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THE REICH TRACE FORMULA

by

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§ 1. The trace formula of Reich [Re] involves analytic elements (in the sense of Krasner) on the complement in a polydisk in p-adic n-space of the neighborhood of a hypersurface

$$g_0(x) = 0$$

where $g_0 \in K[x_1,\dots,x_n]$ and K is a finite extension of Q_p with residue class field, $\mathbb{F}_q$. We may take g_0 to have coefficients in the ring of integers of K and have non-trivial image $\bar{g}$ in $\mathbb{F}_q[x]$.

While this formula has found a number of applications, the proof by Reich requires the following unnatural condition on $\bar{g}$.
(R) The homogeneous component of highest degree of $\bar{g}$ has distinct irreducible factors over the algebraic closure of $\mathbb{F}_q$.

That this condition is superfluous follows from the work of Monsky [Mo] which provides strong generalizations of the Reich formula. Since Reich's treatment is quite transparent as compared to that of Monsky and since the superfluity of (R) has not been well understood [Dw 2, p. 291] we believe it may be useful to reprove Reich's formula along the original lines, eliminating condition (R). The new ingredient is the appendix by J. Fresnel explaining why (R) is not needed to obtain an orthonormal basis $\{\eta_{s,\varepsilon,\Delta}\}$ of the Reich space, $F_{\varepsilon,\Delta,g}$ (for definition see § 2) with the properties :

(1.1) $\eta_{s,\varepsilon,\Delta} = b_{s,\varepsilon,\Delta}\,\eta_{s,0,0}$ where $b_{s,\varepsilon,\Delta} \in \Omega(\varepsilon,\Delta \geqslant 0)$.

(1.2) For each s there exists $m(s) \in \mathbb{N}$ such that $g^{m(s)}\eta_{s,\varepsilon,\Delta}$ is a polynomial.

(1.3) For $\varepsilon' < \varepsilon$, $\Delta' < \Delta$

$$b_{s,\varepsilon,\Delta}/b_{s,\varepsilon',\Delta'} \longrightarrow 0$$

as $s \to \infty$.

(1.4) For each integer N, the space of polynomials of degree bounded by N is spanned by the set of all basis elements $\eta_{s,0,0}$ which lie in that space of polynomial.

The first 3 conditions are used to show that the endomorphism $\alpha_{\varepsilon,\Delta,g}$ is completely continuous for ε, $\Delta > 0$ and permit the determination of the trace by calculations in $F_{0,0,g}$. The fourth condition permits reduction to an endomorphism of the space of polynomials of degree bounded by N.

The notation $\eta_{s,\sigma,\Delta}$, $b_{s,\varepsilon,\Delta}$ will not be used in the following.

§ 2. NOTATION.

Let Ω be an algebraically closed extension field of $\mathbb{Q}_p$ complete under a rank one valuation extending that of $\mathbb{Q}_p$.

Let $\mathcal{O}$ be the ring of integers of Ω.

Let $\bar{g}$ be a non-zero element of $\mathbb{F}_q[x_1,\ldots,x_n]$ and let g be a lifting of $\bar{g}$ having coefficients in an unramified field such that $\deg g = \deg \bar{g} = d$. Let k be the field generated over $\mathbb{Q}_p$ by the coefficients of g.

Let ord designate the valuation of Ω (in additive form) normalized so that $\operatorname{ord} p = 1$.

For ε, $\Delta \geqslant 0$ let

$$D(\varepsilon,\Delta,g) = \{x \in \Omega^n \mid \operatorname{ord} g(x) \leqslant \varepsilon,\ \operatorname{ord} x_i \geqslant -\Delta,\ i = 1,\ldots,n\}.$$

Let

$F(\varepsilon,\Delta,g)$ be the space of analytic elements on $D(\varepsilon,\Delta,g)$, that is the set of all mappings of $D(\varepsilon,\Delta,g)$ into Ω which are in the completion under the sup norm of the ring of all elements of $\Omega(x)$ which can be represented by a ratio ξ/η of polynomials such that η is never zero on $D(\varepsilon,\Delta,g)$. $F(\varepsilon,\Delta,g)$ is a Banach space under the sup norm.

For $z = (z_1, \ldots, z_n)$, $u = (u_1, \ldots, u_n) \in \mathbb{N}^n$, let

$$z^u = z_1^{u_1} z_2^{u_2} \ldots z_n^{u_n} ,$$

a monomial in z. We use

$$|u| = \sum_{i=1}^{n} u_1 .$$

$$S = \{ (u,j) \in \mathbb{N}^n \times \mathbb{Z} \mid u_n < d \} .$$

We assume

$$d\Delta + \varepsilon < 1$$

ε, Δ lie in the (additive) value group of Ω.

