hence there exists a locally finite family $\{\Lambda_\alpha\}$ of real analytic subsets of $T^*X$ satisfying the following conditions.

(3.3.1) $\Lambda_\alpha$ is subanalytic and connected.

(3.3.2) There exists a real analytic submanifold $Y_\alpha$ of X such that $\Lambda_\alpha$ is an open subset of $T^*_{Y_\alpha}X$.

(3.3.3) $\Lambda \subset \bigcup_\alpha \overline{\Lambda}_\alpha$.

(3.3.4) $\Lambda_\alpha \cap \overline{\Lambda}_\beta = \phi$ if $\alpha \neq \beta$ .

Then by proposition 3.3, for $p \in \Lambda_\alpha$ there exists a bounded complex $V_\alpha$ of finite-dimensional vector spaces such that $F \cong \underline{V_\alpha}_{Y_\alpha}$ in $D^+(X;p)$. Then $\chi(V_\alpha) = \sum (-1)^j \dim H^j(V_\alpha)$ is locally constant in p and hence determined by $\alpha$. We set $m_\alpha = \chi(V_\alpha)$.

DEFINITION 3.4 - *We define the* $\pi^{-1}\omega_X$*-valued* n*-chain* $\widetilde{SS}(F)$ *by*

(3.3.5) $$\widetilde{SS}(F) = \sum_\alpha m_\alpha[\Lambda_\alpha]$$

It is almost obvious that this chain does not depend on the choice of $\{\Lambda_\alpha\}$. We shall call this the *characteristic cycle* of F. Later we shall show that $\widetilde{SS}(F)$ is in fact an n-cycle.

## §4. INDEX THEOREM

4.1. Let X be a real analytic manifold of dimension n. For a real valued $C^2$-function $\varphi$ on X we set

(4.1.1) $Y_\varphi = \{ d\varphi(x) \ ; \ x \in X \} \subset T^*X$ and

(4.1.2) $Y_\varphi^a = \{ -d\varphi(x) \ ; \ x \in X \} \subset T^*X$.

Then $Y_\varphi$ and $Y_\varphi^a$ are isomorphic to X and hence we can regard them as $\pi^{-1}\omega_X$-valued n-cycles in $T^*X$.

4.2. Now, we state the following three main theorems, whose proof is given in the next three sections.

THEOREM 4.1 - *For* $F \in Ob(D^b_c(X))$, $\widetilde{SS}(F)$ *is an* n*-cycle, i.e.,* $\partial\widetilde{SS}(F) = 0$.

THEOREM 4.2 - *Let* $\varphi$ *be a* $C^2$*-function and* $F \in Ob(D^b_c(X))$. *We assume*

(4.2.1) *For any* $t \in \mathbb{R}$, $\{x \in \mathrm{Supp}\, F; \varphi(x) \leqq t\}$ *is compact.*

(4.2.2) $Y_\varphi \cap SSF$ *is compact.*

*Then,* $\dim H^j(X;F) < \infty$ *for any* j *and we have*

$$\chi(X;F) \underset{\text{def}}{=} \sum (-1)^j \dim H^j(X;F) = (-1)^{n(n+1)/2}\, \widetilde{SS}(F).Y_\varphi.$$

THEOREM 4.3 - *Let* $\varphi$ *and* F *be as in the preceding. We assume* (4.2.1) *and the following condition.*

(4.2.3) $$Y_\varphi^a \cap SSF \quad \textit{is compact}.$$

*Then* dim $H^j_c(X;F) < \infty$ *for any* j *and we have*

$$\chi_c(X;F) \underset{\mathrm{def}}{=} \sum (-1)^j \dim H^j_c(X;F) = (-1)^{n(n+1)/2}\, \widetilde{SS}(F).Y^a_\varphi \ .$$

Remark that Theorem 4.1, $\pi^{-1}(\omega_X) \otimes \pi^{-1}(\omega_X) \cong \mathbb{Z}_{T^*X}$ and the condition (4.2.2) or (4.2.3) permit us to define the intersection number $\widetilde{SS}(F).Y_\varphi$ or $\widetilde{SS}(F).Y^a_\varphi$ .

