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NEWTON POLYGONS FOR GENERAL HYPERKLOOSTERMAN SUMS

by

Steven SPERBER*

INTRODUCTION

Let $X/\mathbb{F}_q$ be affine, char $\mathbb{F}_q = p$, f a regular function on X, ψ_q a non-trivial additive character of $\mathbb{F}_q$. We define exponential sums

$$S_m(f,X,\psi) = \sum_{x \in X(\mathbb{F}_{q^m})} \psi \circ Tr_{\mathbb{F}_{q^m}/\mathbb{F}_q}(f(x)) ,$$

where $X(\mathbb{F}_{q^m})$ denotes the $\mathbb{F}_{q^m}$-rational points of X. The associated L-function is defined by

$$L(f,X,\psi,T) = \exp\left(\sum_{m=1}^{\infty} S_m(f,X,\psi) T^m / m \right) .$$

In [4], Deligne proved that in the case of the hyperkloosterman sum $S_m(f,X,\psi)$, where X is the algebraic group defined over $\mathbb{F}_q$ by the coordinate equation $x_1x_2\ldots x_{n+1} = 1$ and where $f(x) = x_1+x_2+\ldots+x_{n+1}$, that the associated L-function $L(\mathrm{Kloos}_{n+1})$ has the property that $L(\mathrm{Kloos}_{n+1})^{(-1)^{n+1}}$ is a polynomial of degree $n+1$ having all reciprocal roots of absolute value equal to $q^{n/2}$. In [5], under suitable hypotheses, Katz generalizes Deligne's result, proving a similar result for the L-functions associated with generalized Kloosterman sums $S_m(\bar{g},Y,\psi)$ where Y is the algebraic group defined over $\mathbb{F}_q$ by the equation $x_1^{b_1}x_2^{b_2}\ldots x_{n+1}^{b_{n+1}} = 1$, and where $\bar{g}(x) = \bar{\alpha}_1x_1^k + \bar{\alpha}_2x_2^k + \ldots + \bar{\alpha}_{n+1}x_{n+1}^k$, $\bar{\alpha}_i \in \mathbb{F}_q^+$. (In fact, Katz's result

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is more general proving similar results even when the sums are twisted by multiplicative characters).

In this same work, Katz raises the question of how the Newton polygon of the L-function varies with p. In particular, if $\overline{g}(x)$ above is obtained by reduction from a global situation, say for example from $g(x) \in \mathbb{Z}[x_1, \ldots, x_{n+1}]$, then it makes sense to study the Newton polygon of $L(\overline{g}, Y, \psi_q, T)^{(-1)^{n+1}}$ as p varies. In the case of the zeta function $Z(X_p, T)$ of a projective non-singular variety X_p arising from mod p reduction of a variety X defined over some global field, the Newton polygon of $Z(X_p, T)$ has a well-known relation to the Hodge numbers of X. Are there analogous structures for exponential sums ? For example, in the case of hyperkloosterman sums, we proved in [7] that for $p \geqslant n+3$, the Newton polygon of $L(\mathrm{Kloos}_{n+1})^{(-1)^{n+1}}$ is given by the diagram with vertices

$$(+) \qquad \{(0,0)\} \cup \{(\ell, \ell(\ell-1)/2\}_{\ell=1}^{n+1} .$$

Clearly, in this case, the Newton polygon is independent of p. Katz asks whether this holds also in the case of generalized Kloosterman sums ; more precisely, he asks how the Newton polygon of $L(\overline{g}, Y, \psi_q, T)^{(-1)^{n+1}}$ varies as a function of $(p, b_1, \ldots, b_{n+1})$ for $p \gg 0$.

In the present work, we consider the case $b_{n+1} = k = 1$, which we write in the form

$$S_m = \sum \psi \circ \mathrm{Tr}_{\mathbb{F}_{q^m}/\mathbb{F}_q} (\overline{\alpha}_1 t_1 + \ldots + \overline{\alpha}_n t_n + \overline{x}. t_1^{-b_1} \ldots t_n^{-b_n})$$

where the sum is taken over all $t = (t_1, \ldots, t_n) \in (\mathbb{F}^+_{q^m})^n$. In theorem (5.46), we obtain for $p \equiv 1 \pmod M$, $(M = \mathrm{l.c.m.}\ (b_1, \ldots, b_n))$, a precise description of the Newton polygon of the associated L-function. In theorem (5.31), assuming only $(p, M) = 1$, we show that the Newton polygon always lies over the polygon for $p \equiv 1 \pmod M$. This behavior is not unlike Stickelberger's result for Gauss sums. These results are then generalized in § 6 and § 7.

In terms of Katz's questions then, within the congruence class $p \equiv 1 \pmod M$ the Newton polygon is independent of p. In general this diagram has vertices in the lattice $\mathbb{Z} \times \frac{1}{M}\mathbb{Z}$. However, the fact that $L^{(-1)^{n+1}} \in \mathbb{Q}(\zeta_p)[T]$, where ζ_p is a primitive p^{th} root of 1, together with the fact that $\mathbb{Q}_p(\zeta_p)$ is totally ramified over $\mathbb{Q}_p$ of degree p-1 implies that this same diagram can not in general be the the Newton polygon of $L^{(-1)^{n+1}}$ when $p \not\equiv 1 \pmod{M}$. In § 8, we analyze an example (with $M = 3$) in greater detail showing (theorem 8.9) that if $p \equiv 2 \pmod{3}$, then the Newton polygon varies with p, descending as $p \longrightarrow \infty$ in the congruence classe $p \equiv 2 \pmod{3}$, to the Newton polygon diagram for the case of primes p, $p \equiv 1 \pmod{3}$ given by theorem (5.46).

Throughout we use Dwork's methods. We systematically replace the differential operators that arise in Dwork's theory by simpler operators which we view as perturbations. By this type approximation, we simplify the computations sufficiently to extract very precise estimates. The method also requires a good choice for the basis of the cohomology space. In § 4, we take this opportunity to clarify the process of specialization. We note also that the present techniques eliminate the need to restrict to large p ; in particular, theorem (5.46) shows that (+) is the Newton polygon for $L(Kloos_{n+1})^{(-1)^{n+1}}$ <u>for all</u> p, thus extending the result of [7] quoted above.

Finally, we note that it is possible to give a recipe for the Newton polygon of $L^{(-1)^{n+1}}$ when p 1 (mod M) in which the ingredients for the recipe consist only of the exponents of the deformation equation at the singular point x = O. We believe that this indicates the possibility of using transformations of the polynomial

$$\alpha_1 t_1 + \ldots + \bar{\alpha}_n t_n + x t_1^{-b_1} \ldots t_n^{-b_n} \quad ;$$

the singular fibers of the resulting transforms ; and p-adic analytic continuation to describe analytically the reciprocal zeros of $L^{(-1)^{n+1}}$.

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Let Ω be an algebraically closed field of characteristic zero complete under the extension $|\ |$ of the p-adic valuation of $\mathbb{Q}$. We will also use the additive form "ord" of the valuation, normalized so that ord $p = 1$; if $q = p^r$, then "ord_q" will denote the valuation normalized so that $\mathrm{ord}_q q = 1$. Let $\mathbb{N}$ denote the non-negative integers, and let $\mathbb{N}^+$ denote the natural numbers. Let $a = (a_1,\ldots,a_n) \in (\mathbb{N}^+)^n$. We assume

$$a_1 \geqslant a_2 \geqslant \ldots \geqslant a_n > 0$$

Let $M = \mathrm{l.c.m.}(a_1,a_2,\ldots,a_n)$; $N = \sum_{i=1}^{n} a_i + 1$.

1. DEFINITIONS. Let Ω_0 be a finite extension of $\mathbb{Q}_p(\zeta_p)$ where ζ_p is a primitive p^{th} root of 1 ; let $\mathcal{O}_0$ be the ring of integers of Ω_0. Let $\tau \in \mathrm{Gal}(\Omega_0/\mathbb{Q}_p(\zeta_p))$ denote the Frobenius automorphism. For $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}^n$, $\gamma \in \mathbb{N}$, $m \in \mathbb{N}^+$, define

$$(1.1)\qquad \begin{cases} \sum(\alpha) = \sum_{i=1}^{n} \alpha_i \,, \\ s(\alpha) = \max\{0, \frac{-\alpha_1}{a_1},\ldots,\frac{-\alpha_n}{a_n}\} \\ w(\alpha) = \sum(\alpha) + Ns(\alpha)\,, \\ w_m(\alpha;\gamma) = \sum(\alpha) + N\gamma m^{-1}M^{-1}\,. \end{cases}$$

Let $t_1,\ldots,t_n,Y$ be indeterminates ; let $b,c \in \mathbb{R}$, $b \geqslant 0$. Define

$$(1.2)\qquad L_m(b,c) = \{ \sum_{(\alpha;\gamma)\in S_m} A(\alpha;\gamma)t^{\alpha}Y^{\gamma} \mid A(\alpha;\gamma) \in \Omega_0\,,\ \mathrm{ord}\, A(\alpha;\gamma) > c + w_m(\alpha;\gamma)b \}$$

where the index set S_m is given by

$$(1.3)\qquad S_m = \{(\alpha;\gamma) \in \mathbb{Z}^m \times \mathbb{N} \mid \gamma > mMs(\alpha)\}\,,$$

and where t^{α} denotes $t_1^{\alpha_1}t_2^{\alpha_2}\ldots t_n^{\alpha_n}$.

Let

$$(1.4)\qquad \begin{cases} L_m(b) = \bigcup_{c\in\mathbb{R}} L_m(b,c)\ , \\ R_m(b,c) = \Omega_0[[Y]] \cap L_m(b,c)\ , \\ R_m(b) = \bigcup_{c\in\mathbb{R}} R_m(b,c) = L_m(b) \cap \Omega_0[[Y]]\ . \end{cases}$$

Define

$$(1.5)\qquad f(Y,t) = \sum_{i=1}^{n} c_i t_i + Yt^{-a}\ .$$

Let $\{\gamma_j\}_{j=0}^{\infty} \subset \mathbb{Q}_p(\zeta_p)$ be a sequence of elements with the estimates

$$(1.6)\qquad \begin{cases} \text{ord}\ \gamma_0 = 1/(p-1)\ , \\ \text{ord}\ \gamma_j \geqslant \dfrac{p^{j+1}}{p-1} - (j+1)\ . \end{cases}$$

In terms of these constants, we write

$$(1.7)\qquad \begin{cases} \hat{H}(Y,t) = \gamma_0 \cdot f(Y,t), \\ H(Y,t) = \hat{H}(Y,t) + \sum_{\ell=1}^{\infty} \gamma_\ell \cdot f^{\tau^\ell}(Y^{p^\ell}, t^{p^\ell}), \\ \hat{F}(Y,t) = \exp H(Y,t), \\ E_i = t_i \dfrac{\partial}{\partial t_i}\ , \\ \hat{H}_i^{(m)} (= \hat{H}_i(Y^{mM},t)) = E_i \hat{H}(Y^{mM},t)\ , \\ H_i^{(m)} (= H_i(Y^{mM},t)) = E_i H(Y^{mM},t)\ , \\ D_i^{(m)} = E_i + H_i^{(m)}\ . \end{cases}$$

We note that

$$(1.8)\qquad \hat{H}_i^{(m)} = \gamma_0(c_i t_i - a_i Y^{mM} t^{-a}) \in L_m(b,-e)$$

where $e = b - \frac{1}{p-1}$, and

$$(1.8)'\qquad H_i^{(m)} \in L_m(b,-e)$$

provided $\frac{p}{p-1} \geqslant b$.

We now distinguish for each $J \in \mathbb{Z}$, a unique vector $\sigma^{(J)} \in \mathbb{Z}^n$ such that $\sum(\sigma^{(J)}) = J$. We employ the notation U_i for that element of $\mathbb{Z}^n$ with 1 in the i^{th} position and 0 elsewhere. We define $\sigma^{(J)}$ inductively for $0 \geqslant J \geqslant -(N-1)$. First, define for $J \leqslant 0$,

$$(1.9)\qquad \begin{cases} S(J) = \min\{s(\gamma)\}, \\ W(J) = J + NS(J), \end{cases}$$

where the minimum runs over γ such that $-\gamma \in \mathbb{N}^n$, $\sum(\gamma) = J$. Assume that $\sigma^{(K)}$ has been defined for $0 \geqslant K \geqslant J$, (where $0 > J \geqslant -(N-1)$), with the properties

$$(1.10)\qquad \begin{aligned} &(i)_K. && \textstyle\sum(\sigma^{(K)}) = K; \\ &(ii)_K. && \sigma^{(K)} = \sigma^{(K+1)} - U_{\ell_K}, \\ & && \text{where } K<0, \text{ and } \ell_K \in \{1,2,\dots,n\}; \\ &(iii)_K. && s(\sigma^{(K)}) = S(K); \\ &(i)_K. && \ell_K \text{ is chosen minimally so that the above} \\ & && \text{properties hold.} \end{aligned}$$

We will show that ℓ_J can be chosen so that $\sigma^{(J)}$ and ℓ_J satisfy (1.10) for $K = J$. Let $\sigma^{(J,k)} = \sigma^{(J+1)} - U_k$, for $k \in \{1,2,\dots,n\}$. It suffices to prove $s(\sigma^{(J,k)}) = S(J)$ for some k. Suppose on the contrary

$$(*)\qquad s(\sigma^{(J,k)}) > S(J), \text{ for all } k.$$

Let $\gamma^{(J)} = (\gamma_1^{(J)},\dots,\gamma_n^{(J)}) \in \mathbb{Z}^n$, $-\gamma^{(J)} \in \mathbb{N}^n$, $\sum(\gamma^{(J)}) = J$ and $s(\gamma^{(J)}) = S(J)$. Note that $(*)$ implies

$$(**)\qquad -\gamma_i^{(J)} \leqslant -\sigma_i^{(J+1)}, \text{ for all } i,\ 1 \leqslant i \leqslant n.$$

Otherwise, $-\gamma_i^{(J)} \geqslant -\sigma_i^{(J+1)} + 1$ for some i, $1 \leqslant i \leqslant n$. But then

$$\frac{-\sigma_i^{(J,i)}}{a_i} = \frac{-\sigma_i^{(J+1)} + 1}{a_i} \leqslant \frac{-\gamma_i^{(J)}}{a_i} \leqslant S(J);$$

for $\ell \neq i$,

$$-\frac{\sigma_\ell^{(J,i)}}{a_\ell} = -\frac{\sigma_\ell^{(J+1)}}{a_\ell} \leqslant S(J+1) \leqslant S(J) \ .$$

Thus $s(\sigma^{(J,i)}) < S(J)$, contradicting (*). But summing (**) over i, leads to the contradiction $-J \leqslant -J-1$, and establishes (1.10) for $K = J$.

We have thus defined the sequence

(1.11) $$\{U_{\ell_{-1}}, U_{\ell_{-2}}, \ldots, U_{\ell_{-(N-1)}}\} \ ;$$

for J, $0 > J \geqslant -(N-1)$,

$$\sigma^{(J)} = -\sum_{i=J}^{-1} U_{\ell_i} \ .$$

Define a map

(1.12) $$g : \mathbb{Z} - \{0\} \longrightarrow \{1,2,\ldots,n\}$$

by setting $g(i) = \ell_i$ for $-1 \geqslant i \geqslant -(N-1)$ and requiring periodicity $g(j) = g(j+N-1)$ for $j \leqslant -N$; then set $g(i) = \ell_{i-N}$, for $1 \leqslant i \leqslant N-1$ and $g(j) = g(j-N+1)$ for $j \geqslant N$. Then, if $J < 0$, we define

(1.13) $$\sigma^{(J)} = \sigma^{(J+1)} - U_{g(J)} = -\sum_{i=J}^{-1} U_{g(i)} \ ;$$

if $J > 0$,

$$\sigma^{(J)} = \sigma^{(J-1)} + U_{g(J)} = \sum_{i=1}^{J} U_{g(i)} \ .$$

Observe that with this definition

(1.14) $$s(\sigma^{(J)}) = S(J) \quad \text{for all} \quad J \leqslant 0 \ .$$

It is convenient, as well as consistent with (1.14), to define $S(J) = 0$, $W(J) = J$ for $J > 0$. Observe also that

(1.15) $$S(J + \lambda(N-1)) = S(J) - \lambda$$

for non-positive integers J and λ .

(1.16) DEFINITION. <u>We call the set</u>

$$\Delta = \{\sigma^{(J)} \in \mathbb{Z}^n \mid J \in \mathbb{Z}\}$$

<u>the diagonal weighted by the vector</u> $a = (a_1, a_2, \ldots, a_n) \in (\mathbb{N}^+)^n$.

We distinguish a certain subset of the diagonal Δ which will play an important role in the following sections. Let

$$\tilde{\Delta} = \{\sigma^{(J)} \in \Delta \mid -N+1 \leqslant J \leqslant 0\} . \tag{1.17}$$

Let $V_m(b)$ be the $R_m(b)$-span of

$$\{Y^{mMS(\mu)} t^{\sigma^{(\mu)}} \mid \sigma^{(\mu)} \in \tilde{\Delta}\} \tag{1.18}$$

and

$$V_m(b,c) = V_m(b) \cap L_m(b,c) .$$

2. REDUCTION THEORY.

The purpose of this section is to prove certain explicit reduction formulas modulo the submodule $\sum_{i=1}^{n} \hat{H}_i^{(m)} L_m(b,c+e)$ of $L_m(b,c)$, and to prove under the hypothesis $(p,M) = 1$, that

$$L_m(b,c) = V_m(b,c) + \sum_{i=1}^{n} \hat{H}_i^{(m)} L_m(b,c+e) . \tag{2.1}$$

In the next section we will prove this sum is direct.

(2.2) LEMMA (Reduction to the diagonal). Assume $(p,M) = 1$. Let $\alpha \in \mathbb{Z}^n$, $\sum(\alpha) = J$. Then

$$t^\alpha = u(\alpha)\, t^{\sigma^{(J)}} + \sum_{i=1}^{n} \hat{H}_i^{(m)} (\gamma_0^{-1} p_{i,\alpha})$$

in which $u(\alpha)$ is a unit in Ω_0, and $p_{i,\alpha} \in \mathcal{O}_0[t_1 \ldots, t_n, (t_1,\ldots,t_n)^{-1}]$ has the following properties : if t^β is a monomial of $p_{i,\alpha}$ having non-zero coefficient, then

(i) $\sum(\beta) = \sum(\alpha) - 1$,

and

(ii) $s(\beta) \leqslant s(\alpha)$.

Proof : If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n$, then

$$(2.3)_{i,j} \quad t^{\alpha} = a_i c_j (a_j c_i)^{-1} t^{\alpha - U_i + U_j} + (a_j \hat{H}_i^{(m)} - a_i \hat{H}_j^{(m)})((\gamma_0 a_j c_i)^{-1} t^{\alpha - U_i}) .$$

We describe this process as shifting from i to j. We use this process in either of two cases :

(i) $\quad \alpha_i > 0$;

(ii) $\quad \alpha_i \leqslant 0$, and $\dfrac{-\alpha_i + 1}{a_i} \leqslant s(\alpha)$.

In both cases, we have

$$s(\alpha - U_i) = s(\alpha) , \quad s(\alpha + U_j - U_i) \leqslant s(\alpha) .$$

We use the shift process (2.3) repeatedly to reduce α to the diagonal.

As long as there exists a pair of indices (i,j) with $\alpha_i > 0$ and $\alpha_j < 0$ then we can shift from i to j, as above. Therefore, we may and will assume that either $\alpha_i \geqslant 0$ for all indices i, or that $\alpha_i \leqslant 0$ for all indices i. We treat these cases separately. Assume first that $\alpha_i \geqslant 0$ for all indices i. Then whenever $\alpha_i > 0$, we may shift to j from i, for any j, $j \neq i$, and we obtain the assertion of the lemma after a finite number of steps.

Assume next that $\alpha_i \leqslant 0$ for all indices i. If $s(\alpha) > S(J)$ and $-\dfrac{\alpha_j}{a_j} = s(\alpha)$, then we claim that we can shift from i to j. Note that since $\sum(\alpha) = \sum(\sigma^{(J)}) = J$ and $\alpha \neq \sigma^{(J)}$, therefore $\alpha_i > \sigma_i^{(J)}$ for some i . Thus,

$$\frac{-\alpha_i + 1}{a_i} \leqslant \frac{-\sigma_i^{(J)}}{a_i} \leqslant S(J) < s(\alpha) ,$$

so that we can shift from i to j . In a finite number of steps we can reduce $s(\alpha)$; in fact, the process continues as long as $s(\alpha) > S(J)$. Therefore assume we have reduced α to $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$ with $s(\tilde{\alpha}) = S(J)$. If $\tilde{\alpha} \neq \sigma^{(J)}$, then $\sum(\tilde{\alpha}) = \sum(\sigma^{(J)})$ implies that $\tilde{\alpha}_i > \sigma_i^{(J)}$ and $\tilde{\alpha}_j < \sigma_j^{(J)}$ for some pair of indices (i,j) . Then

$$\frac{-\tilde{\alpha}_i + 1}{a_i} \leqslant \frac{-\sigma_i^{(J)}}{a_i} \leqslant S(j) = s(\tilde{\alpha})$$

so we can shift from i to j . In a finite number of steps, we reduce $\tilde{\alpha}$ to $\sigma^{(J)}$ as desired. ∎

(2.4) THEOREM. <u>Assume</u> $(p,M)=1$. <u>Let</u> $\xi = \sum_{(\alpha;\gamma)\in S_m} A(\alpha;\gamma)t^{\alpha}Y^{\gamma} \in L_m(b,c)$.