For $u \in \mathbb{N}^{\dagger}$, $j \in \mathbb{Z}$, we write

$$w(u,j;\varepsilon,\Delta) = \Delta(|u| + d \sup(0,j)) + \varepsilon \sup(0,-j) .$$

If F, G are banach space over Ω, let

$L(F,G)$ = banach space of continuous linear maps of G into G.

$L_c(F,G)$ = space of completely continuous linear maps of F into G.

For $F = G$ we write

$L(F)$ for $L(F,F)$, $L_c(F)$ for $L_c(F,F)$

$\Omega^N[x]$ = polynomials of degree bounded by N.

If α' is a linear map of $\Omega[x]$ into itself which maps $\Omega^N[x]$ into $\Omega^{N-1}[x]$ for $N > N_0$, then the trace of $\alpha' | \Omega^N[x]$,

$\mathrm{Tr}(\alpha' | \Omega^N[x])$ is independent of N for $N > N_0$.

We define $\mathrm{Tr}\,\alpha'$ to be this common value.

§ 3. The key point is the following lemma (cf. Appendix by J. Fresnel).

LEMMA. There exists ε_0, $\Delta_0 > 0$, and a finite extension field L of K and a matrix, $A \in GL(n, \mathcal{O}_L)$, invertible in the ring of integers of L such that with

$$(z_1, \ldots, z_n) = (x_1, \ldots, x_n)A$$

the banach space $F(\varepsilon, \Delta, g)$ is equivalent to the banach space

$$(3.1) \quad F'(\varepsilon, \Delta, g) = \{ \sum_{(u,j)\in S} A_{u,j} z^u g^j \mid A_{u,j} \in \Omega, \mathrm{ord} A_{u,y} - w(u,j;\varepsilon,\Delta) \to \infty \}$$

with the indicated norm provided $0 \leqslant \varepsilon \leqslant \varepsilon_0$, $0 \leqslant \Delta \leqslant \Delta_0$.

This means that if $\xi \in F(\varepsilon, \Delta, g)$ then ξ has a representation

$$(3.2) \qquad \xi = \sum_{(u,j)\in S} A_{u,j}\, z^u\, g^j$$

where $\mathrm{ord}\, A_{u,j} - w(u,j;\varepsilon,\Delta) \to \infty$ as $(u,j) \to \infty$.

Clearly

$$\frac{-\log|\xi|}{\log p}{}_{\varepsilon,\Delta,g} \geqslant \mathrm{Inf}_{(u,j)\in S} \; (\mathrm{ord}\, A_{u,j} - w(u,j;\varepsilon,\Delta)) .$$

The lemma implies that there exists $k > 1$ (possibly depending on $\varepsilon;\Delta$) such that

$$(3.3) \qquad K|\xi|_{\varepsilon,\Delta,g} \geqslant \mathrm{Sup}_{(u,j)\in S} |A_{u,j}|\, p^{w(u,j;\varepsilon,\Delta)} \geqslant |\xi|_{\varepsilon,\Delta,g} .$$

We observe that $F'(\varepsilon, \Delta, g)$ has orthonormal basis $\{\xi_{u,j,\varepsilon,\Delta}\}_{(u,j)\in S}$ where

$$(3.4) \qquad \xi_{u,j,\varepsilon,\Delta} = C_{u,j;\varepsilon,\Delta}\, z^u g^j = C_{u,j;\varepsilon,\Delta}\, \xi_{u,j;0,0}$$

$$(3.5) \qquad \mathrm{ord}\, C_{u,j;\varepsilon,\Delta} = w(u,j;\varepsilon,\Delta), \quad C_{u,j,\varepsilon,\Delta} \in \Omega .$$

§ 4. THE MAPPING ψ.

PROPOSITION. *Let*

$$(x_1,\dots,x_n) = (y_1^q,\dots,y_n^q)$$

$$1 > d\,\Delta + \varepsilon\,. \tag{4.1}$$

Then $x \in D(\varepsilon,\Delta,g)$ *if and only if* $y \in D(\varepsilon/q\,,\,\Delta/q,g)$.