## §5 - PROOF OF MAIN THEOREMS (I)

5.1 We shall prove first the local version of Theorem 4.2 in a generic case. Let F be an object of $D^b_c(X)$ , and we choose $\{\Lambda_\alpha\}$ and $\{Y_\alpha\}$ as in 3.4. Let $x_\circ$ be a point of X and $\varphi$ a $C^2$-function on X such that

(5.1.1) $\varphi(x_\circ) = 0$ ,

(5.1.2) $d\varphi(x_\circ) \in \Lambda_\alpha$ and $Y_\varphi$ intersects transversally $\Lambda_\alpha$ at $p = d\varphi(x_\circ)$.

PROPOSITION 5.1 - *Under these conditions we have*

$$\chi(\mathbb{R}\Gamma_{\varphi^{-1}(\mathbb{R}^+)}(F)_{x_\circ}) = (-1)^{n(n+1)/2}\,(\widetilde{SS}(F).Y_\varphi)_p \ .$$

*Here the last term means the intersection number of* $\widetilde{SS}(F)$ *and* $Y_\varphi$ *at* $p = d\varphi(x_\circ)$ .

PROOF - We shall take a local coordinate system $(x_1, \dots, x_n)$ of X such that $Y_\alpha$ is given by $x_1 = \dots = x_r = 0$ and $x_\circ = 0$. Then we have

$$T_p(T^*_{Y_\alpha}X) = \{ (x,\xi) \ ; \ x_1 = \dots = x_r = \xi_{r+1} = \dots = \xi_n \}$$

and

$$T_p(Y_\varphi) = \{ (x,\xi) \ ; \ \xi_j = \sum_k \frac{\partial^2\varphi}{\partial x_j \partial x_k}(0)\, x_k \} \ .$$

The transversality condition (5.1.2) implies that the Hessian matrix $\left(\frac{\partial^2\varphi}{\partial x_j \partial x_k}(0)\right)_{r<j,k\leqq n}$ is non-degenerate. Hence by Morse's lemma, after a change of local coordinates, we may assume that

$$\varphi|_{Y_\alpha} = \sum_{j>r} a_j x_j^2 \quad \text{for} \quad a_j \in \mathbb{R}\setminus\{0\} .$$

Let $V$ be a bounded complex of vector spaces such that $F \cong \underline{V}_{Y_\alpha}$ in $D^+(X;p)$ . Then as stated in 3.3, we have

(5.1.3) $$\mathbb{R}\Gamma_{\varphi^{-1}\mathbb{R}^+}(F)_{x_\circ} \cong \mathbb{R}\Gamma_{\varphi^{-1}\mathbb{R}^+}(\underline{V}_{Y_\alpha})_{x_\circ} .$$

Let us note the following lemma.

LEMMA 5.2 - Let $Q(x)$ be a non-degenerate quadratic form on $\mathbb{R}^n$ , $q$ the number of negative eigenvalues of $Q$ . Then for any vector spaces $V$ , we have

$$H^j_{Q^{-1}(\mathbb{R}^+)}(\mathbb{R}^n;V_{\mathbb{R}^n})$$
$$= \underline{H}^j_{Q^{-1}(\mathbb{R}^+)}(V_{\mathbb{R}^n})_\circ = \begin{cases} V & \text{for } j=q \\ 0 & \text{for } j\neq q \end{cases} .$$

Hence we have, by denoting $q = \#\{j;\ a_j < 0\}$ ,

$$H^k(\mathbb{R}\Gamma_{\varphi^{-1}\mathbb{R}^+}(F)_{x_\circ}) \cong \underline{H}^k_{\varphi^{-1}\mathbb{R}^+}(\underline{V}_{Y_\alpha})_{x_\circ} = H^{k-q}(V) .$$

Therefore we obtain

(5.1.4) $$\chi(\mathbb{R}\Gamma_{\varphi^{-1}\mathbb{R}^+}(F)_{x_\circ}) = (-1)^q\chi(V) = (-1)^q\, m_\alpha .$$

On the other hand, we have

$$(\widetilde{SS}(F).Y_\varphi)_p = m_\alpha([T^*_{Y_\alpha}X].Y_\varphi)_p ,$$

and we can easily verify

$$([T^*_{Y_\alpha}X].Y_\varphi)_p = (-1)^{n(n+1)/2 + q}$$

This completes the proof of Proposition 5.1. Q.E.D.