<u>Then for each</u> $k \in \mathbb{Z}$,

$$\sum_{\Sigma(\alpha)=k}\ \sum_{(\alpha;\gamma)\in S_m} A(\alpha;\gamma)t^{\alpha}Y^{\gamma} = a_k(\gamma)t^{\sigma^{(k)}} + \sum_{i=1}^{n} \hat{H}_i^{(m)}\ \eta_i$$

<u>where</u> $a_k(Y) = \sum_{\Sigma(\alpha)=k} u(\alpha) \sum_{(\alpha;\gamma)\in S_m} A(\alpha;\gamma)Y^{\gamma} \in R_m(b,c+kb)$, $a_k(Y)t^{\sigma^{(k)}} \in L_m(b,c)$,

$\eta_i \in L_m(b,c+e)$, $u(\alpha)$ <u>a unit in</u> Ω_0

<u>Proof</u> : The reduction follows from lemma (2.2) (note that Y does not appear in $a_j\hat{H}_i^{(m)} - a_i\hat{H}_j^{(m)}$ so that we may multiply $(2.3)_{i,j}$ by Y^{γ} to reduce $Y^{\gamma}t^{\alpha}$). If $\gamma \geqslant mMs(\alpha)$, then using $s(\alpha) \geqslant s(\beta)$ for any monomial t^{β} of $p_{i,\alpha}$, we see that all monomials of $Y^{\gamma}p_{i,\alpha}$ belong to S_m ; similarly by (1.9) and (1.14), $Y^{\gamma}t^{\sigma^{(k)}} \in S_m$.

Note that if $S_{m,k} = \{(\alpha;\gamma) \in S_m \mid \sum(\alpha) = k\}$, then

$$\text{ord} \sum_{(\alpha;\gamma)\in S_{m,k}} u(\alpha)A(\alpha;\gamma) \geqslant c + kb + \gamma Nbm^{-1}M^{-1} .$$

Thus, $a_k(Y) \in R_m(b,c+kb)$, and $a_k(Y)t^{\sigma^{(k)}} \in L_m(b,c)$. Similarly, in terms of (2.2), $\eta_i = \gamma_0^{-1} \sum_{(\alpha;\gamma)\in S_{m,k}} A(\alpha;\gamma)Y^{\gamma}p_{i,\alpha}$. Thus, if we write

$\eta_i = \sum_{(\beta;\gamma)\in S_{m,k}} B_i(\beta;\gamma)t^{\beta}Y^{\gamma}$, then $B_i(\beta;\gamma)$ has the from

$$B_i(\beta;\gamma) = \gamma_0^{-1} \sum A(\alpha;\gamma)\varepsilon(\alpha)$$

in which the sum runs over $\alpha \in \mathbb{Z}^n$ with $\sum(\alpha) = k = \sum(\beta) + 1$, (so that $w_m(\alpha;\gamma) = w_m(\beta;\gamma) + 1$), and $s(\beta) \leqslant s(\alpha) \leqslant m^{-1}M^{-1}\gamma$, and in which $\varepsilon(\alpha) \in \mathcal{O}_0$. Thus,

$$\text{ord } B_i(\beta;\gamma) \geqslant c + e + bw_m(\beta;\gamma)$$

and $\eta_i \in L_m(b,c+e)$. ∎

To reduce along the diagonal to the set $\tilde{\Delta}$, we will need the following formulas :

(2.5)

(i) $Y^{\gamma}t^{\sigma^{(k)}} = a_{g(k)}c_{g(k)}^{-1}Y^{\gamma+mM}t^{\sigma^{(k-N)}} + \hat{H}_{g(k)}^{(m)}(\gamma_0^{-1}c_{g(k)}^{-1}Y^{\gamma}t^{\sigma^{(k-1)}})$, if $k > 0$;

(ii) $Y^{\gamma}t^{\sigma^{(k)}} = c_{g(k+N)}a_{g(k+N)}^{-1}Y^{\gamma-mM}t^{\sigma^{(k+N)}} - \hat{H}_{g(k+N)}^{(m)}(\gamma_0^{-1}a_{g(k+N)}^{-1}Y^{\gamma-Mm}t^{\sigma^{(k+N-1)}})$,

if $k \leqslant -N$.

(2.6) LEMMA (Reduction along the diagonal). Assume $(p,M) = 1$. Let $(\alpha;\gamma) \in S_m$, with $\alpha = \sigma^{(k)} \in \Delta$, $\sum(\alpha) = \sum(\sigma^{(k)}) = k$. Let $k = N\tau + \mu$ with $-N < \mu \leqslant 0$. Then

$$Y^{\gamma}t^{\sigma^{(k)}} = \omega(k)Y^{\gamma+\tau mM}t^{\sigma^{(\mu)}} + \sum_{i=1}^{n} \hat{H}_i^{(m)}(\gamma_0^{-1}p_{i,\alpha,\gamma}) ,$$

in which $\omega(k)$ is a unit in $\mathcal{O}_0$. Furthermore, for $k > 0$, $p_{i,\alpha,\gamma} \in Y^{\gamma}\mathcal{O}_0[Y,t_1,\dots,t_n]$ such that if $Y^{\nu}t^{\beta}$ is a monomial term of $p_{i,\alpha,\gamma}$ having a non-zero coefficient, then

(i) $\beta \in \Delta$;

(ii) $\sum(\beta) = \sum(\alpha) - \lambda N - 1$ for some λ, $0 \leqslant \lambda < \tau$, and $\nu = \gamma + \lambda mM$, (thus $w_m(\beta;\nu) = w_m(\alpha;\gamma) - 1$) ;

(iii) $s(\beta) = s(\alpha) = 0$.

For $k \leqslant -N$, $p_{i,\alpha,\gamma}$ is a polynomial with coefficients in $\mathcal{O}_0$ in the variables $Y,t_1^{-1},\dots,t_n^{-1}$ such that if $Y^{\nu}t^{\beta}$ is a monomial term of $p_{i,\alpha,\gamma}$ having non-zero coefficient then

(i) $\beta \in \Delta$,

(ii) $\sum(\beta) = \sum(\alpha) + \lambda N - 1$, for some λ, $1 \leqslant \lambda \leqslant |\tau|$, and $\nu = \gamma - \lambda mM$, (thus $w_m(\beta;\nu) = w_m(\alpha;\gamma) - 1)$,

(iii) $s(\beta) = S(k+\lambda-1) - \lambda \leqslant s(\alpha) - \lambda$.

Proof : This is an immediate consequence of the reduction formulas (2.5). Note that if $k \leqslant -N$ we can verify (iii) as follows. Since $\beta \in \Delta$,

$$\begin{aligned} s(\beta) &= S(k+\lambda N-1) \\ &= S(k+\lambda-1+\lambda(N-1)) \\ &= S(k+\lambda-1)-\lambda \\ &\leqslant s(\alpha)-\lambda \end{aligned}$$

by periodicity (1.12). ∎

(2.7) THEOREM. Assume $(p,M)=1$. Let $\xi = \sum_{\alpha\in\Delta} A(\alpha;\gamma)Y^{\gamma}t^{\alpha} \in L_m(b,c)$. Then

$$\xi = \sum_{\sigma^{(\mu)}\in\tilde{\Delta}} b_\mu(Y)t^{\sigma^{(u)}} + \sum_{i=1}^{n} \hat{H}_i^{(m)}\zeta_i$$

where $\sum_{\sigma^{(\mu)}\in\tilde{\Delta}} b_\mu(Y)t^{\sigma^{(\mu)}} \in V_m(b,c)$, and $\zeta_i \in L_m(b,c+e)$.

Proof : In fact, if we define τ_α for each $\alpha\in\Delta$ by $\sum(\alpha) = N\tau_\alpha + \mu$ (with $-N+1 \leqslant \mu \leqslant 0$), then $b_\mu(Y) = \sum_{\nu\geqslant mMS(\mu)} b_\mu(\nu)Y^\nu$, where

$$(*)\qquad b_\mu(\nu) = \sum_{\gamma+\tau_\alpha mM=\nu} A(\alpha;\gamma)\omega(\textstyle\sum(\alpha)).$$

Since $\gamma\geqslant 0$, $\gamma+\tau_\alpha mM \longrightarrow +\infty$ as $\tau_\alpha \longrightarrow \infty$. Since

$s(\alpha) = S(N\tau_\alpha+\mu) = S(\mu+\tau_\alpha) - \tau_\alpha \geqslant s(\sigma^{(\mu)}) - \tau_\alpha$ for $\tau_\alpha < 0$,

$\gamma + \tau_\alpha mM \longrightarrow +\infty$ for $(\alpha;\gamma)\in S_m$, $\alpha\in\Delta$, as $\tau_\alpha \longrightarrow -\infty$. Thus the sum in (*) is finite. Thus

$$\begin{aligned} \operatorname{ord} b_\mu(\nu) &\geqslant \inf_{\gamma+\tau_\alpha mM=\nu} \{c+b(\textstyle\sum(\alpha)+Nm^{-1}M^{-1}\gamma)\} \\ &\geqslant c + bw_m(\sigma^{(\mu)};\nu). \end{aligned}$$

Hence $\sum_{\sigma^{(\mu)}\in\tilde{\Delta}} b_\mu(Y)t^{\sigma^{(\mu)}} \in V_m(b,c)$.

Let $\zeta_i = \sum c_i(\beta;\nu)t^\beta Y^\nu$, in which $\beta\in\Delta$, $(\beta;\nu)\in S_m$. By the lemma, $\zeta_i = \sum_{0\geqslant\mu\geqslant -N+1} \zeta_{i,u}$, where

$$\zeta_{i,\mu} = \gamma_0^{-1} \sum_{\Sigma(\alpha)\equiv\mu \pmod N} A(\alpha;\gamma)p_{i,\alpha,\gamma}.$$

Thus

$$(**) \qquad c_i(\beta) = \gamma_0^{-1} \sum A(\alpha;\gamma)\varepsilon(\alpha)$$

where $\varepsilon(\alpha) \in \mathcal{O}_0$. In the case $\sum(\beta) > 0$, the sum runs over $\alpha \in \Delta$, $(\alpha;\gamma) \in S_m$, $\sum(\alpha) = N\tau_\alpha + \mu = \sum(\beta) + \lambda N + 1$ and $\nu = \gamma + \lambda mM$ for some λ, $0 < \lambda < \tau_\alpha$. Since γ and λ are both non-negative, the sum in (**) is finite. Furthermore since $w_m(\beta;\nu) = w_m(\alpha;\gamma) - 1$,

$$\text{ord}\ c_i(\beta) \geqslant c + e + b\, w_m(\beta;\nu)\ .$$

On the other hand, if $\sum(\beta) \leqslant 0$, then the sum in (**) runs over $\alpha \in \Delta$, $(\alpha;\gamma) \in S_m$, $\sum(\alpha) = N\tau_\alpha + \mu = \sum(\beta) - \lambda N + 1$ and $\nu = \gamma - \lambda mM$ for some λ, $1 \leqslant \lambda \leqslant |\tau_\alpha|$. Since $\nu = \gamma - \lambda mM \geqslant (s(\alpha)+\tau_\alpha)mM$, and $s(\alpha) + \tau_\alpha \longrightarrow +\infty$ as $\sum(\alpha) \longrightarrow -\infty$, the sum in (**) is finite in this case also. Again $w_m(\beta;\nu) = w_m(\alpha;\gamma) - 1$, implies the desired estimate

$$\text{ord}\ c_i(\beta) \geqslant c + e + b\, w_m(\beta;\nu)$$

so that $\zeta_i \in L_m(b,c+e)$ as desired. ∎

Combining theorems (2.4) and (2.7) we have (2.1) in the following precise form.

(2.8) THEOREM. <u>Assume</u> $(p,M) = 1$. <u>Then</u>

$$L_m(b,c) = V_m(b,c) + \sum_{i=1}^{n} \hat{H}_i^{(m)} L_m(b,c+e)\ .$$

<u>In fact, if</u> $\xi = \sum_{(\alpha;\gamma)\in S_m} A(\alpha;\gamma)Y^\gamma t^\alpha \in L_m(b,c)$, <u>then</u>

$$\xi = \hat{v} + \sum_{i=1}^{n} \hat{H}_i^{(m)} \hat{\zeta}_i$$

<u>where</u> $\hat{\zeta}_i \in L_m(b,c+e)$, $\hat{v} = \sum_{-N+1\leqslant\nu\leqslant 0} v_\nu(Y)t^{\sigma^{(\nu)}} \in V_m(b,c)$. <u>Explicitly</u>,

$v_\nu(Y) = \sum_{\delta \geq mMs(\nu)} v_\nu(\delta)Y^\delta$, <u>where</u>

$$v_\nu(\delta) = \sum A(\alpha;\gamma)u(\alpha)\omega(\textstyle\sum(\alpha))$$

<u>in which the sum runs over</u> $(\alpha;\gamma) \in S_m$, $\sum(\alpha) = N\tau_\alpha + \nu$, $\gamma + \tau_\alpha mM = \delta$, <u>and</u> $u(\alpha)$ <u>and</u> $\omega(\sum(\alpha))$ <u>are units</u>. ∎

3. DIRECTNESS OF SUM.

Let A be an arbitrary noetherian unique factorization domain in which c_i and a_i are units for all i . We have in mind the two cases $A = \mathcal{O}_0$ and $A = \mathbb{F}_q$. Let $R = A[Y, t_1, \ldots, t_n, Y^{mM}t^{-a}]$, and $h_i^{(m)} = t_i - \varepsilon_i Y^{mM}t^{-a}$ where ε_i is a unit in A for every i, $i = 1,2,\ldots,n$.

(3.1) THEOREM. <u>The sequence</u> $\{h_i^{(m)}\}_{i=1}^n$ <u>in any order forms an</u> R-<u>sequence in</u> R .

<u>Proof</u> : Let I be a proper subset of $\{1,2,\ldots,n\}$ and define the ideal of R

$$\mathfrak{A}_I = (\ldots, h_i^{(m)} \ldots)_{i \in I} .$$

It suffices to show $h_k^{(m)}$ is not contained in any associated prime ideal of $\mathfrak{A}_I$ for $k \notin I$. For, if so, then

$$(\mathfrak{A}_I : h_k^{(m)}) = \mathfrak{A}_I .$$

We may assume by relabeling that $I = \{1,\ldots,j\}$, in which case we write $\mathfrak{A}_j$ in place of $\mathfrak{A}_I$, and $k = j+1$, (the case $I = \emptyset$ and $k = 1$ is trivial). Let $S = A[Y, t_1, t_{j+1}, \ldots, t_n, Y^{mM}t^{-b}]$ where

$$t^{-b} = t_1^{-b_1} t_{j+1}^{-b_{j+1}} \cdots t_n^{-b_j}$$

with $b_1 = \sum_{\ell=1}^{j} a_\ell$, $b_\ell = a_\ell$ for $j+1 \leq \ell \leq n$. Then the homomorphism $\theta_1 : R \longrightarrow S$ defined by $\theta_1(t_\ell) = \varepsilon_\ell \varepsilon_1^{-1} t_1$, for $1 \leq \ell \leq j$, $\theta_1(t_\ell) = t_\ell$, for $\ell > j$, and $\theta_1(Y) = Y$, induces an isomorphism of rings

$$R/\mathfrak{A}_j \xrightarrow{\ \overline{\theta}_1\ } S/\mathcal{L}$$

where $\mathcal{L} = (t_1 - \varepsilon_0 Y^{mM}t^{-b})$ is a principal ideal , and

$\varepsilon_0 = \varepsilon_1^{b_1 - a_1 + 1} \varepsilon_2^{-a_2} \cdots \varepsilon_j^{-a_j}$ is a unit in A . Let $U = A[Y, t_1, t_{j+1}, \ldots, t_n, t_{n+1}]$. Then the homomorphism $\theta_2 : U \longrightarrow S$ defined by $\theta_2(t_\ell) = t_\ell$ for $\ell \neq n+1$, $\theta_2(Y) = Y$, $\theta_2(t_{n+1}) = Y^{mM}t^{-b}$ induces an isomorphism of rings

$$U/\mathcal{K} \xrightarrow{\ \overline{\theta}_2\ } S/\mathcal{L}$$

where $\mathfrak{K} = (t_1 - \varepsilon_0 t_{n+1}, t_{n+1}t^b - Y^{mM})$. Let $P = A[Y, t_1, t_{j+1}, \ldots, t_n]$. Then the homomorphism $\theta_3 : U \longrightarrow P$ defined by $\theta_3(t_\ell) = t_\ell$, for $\ell \neq n+1$, $\theta_3(Y) = Y$, $\theta_3(t_{n+1}) = \varepsilon_0^{-1} t_1$ induces an isomorphism of rings

$$U/\mathfrak{K} \xrightarrow{\overline{\theta}_3} P/\mathfrak{H}$$

where $\mathfrak{H} = (t_1 t^b - \varepsilon_0 Y^{mM})$. Thus $h_{j+1}^{(m)}$ is not in any associated prime of $\mathfrak{A}_j$ in R if and only if $\overline{\theta}_3\overline{\theta}_2^{-1}\overline{\theta}_1(\overline{h}_{j+1}^{(m)}) = \overline{t}_{j+1} - \overline{\varepsilon}_{j+1}\overline{\varepsilon}_0^{-1}\overline{t}_1$ is not in any associated prime of O in $P/\mathfrak{H}$. This in turn holds if and only if $t_{j+1} - \varepsilon_{j+1}\varepsilon_0^{-1}t_1$ is not in any associated prime of $\mathfrak{H}$ in P, a u.f.d. Let

$$t_1 t^b - \varepsilon_0 Y^{mM} = \varepsilon \prod_{i=1}^{\ell} p_i(t,Y)^{r_i}$$

be the factorization into relatively prime factors in P, ε a unit in P, $r_i > 0$. Then the ideals $(p_i(t,Y))$ are the associated prime of $\mathfrak{H}$ in P. Suppose

$$t_{j+1} - \varepsilon_{j+1}\varepsilon_0^{-1}t_1 \in (p_i(t,Y)) \quad \text{for some } i .$$

Then since $t_{j+1} - \varepsilon_{j+1}\varepsilon_0^{-1}t_1$ is clearly irreducible,

$$t_{j+1} - \varepsilon_{j+1}\varepsilon_0^{-1}t_1 = \varepsilon' p_i(t,Y)$$

where ε' is a unit in P. Thus there is a polynomial $f(t,Y) \in P$ such that

$$(t_{j+1} - \varepsilon_{j+1}\varepsilon_0^{-1}t_1) f(t,Y) = t_1 t^b - \varepsilon_0 Y^{mM} .$$

Specializing $t_{j+1} \longrightarrow \varepsilon_{j+1}\varepsilon_0^{-1}t_1$ in both sides yields a contradiction. ∎

Let $W_m = \Omega[Y, t_1, \ldots, t_n, Y^{mM}t^{-a}]$, $\mathfrak{A}_m = (H_1^{(m)}, \ldots, H_n^{(m)})$. For $k \in m^{-1}M^{-1}\mathbb{N}$, let $W_m^{(k)}$ be the finite dimensional Ω-subspace of W_m spanned by monomials $Y^\gamma t^\alpha$ satisfying $(\alpha;\gamma) \in S_m$ and $w_m(Y^\gamma t^\alpha) = k$. Let

(3.2)
$$\mathfrak{A}_m^{(k)} = \mathfrak{A}_m \cap W_m^{(k)} ,$$
$$V_m^{(k)} = V_m(b) \cap W_m^{(k)} .$$

We claim :

(3.3) $$\mathfrak{A}_m^{(k)} \oplus V_m^{(k)} = W_m^{(k)} .$$

Without the assertion of directness, the claim is a corollary of the results of the previous section. To see directness, we note more generally :

(3.4) THEOREM. $V_m(b) \cap \sum_{i=1}^{n} \hat{H}_i^{(m)} L_m(b) = 0$.

<u>Proof</u> : Assume $v(Y,t) \in V_m(b)$ and $v(Y,t) = \sum_{i=1}^{n} \hat{H}_i^{(m)} \zeta_i$ with $\zeta_i \in L_m(b)$. Then ζ_i converges for

$$(Y,t) \in \{\text{ord } t_i > -b \; ; \; \text{ord } Y^{mM} - \sum_{i=1}^{n} a_i \text{ ord } t_i > -b\} = G_m(b) .$$

In particular, consider

(*) $$t^a v(Y,t) = t^a \sum_{i=1}^{n} \hat{H}_i^{(m)} \zeta_i$$

and set $t_i = (c_i a_1)^{-1} a_i c_1 t_1$.

Set Y equal to a unit u in Ω that does not trivialize the left side of (*), which then becomes a non-trivial polynomial in t_1 of degree at most N-1. However, the right side converges and in fact vanishes for each of the N distinct roots $t_1 \in \Omega$ of

$$t_1^N = u^{mM} (a_1/c_1)^N \prod_{j=1}^{n} (c_j/a_j)^{a_j} .$$

As consequence, both side of (*) vanish identically. ∎

Using these results and the argument of 1, § 3 we obtain the following results

(3.5) THEOREM. <u>Assume</u> $(p,M) = 1$, $\frac{p}{p-1} \geqslant b$.

$$L_m(b,c) = V_m(b,c) + \sum_{i=1}^{n} H_i^{(m)} L_m(b,c+e) . \quad ∎$$

(3.6) THEOREM. <u>Assume</u> $(p,M) = 1$, $\frac{p}{p-1} \geqslant b$.

$$V_m(b) \cap \sum_{i=1}^{n} H_i^{(m)} L_m(b) = (0). \qquad ■$$

(3.7) THEOREM. *Assume* $(p,M) = 1$, $\frac{p}{p\ 1} \geq b > \frac{1}{p-1}$.

$$L_m(b,c) = V_m(b,c) + \sum_{i=1}^{n} D_i^{(m)} L_m(b,c+e).$$

In fact, if $\xi \in L_m(b,c)$, *and*

$$\xi = \tilde{v} + \sum_{i=1}^{n} H_i^{(m)} \tilde{\zeta}_i$$

with $\tilde{v} \in V_m(b,c)$, $\tilde{\zeta}_i \in L_m(b,c+e)$ *as in* (3.5), *then we may express*

$$\xi = v + \sum_{i=1}^{n} D_i^{(m)} \zeta_i$$

with $v \in V_m(b,c)$, $\zeta_i \in L_m(b,c+e)$, $\tilde{v} - v \in V_m(b,c+e)$, $\tilde{\zeta}_i - \zeta_i \in L_m(b,c+2e)$.