Proof : Clearly $\operatorname{ord} x \geq -\Delta$ if and only if $\operatorname{ord} y_i \geq -\Delta/q$. On the other hand

$$g(y)^q = g(y^q) + ph(y)$$

where

$$h \in \underline{O}[x]\,,\ \deg h \leq qd\,.$$

Thus

$$\operatorname{ord} g(y)^q = \operatorname{ord} g(y^q) = \operatorname{ord} g(x) \tag{4.2}$$

if either

$$\operatorname{ord} g(x) < 1+qd\,\operatorname{Min}(0,\operatorname{ord} y) \tag{4.3}$$

or

$$\operatorname{ord} g(y)^q < 1+qd\,\operatorname{Min}(0,\operatorname{ord} y)\,. \tag{4.4}$$

For $x \in D(\varepsilon,\Delta,g)$, condition (4.1) implies (4.3). For $y \in D(\varepsilon/q,\Delta/q,g)$, condition (4.1) implies (4.4). Thus equation (4.2) is valid and this completes the proof.

COROLLARY 4.1. *If* ξ *is a function mapping* $D(\varepsilon/q,\Delta/q,g)$ *into* Ω *then*

$$x \longrightarrow (\psi\xi)(x) = \frac{1}{q^n} \sum_{y^q=x} \xi(y)$$

(the sum is over the q^n *preimages of* x *under* $y \longrightarrow y^q = x$ *(counting multiplicities)) is a well defined function on* $D(\varepsilon,\Delta,g)$. *Furthermore*

$$\sup_{x\in D(\varepsilon,\Delta,g)} |(\psi\xi)(x)| \leq q^n \sup_{x\in D(\varepsilon/q,\Delta/q,g)} |\xi(x)|\,.$$

COROLLARY 4.2. ψ <u>maps</u> $F(\varepsilon/q,\Delta/q,g)$ <u>into</u> $F(\varepsilon,\Delta,g)$.

<u>Proof</u> : Let $\xi \in F(\varepsilon/q,\Delta/q,g)$ then ξ is the uniform limit on $D(\varepsilon/q,\Delta/q,g)$ of rational functions η having denominators never zero on $D(\varepsilon/q,\Delta/q,g)$. Hence $\psi(\xi)$ is the uniform limit of the $\psi(\eta)$ on $D(\varepsilon,\Delta,g)$. Now by the definition of ψ, $\psi\eta \in \Omega(x)$. If η_1 is the denominator of η then $\psi\eta$ has denominator dividing

$$\eta_2(x) = \prod_{y^q=x} \eta_1(y) .$$

The product is clearly a polynomial in x and for $x \in D(\varepsilon,\Delta,g)$ we have $y \in D(\varepsilon/q,\Delta/q,g)$ and so η_2 is never zero on the set in question.

§ 5. THE TRACE FORMULA

We assume (4.1) and that ε,Δ are sufficiently small for the validity of lemma 3.

THEOREM. <u>Let</u> $\varepsilon, \Delta > 0$.

<u>Let</u> $h \in F(\varepsilon/q,\Delta/q,g)$. <u>Let</u> $\alpha\ (=\alpha_{\varepsilon,\Delta,g})$ <u>be the mapping of</u> $F(\varepsilon,\Delta,g)$ <u>given by</u> $\xi \longrightarrow \psi(h.\xi)$. <u>Then</u> α <u>is a completely continuous map of</u> $F(\varepsilon,\Delta,g)$ <u>into itself and</u>

$$(q-1)^n \operatorname{Tr} \alpha = \sum_{\substack{x^q = x \\ |g(x)|=1 \\ \Pi x_i \neq 0}} h(x) \quad . \tag{5.1}$$

<u>Proof</u> :

<u>Step 1</u>. Complete continuity of α.

Let i be the mapping of $F(\varepsilon,\Delta,g)$ into $F(\varepsilon/q,\Delta/q,g)$

$$F(\varepsilon,\Delta,g) \ni \xi \longrightarrow \xi \mid D(\xi/q,\Delta/q,g) .$$

Let h denote the endomorphism of $F(\xi/q,\Delta/q,g)$ given by multiplication by h . Finally ψ is viewed as mapping of $F(\varepsilon/q,\Delta/q,g)$ into $F(\varepsilon,\Delta,g)$ as defined in § 4. Now each of these maps is bounded, hence continuous but i is seen to be completely continuous. To show complete continuity we consider the orthonormal basis (3.4) of $F'(\varepsilon,\Delta,g)$ and so the matrix of i (as mapping of $F'(\varepsilon,\Delta,g)$ into $F'(\varepsilon/q,\Delta/q,g)$) relative to the orthonormal basis is given by

$$i\ \xi_{u,j,\varepsilon,\Delta} = \frac{C_{u,j;\varepsilon/\Delta}}{C_{u,j;\varepsilon/q,\Delta/q}}\quad \xi_{u,j;\varepsilon/q,\Delta/q}\ .$$