5.2. Now we assume the condition (4.2.1) and the following conditions :

(5.2.1) $SSF \cap Y_\varphi \subset \bigsqcup_\alpha \Lambda_\alpha$

(5.2.2) $SSF$ and $Y_\varphi$ intersect transversally.

(5.2.3) $\#(SSF \cap Y_\varphi) < \infty$

PROPOSITION 5.3 - *Under these conditions we have* $\dim H^k(X;F) < \infty$

*and*

$$\chi(X;F) = (-1)^{n(n+1)/2}\ \widetilde{SSF}.Y_{\varphi} .$$

PROOF - Set $\Omega_t = \{x;\varphi(x) < t\}$ and $Z_t = \{x;\varphi(x) \leqq t\}$, and $\varphi\pi(Y_\varphi \cap SSF) = \{t_1, \dots, t_N\}$ with $t_1 < \dots < t_N$. We also set $t_0 = -\infty$, $t_{N+1} = \infty$, $\Omega_j = \Omega_{t_j}$ and $Z_j = Z_{t_j}$. Then by Proposition 3.1, we have

$$H^k(\Omega_{j+1}\ ; F) \cong H^k(\Omega_t\ ; F) \text{ for } t_{j+1} \geqslant t > t_j \text{ and } 0 \leqq j \leqq N.$$

Taking the inductive limit with respect to $t$ we obtain

$$H^k(\Omega_{j+1}\ ; F) \xrightarrow{\sim} H^k(Z_j\ ; F) \tag{5.2.4}$$

Then by the following well-known lemma, we have

$$\dim H^k(\Omega_{j+1}\ ; F) = \dim H^k(Z_j\ ; F) < \infty$$

LEMMA - If K is a compact set and if U is an open neighborhood of K, then the image of $H^k(U;F) \longrightarrow H^k(K;F)$ is finite-dimensional.

Since $\Omega_{N+1} = X$ and $Z_0 = \emptyset$, (5.2.4) implies

$$\chi(X;F) = \sum_{j=1}^{N} (\chi(Z_j\ ;F) - \chi(\Omega_j\ ;F)) . \tag{5.2.5}$$

Now we have a distinguished triangle

$$\mathbb{R}\Gamma(Z_j \setminus \Omega_j\ ;\mathbb{R}\Gamma_{X\setminus\Omega_j}(F)) \to \mathbb{R}\Gamma(Z_j\ ;F) \to \mathbb{R}\Gamma(\Omega_j\ ;F)$$

Hence we obtain

$$\chi(Z_j\ ;F) - \chi(\Omega_j\ ;F) = \chi(\mathbb{R}\Gamma(Z_j \setminus \Omega_j\ ;\mathbb{R}\Gamma_{X\setminus\Omega_j}(F))) . \tag{5.2.6}$$

By the definition of the micro-support, we have

$$\operatorname{supp} \mathbb{R}\Gamma_{X\setminus\Omega_j}(F)|_{\varphi^{-1}(t_j)} \subset \pi(Y_\varphi \cap SSF) .$$

Hence we obtain

$$\mathbb{R}\Gamma(Z_j \setminus \Omega_j\ ; \mathbb{R}\Gamma_{X\setminus\Omega_j}(F)) = \bigoplus_{x \in \pi(Y_\varphi \cap SSF)\cap\varphi^{-1}(t_j)} \mathbb{R}\Gamma_{X\setminus\Omega_j}(F)_x . \tag{5.2.7}$$

The identities (5.2.5), (5.2.6) and (5.2.7) imply

$$\chi(X;F) = \sum_{\substack{x \in \pi(Y_\varphi \cap SSF) \\ \varphi(x)=t_j}} \chi(\mathbb{R}\Gamma_{X \setminus \Omega_j}(F)x) .$$

Thus Proposition 5.3 follows from Proposition 5.1. Q.E.D.

## §6 - PROOF OF MAIN THEOREMS (II)

6.1. We shall prove Theorem 4.1. We give only an outline of the proof.

Since $\widetilde{SS}(F \otimes k_{\{0\}}) = \widetilde{SS}(F) \times T^*_{\{0\}}\mathbb{R}$ , it is sufficient to show that $\widetilde{SS}(F)$ is a cycle outside the zero section.