■

(3.8) THEOREM. *Assume* $(p,M) = 1$, $\frac{p}{p-1} \geq b > \frac{1}{p-1}$.
Let $A \subseteq \{1,2,\ldots n\}$. *If* $\{\zeta_i\}_{i\ A} \subseteq L_m(b)$ *satisfy*

$$\sum_{i \in A} D_i^{(m)} \zeta_i = 0$$

then there exists a skew-symmetric set $\{\eta_{ij}\}_{i,j\in A} \subseteq L_m(b)$ *indexed by* A *such that*

$$\zeta_i = \sum_{j\in A} D_j^{(m)} \eta_{ij}. \qquad ■$$

(3.9) THEOREM. *Assume* $(p,M) = 1$, $\frac{p}{p-1} \geq b > \frac{1}{p-1}$. *Then*

$$V_m(b) \cap \sum_{i=1}^{n} D_i^{(m)} L_m(b) = (0). \qquad ■$$

We wish to compare reduction modulo the submodule $\sum_{i=1}^{n} H_i^{(m)} L_m(b,c+e)$ (respectively, the submodule $\sum_{i=1}^{n} D_i^{(m)} L_m(b,c+e)$) with reduction modulo $\sum_{i=1}^{n} \hat{H}_i^{(m)} L_m(b,c+e)$ studied in section 2. We now specialize our considerations to the case $b = p/(p-1)$, $e = 1$.

(3.10) LEMMA. The following relation holds

$$\hat{H}_i^{(m)} = H_i^{(m)} G_i^{(m)} + \Gamma_i^{(m)}$$

where $\Gamma_i^{(m)} \in L_m(\frac{p}{p-1},0)$, $G_i^{(m)} \in L_m(\frac{p}{p-1})$ is invertible and both $G_i^{(m)}$ and $G_i^{(m)^{-1}}$ belong to $L_m(\frac{p}{p-1},0)$.

Proof : By definition, $\hat{H}_i^{(m)} = \gamma_0(c_i t_i - a_i Y^{mM} t^{-a})$ with $a_i \in \mathbb{N}^*$, $c_i \in \Omega_0$, $c_i^q = c_i$. Thus

$$H_i^{(m)} = \sum_{\ell=0}^{\infty} p^\ell \gamma_\ell (c_i^{\tau^\ell} t_i^{p^\ell} - a_i^{\tau^\ell} Y^{mMp^\ell} t^{-ap^\ell}) .$$

$$= \sum_{\ell=0}^{\infty} p^\ell \gamma_\ell ((c_i t_i)^{p^\ell} - (a_i Y^{mM} t^{-a})^{p^\ell})$$

$$+ \sum_{\ell=1}^{\infty} p^\ell \gamma_\ell (a_i^{\tau^\ell} - a_i^{p^\ell}) Y^{mMp^\ell} t^{-ap^\ell} .$$

Consider

$$\tilde{\Gamma}_i^{(m)} = \sum_{\ell=1}^{\infty} p^\ell \gamma_\ell (a_i^{\tau^\ell} - a_i^{p^\ell}) Y^{mMp^\ell} t^{-ap^\ell} ,$$

$$\tilde{G}_i^{(m)} = 1 + \sum_{\ell=1}^{\infty} \gamma_0^{-1} \gamma_\ell p^\ell \sum_{j=0}^{p^\ell - 1} (c_i t_i)^j (a_i Y^{mM} t^{-a})^{p^\ell - j - 1} .$$

Using the fact $a_i \in \mathbb{Z}_p$, we get p divides $a_i^{\tau^\ell} - a_i^{p^\ell}$ so that $\tilde{\Gamma}_i^{(m)} \in L_m(\frac{p}{p-1},0)$. Similarly $\tilde{G}_i^{(m)} \in L_m(\frac{p}{p-1},0)$.

Finally we note that a series of the form

$$\lambda = 1 - \sum_{w_m(\alpha;\gamma)>0} c(\alpha;\gamma) Y^\gamma t^\alpha \in L_m(b,0)$$

is a unit in $L_m(b)$, with $\lambda^{-1} \in L_m(b,0)$, for the series

$$\sum_{j=0}^{\infty} \Big(\sum_{w_m(\alpha;\gamma)>0} c(\alpha;\gamma) Y^{\gamma} t^{\alpha} \Big)^j$$

is defined, belongs to $L_m(b,0)$, and is an inverse to λ in $L_m(b)$. This shows that $H_i^{(m)} = \hat{H}_i^{(m)} \tilde{G}_i^{(m)} + \tilde{\Gamma}_i^{(m)}$. The lemma follows by solving for $\hat{H}_i^{(m)}$ in terms of $H_i^{(m)}$. ∎

(3.14) THEOREM. Let $\xi \in L_m(\frac{p}{p-1},c)$. Then ξ may be expressed by (2.8) in the form

$$\xi = \hat{v} + \sum_{i=1}^{n} \hat{H}_i^{(m)} \hat{\zeta}_i$$

with $\hat{v} \in V_m(\frac{p}{p-1},c)$, $\hat{\zeta}_i \in L_m(\frac{p}{p-1},c+1)$, and ξ may also be expressed by (3.5) in the form

$$\xi = \tilde{v} + \sum_{i=1}^{n} H_i^{(m)} \tilde{\zeta}_i$$

with $\tilde{v} \in V_m(\frac{p}{p-1},c)$, $\tilde{\zeta}_i \in L_m(\frac{p}{p-1},c+1)$. Then

$$\hat{v} - \tilde{v} \in V_m(\tfrac{p}{p-1},c+1) .$$

Furthermore $\hat{\zeta}_i$ and $\tilde{\zeta}_i$ may be chosen so that

$$\tilde{\zeta}_i - G_i^{(m)} \hat{\zeta}_i \in L_m(\tfrac{p}{p-1},c+2) .$$

Proof : (Cf. [1, lemma (3.6)]). Assume $\xi^{(\ell)} \in L_m(\frac{p}{p-1},c+\ell)$. By (2.8), we may write

$$\xi^{(\ell)} = \hat{v}^{(\ell)} + \sum_{i=1}^{n} \hat{H}_i^{(m)} \hat{\zeta}_i^{(\ell)}$$

with $\hat{v}^{(\ell)} \in V_m(\frac{p}{p-1},c+\ell)$, $\hat{\zeta}_i^{(\ell)} \in L_m(\frac{p}{p-1},c+\ell+1)$.

Then by (3.10),

$$\xi^{(\ell)} = \hat{v}^{(\ell)} + \sum_{i=1}^{n} H_i^{(m)} (G_i^{(m)} \hat{\zeta}_i^{(\ell)}) + \xi^{(\ell+1)} , \tag{3.15}$$

where $\xi^{(\ell+1)} = \sum_{i=1}^{n} \Gamma_i^{(m)} \hat{\zeta}_i^{(\ell)} \in L_m(b,c+\ell+1)$.

Summing (3.15) from $\ell = 0$ to $\ell = K$ and letting $K \longrightarrow \infty$ we obtain the result. ∎

Combining the above result (3.14) with (3.7) we get the following result.

(3.16) THEOREM. Assume $(p,M) = 1$. Let $\xi \in L_m(\frac{p}{p-1},c)$ be expressed by (2.8) as

$$\xi = \hat{v} + \sum_{i=1}^{n} \hat{H}_i^{(m)} \hat{\zeta}_i$$

with $\hat{v} \in V_m(\frac{p}{p-1},c)$, $\hat{\zeta}_i \in L_m(\frac{p}{p-1},c+1)$ and by (3.7) as

$$\xi = v + \sum_{i=1}^{n} D_i^{(m)} \zeta_i$$

with $v \in V_m(\frac{p}{p-1},c)$, $\zeta_i \in L_m(\frac{p}{p-1},c+1)$. Then

$$v - \hat{v} \in V_m(\frac{p}{p-1},c+1)$$

and ζ_i and $\hat{\zeta}_i$ may be chosen so that

$$\zeta_i - G_i^{(m)} \hat{\zeta}_i \in L_m(\frac{p}{p-1},c+2) . \qquad ∎$$

4. SPECIALIZATION.

The previous sections establish the cohomology in the "generic" case. In order to draw arithmetic consequences concerning generalized hyperkloosterman sums, we will need to specialize $L_m(b,c)$ by setting $Y \longrightarrow y$ where $y \in \Omega^*$ satisfies ord $y > - NbM^{-1}m^{-1}$.

(4.1) DEFINITIONS. Assume $x \in \Omega_0^*$, ord $x > - Nb$. Define

(i) $L(x,b,c) = \{ \sum_{\alpha \in \mathbb{Z}^n} A(\alpha) t^\alpha | A(\alpha) \in \Omega_0, \text{ord} A(\alpha) \geq c + w(\alpha) b + s(\alpha) \text{ord } x \}$;

(ii) $L(x,b) = \bigcup_{c \in \mathbb{R}} L(x,b,c)$;

(iii) $\hat{H}_{i,x} = \hat{H}_i(x,t)$; $H_{i,x} = H_i(x,t)$;

(iv) $D_{i,x} = E_i + H_{i,x}$;

(v) $V = \Omega_0$ - span of $\{t^{\sigma(\mu)} \mid \sigma(\mu) \in \tilde{\Delta}\}$

(vi) $V(x,b,c) = V \cap L(x,b,c)$.

Given $x \in \Omega_0^*$, ord $x^m > -Nb$, we fix $y \in \Omega^*$ with $y^M = x$. Let $V_m(b,c)'$, $L_m(b)'$, $R_m(b)'$, $L(x,b,c)'$, $V(x,b,c)'$, $L(x,b)'$, V', be defined exactly as their unprimed counterpart but where the coefficients are allowed to lie in $\Omega_0' = \Omega_0(y)$.

We then define an Ω_0' - linear specialization map S_y (by sending $Y \longrightarrow y$) on various of the space of §§ 1-3 having targets as follows :

$$
(4.2)\qquad
\begin{aligned}
&S_y|_{L_m(b,c)'} &: L_m(b,c)' &\longrightarrow L(x^m,b,c)' \\
&S_y|_{L_m(b)'} &: L_m(b)' &\longrightarrow L(x^m,b)' \\
&S_y|_{R_m(b)'} &: R_m(b)' &\longrightarrow \Omega_0' \\
&S_y|_{V_m(b)'} &: V_m(b)' &\longrightarrow V' \\
&\sum_{i=1}^{n} D_i^{(m)} L_m(b)' & &\longrightarrow \sum_{i=1}^{n} D_{i,x^m} L(x^m,b)'
\end{aligned}
$$

we can also define an Ω_0'-linear section ℓ_y by sending

$$
(4.3)\qquad \ell_y : \sum_{\alpha \in \mathbb{Z}^n} A(\alpha)\, t^\alpha \longrightarrow \sum_{\alpha \in \mathbb{Z}^n} \frac{A(\alpha)\, Y^{mMs(\alpha)}\, t^\alpha}{x^{ms(\alpha)}} .
$$

Clearly $S_y \circ \ell_y = 1$ in the above cases so that the maps S_y in (4.2) are all surjective. The following result describes the kernel of the map S_y .

(4.4) THEOREM . Let x and y be as above . Then

$\ker S_Y|_{L_m(b,c)'} = (Y-y)L_m(b,c-\text{ord}\,y)'$. Thus $\ker S_Y|_{L_m(b)'} = (Y-y)L_m(b)'$, $\ker S_Y|_{R_m(b)'} = (Y-y)R_m(b)'$, and $\ker S_Y|_{V_m(b)'} = (Y-y)V_m(b)'$.

<u>Proof</u> : Let $\xi = \sum_{(\alpha;\gamma)\in S_m} A(\alpha;\gamma)Y^\gamma t^\alpha \in L_m(b,c)'$ and assume $S_y(\xi) = 0$. For each $\alpha \in \mathbb{Z}^n$, we must have

$$\sum_{\gamma \geq mMs(\alpha)} A(\alpha;\gamma)y^\gamma = 0 .$$

Since $y \neq 0$, we may divide by $y^{mMs(\alpha)}$, so that

$$\sum_{\gamma\geqslant 0} A(\alpha;\gamma+mMs(\alpha))y^\gamma = 0 .$$

Thus,

$$\xi = \sum_{\alpha\in\mathbb{Z}^n} \Big(\sum_{\gamma\geqslant 0} A(\alpha;\gamma+mMs(\alpha))(Y^\gamma - y^\gamma)\Big)Y^{mMs(\alpha)}t^\alpha$$

$$= (Y-y) . \sum_{\alpha\in\mathbb{Z}^n} \Big(\sum_{\gamma\geqslant 0} A(\alpha;\gamma+mMs(\alpha)) \sum_{\lambda=0}^{\gamma-1} Y^\lambda y^{\gamma-1-\lambda}\Big)Y^{mMs(\alpha)}t^\alpha$$

and one checks easily that the second factor on the right belongs to $L_m(b,c-\text{ord}\,y)'$, since $NbM^{-1}m^{-1} + \text{ord}\,y > 0$. ∎

By (3.8), the operators $D_i^{(m)}$ form an R-sequence (in any order) on the $R_m(b)$-module $L_m(b)$. We recall the following standard result on Koszul complexes [8, Ch. 8, theorem 7].

(4.5) Let E be an R-module, $\{\delta_j\}_{j=1}^s$ central elements of R. If δ_i is not a zero divisor on $E/\sum_{j=1}^{i-1}\delta_j E$, then $H_\mu(\{\delta_j\}_{j=1}^s|E) = 0$ for all $\mu > 0$.

In particular, setting $R = R_m(b)$, $E = L_m(b)$, $\delta_j = D_j^{(m)}$, this implies the following result.

(4.6) THEOREM. <u>Assume</u> $(p,M) = 1$, $\frac{p}{p-1} \geqslant b > \frac{1}{p-1}$.
<u>Then</u>

$$H_\mu(\{D_j^{(m)}\}_{j=1}^n | L_m(b)) = 0$$

for $\mu > 0$. (*Similarly* $H_\mu(\{D_j^{(m)}\}|L_m(b)') = 0$ *for* $\mu > 0$). ∎

Set $\delta_{n+1} = Y-y$.

(4.7) LEMMA. *Assume* $(p,M) = 1$, $\frac{p}{p-1} \geq b > \frac{1}{p-1}$. *Then* $\delta_{n+1} = Y-y$ *is not a zero divisor on* $L_m(b)'/\sum_{i=1}^n D_i^{(m)} L_m(b)'$.

Proof : Assume

$$(*) \qquad (Y-y)\xi = \sum_{i=1}^n D_i^{(m)} \zeta_i$$

where $\xi \in L_m(b,c)'$, $\zeta_i \in L(b)'$. Then by (3.7)

$$(**) \qquad \xi = v + \sum_{i=1}^n D_i^{(m)} \eta_i$$

where $v \in V_m(b,c)'$. $\eta_i \in L_m(b,c+e)'$. Thus (*), (**), and (3.9) imply that $v = 0$ which completes the proof of the lemma. ∎

As a consequence of the lemma and (4.5) we obtain :

(4.8) THEOREM. *Assume* $(p,M) = 1$, $\frac{p}{p-1} \geq b > \frac{1}{p-1}$.
Then $H_\mu(\{\delta_j\}_{j=1}^{n+1}|L_m(b)') = 0$ *for* $\mu > 0$, *where* $\delta_j = D_j^{(m)}$ *for* $1 \leq j \leq n$ *and* $\delta_{n+1} = Y-y$. ∎

We recall the following result [8, Ch. 8, theorem 4] on Koszul complexes :

(4.9) Let E be an R-module, $\{\delta_j\}_{j=1}^s$ central elements of R. If δ_s is not a zero divisor on E then there is an isomorphism of R-modules

$$H_\mu(\{\delta_i\}_{i=1}^s|E) \cong H_\mu(\{\delta_i\}_{i=1}^{s-1}|E/\delta_s E)$$

for all $\mu \geq 0$.

It is an immediate corollary of (4.3) and (4.4) that $L_m(b)'/(Y-y)L_m(b)' \cong L(x^m,b)'$ (where $y^M = x$) and thus there is an isomorphism over Ω_0' for all $\mu \geq 0$,

(4.10) $H_\mu(\{\delta_i\}_{i=1}^n | L_m(b)'/\delta_{n+1}L_m(b)') \cong H_\mu(\{D_{i,x^m}\}_{i=1}^n | L(x^m,b)')$.

Combining (4.8), (4.9) and (4.10) yields

(4.11) THEOREM. *Assume* $(p,M) = 1$, $\frac{p}{p-1} \geq b > \frac{1}{p-1}$, $x \in \Omega_0$, ord $x^m > -Nb$. *Then*

$$H_\mu(\{D_{i,x^m}\}_{i=1}^n | L(x^m,b)') = 0$$

for $\mu > 0$, $m \geq 1$. ∎

It remains to examine $H_0 = L(x^m,b)'/\sum_{i=1}^n D_{i,x^m} L(x^m,b)'$, of the specialized complex, which by (4.9) and (4.10) is isomorphic as an Ω_0'-vector space to $L_m(b)'/((Y-y)L_m(b)' + \sum_{i=1}^n D_i^{(m)} L_m(b)')$.

(4.12) THEOREM. *Assume* $(p,M) = 1$, $\frac{p}{p-1} \geq b > \frac{1}{p-1}$.

Let $V_0' = \Omega_0'$ - *span of* $\{Y^{mMs(\mu)} t^{\sigma^{(\mu)}} | \sigma^{(\mu)} \in \tilde{\Delta}\}$.

Then

$$L_m(b)' = V_0' \oplus \{\sum_{i=1}^n D_i^{(m)} L_m(b)' + (Y-y)L_m(b)'\} .$$

Proof : Let $\xi \in L_m(b)'$. Then by (3.7),

$$\xi = v + \sum_{i=1}^n D_i^{(m)} \zeta_i$$

where $v \in V_m(b)'$, $\zeta_i \in L_m(b)'$. Since

$$v = \sum_{-N+1 \leq \mu \leq 0} a_\mu(Y) Y^{mMS(\mu)} t^{\sigma^{(\mu)}}$$

with $a_\mu(Y) \in R_m(b)'$, we obtain as in (4.4)

$$v = \sum_{-N+1 \leq \mu \leq 0} a_\mu(y) Y^{mMS(\mu)} t^{\sigma^{(\mu)}} + \delta$$

with $\delta \in \operatorname{Ker} S_y \cap V_m(b)' = (Y-y)V_m(b)'$. This establishes everything except for the directness of the sum above. Assume $v_0 \in V_0'$

$$v_0 = \sum_{-N+1 \leq \mu \leq 0} a_\mu Y^{mMS(\mu)} t^{\sigma^{(\mu)}} = \sum_{i=1}^{n} D_i^{(m)} \zeta_i + (Y-y)\eta$$

with ζ_i, $\eta \in L_m(b)'$. By (3.7), we have

$$\eta = v + \sum_{i=1}^{n} D_i^{(m)} \eta_i$$

with $v \in V_m(b)'$, $\eta_i \in L_m(b)'$. Using directness of the sum (3.9), we have

$$\sum_{-N+1 \leq \mu \leq 0} a_\mu Y^{mMS(\mu)} t^{\sigma^{(\mu)}} = (Y-y)v \ .$$

Applying S_y to both sides, and recalling $y \neq 0$, we get $a_\mu = 0$ for all $-N+1 \leq \mu \leq 0$ so that $v_0 = 0$. ∎

We summarize the above results in terms of the specialized Koszul complex.

(4.13) THEOREM. Assume $(p,M) = 1$, $\frac{p}{p-1} \geq b > \frac{1}{p-1}$. Assume $x \in \Omega_0^*$, $\operatorname{ord} x > -Nb$. Then

$$L(x,b,c) = V(x,b,c) \oplus \sum_{i=1}^{n} D_{i,x} L(x,b,c,+e)$$

so that

$$H_0(\{D_{i,x}\}_{i=1}^{n} | L(x,b)) \cong \operatorname{Span}_{\Omega_0}\{t^{\sigma^{(\mu)}} | \sigma^{(\mu)} \in \tilde{\Delta}\}$$

(as vector spaces over Ω_0) .

Furthermore, $H_\mu(\{D_{i,x}\}_{i=1}^{n} | L(x,b)) = 0$, for $\mu \geq 1$.

Proof : We emphasize that the above assertion has Ω_0 (not Ω_0') as the field of definition. First of all, we have already observed that $H_\mu(\{D_{i,x}\}_{i=1}^{n} | L(x,b)') = 0$ for $\mu \geq 1$, so that the last assertion follows from

(4.14) $$H_\mu(\{D_{i,x}\}_{i=1}^n | L(x,b)') = H_\mu(\{D_{i,x}\}_{i=1}^n | L(x,b)) \underset{\Omega_0}{\otimes} \Omega_0'$$

and the fact that $H_\mu(\{D_{i,x}\}_{i=1}^n | L(x,b))$ is a vector space over Ω_0. For the first assertion, we note that from (4.12) we can conclude

(4.15) $$L(x,b,c)' = V(x,b,c)' \oplus (\sum_{i=1}^n D_{i,x} L(x,b,c+e)') .$$

Let $\{\eta_i\}_{i=1}^K$ be a basis for Ω_0'/Ω_0 with $\eta_1 = 1$, $\{\eta_i\} \subseteq \mathcal{O}_0'$ (= ring of integers of Ω_0'), and the property that if

$$\omega = \sum \omega_i \, \eta_i , \quad \omega_i \in \Omega_0$$

then ord $\omega_1 \geq$ ord ω. For example, if $e(\Omega_0'/\Omega_0) = s$, $f(\Omega_0'/\Omega_0) = f$ π' a uniformizer for Ω_0', $\{\bar{\zeta}_1, \bar{\zeta}_2, \ldots, \bar{\zeta}_f\}$ a basis for $\bar{\Omega}_0'/\bar{\Omega}_0$, with $\bar{\zeta}_1 = \bar{1}$, $\zeta_1 = 1$, ζ_i an arbitrary lifting of $\bar{\zeta}_i$ for $i \neq 1$, then as is well-known $\{\zeta_i(\pi')^j\}_{i=1,\ldots,f\,;\, j=1,2,\ldots,s}$ is a basis for Ω_0'/Ω_0. But then if

$$\omega = \sum_{i=1}^f (\sum_{j=1}^s \omega_{ij}(\pi')^j)\zeta_i$$

we obtain from the linear independence of the $\bar{\zeta}_i$'s that

ord $\omega = \inf_i \text{ord}(\sum_{j=1}^s \omega_{ij}(\pi')^j)\}$. From the fact that π' is a uniformizer for Ω_0' and $e(\Omega_0'/\Omega_0) = s$, we obtain

$$\text{ord}(\sum_{j=1}^s \omega_{ij}(\pi')^j) = \inf_{1 \leq j \leq s} \text{ord}(\omega_{ij}(\pi')^j) .$$

Thus ord $\omega_{1,0} \geq$ ord ω as required.