The matrix is thus diagonal and the diagonal coefficient goes to zero as $(u,j) \longrightarrow \infty$. This shows that i is completely continuous in terms of the norms of $F(\varepsilon,\Delta,g)$ and $F'(\varepsilon/q,\Delta/q,g)$. This shows that the composition $\alpha = \psi\circ h\circ i$ is completely continuous as endomorphism of $F(\varepsilon,\Delta,g)$ and hence [Se], α has a trace.

Note : If we consider the corresponding endomorphism of $F(\varepsilon',\Delta',g)\,(\varepsilon',\Delta'>0)$ we see easily that the orthonormal basis of $F'(\varepsilon',\Delta',g)$ is the same as that of $F'(\varepsilon,\Delta,g)$ except for constant factors and hence the trace is the same.

Step 2. Reduction to h = polynomial/g^s .

The mapping

$$\begin{aligned} F(\varepsilon/q,\Delta/q,g) &\longrightarrow L_c(F(\varepsilon,\Delta,g)) \\ h &\longrightarrow (\psi\circ h)_{\varepsilon,\Delta,g} \end{aligned}$$

is continuous. The map

$$\begin{aligned} L_c(F(\varepsilon,\Delta,g)) &\longrightarrow \Omega \\ \phi &\longrightarrow \text{trace } \phi \end{aligned}$$

is also continuous. Again the map

$$\begin{aligned} F(\varepsilon,q,\Delta/q,g) &\longrightarrow \Omega \\ h &\longrightarrow \sum_{\substack{x^q = x \\ |g(x)|=1 \\ \pi x_i \neq 0}} h(x) \end{aligned}$$

is continuous. Hence both sides of (5.1) may be approximated with h replaced by rational functions with powers of g as denominator. This completes the reduction.

Step 3. Reduction to $F(o,o,g)$.

Strictly speaking $\alpha_{o,o,g}$ is not completely continuous and so does not have a unique trace. Il fowever we fix the basis $\{\xi_{u,o,o}\}$ of $F(o,o,g)$ and we write

(5.2) $$\alpha_{o,o,g}\ \xi_{u,j;o,o} = \sum_{u',j'} A_{u,j;u',j'}\ \xi_{u',j';o,o}$$

then by (3.4)

$$\alpha_{\varepsilon,\Delta,g}\ \xi_{u,j;\varepsilon,\Delta} = \sum_{u',j'} A_{u,j,u',j'}\ \frac{C_{u,j,\varepsilon,\Delta}}{C_{u',j',\varepsilon,\Delta}}\ \xi_{u',j';\varepsilon,\Delta}\ .$$

This shows that

(5.3) $$\mathrm{Tr}(\alpha_{\varepsilon,\Delta,g}) = \sum_{u,y} A_{u,j;u,j}\ .$$

The right side may be viewed as the trace of $\alpha_{o,o,g}$ relative to the fixed basis. In this sense the calculation of trace is reduced to a calculation in $F(o,o,g)$.

Step 4. Reduction to endomorphism of $\Omega[x]$.

We construct a sequences of maps $\alpha_r, \tilde{\alpha}_r$ indexed by $r \in \mathbb{N}$ and will restrict our attention to r large. We assume

$$h\ g(x)^{(q-1)p^r} \in \Omega[x]\ .$$

Let π be a generator of the prime ideal of $K \cap \underline{\mathcal{O}}$, then

$$g(x)^q \equiv g(x^q) \quad \bmod\ \pi\ \underline{\mathcal{O}}[x]$$

and hence

$$g(x)^{qp^r} \equiv g(x^q)^{p^r} \quad \bmod\ \pi_r\ \underline{\mathcal{O}}[x]$$

where $|\pi_r| \longrightarrow 0$ as $r \longrightarrow \infty$.