The support of $\beta = \partial\widetilde{SS}(F)$ is an $(n-1)$-dimensional subanalytic subset contained in $\bigsqcup_\alpha \partial\Lambda_\alpha$ . Taking a smooth point p of supp $\beta \setminus T^*_XX$, we shall derive the contradiction by the use of Lemma 1.1 and Proposition 5.3 .

6.2. Let us take a local coordinate system $(x_1, \ldots ,x_n)$ of X such that $p = (0,\xi_0)$ and that the map $(x,\xi) \longmapsto \xi$ from $T^*X$ to $\mathbb{R}^n$ gives a local embedding from supp $\beta$ into $\mathbb{R}^n$ and a finite map from SSF into $\mathbb{R}^n$ .

Set $\varphi(x,y) = \frac{1}{2} x^2 + xy$ and $\varphi_y(x) = \varphi(x,y)$ .

Then we have

$$SSF \cap Y_{\varphi_y} \cap \{x;\ |x| = \varepsilon\} = \emptyset \text{ for } |y| \leqq \varepsilon \text{ and } 0 < \varepsilon \ll 1 .$$

Therefore, if $|y| \ll \varepsilon$ and if $Y_{\varphi_y}$ satisfies the conditions (5.2.1) - (5.2.3), then we have, by Proposition 5.3

$$\chi(\{x\ ;\ |x| < \varepsilon\};\ F) = (-1)^{n(n+1)/2}\ \widetilde{SS}(F).Y_{\varphi_y} .$$

In particular, $\widetilde{SS}(F).Y_{\varphi_y}$ does not depend on y .

The relation $\xi = \mathrm{grad}_x \varphi_y = x + y$ gives the projection $g : T^*X \to \mathbb{R}^n$ by $g(x,\xi) = \xi - x$ . Since $g^{-1}(y) = Y_{\varphi_y}$ , $g^{-1}(y).\widetilde{SS}(F)$ is constant in y .

Therefore we can apply Lemma 1.1 to see $\partial\widetilde{SS}(F) = 0$ .

## §7 - PROOF OF MAIN THEOREMS (III)

7.1. In order to prove Theorem 4.2, we shall note the following

LEMMA 7.1. (i) *Let* $\Lambda$ *be an n-dimensional subanalytic conic real analytic submanifold of* $T^*X$. *Then* $\{\varphi\,;\ Y_\varphi$ *and* $\Lambda$ *intersect transversally* $\}$ *is dense in the space* $C^\infty(X)$ *of* $C^\infty$*-functions on* X *with respect to the* $C^2$*-topology.*

(ii) *Let* Z *be an* (n-1)*-dimensional subanalytic conic subset of* $T^*X$. *Then* $\{\varphi\,;\ Y_\varphi \cap Z = \emptyset\}$ *is a dense subset of* $C^\infty(X)$ .

They can be shown by using Baire's category theorem similarly to the proof of the existence theorem of Morse's function.

Let $\varphi$ and F satisfy the conditions in Theorem 4.2. Then there exists a function $\varphi'$ close to $\varphi$ which satisfies the conditions (5.2.1) - (5.2.3). Hence Proposition 5.3 can be applied to see $\chi(X;F) = (-1)^{n(n+1)/2}\ \widetilde{SS}(F).Y_{\varphi'}$ .

Since $Y_\varphi$ and $Y_{\varphi'}$ are homotopic, we have

$$\widetilde{SS}(F).Y_\varphi = \widetilde{SS}(F).Y_{\varphi'}\ .$$

This shows Theorem 4.2.

7.2. Theorem 4.3 can be proven in a similar argument or by reducing to Theorem 4.2 by the use of the Poincaré duality and the following proposition, which can be shown easily.

PROPOSITION 7.2 - *For* $F \in \mathrm{Ob}(D^b_c(X))$ *,we have*

$$\widetilde{SS}(\mathbb{R}\underline{\mathcal{H}om}_k(F,k_X)) = a^*(\widetilde{SS}(F)) \quad ,$$

*where* a *is the antipodal map of* $T^*X$ .