Now if $\xi \in L(x,b,c)$, then by (4.15), we may conclude that

(4.16) $$\xi = \sum_{\sigma(\mu) \in \tilde{\Delta}} v_\mu t^{\sigma(\mu)} + \sum_{i=1}^n D_{i,x} \sum_{\alpha \in \mathbb{Z}^n} A_i(\alpha) t^\alpha$$

where v_μ, $A_i(\alpha) \in \Omega_0'$ and ord $A_i(\alpha) \geq c+e+w(\alpha)b + s(\alpha)\text{ord } x$. Writing

$$v_\mu = \sum_{j=1}^K v_\mu^{(j)} \eta_j ,$$

$$A_i(\alpha) = \sum_{j=1}^{K} A_i^{(j)}(\alpha)\eta_j$$

we obtain since ξ is defined over Ω_0

$$(4.17) \qquad \xi = \sum_{\sigma^{(\mu)}\in\tilde{\Delta}} v_\mu^{(1)} t^{\sigma^{(\mu)}} + \sum_{i=1}^{n} D_{i,x} \sum_{\alpha\in\mathbb{Z}^n} A_i^{(1)}(\alpha)t^\alpha .$$

Since ord $v_\mu^{(1)} \geqslant$ ord v_μ , ord $A_i^{(1)}(\alpha) \geqslant$ ord $A_i(\alpha)$ we obtain from (4.17) that

$$L(x,b,c) = V(x,b,c) + \sum_{i=1}^{n} D_{i,x}L(x,b,c+e) .$$

The directness of this sum is as immediate consequence of (4.15). We observe that (4.16) also implies

$$(4.18) \qquad \sum_{\sigma^{(\mu)}\in\tilde{\Delta}} v_\mu^{(j)} t^{\sigma^{(\mu)}} = - \sum_{i=1}^{n} D_{i,x} \sum_{\alpha\in\mathbb{Z}^n} A_i^{(j)}(\alpha)t^\alpha$$

which by directness gives that both sides of (4.18) are zero, hence we have the additional information that in the reductions (4.16) and (4.17)

$$(4.19) \qquad v_\mu = v_\mu^{(1)} . \qquad ■$$

5. FROBENIUS MAP.

Let $q = p^r$. In the present section, we apply the previous results to the study of the Kloosterman-like exponential sums :

$$(5.1) \qquad S_m(\bar{f}(x,t)) = \sum \Psi\circ Tr_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\bar{f}(x,\bar{t})) ,$$

where $\bar{f}(\bar{x},\bar{t})$ is the reduction of (1.5) (with $Y=\bar{x}$), and where the outer sum on the right runs over $t = (t_1,t_2,\dots,t_n) \in (\mathbb{F}^*_{q^m})^n$; Ψ is an arbitrary non-trivial additive character of $\mathbb{F}_q$; and $\{\bar{c}_i\}_{i=1}^n \cup \{x\} \subseteq \mathbb{F}_q^*$. Let c_i, x denote Teichmuller liftings of $\bar{c}_i$ and $\bar{x}$ ($c_i^q = c_i$, $x^q = x$). Set $\Omega_1 = \mathbb{Q}_p(\zeta_p)$; K_r = the unramified extension of $\mathbb{Q}_p$ in Ω of degree r ; $\mathcal{T}$ = the completion of the maximal unramified extension of $\mathbb{Q}_p$ in Ω ; $\Omega_0 = K_r(\zeta_p)$. Note c_i, $x \in \Omega_0$. We will choose Ψ special - we assume that $\Psi = \oplus \circ \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$, where $\oplus$ is a non-trivial additive character of $\mathbb{F}_p$. However, as $\bar{b}$ runs over $\mathbb{F}_q^+$, $\oplus \circ \bar{b}$ runs overs the non-trivial additive characters of $\mathbb{F}_q$; since our results will be independent of the constants $\{\bar{c}_i\}_{i=1}^n$, there is no loss of generality in choosing Ψ special - thus, results (5.31) and (5.46) are independent of the choice of non-trivial additive characters Ψ of $\mathbb{F}_q$.

Let $E(z) = \exp(\sum_{j=0}^{\infty} z^{p^j}/p^j)$ be the Artin-Hasse exponential series; fix $\gamma \in \mathbb{Q}_p(\zeta_p)$, ord $\gamma = 1/(p-1)$, satisfying $\sum_{j=0}^{\infty} \gamma^{p^j}/p^j = 0$. Dwork calls the function

$$\Theta_\infty(z) = E(\gamma z) \tag{5.2}$$

a splitting function. As a power series in z,

$$\Theta_\infty(z) = \sum_{m=0}^{\infty} B_m z^m \tag{5.3}$$

with ord $B_m \geq m/(p-1)$, for all $m \geq 0$; and $B_m = \gamma^m/m!$, for $0 \leq m \leq p-1$. In terms of Θ_∞ we define

$$F_0(Y,t) = \Theta_\infty(Yt^{-a}) \prod_{i=1}^{n} \Theta_\infty(c_i t_i), \tag{5.4}$$

so that

$$F_0(Y^{mM},t) \in L_m(\tfrac{1}{p-1},0). \tag{5.5}$$

In terms of F_0 we define

$$F(Y,t) = \prod_{j=0}^{r-1} F_0^{\tau^j}(Y^{p^j},t^{p^j}), \tag{5.6}$$

so that

$$F(Y^{mM},t) \quad L_m(\tfrac{p}{q(p-1)},0).$$

We also define an $R_1(b)$-linear map ψ,

$$\psi : L_1(b,c) \longrightarrow L_p(pb,c),$$

defined on monomials by

(5.7) $$\psi(t^{\alpha}) = \begin{cases} t^{\alpha/p}, & \text{when } p|\alpha_i,\ 1 \leqslant i \leqslant n\ ; \\ 0, & \text{otherwise}\ ; \end{cases}$$

and extended "linearly" to arbitrary elements of $L_1(b,c)$. Then for $\frac{p}{p-1} \geqslant b > \frac{1}{p-1}$,

(5.8) $$\alpha_{Y^M} = \psi^r \circ F(Y^M,t) \ : \ L_1(b,c) \longrightarrow L_q(b,c)$$

is an $R_1(b)$-linear map. Choose $y \in \Omega$ such that $y^M = x$. By definition, $S_y \circ \psi = \psi \circ S_y$, so that

(5.9) $$S_y \circ \alpha_{Y^M} = \alpha_x \circ S_y$$

where $\alpha_x = \psi^r \circ F(x,t)$ acts on $L(x,b,c)$. The significance of the map α_x arises from the Dwork trace formula. Let

(5.10) $$L(\overline{f},T) = \exp\Big(\sum_{m=1}^{\infty} S_m(\overline{f})T^m/m\Big)$$

be the L-function associated with the exponential sum (5.1). Then a consequence of Dwork's trace formula is

(5.11) $$\det(I - T\alpha_x)^{\delta^n} = L(\overline{f},T)^{(-1)^{n+1}}$$

where $\det(I-T\alpha_x)$ denotes the Fredholm determinant of the completely continous endomorphism α_x acting on $L(x,b)$, and where δ acts on $g(T) \in 1 + T\Omega[[T]]$ by $g(T)^{\delta} = g(T)/g(qT)$.

We now fix the choice of constants (1.6), $\{\gamma_j\}_{j=0}^{\infty}$, by setting

(5.12) $$\begin{cases} \gamma_0 = \gamma \\ \gamma_j = \sum_{\ell=0}^{j} \gamma^{p^\ell}/p^\ell = - \sum_{\ell=j+1}^{\infty} \gamma^{p^\ell}/p^\ell\ . \end{cases}$$

We recall [1, § 4] that

$$\begin{cases} F_0(Y^M,t) = \hat{F}(Y^M,t)/\hat{F}^{\tau}(Y^{Mp},t^p)\ ; \\ F(Y^M,t) = \hat{F}(Y^M,t)/\hat{F}(Y^{Mq},t^q)\ , \end{cases}$$

where $\hat{F}$ is given in (1.7). The following commutativity relation may then be derived

$$(5.13)\qquad \alpha_{Y^M}\circ D_i^{(1)} = qD_i^{(q)}\circ\alpha_{Y^M}\ .$$

Relation (5.13) specializes via S_Y to

$$(5.14)\qquad \alpha_x\circ D_{i,x} = qD_{i,x}\circ\alpha_x\ .$$

As a consequence, α_x acts on the Koszul complex $K(\{D_{i,x}\}_{i=1}^n \mid L(x,b))$ yielding via the results of the previous sections and [1, § 4],

$$(5.15)\qquad \det(I - T\overline{\alpha}_x) = L(\overline{f}(\overline{x},\overline{t}),T)^{(-1)^{n+1}}$$

where $\overline{\alpha}_x$ is the map α_x acting on the quotient

$$\mathcal{W}_x\ (= H_O(\{D_{i,x}\}_{i=1}^n \mid L(x,b)) = L(x,b) \Big/ \sum_{i=1}^n D_{i,x}L(x,b)\ .$$

Note that if

$$(5.16)\qquad \begin{cases} \alpha_{Y^M}^{(0)} = \hat{F}^\tau(Y^{pM},t)^{-1}\circ\psi\circ\hat{F}(Y^M,t) = \psi\circ F_O(Y^M,t) \\ \alpha_x^{(0)} = S_Y\circ\alpha_{Y^M}^{(0)} = \hat{F}^\tau(x^p,t)^{-1}\circ\psi\circ\hat{F}(x,t) = \psi\circ F_O(x,t) \end{cases}$$

then $\alpha_{Y^M}^{(0)}$ is an $R_1(b)$ linear map from $L_1(b)$ to $L_p(b)$, and $\alpha_x^{(0)}$ is an Ω_O-linear map,

$$\begin{cases} \alpha_{Y^M}^{(0)} : L_1(b,c) \longrightarrow L_p(pb,c), \\ \alpha_x^{(0)} : L(x,b,c) \longrightarrow L(x^p,pb,c), \end{cases}$$

$(\frac{p}{p-1} \geqslant b > \frac{1}{p-1})$ satisfying

$$(5.17)\qquad \begin{cases} \alpha_{Y^M}^{(0)}\circ D_i^{(1)} = pD_{i,\tau}^{(p)}\circ\alpha_{Y^M}^{(0)} \\ \alpha_x^{(0)}\circ D_{i,x} = pD_{i,x^p}^{(\tau)}\circ\alpha_x^{(0)} \end{cases}$$

in which $D_{i,\tau}^{(m)} = E_i + H_i^\tau(Y^{Mm},t)$, $D_{i,x}^{(\tau)} = E_i + H_i^\tau(x,t)$. Thus $\alpha_{Y^M}^{(0)}$ and $\alpha_x^{(0)}$ define obvious quotient maps

$$(5.18)\quad \begin{cases} \alpha^{(0)}_{Y^M} : \mathcal{W}_Y \longrightarrow \mathcal{W}^{(\tau)}_{Y^p} \left(=L_p(b) \Big/ \sum_{i=1}^{n} D^{(p)}_{i,\tau} L_p(b)\right) , \\ \alpha^{(0)}_x : \mathcal{W}_x \longrightarrow \mathcal{W}^{(\tau)}_{x^p} \left(=L(x^p,b) \Big/ \sum_{i=1}^{n} D^{(\tau)}_{i,x^p} L(x^p,b)\right) . \end{cases}$$

The following factorizations hold

$$(5.19)\quad \begin{cases} \bar{\alpha}_{Y^M} = \bar{\alpha}^{(0)}_{Y^{Mq/p}} \circ \dots \circ \bar{\alpha}^{(0)}_{Y^{Mp}} \circ \bar{\alpha}^{(0)}_{Y^M} \\ \bar{\alpha}_x = \bar{\alpha}^{(0)}_{x^{q/p}} \circ \dots \circ \bar{\alpha}^{(0)}_{x^p} \circ \bar{\alpha}^{(0)}_x \quad . \end{cases}$$

Finally for $x \in \mathcal{T}(\zeta_p)$, ord $x \geqslant 0$, $\tau(x) = x^p$, we can define

$$(5.20)\qquad \tau^{-1} : \mathcal{W}^{(\tau)}_{x^p} \longrightarrow \mathcal{W}_x$$

by sending $\xi = \sum_\alpha A(\alpha)t^\alpha \in L(x^p,b,c)$ into

$$\tau^{-1}(\xi) = \sum_\alpha \tau^{-1}(A(\alpha))t^\alpha \in L(x,b,c) \ ;$$

clearly

$$\tau^{-1}\left(\sum_{i=1}^{n} D^{(\tau)}_{i,x^p} L(x^p,b)\right) \subset \sum_{i=1}^{n} D_{i,x} L(x,b) ,$$

so τ^{-1} is defined on the quotient. In the rest of this section, we fix $b = p/(p-1)$, $e = 1$. Let $x = x^q$, $q = p^r$; let

$$(5.21)\qquad \bar{\alpha}'_x = \tau^{-1} \circ \bar{\alpha}^{(0)}_x : \mathcal{W}_x \longrightarrow \mathcal{W}_x$$

a τ^{-1} (semi)-linear map. Then $\bar{\alpha}'_x$ is a completely continuous endomorphism of $L(x,\frac{p}{p-1})$ over $\Omega_1 = \mathbb{Q}_p(\xi_p)$, and

$$\bar{\alpha}_x = (\bar{\alpha}'_x)^r \ .$$

<u>Remark</u> : The following result of Dwork [2, lemma 7.1] will be instrumental in obtaining a general lower bound for the Newton polygon of $\det_{\Omega_0}(I - T\bar{\alpha}_x) (=L(\bar{f},T)^{(-1)^{n+1}})$.

(5.22) LEMMA (Dwork). Let $x^q = x$, $q = p^r$. The Newton polygon of $\det_{\Omega_0}(I - T\alpha_x)$ can be obtained from that of $\det_{\Omega_1}(I - T\bar{\alpha}'_x)$ by reducing both ordinates and absissas by the factor $1/r$ and interpreting the ordinates as normalized so that ord $q = 1$. ∎

We need estimates for the matrix of $\bar{\alpha}_x^{(0)}$.
Observe that

$$Y^{MS(\mu)} t^{\sigma^{(\mu)}} \in L_1\left(\frac{1}{p-1}, \frac{-W(\mu)}{p-1}\right), \tag{5.23}$$

for $\sigma^{(\mu)} \in \Delta$, so that $\alpha^{(0)}_{Y^M}(Y^{MS(\mu)} t^{\sigma^{(\mu)}}) \in L_p\left(\frac{p}{p-1}, \frac{-W(\mu)}{p-1}\right)$, and we may write

$$\alpha^{(0)}_{Y^M}(Y^{MS(\mu)} t^{\sigma^{(\mu)}}) = \sum_{\sigma^{(\nu)} \in \tilde{\Delta}} \tilde{A}_{\nu\mu}(Y) Y^{pMS(\nu)} t^{\sigma^{(\nu)}} + \sum_{i=1}^{n} D_{i,\tau}^{(p)} \zeta_i \tag{5.24}$$

with $\tilde{A}_{\nu,\mu}(Y) \in R_p\left(\frac{p}{p-1}, \frac{pW(\nu)-W(\mu)}{p-1}\right)$, and $\zeta_i \in L_p\left(\frac{p}{p-1}, \frac{-W(\mu)}{p-1} + 1\right)$.
Applying S_y to (5.24) and multiplying the result by $x^{-S(\mu)}$, we obtain

$$\alpha_x^{(0)}(t^{\sigma^{(\mu)}}) = \sum_{\sigma^{(\nu)} \in \tilde{\Delta}} A_{\nu,\mu}(y) x^{pS(\nu)-S(\mu)} t^{\sigma^{(\nu)}} + \sum_{i=1}^{n} D^{(\tau)}_{i,x^p}(S_y(\zeta_i) x^{-S(\mu)}. \tag{5.25}$$

Note that by the results of (4.12) and particularly (4.19),

$$\tilde{A}_{\nu,\mu}(y) x^{pS(\nu)-S(\mu)} \in \Omega_0$$

so that it makes sense to write

$$A_{\nu,\mu}(x) = \tilde{A}_{\nu,\mu}(y) x^{pS(\nu)-S(\mu)} \tag{5.26}$$

In particular, if $[\Omega'_0 : \Omega_0] = K$ and if $\tilde{A}_{\nu,\mu}(Y) = \sum_{j \geq 0} \tilde{A}^{(j)}_{\nu,\mu} Y^j, \tilde{A}^{(j)}_{\nu,\mu} \in \Omega_0$, then for $0 < \ell < K$,

$$\sum \tilde{A}^{(j)}_{\nu,\mu} Y^j = 0,$$

where the sum runs over $j \in \mathbb{N}$, $j + M(pS(\nu) - S(\mu)) \equiv \ell \pmod K$.
Furthermore, by the same reason,

$$(5.27) \qquad A_{\nu,\mu}(x) = \sum \tilde{A}^{(j)}_{\nu,\mu} Y^{j+M(pS(\nu)-S(\mu))}$$

where the sum runs over $j \in \mathbb{N}$, $j + M(pS(\nu) - S(\mu)) \equiv 0 \pmod K$.

One obtains the matrix of $\bar{\alpha}'_x$ by applying τ^{-1} to (5.25). Summarizing :

(5.28) THEOREM. Assume $(p,M) = 1$.

(i) Let $x \in \Omega^*$, $\operatorname{ord} x > -N/(p-1)$. Then $\bar{\alpha}^{(0)}_x : \mathcal{W}_x \longrightarrow \mathcal{W}^{(\tau)}_{x^p}$ is an Ω-linear map

$$\bar{\alpha}^{(0)}_x(\overline{t^{\sigma^{(\mu)}}}) = \sum_{\sigma^{(\nu)} \in \tilde{\Delta}} A_{\nu,\mu}(x)\overline{t^{\sigma^{(\nu)}}}$$

with matrix $A = (A_{\nu,\mu}(x))$, $A_{\nu,\mu} = S_Y(A_{\nu,\mu}(Y))x^{pS(\nu)-S(\mu)}$, with respect to the bases $\{\overline{t^{\sigma^{(\mu)}}} \mid \sigma^{(\mu)} \in \tilde{\Delta}\}$ of $\mathcal{W}_x$ and $\mathcal{W}^{(\tau)}_{x^p}$ respectively. For $\operatorname{ord} x > \frac{-N}{p-1}$, $\operatorname{ord} S_Y(A_{\nu\mu}(Y)) \geq (pW(\nu) - W(\mu))/(p-1)$. Thus, if $\operatorname{ord} x = 0$, then

$$\operatorname{ord} A_{\nu,\mu}(x) \geq \frac{pW(\nu) - W(\mu)}{p-1}$$

for $\sigma^{(\nu)}, \sigma^{(\mu)} \in \tilde{\Delta}$.

(ii) Let $x \in \mathcal{J}(\zeta_p)$, $\operatorname{ord} x = 0$, $\tau(x) = x^p$. Then $\bar{\alpha}'_x : \mathcal{W}_x \longrightarrow \mathcal{W}_x$ is a τ^{-1} (semi)-linear endomorphism

$$\bar{\alpha}'_x(\overline{t^{\sigma^{(\mu)}}}) = \sum_{\sigma^{(\nu)} \in \tilde{\Delta}} \mathfrak{A}_{\nu,\mu}(x)(\overline{t^{\sigma^{(\nu)}}})$$

with matrix $\mathfrak{A} = (\mathfrak{A}_{\nu,\mu}(x))$ with respect to the basis $\{\overline{t^{\sigma^{(\mu)}}} \mid \sigma^{(\mu)} \in \tilde{\Delta}\}$; $\mathfrak{A}_{\nu,\mu} = \tau^{-1}(A_{\nu,\mu}(x))$; $\operatorname{ord} \mathfrak{A}_{\nu,\mu} \geq (pW(\nu)-W(\mu))/(p-1)$ for $-N+1 \leq \mu,\nu \leq 0$. ∎

We now fix an integral basis $\{\eta_i\}_{i=1}^r$ of Ω_0/Ω_1 with the property that $\{\bar{\eta}_i\}_{i=1}^r$ is a basis for $\mathbb{F}_q/\mathbb{F}_p$. Then $\{\eta_i\}_{i=1}^r$ has the property that if $\omega \in \Omega_0$, $\omega = \sum_{i=1}^{r} \alpha_i \eta_i$, $\alpha_i \in \Omega_1$, then $\operatorname{ord} \omega = \inf_{1 \leq i \leq r} \{\operatorname{ord} \alpha_i\}$. Using this property, the following result is easily obtained.