Thus in the sup norm on $D(o,o,g)$, written $|\ \ |_{o,o,g}$ we have

(5.4) $$\left| g(x)^{(q-1)p^r} - \left(\frac{g(x^q)}{g(x)}\right)^{p^r} \right|_{o,o,g} \leq |\pi_r|\ .$$

We put

$$\alpha_r = \psi \circ h g(x)^{(q-1)p^r}$$

$$(5.5) \qquad \beta_r = \psi \circ h\left(\frac{g(x)^q}{g(x)}\right)^{p^r} = g(x)^{p^r} \circ \alpha_{o,o,g} \circ \frac{1}{g(x)^{p^r}} .$$

Both may be viewed as endomorphisms of $F(o,o,g)$ but if we let $\alpha'_r = \alpha_r \| \mathfrak{C}[x]$ we obtain an endomorphism of $\mathfrak{R}[x]$. It is well known [Dw 1] that α'_r has a well defined trace and that

$$(5.6) \qquad (q-1)^n \text{ Trace } \alpha'_r = \sum_{\substack{x^q = x \\ \pi x_i \neq 0}} h(x) g(x)^{(q-1)p^r} .$$

It follows from (5.4) that as elements of $L(F_{o,olg})$,

$$(5.7) \qquad |\alpha_r - \beta_r|_{o,o,g} \leq |\pi_r| \, q^n .$$

Let us write

$$(5.8) \qquad \alpha_r \, \xi_{u,j,o,o} = \sum_{u',j'} A^{(r)}_{u,j,u',j'} \, \xi_{u',j',o,o} .$$

Using (5.2) and (5.5)

$$(5.9) \qquad \beta_r \, \xi_{u,j,o,o} = \sum_{u',j'} A_{u,j-p^r;u',j'} \, \xi_{u',j',o,o} \, g(x)^{p^r}$$

$$= \sum_{u',j'} A_{u,j-p^r;u',j'-p^r} \, \xi_{u',j',o,o} .$$

It follows from (5.8), (5.9) and (5.7) that

$$|(\alpha_n - \beta_n)\xi_{u,j;o,o}|_{o,o,g} \leq |\pi_r| \, q^n$$

and hence by (3.3)

$$(5.10) \qquad |A^{(r)}_{u,j;u',j'} - A_{u,j-p^r;u',j'-p^r}| \leq c|\pi_r|q^n \underline{\underline{\text{def}}} \ \ell_r$$

where c is a strictly positive real number independent of r,u,u',j,j'.

Now $\{\xi_{u,j,o,o}\}_{\substack{u\in\mathbb{N},j\geqslant o\\ u_n<d}}$ is a basis of $\Omega[x]$ with property (1.4) and hence

$$t_r\ \alpha'_r = \sum_{j\geqslant o,u} A^{(r)}_{u,j;u,j}\ . \tag{5.11}$$

Equation (5.10) shows

$$\Big|\mathrm{tr}\ \alpha'_r - \sum_{u\in\mathbb{N}^r,u_n<d,j>o} A_{u,j-p^r;u,j-p^r}\Big| \leqslant \ell_r \tag{5.12}$$

where $\ell_r \longrightarrow o$ as $r \longrightarrow \infty$.

Now

$$\lim_{r\to\infty} \sum_{j\geqslant o,u} A_{u,j-p^r;u,j-p^r} = \sum_{j\in\mathbb{Z},u} A_{u,j;u,j} = \mathrm{Tr}\ \alpha\ .$$

To compute the limit of $\mathrm{tr}\ \alpha'_r$ we use the right hand side of (5.6) and observe that for $x^q = x$.

$$\lim_{r\to\infty} g(x)^{(q-1)p^r} = \begin{cases} 1 & \text{if } |g(x)| = 1 \\ 0 & \text{if } |g(x)| < 1\ . \end{cases}$$

Thus $(q-1)^n\ \mathrm{Tr}\ \alpha'_n$ has the right side of (5.1) as limit. Equation (5.1) now follows from (5.12).

COROLLARY. <u>Let</u> g_o <u>be an arbitrary lifting of</u> $\bar{g}$ <u>to</u> $\mathcal{O}[x]$ <u>then for</u> ε,Δ <u>strictly positive and sufficiently small and for</u> $h \in F_{\varepsilon,\Delta,g_o}$ <u>the mapping</u> $\psi\circ h$ <u>of</u> $F(\varepsilon,\Delta,g_o)$ <u>into itself is completely continuous and has trace</u>

$$\frac{1}{(q-1)^n} \sum_{\substack{|g_o(x)|=1\\ \Pi x_i\neq 0\\ x^q=x}} h(x)\ .$$

<u>Proof</u> : For ε,Δ small, the space $F(\varepsilon,\Delta,g_o)$ and $F(\varepsilon,\Delta,g)$ coincide.

The condition $|g_o(x)| = 1$ is the same as $|g(x)| = 1$ for $x^q = x$. This completes the proof.

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