## §8 - APPLICATIONS

8.1. The following theorem follows immediately from Theorem 4.2.

THEOREM 8.1 - *Let* X *be a compact complex manifold, and* $F \in \mathrm{Ob}(D^b_c(X))$. *Then*

$$\chi(X;F) = (-1)^{n(n+1)/2}\ \widetilde{SS}(F).T^*_X X .$$

8.2. When X is a complex manifold and $\mathcal{M}$ is a holonomic module over the ring $\mathcal{D}_X$ of differential operators. Then $SS(DR(\mathcal{M}))$ coincides with the characteristic variety $Ch(\mathcal{M})$ of $\mathcal{M}$ and $\widetilde{SS}(DR(\mathcal{M}))$ coincides with the characteristic cycle $\underline{Ch}(\mathcal{M})$ of $\mathcal{M}$. Hence the results in this paper can be easily applied to holonomic modules.

8.3. Let $\varphi$ be a real -valued real analytic function defined on X and $x_o \in X$ .

(8.3.1) $$\varphi(x) > 0 \quad \text{for} \quad x \in X \setminus \{x_o\} .$$

LEMMA 8.2. *For any subanalytic closed conic Lagrangian set* $\Lambda$ , $d\varphi(x_o)$ *is an isolated point of* $\Lambda \cap Y_\varphi$ .

PROOF - Otherwise there exists a real analytic path $x = x(t)$ such that $x(0) = x_o$ ; $x(t) \neq x_o$ for $t \neq 0$ and $d\varphi(x(t)) \in \Lambda$ . Since $\Lambda$ is Lagrangian, $\theta = d\varphi(x(t)) = 0$ . Hence $\varphi(x(t))$ is a constant function, which is a contradiction. Q.E.D.

Along with this lemma, the following theorem follows immediately from Theorems 8.2 and 8.3.

THEOREM 8.3 - *Let* $F \in \mathrm{Ob}(D^b_c(X))$ *and let* $\varphi$ *satisfy* (8.3.1). *Then we have*

(8.3.1) $$\chi(F_{x_o}) = (-1)^{n(n+1)/2}\ (\widetilde{SS}(F).Y_\varphi)_{x_o} ,$$

(8.3.2) $$\chi(\mathbb{R}\Gamma_{\{x_o\}}(X;F)) = (-1)^{n(n+1)/2}\ (\widetilde{SS}(F).Y^a_\varphi)_{x_o} .$$

*Here* (.) *means the intersection number of two cycles at* $x_o \in T^*_X X \cong X \subset T^*X$ .

8.4. A $\mathbb{Z}$-valued function $\varphi$ on X is called *constructible* if there exists a subanalytic stratification $X = \bigsqcup X_\alpha$ of X such that $\varphi|_{X_\alpha}$ is constant. We define the $\pi^{-1}\omega_X$-valued n-cycle

(8.4.1) $$c(\varphi) = \sum_\alpha \varphi(X_\alpha)\widetilde{SS}(Q_{X_\alpha}) .$$

Then it is immediate that this does not depend on the choice of stratification.

Let us denote by $C(X)$ the space of $\mathbb{Z}$-valued constructible functions on $X$. Let $K(D_c^b(X))$ be the additive group generated by $Ob(D_c^b(X))$ with the relation

$$[F] = [F'] + [F'']$$

for distinguished triangles $F' \to F \to F'' \to F'[1]$.

For $F \in Ob(D_c^b(X)$ we define the constructible function $\chi(F)$ by $X \ni x \mapsto \chi(F_x)$. Then this passes through the quotient and we obtain the commutative diagram

$$\begin{tikzcd} K(D_c^b(X)) \arrow[rr, "\chi"] \arrow[dr, "\widetilde{SS}"'] & & C(X) \arrow[dl, "c"] \\ & Z_n(T^*X\ ;\ \pi^{-1}\omega_X) & \end{tikzcd} \tag{8.4.2}$$

Here $Z_n(T^*X\ ;\ \pi^{-1}\omega_X)$ denotes the space of $\pi^{-1}\omega_X$-valued subanalytic n-cycles.

EXAMPLE 8.5.