(5.29) COROLLARY. <u>Assume</u> $(p,M) = 1$. <u>Let</u> $x \in \Omega_0^*$, $x^q = x$. <u>Thus</u> $\bar{\alpha}'_x : \mathcal{W}_x \longrightarrow \mathcal{W}_x$ <u>is an</u> Ω_1<u>-linear map</u>.

$$\bar{\alpha}'_x(\overline{\eta_i t^{\sigma^{(\mu)}}}) = \sum \mathcal{A}((\nu,j);(\mu,i))(\overline{\eta_j t^{\sigma^{(\nu)}}})$$

<u>with matrix</u> $\mathcal{A}' = (\mathcal{A}((\nu,j);(\mu,i))$ <u>with respect to the basis</u> $\{\overline{\eta_i t^{\sigma^{(\mu)}}} \mid -N+1 \leq \mu \leq 0 ;\ 1 \leq i \leq r\}$ <u>of</u> $\mathcal{W}_x$ <u>over</u> Ω_1. <u>The estimate</u>

$$\operatorname{ord} \mathcal{A}((\nu,j);(\mu,i)) > (pW(\nu) - W(\mu))/(p-1)$$

<u>holds for all entries of the matrix</u>. ∎

We now proceed as in [2, § 7] to estimate the Newton polygon of $\det_{\Omega_1}(I - T\bar{\alpha}_x^{-1}) = 1 + \sum_{j=1}^{rN} d_j T^j$. Here d_j is (up to sign) the sum of the $j \times j$ principal minors of the matrix $\mathcal{A}'$. As a consequence,

(5.30) $\operatorname{ord} d_j \geq \inf$ of all j-fold sums $\sum_{\ell=1}^{j} W(\nu_\ell)$ in which $\{(\nu_\ell, i_\ell)\}_{\ell=1}^{j}$ are j distinct elements in $\{(\nu,i) \mid -N+1 \leq \nu \leq 0 ;\ 1 \leq i \leq r\}$.

This yields

(5.31) THEOREM. <u>Assume</u> $(p,M) = 1$. Let $\tilde{H}(\bar{f},T) = \prod_{-N+1 \leq \mu \leq 0} (1 - q^{W(\mu)}T)$. <u>Then the Newton polygon of</u> $\det(I - T\bar{\alpha}_x)$ $(= L(\bar{f},T)^{(-1)^{n+1}})$ <u>lies over the Newton polygon of</u> $\tilde{H}(\bar{f},T)$. ∎

Thus $\tilde{H}(\bar{f},T)$ is a type of "Hodge" polygon for this example. Note that $\tilde{H}(\bar{f},T)$ is independent of the constants

$$\{\bar{c}_i\}_{i=1}^n \cup \{\bar{x}\} \subseteq \mathbb{F}_q^* .$$

We will now prove that when $p \equiv 1 \pmod M$ the Newton polygon of $L(\bar{f},T)^{(-1)^{n+1}}$ is equal to the Newton polygon of $\tilde{H}(\bar{f},T)$.

(5.32) THEOREM. <u>Assume</u> $(p,M) = 1$. <u>Write</u> (<u>using</u> (2.8))

$$\alpha^{(0)}_{Y^M}(Y^{MS(\mu)}t^{\sigma^{(\mu)}}) = \sum_{\sigma^{(\nu)}\in\tilde{\Delta}} \hat{A}_{\nu,\mu}(Y)Y^{pMS(\nu)}t^{\sigma^{(\nu)}} + \sum_{i=1}^n \hat{H}_i^\tau(Y^{pM},t)\hat{\zeta}_i ,$$

<u>with</u> $\hat{A}_{\nu,\mu}(Y) = \sum_{j=0}^\infty \hat{A}^{(j)}_{\nu,\mu}Y^j \in R_p(\frac{p}{p-1}, \frac{pW(\nu)-W(\mu))}{p-1}$, <u>and</u> $\hat{\zeta}_i \in L_p(\frac{p}{p-1}, -\frac{W(\mu)}{p-1}+1)$. <u>If</u> $\tilde{A}_{\nu,\mu}(Y)$ <u>is given by</u> (5.24), <u>then</u>

$$\hat{A}_{\nu,\mu}(Y) - \tilde{A}_{\nu,\mu}(Y) \in R_p(\frac{p}{p-1}, \frac{pW(\nu)-W(\mu)}{p-1}+1) .$$

<u>Furthermore if we write by abuse of notation</u>, $\sigma^{(\mu)} = (\mu_1,\dots,\mu_n)$, <u>then</u>

$$\hat{A}^{(j)}_{\nu,\mu} = \sum \tilde{v}(\beta,\gamma,j)B_\gamma \prod_{i=1}^n B_{p\beta_i-\mu_i+a_i\gamma}$$

<u>where</u> B_m <u>is defined in</u> (5.3), $\tilde{v}(\beta,\gamma,j)$ <u>is a unit in</u> Ω_0 <u>and the sum runs over</u> $\beta = (\beta_1,\dots,\beta_n) \in \mathbb{Z}^n$, $\gamma \in \mathbb{N}$ <u>satisfying</u> $\gamma = (\tilde{\gamma} - MS(\mu))M^{-1}$, (<u>where</u> $\tilde{\gamma} \in \mathbb{N}$, $\tilde{\gamma} \geq pMs(\beta)$), <u>and</u> $\Sigma(\beta) = N\tau_\beta + \nu$ <u>for some</u> $\tau_\beta \in \mathbb{Z}$, <u>and</u> $\tilde{\gamma} + p\tau_\beta M = j + pMS(\nu)$.

<u>Proof</u> : The first assertion is simply a statement of (3.16) in the constext of $\xi = \alpha^{(0)}_{Y^M}(Y^{MS(\mu)}t^{\sigma^{(\mu)}}) \in L_p(\frac{p}{p-1}, \frac{-W(\mu)}{p-1})$. To establish the second assertion, we may write by (5.4)

$$F_0(Y^M,t) = \sum_{(\alpha;M\gamma)\in S_1} v(\alpha;M\gamma)B(\alpha;M\gamma)Y^{M\gamma}t^\alpha$$

in which $v(\alpha,M\gamma)$ is a unit in Ω_0 and where

$$(5.33) \qquad B(\alpha;M\gamma) = B_\gamma \prod_{i=1}^{n} B_{\alpha_i + a_i\gamma}$$

with the B_m's defined in (5.3). The second assertion is then the explicit reduction of

$$\alpha^{(0)}_{Y^M}(Y^{MS(\mu)}t^{\sigma^{(\mu)}}) = \sum_{(\beta;\tilde{\gamma})\in S_p} B(p\beta - \sigma^{(\mu)}, \tilde{\gamma} - MS(\mu))Y^{\tilde{\gamma}}t^\beta$$

modulo $\sum_{i=1}^{n} \hat{H}_i^\tau(Y^{pM},t)L_p(\frac{p}{p-1})$ given in (2.8).

(5.34) THEOREM. <u>Assume</u> $(p,M) = 1$. <u>Then in</u> (5.32),

$$\operatorname{ord} \hat{A}^{(j)}_{\nu,\mu} \geqslant \frac{pW(\nu)-W(\mu)}{p-1} + \frac{jNM^{-1}}{p-1}$$

<u>for all</u> $j \geqslant 0$. <u>Furthermore, assume</u> $p \equiv 1 \pmod M$. <u>Then</u>

$$\operatorname{ord} \hat{A}^{(0)}_{\nu,\nu} = W(\nu) \;;$$

if $\mu \neq \nu$ and $W(\mu) \geqslant W(\nu)$, then

$$\operatorname{ord} \hat{A}^{(0)}_{\nu,\mu} > (pW(\nu) - W(\mu))/(p-1) .$$

<u>Proof</u> : Let

$$(5.35) \qquad b(\beta,\gamma,j) = \tilde{V}(\beta,\gamma,j)B(\alpha;M\gamma)$$

be a typical term in the sum for $\hat{A}^{(j)}_{\nu,\mu}$. Then

$$(5.36) \qquad \operatorname{ord} b(\beta,\gamma,j) \geqslant \frac{p\Sigma(\beta) - \mu + N\gamma}{p-1}$$

and by the conditions on $\Sigma(\beta)$ and γ,

$$p\Sigma(\beta) - \mu + N\gamma = pW(\nu) - W(\mu) + jNM^{-1} .$$

This proves the first assertion.

For the second assertion, we need a finer analysis of $b(\beta,\gamma,j)$. Observe that by definition, the indices $p\beta_i - \mu_i + a_i\gamma$ must be non-negative integers so that

$$(5.37) \qquad \gamma \geqslant s(p\beta - \sigma^{(\mu)}) .$$

Using $\Sigma(\beta) = N\tau + \nu$, (5.36) becomes (in the case $j=0$)

(5.38) $$\operatorname{ord} b(\beta,\gamma,0) \geqslant \frac{p\nu - \mu + pN\tau + N\gamma}{p-1}.$$

Consider first the case $\tau \geqslant 1$. Then $\operatorname{ord} b(\beta,\gamma,0) \geqslant \nu+N+(\nu-\mu+N)/(p-1)$. Using $0 < S(\nu) \leqslant 1$ for $-N+1 \leqslant \nu < 0$, we obtain for $\tau \geqslant 1$

$$\operatorname{ord} b(\beta,\gamma,0) \geqslant \frac{pW(\nu)-W(\mu)}{p-1} + \frac{N(1 - S(\nu) + S(\mu)}{p-1}) .$$

Thus, in the case $\tau \geqslant 1$, $\operatorname{ord} b(\beta,\gamma,0) \geqslant (pW(\nu) - W(\mu)/(p-1)$ and this inequality is strict unless the following situation holds

(5.39) $$\tau = 1, \quad \mu = 0, \quad S(\nu) = 1 .$$

Now assume $\tau \leqslant 0$. Since

$$\sum_{i=1}^{n} \beta_i = N\tau + \nu$$
$$= \sum_{i=1}^{n} (a_i\tau + \nu_i) + \tau ,$$

unless $\tau = 0$ and $\beta_i = \nu_i$ for all i, $1 \leqslant i \leqslant n$, there is some index i such that

$$\beta_i < a_i\tau + \nu_i .$$

But then by (5.37)

(5.40) $$\gamma \geqslant \frac{-p\beta_i + \mu_i}{a_i} \geqslant -p\tau + (p-1)\left(\frac{-\nu_i+1}{a_i}\right) + \left(\frac{-\nu_i+\mu_i+1}{a_i}\right) .$$

We claim that for any index i,

(5.41) $$\frac{-\nu_i+1}{a_i} \geqslant S(\nu) .$$

Suppose not, then $(-\nu_i+1)/a_i < S(\nu)$. We will show that this violates the definition of the diagonal sequence Δ, (1.10). Let $\sigma^{(\nu)} = -\sum_{\nu\leqslant j\leqslant -1} U_{g(j)}$; let $\ell = g(\nu)$. We will also establish for future use

(5.42) $$S(\nu) = -\nu_\ell/a_\ell \ .$$

Suppose $-\nu_\ell/a_\ell < S(\nu)$, and δ is the smallest index $\nu < \delta \leq 0$ such that $S(\sigma^{(\delta)}) < S(\nu)$. Since $\sigma^{(\nu)} = \sigma^{(\delta)} - \sum_{\nu \leq j \leq \delta-1} U_{g(j)}$, we obtain $\nu_i \leq \delta_i$ for all i, $1 \leq i \leq n$. In particular $\nu_\ell < \delta_\ell$, so that if $-\nu_\ell/a_\ell < S(\nu)$, then

$$s(\sigma^{(\delta)} - U_\ell) < S(\delta - 1) = S(\nu)$$

contradicting (1.9). Note that with δ defined as above and with the index i violating (5.41), then

$$s(\sigma^{(\delta)} - U_i) < S(\delta - 1) = S(\nu) \ .$$

But, this violates (1.9) again.

Now (5.40) and (5.41), imply

$$\gamma \geq -p\tau + (p-1)S(\nu) + \frac{\mu_i - \nu_i + 1}{a_i} \ .$$

If $\mu \geq \nu$, then $\sigma^{(\nu)} = \sigma^{(\mu)} - \sum_{\nu \leq j \leq \mu-1} U_{g(j)}$, so that $\nu_i \leq \mu_i$ for every index i. Thus $\gamma \in \mathbb{N}$ and our hypothesis $p \equiv 1 \pmod{M}$, imply that

$$\gamma \geq -p\tau + (p-1)S(\nu) + 1$$

which yields from (5.38)

$$\text{ord } b(\beta,\gamma,0) \geq \frac{pW(\nu)-W(\mu)}{p-1} + \frac{N(1-S(\nu)+S(\mu))}{p-1} \ .$$

Note that $1-S(\nu)+S(\mu) > 0$ in the case $\mu \geq \nu$. Thus we have strict inequality $\text{ord } b(\beta,\gamma,0) > (pW(\nu) - W(\mu))/(p-1)$ in the case $\tau \leq 0$, $\mu \geq \nu$, unless the following situation holds

(5.43) $$\tau = 0 \ ; \ \beta = \sigma^{(\nu)} \ .$$

We consider now the case (5.43). Then (5.37) implies

$$\gamma > (p-1) \frac{(-\nu_i)}{a_i} + \frac{\mu_i - \nu_i}{a_i}$$

for all i, $i = 1,\ldots,n$. If $\nu < \mu$, and $\ell = g(\nu)$, then we have seen above $\nu_\ell < \mu_\ell$. With (5.42), this implies

$$\gamma \geq (p-1)S(\nu) + 1 \ ,$$

so that by (5.38), we again deduce

$$\text{ord } b(\beta,\gamma,0) > \frac{pW(\nu)-W(\mu)}{p-1} .$$

This leaves the case $\nu = \mu$, $\tau = 0$, $\beta = \sigma^{(\nu)}$. Clearly (5.37) implies $\gamma \geqslant (p-1)S(\nu)$. Note that for $\gamma = (p-1)S(\nu)$, $\nu = \mu$,

$$p\nu_i - \mu_i + a_i\gamma = (p-1)(\nu_i+S(\nu)a_i) .$$

By (5.41), $0 \leqslant \nu_i + S(\nu)a_i \leqslant 1$. Thus by (5.3),

$$\text{ord } b(\sigma^{(\nu)},(p-1)S(\nu),0) = W(\nu) ;$$

clearly by (5.38)

$$\text{ord } b(\sigma^{(\nu)}\gamma) > W(\nu) \quad \text{for} \quad \gamma > (p-1)S(\nu) .$$

Consider now $\tau \leqslant 0$ and $\nu > \mu$. Since $-a_i \leqslant \mu_i$, for all i, (5.40) implies that

$$\gamma \geqslant -p\tau + (p-1)S(\nu) .$$

But then (5.38) yields

$$\text{ord } b(\beta,\gamma,0) \geqslant \frac{pW(\nu)-W(\mu)}{p-1} + \frac{N(S(\mu)-S(\nu))}{p-1} .$$

Since $0 \leqslant \mu < \nu \leqslant -N+1$, clearly $S(\mu) \geqslant S(\nu)$.
Thus

$$\text{ord } b(\beta,\gamma,0) > (pW(\nu)-W(\mu))/(p-1) ,$$

unless

(5.44) $$\nu > \mu \quad \text{and} \quad S(\mu) = S(\nu) .$$

In summary, $\text{ord } \hat{A}^{(0)}_{\nu,\nu} = W(\nu)$ and $\text{ord } \hat{A}^{(0)}_{\nu,\mu} \geqslant (pW(\nu)-W(\mu))/(p-1)$ and the inequality is a strict one unless (5.39) or (5.44) holds. In particular, we observe that $W(\mu) \geqslant W(\nu)$ precludes both (5.39) and (5.44). ∎

Our main result now follows :

(5.45) THEOREM. <u>Assume</u> $p \equiv 1 \pmod M$, $x \in \Omega_0^*$, $x^q = x$. <u>Then the following estimates hold for the entries of the matrix</u> $\mathcal{A} = (\mathcal{A}_{\nu,\mu})$ <u>of the semi-linear map</u> $\bar{\alpha}'_x : \mathcal{W}_x \longrightarrow \mathcal{W}_x$:

(i) ord $\mathcal{A}_{\nu,\mu} \geq (pW(\nu)-W(\mu))/(p-1)$,

(ii) ord $\mathcal{A}_{\nu,\nu} = W(\nu)$,

(iii) if $\mu \neq \nu$, and if $W(\mu) \geq W(\nu)$, then

ord $\mathcal{A}_{\nu,\mu} > (pW(\nu)-W(\mu))/(p-1)$.

Proof : This result is an immediate consequence of (5.32) and (5.34). ∎

(5.46) THEOREM. Assume $p \equiv 1 \pmod M$, $x \in \Omega_0^*$, $x^q = x$. Then the reciprocal zeros ω_μ , $-N+1 \leq \mu \leq 0$, of $L(\bar{f}(\bar{x},\bar{t}),T)^{(-1)^{n+1}}$ belong to $\mathbb{Q}_p(\zeta_p)$ and may be arranged so that

$$\mathrm{ord}_q\, \omega_\mu = W(\mu) .$$

In other words the Newton polygons of $L(\bar{f}(\bar{x},\bar{t}),T)^{(-1)^{n+1}}$ and $\tilde{H}(\bar{f}(\bar{x},\bar{t}),T)$ coincide.

Proof : It is useful to allow somewhat greater ramification than that of $\mathcal{J}(\zeta_p)$. Let π be a uniformizer in $\mathcal{J}(\zeta_p)$, and let $\mathcal{J}'$ be the extension of $\mathcal{J}(\zeta_p)$ defined by adjunction of a root say π' of $X^M-\pi$; extend τ to $\mathcal{J}'$ by setting $\tau(\pi') = \pi'$. Let $\mathcal{C} = \mathrm{diag}(\pi^{-W(0)},\ldots,\pi^{-W(-N+1)})$ be the $N\times N$ diagonal matrix with entries $\{\pi^{-W(\mu)}\}_{-N+1\leq\mu\leq 0}$ in the order shown. If $\mathcal{C}^\tau \mathcal{A} \mathcal{C}^{-1} = \mathcal{A}^{(1)} = (\mathcal{A}^{(1)}_{\nu,\mu})$, then

$$\mathcal{A}^{(1)}_{\nu,\mu} = \pi^{W(\nu)-W(\mu)}\mathcal{A}_{\nu\mu}$$

so that

$$\mathrm{ord}\ \mathcal{A}^{(1)}_{\nu,\mu} \geq W(\nu) ,$$

(5.47) $$\mathrm{ord}\ \mathcal{A}^{(1)}_{\nu,\nu} = W(\nu) ,$$

$$\mathrm{ord}\ \mathcal{A}^{(1)}_{\nu,\mu} > W(\nu),\ \text{if } \nu\neq\mu\ ,\ \text{and } W(\mu) \geq W(\nu) .$$

Now by [7, proposition 2.20], there exists an $N\times N$ matrix $\mathcal{M} \in GL(N,\mathcal{O}'_\infty)$, (where $\mathcal{O}'_\infty$ is the ring of integer of $\mathcal{J}'$) satisfying

(5.48) $$\mathcal{M}^{\tau}\, \mathcal{E}^{\tau}\, \mathcal{A}\, \mathcal{E}^{-1}\, \mathcal{M}^{-1} = \mathcal{E}^{-1} .$$

In fact, we can rework the proof of [7, proposition 2.20] to show the existence of an N×N matrix $\mathcal{Y} \in GL(N, \mathcal{O}_\infty)$ satisfying

(5.49) $$\mathcal{Y}^{\tau}\, \mathcal{A}\, \mathcal{Y}^{-1} = \mathcal{E}^{-1} .$$

Then the argument of [7, (2.28) and (2.29)] yields the additional information $\omega_\mu \in \mathbb{Q}_p(\zeta_p)$, $-N+1 \leqslant \mu \leqslant 0$. ∎

6. THE ONE-VARIABLE CASE.

The methods we have employed above can be used to treat somewhat more general exponential sums of a "Kloosterman type". In this section and the next, we will expand the set of examples we can study in this way. It seems likely that the same approach may be employed in the investigation of the exponential sums $S_m(\bar{f})$, (5.1), in the more general setting where $d \in \mathbb{N}^*$, and

(6.1) $$\bar{f}(\bar{x},t) = \sum_{i=1}^{n} \bar{c}_i t_i^d + \bar{x}\,\bar{t}^{-a} .$$

In fact, if $\bar{g} \in \mathbb{F}_q[\bar{t}_1,\ldots,\bar{t}_n]$ is any homogeneous polynomial of degree d, "regular" in Dwork's sense [1], then we believe that the exponential sums $S_m(\bar{f})$ with

(6.2) $$\bar{f}(\bar{x},\bar{t}) = \bar{g}(\bar{t}) + \bar{x}\,\bar{t}^{-a}$$

should be amenable to this type of analysis.

In this section we will restrict ourselves to the simpler one variable case. Let

(6.3) $$\bar{\hat{f}}(\bar{x},\bar{t}) = \bar{c}\,\bar{t}^{d_1} + \bar{x}\,\bar{t}^{-d_2}$$

with $\bar{c} \in \mathbb{F}_q^*$, $\bar{x} \in \bar{\mathbb{F}}_q^*$, $d_1, d_2 \in \mathbb{N}^*$ be the reduction of $\hat{f}(x,t)$ where $\hat{f}(Y,t) = ct^{d_1} + Yt^{-d_2}$ and $c = c^q$, $x = x^q$ belong to Ω_0.

We proceed as in §§ 1-3. Let $Y, t_1, \ldots, t_n$ be algebraically independent, $M = \text{l.c.m.}\{d_1, d_2\}$. Then we define for $\ell \in \mathbb{Z}$, $\gamma \in \mathbb{N}$

$$(6.4)\qquad \begin{cases} \hat{s}(\ell) = \max\{0, \frac{-\ell}{d_2}\} \\ \hat{w}(\ell) = \frac{\ell}{d_1} + (\frac{d_2}{d_1} + 1)s(\ell) = \max\{\frac{\ell}{d_1}, \frac{-\ell}{d_2}\} , \\ \hat{w}_m(\ell;\gamma) = \frac{\ell}{d_1} + (\frac{d_2}{d_1} + 1)\gamma m^{-1} M^{-1} . \end{cases}$$

Define

$$(6.5)\qquad L_m(b,c) = \{ \sum_{(\ell;\gamma)\in\hat{S}_m} A(\ell;\gamma)t^{\ell}Y^{\gamma} \mid A(\ell;\gamma) \in \Omega_0 , \quad \text{ord } A(\ell;\gamma) \geq c + b\hat{w}_m(\ell;\gamma) \} ,$$

where the index set $\hat{S}_m$ is given by

$$(6.6)\qquad \hat{S}_m = \{ (\ell;\gamma) \in \mathbb{Z} \times \mathbb{N} \mid \gamma \geq mM\hat{s}(\ell) \} .$$

We define $\hat{H}(Y,t)$, $H(Y,t)$, $\hat{F}(Y,t)$, $E_0 = t\frac{d}{dt}$, $\hat{H}_0^{(m)}$, $H_0^{(m)}$, $D_0^{(m)}$ as in (1.7) using $\hat{f}(Y,t)$ in place of $f(Y,t)$, and using the sequence $\{\gamma_\ell\}_{\ell\geq 0}$ given in (5.12). We define $R_m(b,c)$, $R_m(b)$, $L_m(b)$ just as in (1.4). Finally, we define

$$(6.7)\qquad \tilde{\Delta} = \{ \ell \mid -d_2 \leq \ell < d_1 \} .$$

$V_m(b)$ is the $R_m(b)$-span of

$$(6.8)\qquad \{ Y^{mM\hat{s}(\ell)} t^{\ell} \mid \ell \in \tilde{\Delta} \}$$

and

$$(6.9)\qquad V_m(b,c) = V_m(b) \cap L_m(b,c) .$$

The reduction formulas mod $\hat{H}_0^{(m)}$ are extremely simple :

$$(6.10)\qquad \begin{aligned} Y^{\gamma}t^{\ell} &= d_2 d_1^{-1} Y^{\gamma+mM} t^{\ell-d_1-d_2} + H_0^{(m)}(\gamma_0^{-1} d_1^{-1} Y^{\gamma} t^{\ell-d_1}) , \quad \text{if } \ell \geq d_1 , \\ Y^{\gamma}t^{\ell} &= -d_1 d_2^{-1} Y^{\gamma-mM} t^{\ell+d_1-d_2} - H_0^{(m)}(\gamma_0^{-1} d_2^{-1} Y^{\gamma-mM} t^{\ell+d_2}) , \quad \text{if } \ell < -d_2 . \end{aligned}$$

Thus we have

(6.11) THEOREM. Assume $(p,M) = 1$. Then for $\frac{p}{p-1} \geq b$,

$$L_m(b,c) = V_m(b,c) + \hat{H}_0^{(m)} L_m(b,c+e) .$$

If $\xi = \sum_{(\ell;\gamma)\in \hat{S}_m} A(\ell;\gamma)t^{\ell}Y^{\gamma} \in L_m(b,c)$ then $\xi = \hat{v} + \hat{H}_0^{(m)}\hat{\zeta}$ with $\hat{v} \in L_m(b,c)$, $\hat{\zeta} \in L_m(b,c+e)$, and

$$\hat{v} = \sum_{\ell \in \tilde{\Delta}} \hat{v}_{\ell}(Y) Y^{mM\hat{s}(\ell)} t^{\ell}$$

$$\hat{v}_{\ell}(Y) = \sum_{j \geq 0} \hat{v}_{\ell,j} Y^j ,$$

with explicit reduction formulas given by

$$(6.12) \qquad \hat{v}_{\ell,j} = \sum u(r,\alpha)A(r;\alpha)$$

where $u(r;\alpha)$ is a unit in Ω_0 and the sum in (6.12) runs over $(r;\alpha) \in \hat{S}_m$, $r = \ell+\tau(d_1+d_2)$, $\alpha = j + mM\hat{s}(\ell) - \tau mM$, $\tau \in \mathbb{Z}$. (Note that $(r;\alpha) \in \hat{S}_m$ implies the sum (6.12) is finite). ∎

It is also quite clear that if A is a noetherian, unique factorization domain, and we set $h_0^{(m)} = t^{d_1} - \varepsilon Y^{mM} t^{-d_2}$, with ε a unit in A, then $h_0^{(m)}$ is not a zero-divisor in $R = A[Y,t,Y^{mM}t^{-d_2}]$. This ensures the directness of the sum (6.11). We summarize :

(6.13) THEOREM. Assume $(p,M) = 1$, $\frac{p}{p-1} \geq b > \frac{1}{p-1}$ so that $e = b - \frac{1}{p-1} > 0$. Then

$$L_m(b,c) = V_m(b,c) \oplus D_0^{(m)} L_m(b,c+e) .$$

Furthermore, if we set $b = \frac{p}{p-1}$ and if we write $\xi \in L_m(\frac{p}{p-1},c)$ according to (6.11) as

$$\xi = \hat{v} + \hat{H}_0^{(m)}\hat{\zeta}$$

with $\hat{v} \in V_m(\frac{p}{p-1},c)$, $\hat{\zeta} \in L_m(\frac{p}{p-1},c+1)$, then we can express

$$\xi = v + D_0^{(m)}\zeta$$

<u>where</u> $v \in V_m(\frac{p}{p-1},c)$, $\zeta \in L_m(\frac{p}{p-1},c+1)$ <u>and</u> $v - \hat{v} \in V_m(\frac{p}{p-1},c+1)$. ∎

The situation above can be specialized. We observe that elements of $L_m(b,c)$ converge on the region

$$(6.14)\quad \hat{G}_m(b) = \{(t,Y) \in \Omega^2 \mid \text{ord } t > -bd_1^{-1},\ \text{ord } Y > -bd_1^{-1}(d_1+d_2)m^{-1}M^{-1},\ \text{ord } t - mMd_2^{-1}\,\text{ord } Y > db_2^{-1}\}.$$

We fix $x \in \Omega_0^*$, ord $x > -bd_1^{-1}(d_1+d_2)$, and $y \in \Omega$, $y^M = x$. Set $\Omega_0' = \Omega_0(y)$. We define

$$(6.15)\quad L(x,b,c) = \{\sum_{\ell \in \mathbf{Z}} A(\ell)t^\ell \mid A(\ell) \in \Omega_0,\ \text{ord } A(\ell) \geq c + \hat{w}(\ell)b + \hat{s}(\ell)\,\text{ord } x\}.$$

We will employ a dash to denote that the field of definition of a space under consideration has been extended from Ω_0 to Ω_0'. The specialization maps S_y, as in § 4, then are surjections. For example,

$$S_y : L_m(b,c)' \longrightarrow L(x^m,b,c)',$$

if ord $x^m > -bd_1^{-1}(d_1+d_2)$. We also define Frobenius maps α_{Y^M}, $\alpha_{Y^M}^{(0)}$, $\bar{\alpha}_x$, $\bar{\alpha}_x^{(0)}$, $\bar{\alpha}_x'$ as in § 5. Since $\alpha_{Y^M}^{(0)}(Y^{\hat{s}(j)M}t^j) \in L_p(\frac{p}{p-1},\frac{-\hat{w}(j)}{p-1})$, we may write

$$(6.16)\qquad \alpha_{Y^M}^{(0)}(Y^{\hat{s}(j)M}t^j) = \sum_{i \in \tilde{\Delta}} \tilde{A}_{ij}(Y)Y^{p\hat{s}(i)M}t^i + D_{0,p}^{(\tau)}\zeta$$

where $\zeta \in L_p(\frac{p}{p-1},\frac{-\hat{w}(j)}{p-1}+1)$, and

$$\sum_{i \in \tilde{\Delta}} \tilde{A}_{ij}(Y)Y^{p\hat{s}(i)M}t^i \in V_p(\frac{p}{p-1},\frac{-\hat{w}(j)}{p-1}).$$

(6.17) THEOREM. <u>Assume</u> $(p,M) = 1$.

(i) <u>Let</u> $x \in \Omega^*$, ord $x > -d_1^{-1}(d_1+d_2)/(p-1)$. <u>Then</u> $\bar{\alpha}_x^{(0)} : \mathcal{W}_x \longrightarrow \mathcal{W}_{x^p}^{(\tau)}$ <u>is an</u> Ω-<u>linear map</u>

$$\bar{\alpha}_x^{(0)}(\bar{t}^j) = \sum_{-d_2 \leq i < d_1} A_{ij}(x)\bar{t}^i$$

with matrix $A = (A_{ij}(x))$, $A_{ij}(x) = S_y(\tilde{A}_{ij}(Y))x^{ps(i)-s(j)}$, with respect to the bases $\{\bar{t}^i \mid -d_2 \leq i < d_1\}$ of $\mathcal{W}_x$ and $\mathcal{W}^{(\tau)}_{x^p}$ respectively. For $\operatorname{ord} x > \frac{-(d_1+d_2)}{d_1(p-1)}$, $\operatorname{ord} S_y(\tilde{A}_{ij}(Y)) \geq (p\hat{w}(i)-\hat{w}(j))/(p-1)$. Thus, if $\operatorname{ord} x = 0$,

$$\operatorname{ord} A_{ij}(x) \geq (p\hat{w}(i)-\hat{w}(j))/(p-1)$$

for $-d_2 \leq i,j < d_1$.

(ii) Let $x \in \mathcal{O}(\zeta_p)$, $\operatorname{ord} x = 0$, $\tau(x) = x^p$. Then $\bar{\alpha}'_x : \mathcal{W}_x \longrightarrow \mathcal{W}_x$ is a τ^{-1} (semi)-linear endomorphism

$$\bar{\alpha}'_x(\bar{t}^j) = \sum_{-d_2 \leq i < d_1} \mathcal{A}_{ij}(x)\bar{t}^i$$

with matrix $\mathcal{A} = (\mathcal{A}_{ij}(x))$ with respect to the basis $\{t^{-i} \mid -d_2 \leq i < d_1\}$ of $\mathcal{W}_x$; $\mathcal{A}_{ij}(x) = \tau^{-1}(A_{ij}(x))$;

$$\operatorname{ord} \mathcal{A}_{ij}(x) \geq (p\hat{w}(j)-\hat{w}(i))/(p-1)$$

for $-d_2 \leq i,j < d_1$. ∎

Just as in § 5, the above result yields.

(6.18) THEOREM. Assume $(p,M) = 1$, let $\hat{\bar{f}}(\bar{x},\bar{t}) = \bar{c}\bar{t}^{d_1} + \bar{x}\bar{t}^{-d_2} \in \mathbb{F}_q[\bar{t}]$. Let $\tilde{H}(\hat{\bar{f}},T) = \prod_{-d_2 \leq i < d_1} (1-q^{\hat{w}(i)}T)$. Then the Newton polygon of $L(\hat{\bar{f}},T)$ lies over the Newton polygon of $\tilde{H}(\hat{\bar{f}},T)$.

Just as in § 5, we will now show that when $p \equiv 1 \pmod M$ the Newton polygons of $L(\hat{\bar{f}},T)$ and of $\tilde{H}(\hat{\bar{f}},T)$ coincide. Just as § 5, the following holds.

(6.19) THEOREM. Assume $(p,M) = 1$. Write (using (6.11))

$$\alpha^{(0)}_{Y^M}(Y^{M\hat{s}(j)}t^j) = \sum_{-d_2 \leq i < d_1} \hat{A}_{ij}(Y)Y^{pM\hat{s}(i)}t^i + \hat{H}^{(\tau)}_0(Y^{pM},t)\hat{\zeta}$$

with $\hat{A}_{ij}(Y) = \sum_{\ell=0}^{\infty} \hat{A}_{ij}^{(\ell)} Y^{\ell} \in R_p(\frac{p}{p-1}, \frac{p\hat{w}(i)-\hat{w}(j)}{p-1})$, *and* $\hat{\zeta} \in L_p(\frac{p}{p-1}, -\frac{\hat{w}(j)}{p-1}+1)$. *If* $\tilde{A}_{ij}(Y)$ *is given by* (6.16) *then*

$$\hat{A}_{ij}(Y) - \tilde{A}_{ij}(Y) \in R_p(\frac{p}{p-1}, \frac{p\hat{w}(j)-\hat{w}(i)}{p-1} + 1) .$$

Furthermore,

$$\hat{A}_{ij}^{(\ell)} = \sum \tilde{u}(r,\gamma,\ell) B_{\gamma} B_{d_1^{-1}(pr-j+d_2\gamma)} \tag{6.20}$$

where $\tilde{u}(r,\gamma,\ell)$ *is a unit in* Ω_0 , *where* B_n *is defined in* (5.3), *and where the indices* γ *and* $d_1^{-1}(pr-j+d_2\gamma)$ *are both non-negative integers. The sum* (6.20) *runs over* $(r,\gamma) \in \mathbb{Z} \times \mathbb{N}$ *satisfying* $\gamma = (\tilde{\gamma} - Ms(j))M^{-1}$ (*where* $\tilde{\gamma} \in \mathbb{N}$, $\tilde{\gamma} \geq pM\hat{s}(r)$), *and* $r = i+\tau(d_1+d_2)$ *for some* $\tau \in \mathbb{Z}$, *and* $\tilde{\gamma} + \tau pM = \ell + pM\hat{s}(i)$. ∎

The key result then is the following.

(6.21) THEOREM. *Assume* $(p,M) = 1$. *Then in* (6.19)

$$\text{ord}\ \hat{A}_{ij}^{(\ell)} \geq \frac{p\hat{w}(i)-\hat{w}(j)}{p-1} + \frac{\ell M^{-1}d_1^{-1}(d_1+d_2)}{p-1}$$

for all $\ell \geq 0$. *Furthermore, if* $p \equiv 1 \pmod{M}$,

$$\text{ord}\ \hat{A}_{ii}^{(0)} = \hat{w}(i) ;$$

if $i \neq j$, *then*

$$\text{ord}\ \hat{A}_{ij}^{(0)} > (p\hat{w}(i)-\hat{w}(j))/(p-1) ,$$

except possibly for the case $i = -d_2$, $j \geq 0$.

Proof : Note that by (6.20),

$$\text{ord}\ \hat{A}_{ij}^{(\ell)} \geq d_1^{-1}(pr - j + (d_1+d_2)\gamma)/(p-1) .$$

Using the conditions on r, γ, and $\tilde{\gamma}$, we obtain the first assertion.

If we set $m_2 = \gamma$, $m_1 = d_1^{-1}(pr-j+d_2m_2)$, the situation appears more symmetric. Let

$$b(m_1,m_2) = B_{m_1}B_{m_2}$$

subject to the conditions :

$$(6.22) \qquad \begin{cases} m_1,m_2 \in \mathbb{N} \\ d_1m_1-d_2m_2 = pi - j+p\tau(d_1+d_2)\ , \quad \text{for} \quad \tau \in \mathbb{Z}\ . \end{cases}$$

Assume first that $i \geqslant 0$ so that $\hat{w}(i) = i/d_1$. Then (6.22) gives (since $m_2 \geqslant 0$)

$$(6.23) \qquad \begin{cases} m_1 \geqslant p(\hat{w})i - \dfrac{j}{d_1} + \dfrac{p\tau(d_1+d_2)}{d_1} \\[2ex] m_2 = \dfrac{d_1}{d_2}\, m_1 + \dfrac{-p\tau(d_1+d_2)-pi+j}{d_2}\ . \end{cases}$$

Therefore, if $\tau > 0$,

$$(6.24) \qquad m_1 + m_2 > p\hat{w}(i) - \hat{w}(j)\ .$$

Furthermore if $\tau = 0$, we also obtain (6.24) when $j < 0$. Suppose now that $j \geqslant 0$, and $\tau = 0$. Then

$$m_1 \geqslant p\hat{w}(i) - \hat{w}(j)$$

so (6.24) holds unless $m_1 = p\hat{w}(i) - \hat{w}(j)$ and $m_2 = 0$. But $0 \leqslant i,j < d_1$, together with $m_1 = (pi-j)/d_1 \in \mathbb{N}$, imply $i = j$. If $i = j \geqslant 0$, then

$$\operatorname{ord} b((p-1)i/d_1,0) = \hat{w}(i)\ .$$

We consider $i \geqslant 0$, $\tau < 0$. Then

$$\begin{aligned} m_1 + m_2 &= \frac{(d_1+d_2)m_1-p\tau(d_1+d_2)-pi+j}{d_2} \\ &\geqslant \frac{p(d_1+d_2)}{d_2} - \frac{pi}{d_2} + \frac{j}{d_2} \\ &> \frac{pi}{d_1} + \frac{j}{d_2} \geqslant p\hat{w}(i) - \hat{w}(j)\ , \end{aligned}$$

the strict inequality being a simple consequence of $i < d_1$. This completes the proof of the theorem in the case $i \geqslant 0$.

In the case $i < 0$, we interchange the roles of m_1 and m_2 in (6.23) and τ and $-\tau$ in the subsequent argument. We may then conclude that (6.24) holds in all cases except $\tau = 0$, $i = j$, (in which case $\text{ord } b(0,-(p-1)i/d_2) = \hat{w}(i))$, or when $\tau = 0$, $i = -d_2$, $j = 0$, or $\tau = 1$, $i = -d_2$, $j \geq 0$, (in which cases we only obtain the weak inequality $m_1+m_2 \geq p\hat{w}(i)-\hat{w}(j))$. ■

The following result is then an immediate consequence.

(6.25) THEOREM. Assume $p \equiv 1 \pmod M$, where $M = \text{l.c.m.}(d_1,d_2)$. Then $L(\overline{\hat{f}},T)$ is a polynomial of degree d_1+d_2, with reciprocal zeros $\{\omega_i\}_{i=-d_2}^{d_1-1}$ algebraic integers lying in $\mathbb{Q}_p(\zeta_p)$ which can be arranged so that

$$\text{ord}_q\ \omega_i = \hat{w}(i) .$$

In other words, provided $p \equiv 1 \pmod M$, the Newton polygons of $L(\overline{\hat{f}},T)$ and $\tilde{H}(\overline{\hat{f}},T)$ coincide.

7. FURTHER APPLICATIONS.

We will extend the results of the two previous sections. We consider here the two cases

$$(7.1)\quad \text{(i)}\quad \overline{h}(\overline{x},\overline{t}) = \overline{f}(\overline{x},\overline{t}) + \sum_{\alpha\in J} \overline{b}_\alpha \overline{t}^\alpha$$

$$\text{(ii)}\quad \overline{\hat{h}}(\overline{x},\overline{t}) = \overline{\hat{f}}(\overline{x},\overline{t}) + \sum_{i\in K} \overline{b}_i \overline{t}^i$$

where $\overline{f}(\overline{x},\overline{t})$ is the reduction modulo the maximal ideal of $\mathcal{O}_0$ of (1.5); $\overline{\hat{f}}(\overline{x},\overline{t})$ is given by (6.1) ; $\{\overline{b}_\alpha\}_{\alpha\in J} \cup \{\overline{b}_i\}_{i\in k} \subseteq \mathbb{F}_q^*$; J is a finite sum of monomials α satisfying

$$(7.2)\quad \text{(i)}\quad 0 < w(\alpha) < 1 ;$$

$K \subseteq \{i \in \mathbb{Z} \mid -d_2 < i < d_1\}$, and the inequalities

$$\text{(ii)}\quad 0 < \hat{w}(i) < 1 ,$$

are obvious for $i \in K$.

We give a few examples to show that the notion is not an empty one :

(7.3)

(i) $\bar{h}(t_1,t_2) = t_1+t_2+t_1^{-1}+t_1^{-2}t_2^{-1}+t_1^{-7}t_2^{-2}+t_1^{-10}t^{-3}$;

(ii) $\bar{h}(t_1,\dots,t_n) = \sum_{i=1}^{n} \bar{c}_i r_i + \sum_{j=1}^{m} \bar{b}_j t^{-aj}$,

where

$$a = (a_1,\dots,a_n) \in (\mathbb{N}^*)^n , \quad \{\bar{c}_i\}_{i=1}^{n} \cup \bar{b}_m \subsetneq \mathbb{F}_q^* ,$$

$$\{\bar{b}_j\}_{j=1}^{m-1} \subseteq \mathbb{F}_q .$$

We will work primarily with case (7.1)(i). Case (ii) is entirely analogous. Consider

$$h(x,t) = \sum_{i=1}^{n} c_i t_i + \sum_{\alpha\in J} b_\alpha t^\alpha + xt^{-a} \tag{7.4}$$

where c_i , b_α and x are Teichmüller units in Ω_0 , and the reduction modulo the maximal ideal of h is $\bar{h}(\bar{x},\bar{t})$. We define as usual

$$F_0^*(x,t) = \theta_\infty(xt^{-a}) \prod_{i=1}^{n} \theta_\infty(c_i t_i) \prod_{\alpha\in J} \theta_\infty(b_\alpha t^\alpha)$$

(7.5)

$$= \sum_{\gamma\in\mathbb{Z}^n} (u(\gamma) \sum B_\ell \prod_{i=1}^{n} B_{m_i} \prod_{\alpha\in J} B_{n_\alpha})t^\gamma$$

where $u(\gamma)$ is a unit for each $\gamma \in \mathbb{Z}^n$, and the inner sum runs over the set $\mathcal{J}(\gamma)$, where

(7.6)

$$\mathcal{J}(\gamma) = \{ (\{m_i\}_{i=1}^{n}, \{n_\alpha\}_{\alpha\in J}, \ell) \in \mathbb{N}^{n+|J|+1} \quad \text{such that}$$

$$m_i + \sum_{\alpha\in J} n_\alpha \alpha_i - a_i \ell = \gamma_i \quad \text{for} \quad i=1,2,\dots,n \}$$

where we have systematically written $\alpha = (\alpha_1,\dots,\alpha_n)$ for $\alpha \in \mathbb{Z}^n$. Let

$$F(\gamma) = u(\gamma) \sum_{\mathcal{J}(\gamma)} B_\ell \sum_{i=1}^{n} B_{m_i} \sum_{\alpha\in J} B_{n_\alpha} \tag{7.7}$$

then

$$\text{ord } F(\gamma) \geqslant \inf_{\mathcal{J}(\gamma)} \Big(\sum_{i=1}^{n} m_i + \sum_{\alpha\in J} n_\alpha + \ell\Big)/(p-1) .$$

Note that the requirement $w(\alpha) < 1$ implies that $\Sigma(\alpha) \leqslant 0$, because $s(\alpha) \geqslant 0$, and therefore $\Sigma(\alpha) \geqslant 1$ implies $w(\alpha) \geqslant 1$. Note also that for $(\{m_i\}_{i=1}^{n}, \{n_\alpha\}_{\alpha\in J}, \ell) \in \mathcal{J}(\gamma)$,

$$(7.8) \qquad \sum_{i=1}^{n} m_i + \sum_{\alpha\in J} n_\alpha + \ell = \Sigma(\gamma) + N\ell + \sum_{\alpha\in J} n_\alpha(1-\Sigma(\alpha)) .$$

We claim

$$(7.9) \qquad \inf_{\mathcal{J}(\gamma)} \{N\ell + \sum_{\alpha\in J} n_\alpha(1-\Sigma(\alpha))\} \geqslant Ns(\gamma) ,$$

where $s(\gamma)$ is defined in (1.1). Clearly the left side is non-negative. It remains to show that

$$(*) \qquad N\ell + \sum_{\alpha\in J} n_\alpha(1-\Sigma(\alpha)) \geqslant -N\gamma_i/a_i$$

for any i, and any $(\{m_i\}_{i=1}^{n}, \{n_\alpha\}_{\alpha\in J}, \ell) \in \mathcal{J}(\gamma)$. Substituting in (*) for γ_i from (7.6) the desired inequality will hold, provided

$$1 - \Sigma(\alpha) \geqslant - N\alpha_i/a_i$$

for any $\alpha \in J$. However $w(\alpha) < 1$, implies

$$\Sigma(\alpha) + Ns(\alpha) < 1$$

which completes the proof of (7.9) and the following result.