(i) Let $Y$ be a closed r-codimensional submanifold of $X$ and $\chi_Y$ the characteristic function of $Y$. Then

$$c(\chi_Y) = [T_Y^*X]$$

(ii) Set $X = \mathbb{R}$, $Z_\pm = \{x\ ;\ \pm x > 0\}$, $Z_\circ = \{0\}$.
We define the 1-cycles $\alpha_\pm$ and $\beta_\pm$ by

$$\alpha_\pm = \{(x,\xi)\ ;\ \xi = 0,\ \pm x > 0\} \text{ with } \operatorname{sgn} dx \otimes \operatorname{sgn} dx\ ,$$

$$\beta_\pm = \{(x,\xi)\ ;\ x = 0,\ \pm\xi > 0\} \text{ with } \operatorname{sgn} d\xi \otimes \operatorname{sgn} dx\ .$$

Then we have

$$c(\chi_{Z_+}) = \alpha_+ + \beta_-\ ,$$
$$c(\chi_{Z_-}) = \alpha_- + \beta_+ \text{ and}$$
$$c(\chi_{Z_\circ}) = -\beta_+ - \beta_-\ .$$

(iii) Set $X = \mathbb{R}^n$, $q(x) = x_1^2 - x_2^2 - \dots - x_n^2$ $(n \geqslant 2)$,

$$dx' = dx_2 \wedge \ldots \wedge dx_n \ , \ dx = dx_1 \wedge dx' \ ,$$
$$Z_\pm = \{x \in X \ ; \ q(x) \geqq 0, \ \pm x_1 \geqslant 0\} \ ,$$
$$Z_\circ = \{x \in X \ ; \ q(x) \leqq 0\} \ ,$$
$$\text{and} \quad U_\varepsilon = \operatorname{Int} Z_\varepsilon \quad (\varepsilon = \pm \ , \ 0) \ .$$

We define the n-cycles in $T^*X$ by

$$\alpha_\varepsilon = \{(x,\xi) \ ; \ x \in U_\varepsilon \ , \xi = 0\} \quad \text{with} \quad \operatorname{sgn} dx \otimes \operatorname{sgn} dx \ ,$$
$$\beta_\varepsilon = \{(x,\xi) \ ; \ x = 0, \xi \in U_\varepsilon\} \quad \text{with} \quad \operatorname{sgn} d\xi \otimes \operatorname{sgn} dx \ ,$$
$$\text{for} \quad \varepsilon = \pm \ , \ 0 \ , \text{ and}$$
$$\gamma_{\varepsilon_1, \varepsilon_2} = \{(x,\xi) \ ; \ \varepsilon_1 x_1 > 0, \ \varepsilon_2 \xi_1 > 0, \ \xi_j / x_j = -\xi_1 / x_1 \ \text{for} \ j \geqslant 2, \ q(x) = 0\}$$

with $\operatorname{sgn}(d\xi_1 \wedge dx') \otimes \operatorname{sgn} dx$ , for $\varepsilon_1 , \varepsilon_2 = \pm 1$ .

Then we have

$$c(\chi_{Z_\pm}) = \alpha_\pm - \gamma_{\pm,\pm} + (-)^n \beta_\pm \ ,$$
$$c(\chi_{U_\pm}) = \alpha_\pm + \gamma_{\pm,\mp} + \beta_\mp \ ,$$
$$c(\chi_{Z_\circ}) = \alpha_\circ - \gamma_{+,-} - \gamma_{-,+} - \beta_+ - \beta_- \quad \text{and}$$
$$c(\chi_{U_\circ}) = \alpha_\circ + \gamma_{+,+} + \gamma_{-,-} - (-)^n \beta_+ - (-)^n \beta_- \ .$$

## §9 - VARIATIONS OF MAIN THEOREMS

9.1. Let f be a real analytic function on X . We define, for $F \in \operatorname{Ob}(D(X))$ ,

(9.1.1) $$\mu_f(F) = \mathbb{R}\Gamma_{f^{-1}(\mathbb{R}^+)}(F)|_{f^{-1}(0)} \ .$$

Let $F \in \operatorname{Ob}(D_c^b(X))$ and $\Omega$ an open subset of $f^{-1}(0)$ . We assume

(9.1.2) $\Omega \cap \operatorname{supp} F$ is relatively compact.