(7.10) THEOREM. _If_ J _is a subset of monomials_ α _satisfying_ $0 < w(\alpha) < 1$, _then_

$$F_0^*(x,t) \in L(x, \tfrac{1}{p-1}, 0)$$

where L _is defined in_ (4.1 (i)) . ∎

Set

$$(7.11) \qquad \mathcal{H}(x,t) = \sum_{\ell=0}^{\infty} \gamma_\ell h^{\tau^\ell}(x^{p^\ell}, t^{p^\ell})$$

where $\{\gamma_\ell\}_{\ell=0}^{\infty}$ is given by (5.12) ; let

$$(7.12) \qquad \mathcal{H}_{i,x} = E_i\, \mathcal{H}(x,t) = H_{i,x} + \Lambda_{i,x}$$

where $H_{i,x}$ is given by (4.1 (iii)) and

(7.13) $$\Lambda_{i,x} = \sum_{\ell=0}^{\infty} \gamma_\ell p^\ell \sum_{\alpha\in J} \alpha_i b_\alpha t^{\alpha p^\ell} .$$

Note that since $w(\alpha) \in M^{-1}\mathbb{N}$, and $w(\alpha) < 1$ for $\alpha \in J$, therefore

(7.14) $$w(\alpha) \leqslant 1 - \frac{1}{M}, \quad \text{for} \quad \alpha \in J .$$

It follows from (7.13) and (7.14) that for $b \leqslant \frac{p}{p-1}$

(7.15) $$\Lambda_{i,x} \in L(x,b,-e+e')$$

where $e' = b/M$; recall also that for $b \leqslant \frac{p}{p-1}$, $H_{i,x} \in L(x,b,-e)$. Define

(7.16) $$\mathcal{D}_{i,x} = E_i + \mathcal{H}_{i,x} = D_{i,x} + \Lambda_{i,x} .$$

It is not hard to prove that for ord $x > -Nb$, $(p,M) = 1$, the $\{\mathcal{D}_{i,x}\}_{i=1}^{n}$ form an R-sequence on $L(x,b)$ and furthermore that for $\frac{p}{p-1} \geqslant b > \frac{1}{p-1}$

$$L(x,b,c) = V(x,b,c) \oplus \sum_{i=1}^{n} \mathcal{D}_{i,x} L(x,b,c+e) .$$

From (7.16) we derive in the usual manner :

(7.17) THEOREM. <u>Assume</u> $(p,M) = 1$, $\frac{p}{p-1} \geqslant b > \frac{1}{p-1}$, ord $x > -Nb$. <u>If we express</u> $\xi \in L(x,b,c)$ <u>in the forms</u> $\xi = v + \sum_{i=1}^{n} D_{i,x}\zeta_i$ <u>and</u> $\xi = v^* + \sum_{i=1}^{n} \mathcal{D}_{i,x}\zeta_i^*$ <u>with</u> v, $v^* \in V(x,b,c)$, ζ_i, $\zeta_i^* \in L(x,b,c+e)$ <u>then</u>

$$v - v^* \in V(x,b,c,+e') . \qquad \blacksquare$$

Define as in § 5, $\alpha_x^{(0)*} = \psi\circ F_0^*(x,t)$, $\alpha_x'^{*} = \tau^{-1}\circ\psi\circ F_0^*(x,t)$, $\alpha_x^* = \psi^r\circ F^*(x,t)$ where $F^*(x,t) = \prod_{j=0}^{r-1} F_0^{*\tau^j}(x^{p^j}, t^{p^j})$. Define also

$$\mathcal{W}_x^* = L(x,b)/\sum_{i=1}^{n} \mathcal{D}_{i,x}L(x,b) , \quad \mathcal{W}_{x^p}^{(\tau)*} = L(x^p,b) / \sum_{i=1}^{n} \mathcal{D}_{i,x^p}^{(\tau)} L(x^p,b) .$$

We shall work exclusively with b fixed, $b = p/(p-1)$, $e = 1$, $e' = p/M(p-1)$.

Writing

$$(7.18) \qquad F_0^*(x,t) = \sum_{\gamma \in \mathbb{Z}^n} F(\gamma) t^\gamma$$

with $F(\gamma)$ defined in (7.7) above, we may write

$$(7.19) \qquad \alpha_x^{(0)*}(x^{S(\mu)} t^{\sigma^{(\mu)}}) = \sum_{\beta \in \mathbb{Z}^n} A(\beta) t^\beta \in L(x^p, \frac{p}{p-1}, -\frac{W(\mu)}{p-1})$$

where $A(\beta) = x^{S(\mu)} F(p\beta - \sigma^{(\mu)})$. We know

$$(7.20) \qquad \alpha_x^{(0)*}(x^{S(\mu)} t^{\sigma^{(\mu)}}) = S_Y\Big(\sum_{\beta \in \mathbb{Z}^n} A(\beta) x^{-ps(\beta)} Y^{pMs(\beta)} t^\beta \Big)$$

and by (2.8) we define $\hat{A}_{\nu\mu}^{(j)}$ by

$$(7.21) \qquad \sum_{\beta \in \mathbb{Z}^n} A(\beta) x^{-ps(\beta)} Y^{pMs(\beta)} t^\beta = \sum_{\sigma^{(\nu)} \in \tilde{\Delta}} \Big(\sum_{j=0}^{\infty} \hat{A}_{\nu,\mu}^{(j)} Y^j \Big) Y^{pMs(\nu)} t^{\sigma^{(\nu)}} + \sum_{i=1}^{n} \hat{H}_{i,p}^{(\tau)}(Y^{pM}, t) \zeta_i$$

where $\zeta_i \in L_p(\frac{p}{p-1}, \frac{-W(\mu)}{p-1} + 1)$, $\hat{A}_{\nu\mu}(Y) = \sum_{j=0}^{\infty} \hat{A}_{\nu,\mu}^{(j)} Y^j \in R_p(\frac{p}{p-1}, \frac{pW(\nu)-W(\mu)}{p-1})$ and we have explicit formulas from (2.8)

$$(7.22) \qquad \hat{A}_{\nu,\mu}^{(j)} = \sum A(\beta) w^{-ps(\beta)} u(\beta,j)$$

in which $u(\beta,j)$ is a unit in Ω_0 , and in which the sum runs over $\beta \in \mathbb{Z}^n$ where $\Sigma(\beta) = N\tau_\beta + \nu$, $pMs(\beta) + \tau_\beta pM = j + pMS(\nu)$. Thus $\hat{A}_{\nu,\mu}^{(j)} = \sum F(p\beta - \sigma^{(\mu)}) x^{S(\mu)-ps(\beta)} u(\beta,j)$ where β runs over the above index set.

(7.23) THEOREM. <u>Assume</u> $(p,M) = 1$, ord $x > -N/(p-1)$

$$\text{ord } \hat{A}_{\nu,\mu}^{(j)} > \frac{pW(\nu)-W(\mu)}{p-1} + \frac{jNM^{-1}}{p-1}$$

<u>for all</u> $j \geqslant 0$. <u>Furthermore</u>, <u>if</u> $p \equiv 1 \pmod M$,

$$\text{ord } \hat{A}^{(0)}_{\nu,\nu} = W(\nu) \;;$$

if $\mu \neq \nu$ <u>and</u> $W(\mu) \geqslant W(\nu)$ <u>then</u>

$$\text{ord } \hat{A}^{(0)}_{\nu,\mu} > (pW(\nu)-W(\mu)/(p-1)) .$$

<u>Proof</u> : Let

$$b((\{m_i\}_{i=1}^n,\{n_\alpha\}_{\alpha\in J},\ell) \;;\; j) = u(\beta,j)x^{S(\mu)-ps(\beta)}B_\ell \prod_{i=1}^n B_{m_i} \prod_{\alpha\in J} B_{n_\alpha}$$

for $(\{m_i\}_{i=1}^n, \{n_\alpha\}_{\alpha\in J},\ell) \in \mathcal{J}(p\beta-\sigma^{(\mu)})$ (where β satisfies the conditions in (7.22)) be a typical term in the series for $A^{(j)}_{\nu,\mu}$. Then the first assertion is a straighforward computation using ord $x > -N/(p-1)$. We obtain for $j = 0$,

$$(7.24) \quad \text{ord } b((\{m_i\}_{i=1}^n,\{n_\alpha\}_{\alpha\in J},\ell);0) \geqslant \frac{pW(\nu)-W(\mu)}{p-1} + \frac{\sum_{\alpha\in J} n_\alpha(1-\Sigma(\alpha))}{p-1} .$$

Since $\Sigma(\alpha) \leqslant 0$, we get a strict inequality in (7.23) if $n_\alpha \neq 0$ for any $\alpha \in J$. Therefore, we can restrict our attention to

$$(\{m_i\}_{i=1}^n \;,\; \{0\}_{\alpha\in J} \;,\; \ell) \in \mathcal{J}(p\beta-\sigma^{(\mu)})$$

where β satisfies the conditions of (7.21). Thus

$$b((\{m_i\}_{i=1}^n \;,\; \{0\}_{\alpha\in J} \;,\; \ell) \;;\; 0) = u(\beta;0)B_0^{|J|}B_\ell \prod_{i=1}^n B_{p\beta_i-\mu_i+a_i\ell}$$

by definition (7.6) of the set $\mathcal{J}$. But then the argument is clearly the same as (5.34). ∎

We can use the result above to obtain estimates for the reduction of $\alpha_x^{(0)*}(t^{\sigma^{(\mu)}})$ modulo $\sum_{i=1}^n \hat{H}^\tau_{i,x}L(x^p,b)$ (here $\hat{H}_{i,x}$ is given by (4.1)). By (7.20) and (7.21) we write

$$\alpha_x^{(0)*}t^{\sigma^{(\mu)}} = \sum_{\sigma^{(\nu)}\in\tilde{\Delta}} \hat{A}^*_{\nu,\mu}(x)t^{\sigma^{(\mu)}} + \sum_{i=1}^n \hat{H}^\tau_{i,x^p}\,\hat{\zeta}^*_i$$

where $\hat{\zeta}^*_i \in L(x^p \,,\, \frac{p}{p-1} \,,\, \frac{-W(\mu)}{p-1} - S(\mu) \text{ ord } x + 1)$.

(7.25) COROLLARY. <u>Assume</u> $(p,M) = 1$, ord $x > -N/(p-1)$

$$\text{ord } S_y(\hat{A}_{\nu\mu}(Y) \geq (pW(\nu)-W(\mu))/(p-1) .$$

<u>Assume in addition</u>, ord $x = 0$, <u>and set</u>

$$\hat{A}^*_{\nu,\mu}(x) = x^{pS(\nu)-S(\mu)} S_y(\hat{A}_{\nu,\mu}(Y)) .$$

<u>Then clearly</u> ord $\hat{A}^*_{\nu\mu}(x) \geq (pW(\nu)-W(\mu))/(p-1)$. <u>If we also assume</u> $p \equiv 1 \pmod{M}$, <u>then</u>

$$\text{ord } \hat{A}^*_{\nu\nu}(x) = W(\nu) ;$$

<u>if</u> $\mu \neq \nu$ <u>and</u> $W(\mu) \geq W(\nu)$, <u>then</u>

$$\text{ord } \hat{A}^*_{\nu,\mu} > (pW(\nu) - W(\mu))/(p-1) . \qquad \blacksquare$$

If we combine (7.17), (3.16) and (7.25), we obtain the following result.

(7.26) THEOREM. <u>Assume</u> $(p,M) = 1$.

(i) <u>Let</u> $x \in \mathcal{T}(\zeta_p)$, ord $x = 0$, $\tau(x) = x^p$. <u>Then</u> $\bar{\alpha}'^*_x = \tau^{-1} \circ \bar{\alpha}^{(0)*}_x : \mathcal{W}^*_x \longrightarrow \mathcal{W}^*_x$ is a τ^{-1} <u>semi-linear endomorphism</u>

$$\bar{\alpha}'^*_x(\bar{t}^{\sigma^{(\mu)}}) = \sum_{\sigma^{(\mu)} \in \tilde{\Delta}} \mathcal{A}^*_{\nu\mu}(x)\bar{t}^{\sigma^{(\mu)}}$$

<u>with matrix</u> $\mathcal{A}^* = (\mathcal{A}^*_{\nu,\mu}(x))$ <u>with respect to the basis</u> $\{\bar{t}^{\sigma^{(\mu)}} | \sigma^{(\mu)} \in \tilde{\Delta}\}$; <u>the estimate</u> ord $\mathcal{A}^*_{\nu\mu} \geq (pW(\nu)-W(\mu))/(p-1)$ <u>holds for all</u> $-N+1 \leq \nu,\mu \leq 0$.

(ii) <u>If</u> $p \equiv 1 \pmod{M}$, <u>then</u>

$$\text{ord } \mathcal{A}^*_{\nu\nu}(x) = W(\nu) ;$$

<u>if</u> $\mu \neq \nu$ <u>and</u> $W(\mu) \geq W(\nu)$, <u>then</u>

$$\text{ord } \mathcal{A}^*_{\nu\mu}(x) > (pW(\nu) - W(\mu))/(p-1) . \qquad \blacksquare$$

From this we deduce the usual consequences for the Newton polygon of $L(\bar{h},T)^{(-1)^{n+1}}$.

(7.27) THEOREM. Assume $(p,M) = 1$. The Newton polygon of $L(\overline{h},t)^{(-1)^{n+1}}$ lies over the Newton polygon of $\tilde{H}(\overline{f},T)$ (defined in (5.31)). If $p \equiv 1 \pmod M$, the reciprocal zeros of $L(\overline{h},T)^{(-1)^{n+1}}$ are algebraic integers in $\mathbb{Q}_p(\zeta_p)$ and the Newton polygons of $L(\overline{h},T)^{(-1)^{n+1}}$, $L(\overline{f},T)^{(-1)^{n+1}}$ and $\tilde{H}(f,T)$ all coincide. ∎

A similar result holds in case (ii) by applying the above technique. Robba [6] has obtained a similar result by using his index theory.

(7.28) THEOREM. Let $M = \text{l.c.m.}(d_1,d_2)$. Assume $(p,M) = 1$. The Newton polygon of $L(\overline{\hat{h}},T)^{(-1)^{n+1}}$ lies over the Newton polygon of $H(\overline{\hat{f}},T)$ defined in (6.17). If $p \equiv 1 \pmod M$ the reciprocal zeros of $L(\overline{\hat{h}},T)^{(-1)^{n+1}}$ are algebraic integers in $\mathbb{Q}_p(\zeta_p)$ and the Newton polygons of $L(\overline{\hat{h}},T)^{(-1)^{n+1}}$, $L(\overline{\hat{f}},T)^{(-1)^{n+1}}$, and $\tilde{H}(\overline{\hat{f}},T)$ all coincide. ∎

8. OTHER CONGRUENCE CLASSES.

In this section, we examine the situation when we drop the hypothesis $p \equiv 1 \pmod M$. We consider the exponential sums

$$S_m(\overline{f}) = \sum_{t \in \mathbb{F}_{q^m}^*} \Psi \circ Tr_{\mathbb{F}_{q^m}/\mathbb{F}_q}(\overline{f}(t))$$

where

$$\overline{f}(t) = \overline{\alpha}t + \overline{x}t^{-3} \tag{8.1}$$

$\overline{\alpha}, \overline{x} \in \mathbb{F}_q^*$, when $p \equiv 2 \pmod 3$. (The case $p \equiv 1 \pmod 3$ is already included in the results of § 5). The results of the present section show that for the example (8.1), the Newton polygon of the associated L-function when $p \equiv 2 \pmod 3$, $(p > 5)$ lies over the Newton polygon for the associated L-function when $p \equiv 1 \pmod 3$, and it approaches this Newton polygon in the limit when p varies in the congruence class $p \equiv 2 \pmod 3$ and $p \to \infty$.

It is convenient to introduce the following notation.

(8.2) $$\lambda(\nu) = \begin{cases} \text{least non-negative residue mod 3 of } p\nu, & \text{for } 0 \leqslant \nu \leqslant 2 ; \\ 3, & \text{for } \nu = 3. \end{cases}$$

$$\eta(\nu) = \frac{p\nu - \lambda(\nu)}{3}$$

$$\delta(\nu,\mu) = \text{least non-negative residue mod 4 of } \mu - \lambda(\nu) ;$$

$$\varepsilon(\nu,\mu) = \eta(\nu) + \delta(\nu,\mu) .$$

(8.3) LEMMA. *For* ord $x = 0$, $p > 5$, $p \equiv 2 \pmod 3$, $0 \leqslant \mu, \nu \leqslant 3$

$$\frac{pw(-\nu) - w(-\mu)}{p-1} + 1 > \frac{\varepsilon(\nu,\mu)}{p-1}$$

where $w(-\nu)$ *is defined in* (1.1).

Proof : Using the definitions in (8.2), and the definition of $w(-\nu) = \nu/3$ for $3 \geqslant \nu \geqslant 0$ (from (1.1)), the asserted inequality in the statement of the lemma is equivalent to

$$p-1 > \frac{\mu - \lambda(\nu)}{3} + \delta(\nu,\mu) ,$$

which holds by definition of $\delta(\nu,\mu)$ and our restriction to primes $p > 5$.

(8.4) THEOREM. *Let* α, x *be Teichmüller liftings of* $\bar{\alpha}, \bar{x}$. *If* $p > 5$, $p \equiv 2$ (mod 3), ord $x = 0$, *then the following estimates for the size of the entries of the Frobenius matrix hold*

(8.5) $$A_{-\nu,-\mu} = \pi^{\varepsilon(\nu,\mu)} \mathcal{a}_{\nu,\mu} , \quad 0 \leqslant \nu, \ \mu \leqslant 3,$$

where $\mathcal{a}_{\nu,\mu}$ *is an integer in* $\mathbb{Q}_p(\pi, x, \alpha)$, *and, in fact, is a unit for all pairs* (ν,μ) *with the possible exceptions of* (3,0), (3,1) *and* (3,2).

Proof : By lemma (8.3) and theorem (5.32), it is sufficient to prove the estimates (8.5) for the matrix entries $\hat{A}_{-\nu,-\mu}$ where $\hat{A}_{-\nu,-\mu}$ is defined by

$$\alpha_x^{(0)} t^{-\mu} = \sum \hat{A}_{-\nu,-\mu} t^{-\nu} \pmod{\hat{H}_{0,x}^{(\tau)} L(x,b)},$$

and where $\hat{A}_{-\nu,-\mu}$ is given explicity in the notation of theorem (5.32), by

$$\hat{A}_{-\nu,-\mu} = \sum_{\tau \in \mathbb{Z}} u(\tau,x) \sum B_{4p\tau-p\nu+\mu+3\gamma} B_\gamma \tag{8.6}$$

where $u(\tau,x)$ is a unit for all $\tau \in \mathbb{Z}$ and the inner sum runs over all non-negative integers γ such that

$$4p\tau - p\nu + \mu + 3\gamma \geqslant 0 .$$

For real x let $\langle x \rangle$ denote the smallest integer greater than or equal to x. For $\tau \leqslant 0$,

$$\begin{aligned} \gamma &\geqslant \max \{0 , \langle \tfrac{-4p\tau+p\nu-\mu}{3} \rangle \} \\ &= \max \{0 , -p\tau + \langle \tfrac{p\nu-\mu-p\tau}{3} \rangle \} . \end{aligned} \tag{8.7}$$

Let

$$b_{\nu,\mu}(\tau;\gamma) = u(\tau,x) B_\gamma B_{4p\tau-p\nu+\mu+3\gamma} \tag{8.8}$$

be a typical term in the series (8.6) for $\hat{A}_{-\nu,-\mu}$. Thus

$$\text{ord } b_{\nu,\mu}(\tau;\gamma) \geqslant \frac{4p\tau-p\nu+\mu+4\gamma}{p-1} . \tag{$*$}$$

Consider first the case $\tau > 0$. If $\tau \geqslant 2$ or if $\tau = 1$ and $\nu < 3$, we obtain from $(*)$

$$\text{ord } b_{\nu,\mu}(\tau;\gamma) > 1 + \frac{3}{p-1} .$$

If $\tau = 1$ and $\nu = 3$, we obtain from $(*)$

$$\text{ord } b_{\nu,\mu}(\tau;\gamma) \geqslant \frac{p+\mu}{p-1} \geqslant \frac{\varepsilon(3,\mu)}{p-1}$$

in which the last inequality is sharp if $\mu = 3$. If $\tau < 0$, or $\nu > 0$, or $\mu < 3$, then

$$\begin{aligned} &\max \{0 , -p\tau + \langle \tfrac{p\nu-\mu-p\tau}{3} \rangle \} \\ &= -p\tau + \langle \tfrac{p\nu-\mu-p\tau}{3} \rangle . \end{aligned}$$

Thus by (8.7),

$$4p\tau - p\nu + \mu + 4\gamma \geqslant -p\nu + \mu + 4 \left\langle \frac{p\nu-\mu}{3} \right\rangle$$
$$= \eta(\nu) + \mu - \lambda(\nu) + 4 \left\langle \frac{\lambda(\nu)-\mu}{3} \right\rangle$$

where the first inequality is sharp if $\tau < 0$ or if $\tau = 0$ and $\gamma > \left\langle \frac{p\nu-\mu}{3} \right\rangle$. Note that if $\nu \neq 0$ or if $\mu \neq 3$, then

$$\mu - \lambda(\nu) + 4 \left\langle \frac{\lambda(\nu)-\mu}{3} \right\rangle = \delta(\nu,\mu) ,$$

(since $-3 < \lambda(\nu) - \mu \leqslant 0$ gives $\mu-\lambda(\nu)$ on both sides, and $0 < \lambda(\nu) - \mu \leqslant 3$ gives $\mu - \lambda(\nu) + 4$ on both sides). In case $\tau = 0$ and $\gamma = \left\langle \frac{p\nu-\mu}{3} \right\rangle$, we obtain

$$\operatorname{ord} b_{\nu,\mu}(0;\gamma) = \frac{\eta(\nu)+\delta(\nu,\mu)}{p-1} .$$

In case $\nu = 0$ and $\mu = 3$, $b_{0,3}(0;0) = u(0,x)B_0B_3$ so that

$$\operatorname{ord} b_{0,3}(0;0) = \frac{\delta(0,3)}{p-1} . \qquad \blacksquare$$

Let $\bar{\alpha}_x^{(0)}$ denote the Frobenius map

$$\bar{\alpha}_x^{(0)} : \mathcal{W}_x \longrightarrow \mathcal{W}_{x^p}^{(\tau)} .$$

If $q = p^r$, we wish to compute the ord of the eigenvalues of the map $\alpha_x = (\tau^{-1} \circ \alpha_x^{(0)})^r$. We will modify the argument of [7, § 2] .

(8.9) THEOREM. <u>Let</u> $p > 5$, $p \equiv 2 \pmod 3$. <u>Let</u> $x \in \mathbb{F}_q^*$, $(q = p^r)$, <u>and let</u> x <u>be the Teichmüller lifting of</u> x <u>in</u> Ω. <u>Then</u>

(i) <u>for</u> $p > 11$, <u>the eigenvalues</u> $\{\omega_i\}_{i=0}^3$ <u>of the Frobenius map</u> $\bar{\alpha}_x$ <u>can be arranged so that</u>

$$\operatorname{ord}_q \omega_0 = 0 ,$$
$$\operatorname{ord}_q \omega_1 = \frac{1}{p-1} \{\eta(1) + 3\} ,$$
$$\operatorname{ord}_q \omega_2 = \frac{1}{p-1} \{\eta(2) - 3\} ,$$
$$\operatorname{ord}_q \omega_3 = 1 .$$

(ii) <u>For</u> $p = 11$, <u>the eigenbalues</u> $\{\omega_i\}_{i=0}^3$ <u>of the Frobenius map</u> $\bar{\alpha}_x$ <u>can be arranged so that</u>

$$\operatorname{ord}_q \omega_0 = 0 ,$$
$$\operatorname{ord}_q \omega_1 = \operatorname{ord}_q \omega_2 = \frac{1}{2} ,$$
$$\operatorname{ord}_q \omega_3 = 1 .$$

Proof : We will follow the argument of [7, § 2] working with the matrix $A = (A_{-\nu,-\mu})_{0\leq \nu,\mu\leq 3}$ of the Frobenius map $\bar{\alpha}_x^{(0)}$ with respect to the basis $\{t^{-\nu}\}_{\nu=0}^{3}$. We use an argument from (semi)-linear algebra via the usual formalism (cf. [3]). In both cases in the statement of (8.9), we can find a matrix $C \in GL(4,\mathcal{O}_\infty)$ (here $\mathcal{O}_\infty$ is the ring of integers in $\mathfrak{F}(\zeta_p)$, π is a uniformizer for $\mathcal{O}_\infty$) having the property that $C\,A\,(C^{-1})^\tau$ has the block form

$$\begin{bmatrix} 1 & 0 \\ 0 & A^{(1)} \end{bmatrix}$$

where $A^{(1)}$ is a 3×3 matrix with entries in $\mathcal{O}_\infty$ and

$$A^{(1)}_{\nu,\mu} \equiv A_{-\nu,-\mu} \bmod \pi^{\varepsilon(\nu,\mu)+1}$$

for any $1 \leq \nu,\ \mu \leq 3$. We will use the notation $X^{-\tau}$ to denote $(X^{-1})^\tau$.

We will now deal with case (i), i.e. $p > 11$. We claim that there exists $\xi = (\xi_1,\xi_2,\xi_3) \in \mathcal{O}_\infty^3$ such that

$$\xi\, A^{(1)} = \pi^{\eta(1)+3}\xi^\tau , \tag{8.10}$$

and furthermore $\xi = (\pi^{\varepsilon(1,1)}\tilde{\xi}_1,\ \pi^{\varepsilon(1,2)}\tilde{\xi}_2,\ \pi^{\varepsilon(1,3)}\tilde{\xi}_3)$ where $\tilde{\xi}_1,\tilde{\xi}_2,\tilde{\xi}_3$ are units in $\mathcal{O}_\infty$. We write

$$A^{(1)}_{\nu,\mu} = \pi^{\varepsilon(\nu,\mu)}\ \mathcal{A}^{(1)}_{\nu,\mu}$$

for $1 \leq \nu,\ \mu \leq 3$. The existence of ξ is equivalent to the existence of $\tilde{\xi} = (\tilde{\xi}_1,\tilde{\xi}_2,\tilde{\xi}_3) \in \mathcal{O}_\infty^3$ with unit co-ordinates such that

$$\begin{aligned}
&\tilde{\xi}_1\,\mathcal{A}^{(1)}_{1,1} + \pi^{\eta(1)-5}\tilde{\xi}_2\,\mathcal{A}^{(1)}_{2,1} + \pi^{\eta(2)-3}\tilde{\xi}_3\,\mathcal{A}^{(1)}_{3,1} = \tilde{\xi}_1^\tau \\
&\tilde{\xi}_1\,\mathcal{A}^{(1)}_{1,2} + \pi^{\eta(1)-1}\tilde{\xi}_2\,\mathcal{A}^{(1)}_{2,2} + \pi^{\eta(2)+1}\tilde{\xi}_3\,\mathcal{A}^{(1)}_{3,2} = \tilde{\xi}_2^\tau \\
&\tilde{\xi}_1\,\mathcal{A}^{(1)}_{1,3} + \pi^{\eta(1)-1}\tilde{\xi}_2\,\mathcal{A}^{(1)}_{2,3} + \pi^{\eta(2)-3}\tilde{\xi}_3\,\mathcal{A}^{(1)}_{3,3} = \tilde{\xi}_3^\tau \ .
\end{aligned} \tag{8.11}$$

Note that $\eta(2) - 3 \geqslant 1$ and $\eta(1) - 5 \geqslant 0$ provided p is a prime, $p \equiv 2 \pmod 3$, and $p > 11$. Consider the reduced system

$$\bar{\xi}_1 \, \bar{u}_{1,1} + \bar{\xi}_2 \, \bar{u}_{2,1} = \bar{\xi}_1^p$$

$$(8.12) \qquad \bar{\xi}_1 \, \bar{u}_{1,2} = \bar{\xi}_2^p$$

$$\bar{\xi}_1 \, \bar{u}_{1,3} = \bar{\xi}_3^p ,$$

where $\bar{u}_{1,1}$, $\bar{u}_{1,2}$, $\bar{u}_{1,3}$ are non-zero and $\bar{u}_{2,1}$ is zero if $p > 17$, but is non-zero in case $p = 17$. In either case, the equations may be solved simultaneously in $\bar{\mathbb{F}}_p$, the algebraic closure of $\mathbb{F}_p$, for a solution $\bar{\xi} = (\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3) \in (\mathbb{F}_p^*)^3$ of (8.12). Lifting, we obtain $\xi^{(1)} = (\xi_1^{(1)}, \xi_2^{(1)}, \xi_3^{(1)}) \in (\mathcal{O}_\infty^*)^3$, a solution mod π of (8.11).

Assume now that $\xi^{(i)} = (\xi_1^{(i)}, \xi_2^{(i)}, \xi_3^{(i)}) \in (\mathcal{O}_\infty^*)^3$ has been constructed for $i \leqslant \ell$ so that (8.11) holds mod π^i and so that

$$\xi_j^{(i)} \equiv \xi_j^{(i+1)} \pmod{\pi^i}$$

for $j = 1,2,3$ and $i < \ell$. We construct $\xi^{(\ell+1)}$ as follows. Let

$$\xi_j^{(\ell+1)} = \xi_j^{(\ell)} + \pi^\ell w_j$$

$(j = 1,2,3)$, let

$$\tilde{\mathcal{A}}_{\nu,\mu}^{(1)} = \theta(\nu,\mu) \, \mathcal{A}_{\nu,\mu}^{(1)}$$

where $\theta(1,\mu) = 1$, for $\mu = 1,2,3$; $\theta(2,1) = \pi^{\eta(1)-5}$; $\theta(2,\mu) = \pi^{\eta(1)-1}$, for $\mu = 2,3$; $\theta(3,\mu) = \pi^{\eta(2)-3}$, for $\mu = 1,3$; $\theta(3,2) = \pi^{\eta(2)+1}$. Finally, let

$$\beta_j = \frac{1}{\pi^\ell} \left(\xi_j^{(\ell)^\tau} - \sum_{k=1}^{3} \xi_k^{(\ell)} \hat{\mathcal{A}}_{kj} \right) \in \mathcal{O}_\infty$$

for $j = 1,2,3$. Then we obtain the following equations for w_j :

$$\sum_{k=1}^{3} w_k \hat{\mathcal{A}}_{kj} = w_j^\tau + \beta_j$$

$(j = 1,2,3)$. This system may be solved for w_j by reducing (mod π) and solving for $(\bar{w}_1,\bar{w}_2,\bar{w}_3) \in (\bar{\mathbb{F}}_p)^3$ (just as we solved (8.12)), then lifting the solution to $\mathcal{O}_\infty^3$. We then obtain $\tilde{\xi}$ by taking $\tilde{\xi}_j = \lim_{\ell\to\infty} \xi_j^{(\ell)}$.

Having found ξ as desired, we note $\zeta = \pi^{-\eta(1)}\xi\,(=(\zeta_1,\zeta_2,\zeta_3)) \in \mathcal{O}_\infty^3$ satisfies (8.10) with ord $\zeta_1 = \frac{3}{p-1}$, ord $\zeta_2 = 0$, ord $\zeta_3 = 1/(p-1)$. Note that if we set

$$\Lambda = \begin{pmatrix} \zeta_1 & \zeta_2 & \zeta_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$\mathcal{D} = \Lambda . A^{(1)} . \Lambda^{-\tau} ,$$

then $\mathcal{D}$ has the block form

$$\mathcal{D} = \begin{pmatrix} \pi^{\eta_1+3} & 0 \\ \Gamma & \mathcal{D}^{(1)} \end{pmatrix}$$

where $\Gamma = (\Gamma_2,\Gamma_3)^t$ is a 2×1 submatrix, and $\mathcal{D}^{(1)} = (\mathcal{D}^{(1)}_{\nu,\mu})_{2\leqslant\nu,\mu\leqslant 3}$ is a 2×2 submatrix. Furthermore

$$\Gamma_2 = A^{(1)}_{2,1}\zeta_1^{-\tau}, \text{ord}\,\Gamma_2 = \frac{1}{p-1}(\eta(2)-3) \;;$$

$$\Gamma_3 = A^{(1)}_{3,1}\zeta_1^{-\tau}, \text{ord}\,\Gamma_3 \geqslant 1 - \frac{1}{p-1} \;;$$

(8.13) $$\mathcal{D}^{(1)}_{2,2} = -A^{(1)}_{2,1}\zeta_1^{-\tau}\zeta_2^{\tau} + A^{(1)}_{2,2}, \quad \text{ord}\ \mathcal{D}^{(1)}_{2,2} = \frac{1}{p-1}(\eta(2)-3) \;;$$

$$\mathcal{D}^{(1)}_{2,3} = -A^{(1)}_{2,1}\zeta_1^{-\tau}\zeta_3^{\tau} + A^{(1)}_{2,3}, \quad \text{ord}\ \mathcal{D}^{(1)}_{2,3} = \frac{1}{p-1}(\eta(2)-2) \;;$$

$$\mathcal{D}^{(1)}_{3,2} = -A^{(1)}_{3,1}\zeta_1^{-\tau}\zeta_2^{\tau} + A^{(1)}_{3,2}, \quad \text{ord}\ \mathcal{D}^{(1)}_{3,2} \geqslant 1 - \frac{1}{p-1} \;;$$

$$\mathcal{D}^{(1)}_{3,3} = -A^{(1)}_{3,1}\zeta_1^{-\tau}\zeta_3^{\tau} + A^{(1)}_{3,3}, \quad \text{ord}\ \mathcal{D}^{(1)}_{3,3} \geqslant 1 .$$

We construct a matrix Z of the form

$$Z = \begin{pmatrix} 1 & 0 \\ \Theta & I \end{pmatrix}$$

where $\Theta = (\Theta_2,\Theta_3)^t$ is a 2×1 submatrix, I is the 2 2 identity, and Z satisfies

$$Z \mathcal{D} Z^{-\tau} = \begin{pmatrix} \pi^{\eta(1)+3} & 0 \\ 0 & \mathcal{D}^{(1)} \end{pmatrix} .$$

This is equivalent to solving the system

$$\text{(8.14)} \qquad \begin{aligned} &\Theta_2 \, \pi^{\eta(1)+3} + \Gamma_2 - \mathcal{D}^{(1)}_{2,2} \, \Theta_2^{\tau} - \mathcal{D}^{(1)}_{2,3} \, \Theta_3^{\tau} = 0 \\ &\Theta_3 \, \pi^{\eta(1)+3} + \Gamma_3 - \mathcal{D}^{(1)}_{3,2} \, \Theta_2^{\tau} - \mathcal{D}^{(1)}_{3,3} \, \Theta_3^{\tau} = 0 . \end{aligned}$$

We write $\Gamma_2 = \pi^{\eta(2)-3}\gamma_2$, $\Gamma_3 = \pi^{\eta(3)-1}\gamma_3$, $\mathcal{D}^{(1)}_{2,2} = \pi^{\eta(2)-3}d_{2,2}$, $\mathcal{D}^{(1)}_{2,3} = \pi^{\eta(2)-2}d_{2,3}$, $\mathcal{D}^{(1)}_{3,2} = \pi^{\eta(3)-1}d_{3,2}$, $\mathcal{D}^{(1)}_{3,3} = \pi^{\eta(3)}d_{3,3}$ and $\gamma_2, \gamma_3, d_{2,2}, d_{2,3}, d_{3,2}$ and $d_{3,3}$ belong to $\mathcal{O}_\infty$ and γ_2, $d_{2,2}$, and $d_{2,3}$ are, in fact, units. For $p > 11$, $p \equiv 2 \pmod 3$, $\eta(2) - 2 \geqslant \eta(1) + 3$ and $\eta(3) - 2 > \eta(1) + 3$, so (8.14) reduces to the following system

$$\text{(8.15)} \qquad \begin{aligned} &\Theta_2 + \pi^{\eta(1)-5}\gamma_2 - \pi^{\eta(1)-5}d_{2,2}\Theta_2^{\tau} - \pi^{\eta(1)-4}d_{2,3}\Theta_3^{\tau} = 0 \\ &\Theta_3 + \pi^{\eta(2)-4}\gamma_3 - \pi^{\eta(2)-4}d_{3,2}\Theta_3^{\tau} - \pi^{\eta(2)-3}d_{3,3}\Theta_3^{\tau} = 0 . \end{aligned}$$

This system can be solved for Θ_2 and Θ_3 by a similar argument to that used for the solution of (8.11).

We can now again apply the result of [7, § 2] to obtain $\sigma = (\sigma_2, \sigma_3) \in \mathcal{O}_\infty^2$, with σ_2 a unit, σ_3 divisible by π, satisfying

$$\text{(8.16)} \qquad \sigma \mathcal{D}^{(1)} = \pi^{\eta(2)-3} \, \sigma^{\tau}.$$

Just as before this enables us to define a 2×2 matrix $\xi \in GL(2, \mathcal{O}_\infty)$ satisfying

$$\xi \, \mathcal{D}^{(1)} \xi^{-\tau} = \mathrm{diag}(\pi^{\eta(2)-3}, \pi^{y})$$

where the right-side denotes a 2×2 diagonal matrix with diagonal entries indicated. By the well-known functional relation, it follows that $y = \eta(3)$.

In case (ii), i.e. $p = 11$, the above argument is modified as follows. We can find $\xi = (\xi_1, \xi_2, \xi_3)$, $\xi_1 = \pi^2 \tilde{\xi}_1$, $\xi_2 = \tilde{\xi}_2$, $\xi_3 = \pi \tilde{\xi}_3$

with $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2, \tilde{\xi}_3)$ having unit co-ordinates in $\mathcal{O}_\infty$ satisfying

(8.17) $$\xi\, A^{(1)} = \pi^5\, \xi^\tau \,.$$

Then letting

$$\hat{\Lambda} = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and setting

$$\hat{\mathfrak{D}} = \hat{\Lambda}\, A^{(1)}\, \hat{\Lambda}^{-\tau} \,,$$

yields $\hat{\mathfrak{D}}$ in the block form

$$\hat{\mathfrak{D}} = \begin{pmatrix} \pi^5 & 0 \\ \Gamma & \hat{\mathfrak{D}}^{(1)} \end{pmatrix}$$

where $\hat{\Gamma} = (\hat{\Gamma}_1, \hat{\Gamma}_2)^t$ is a 2×1 submatrix and $\hat{\mathfrak{D}}^{(1)} = (\hat{\mathfrak{D}}^{(1)}_{\nu,\mu})_{2\leqslant\nu,\mu\leqslant 3}$ is a 2×2 submatrix. Furthermore

$$\hat{\Gamma}_2 = A^{(1)}_{2,1}\xi^{-\tau}, \quad \text{ord } \hat{\Gamma}_2 = \tfrac{1}{2} \,;$$

$$\hat{\Gamma}_3 = A^{(1)}_{3,1}\xi_1^{-\tau}, \quad \text{ord } \hat{\Gamma}_3 \geqslant 1 \,;$$

(8.18)
$$\begin{aligned}
\hat{\mathfrak{D}}^{(1)}_{2,2} &= -\, A^{(1)}_{2,1}\xi_1^{-\tau}\xi_2^{\tau} + A^{(1)}_{2,2}\,, && \text{ord } \hat{\mathfrak{D}}^{(1)}_{2,2} = \tfrac{1}{2} \,;\\
\hat{\mathfrak{D}}^{(1)}_{2,3} &= -\, A^{(1)}_{2,1}\xi_1^{-\tau}\xi_3^{\tau} + A^{(1)}_{2,3}\,, && \text{ord } \hat{\mathfrak{D}}^{(1)}_{2,3} = \tfrac{3}{5} \,;\\
\hat{\mathfrak{D}}^{1}_{3,2} &= -\, A^{(1)}_{3,1}\xi_1^{-\tau}\xi_2^{\tau} + A^{(1)}_{3,2}\,, && \text{ord } \hat{\mathfrak{D}}^{(1)}_{3,2} \geqslant 1 \,;\\
\hat{\mathfrak{D}}^{(1)}_{3,3} &= -\, A^{(1)}_{3,1}\xi_1^{-\tau}\xi_3^{\tau} + A^{(1)}_{3,3}\,, && \text{ord } \hat{\mathfrak{D}}^{(1)}_{3,3} = 1 \,.
\end{aligned}$$

We can now find $\hat{\Theta} = (\hat{\Theta}_1, \hat{\Theta}_2)^t \in \mathcal{O}_\infty^2$ such that if

$$\hat{Z} = \begin{pmatrix} 1 & 0 \\ \hat{\Theta} & I \end{pmatrix}$$

with I the 2×2 identity matrix, then

$$\hat{Z}\,\hat{\mathcal{D}}\,\hat{Z}^{-\tau} = \begin{pmatrix} \pi^5 & 0 \\ 0 & \hat{\mathcal{D}}^{(1)} \end{pmatrix}.$$

The rest of the argument goes through unaltered. This completes the proof of (8.9).

Remark : Let $\overline{h}(t) = \overline{\alpha}_1 t + \overline{\alpha}_{-1} t^{-1} + \overline{\alpha}_{-2} t^{-2} + \overline{\alpha}_{-3} t^{-3}$ with $\overline{\alpha}_i \in \mathbb{F}_q$, $\overline{\alpha}_1, \overline{\alpha}_{-3} \in \mathbb{F}_q^*$. By the method of § 7, we can conclude that for $p > 11$, $p \equiv 2 \pmod 3$, the Newton polygon of $L(\overline{h},t)$ and $L(\overline{f},t)$ (given in theorem (8.9 (i))) coincide.

REFERENCES

[1] B. DWORK, "On the zeta function of a hypersurface", Publ. Math. I.H.E.S., n° 12, Paris, 1962.

[2] B. DWORK, "On the zeta function of a hypersurface, II", Ann. of Math., Vol. 80 (1964), 227-299.

[3] B. DWORK, "Bessel functions as p-adic functions of the argument", Duke Math. Jour., Vol. 41 (1974), 711-738.

[4] P. DELIGNE, "Applications de la formule des traces aux sommes trigonometrique", in SGA 4 1/2 Cohomologie Etale, Lecture Notes 569, Springer Verlag (1977) Berlin.

[5] N. KATZ, "Sommes Exponentielles", Asterisque 79 (1980), Paris.

[6] P. ROBBA, "Index of p-adic differential operators, III. Applications to twisted exponential sums", to appear.

[7] S. SPERBER, "Congruence properties of the hyperkloosterman sum", Compositio Math., Bol. 40 (1980), 3-33.

[8] D. G. NORTHCOTT, Lessons on rings, modules and multiplicities, Chapter 8, Cambridge University Press (1968), London.

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