(9.1.3) $SSF \cap Y_f \cap \pi^{-1}(\partial\Omega) = \emptyset$ .

Then we have the following

THEOREM 9.1 - *Under these conditions we have* $\dim H^k(\Omega \ ; \ \mu_f(F)) < \infty$

*and*

$$\chi(\Omega\ ;\ \mu_f(F)) = (-1)^{n(n+1)/2}\ (\widetilde{SS}F \cap \Omega).(Y_f \cap \Omega) \quad .$$

This theorem can be shown by deforming $f$ to a generic position with respect to $SSF$ .

9.2. Let $F$ and $F'$ be two objects of $D_c^b(X)$ and $\varphi$ a $C^1$-function on $T^*X$ . We assume the following

(9.2.1) $\Omega = \{p \in T^*X\ ;\ \varphi(p) < 0\}$ is relatively compact in $T^*X$ .

(9.2.2) $C_p(SS(F'),SS(F)) \not\ni -H_\varphi(p)$ for any $p \in \varphi^{-1}(0)$ .

Here $C_p$ means the normal cone (see [4]), and $H_\varphi$ means the Hamiltonian vector field of $\varphi$. We set

$$SS(F)^{\varepsilon} = e^{-\varepsilon H_\varphi}(SSF)$$

$$\text{and}\quad \widetilde{SS}(F)^{\varepsilon} = e^{-\varepsilon H_\varphi}(\widetilde{SS}F)$$

Then (8.6.2) implies for $0 < \varepsilon << 1$

$$(SS(F)^{\varepsilon} \cap \Omega) \cap (SS(F') \cap \Omega) = \emptyset\ .$$

THEOREM 9.2 - *Under these conditions we have*

$$\dim H^k(\Omega\ ;\ \mu hom(F,F')) < \infty$$

*and*

$$\chi(\Omega\ ;\ \mu hom(F,F')) = (-1)^{n(n+1)/2}\ (\widetilde{SS}(F') \cap \Omega).(\widetilde{SS}(F)^{\varepsilon} \cap \Omega)\ .$$

For the definition of $\mu hom$ , we refer to [4] . This theorem can be shown by reducing to Theorem 9.1 with the aid of contact transformations.

If we assume instead of (9.2.2)

(9.2.3) $C_p(SS(F'),SSF) \not\ni H_\varphi(p)$ for any $p \in \varphi^{-1}(0)$ .

Then we have

THEOREM 9.3 - *Under* (9.2.1) *and* (9.2.3) *we have*

$$\dim H_c^k(\Omega\ ;\ \mu hom(F,F')) < \infty$$

*and*

$$\chi_c(\Omega\ ;\ \mu hom\ (F,F')) = (-1)^{n(n+1)/2}\ (\widetilde{SS}(F') \cap \Omega).(\widetilde{SS}(F)^{-\varepsilon} \cap \Omega).$$

Remark that if we take as $F$ the constant sheaf $k_X$ , then we can recover Theorems 4.2 and 4.3.

## REFERENCES


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[4] M. KASHIWARA and P. SCHAPIRA, Micro-local study of sheaves, RIMS-469 (1984), Astérisque n° 128, (1985).


<u>Corrections to "Microlocal study of Sheaves"</u>, M.Kashiwara,P.Schapira. Astérisque 128, 1985.

1) p.48, 1.-6 ; p.85, 1.-8,-9 ; p.86, 1.-2 ; p.191, 1.-8,-5 :
read "... $\overset{\mathbb{L}}{\otimes}\, \varpi^{!}\, \underline{\mathbb{Z}}_{T^*_M X}$"

2) p.40, 1.-3 : p.47, 1.-9 :
read "... convex proper cone of..."

3) p.40, 1.-2 : read "...$\mathbb{R}\Gamma(\mathrm{Int}(A^{oa}), \underline{F})$..."

4) p.47, 1.-6 : read "... $\cap\ \mathrm{Int}\ Z^{oa}$..."

5) p.189, 1.4 : read "... is punctually endowed..."

6) p.119, 1. 4, 1.6 : read " $\alpha \geqslant 3$ " , " a $C^2$-function"