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HIGHER p-ADIC GAMMA FUNCTIONS AND DWORK COHOMOLOGY

by

Francesco BALDASSARRI

0. INTRODUCTION.

Let p be a prime $\neq 2$ and let Ω be a universal p-adic domain. Let $\pi \in \Omega$ be a fixed root of $-p$ order $p-1$; let $\zeta = \zeta_\pi \in \Omega$ be the p-th root of unity which is closest to $1+\pi$. For $a \in \mathbb{Q} \cap \mathbb{Z}_p$ let $a' \in \mathbb{Q} \cap \mathbb{Z}_p$ be such that $pa' - a \in S = \{0,1,\dots,p-1\}$; also define, recursively, $a^{(0)} = a$, $a^{(i+1)} = (a^{(i)})'$, for $i = 0,1,\dots$.

Suppose that $a^{(f)} = a$, $f \geqslant 1$, and let $q = p^f$; the Gross-Koblitz formula ([5]) asserts that :

$$(0.1) \qquad g_f(a,\pi) = -\sum_{x^{q-1}=1} x^{-a(q-1)} \zeta^{x+x^p+\dots+x^{p^{f-1}}} = \pi^{S(a(q-1))} \prod_{i=0}^{f-1} \Gamma_p(a^{(i)})$$

where $S(n)$ = sum of the digits in the p-adic expansion of $n \in \mathbb{N}$ (notice that $a(q-1) \in \mathbb{N}$) and Γ_p denotes the p-adic gamma function introduced by Morita. Boyarsky ([1]) interpreted formula (0.1) in terms of Dwork's cohomology and proved that Γ_p is analytic on the set $\bigcup_{i \in S} D(i,(p^{-1/p-1/p-1})^-)$, where as usual, for $t \in \Omega$ and $\rho > 0$, $D(t,\rho^-) = \{x \in \Omega \mid |x-t| < \rho\}$

We take the viewpoint of Boyarski and Dwork ([1], [2], [3]) but, following a suggestion of Dwork's (see section 21 of [2]), we use his more complicated analytic liftings of an additive character of a finite field (see [4], § 4, a)) to obtain a family of formulae indexed by $s = 1,2,\dots,\infty$:

$$(0.1)_s \qquad g_f(a,\pi) = \gamma_s^{S(a(q-1))} \prod_{i=0}^{f-1} \Gamma_{D,s}(a^{(i)}) \ .$$

In $(0.1)_s$, π is replaced by $\gamma_s \in \Omega$, where π/γ_s is a unit in $\mathbb{Z}_p$ and Γ_p by $\Gamma_{D,s}$, the s-th Dwork gamma function (see (1.21)), which is analytic on $\bigcup_{i \in S} D(i,(p^{e_s-1})^-)$, $e_s = 1 - p^{-s}(s+1+\frac{1}{p-1})$. For $s=1$, $\Gamma_{D,1} = \Gamma_p$ (see [2], § 21), while for $s=\infty$ we obtain a function $\Gamma_{D,\infty}$ analytic on $\bigcup_{i \in S} D(i,1^-)$. This improvement in the radius of local analyticity of $\Gamma_{D,s}$ for higher $\underline{s}$ is obtained at the expense of simplicity in the functional equations (of translation (1.25) and of reflection (2.12)) for $\Gamma_{D,s}$, which involve a function $\Lambda_s(y)$ that gets more complicated when $\underline{s}$ is increased. The values of Λ_s at negative integers can however be effectively computed (in terms of π/γ_s) (see (1.13)) and from that one can effectively compute the values of $\Gamma_{D,s}$ at negative integers. It has been shown by Adolphson that any continuous non-vanishing function g on $\mathbb{Z}_p$ such that equation (0.1) remains valid with Γ_p replaced by g must be of the form $g(a) = \Gamma_p(a) \frac{h(a')}{h(a)}$ where h is again a continuous non vanishing function on $\mathbb{Z}_p$. We believe that in the above statement "continuous" may be replaced by 'locally analytic". Our results give a class of functions of this latter type (see (1.27)).

An interesting feature of the cohomology theory presented here is that it uses as cohomology space the analytic cokernel of a differential operator whose analytic index differs from the algebraic index.

This explains why the continguity (cohomology) relations among differentials ((1.19)) and the functional equations for $\Gamma_{D,s}$ ((1.25), (2.12)) involve a trascendental function $\Lambda_s(y)$, and also why this function $\Lambda_s(y)$ itself satisfies a functional equation (1.13) involving only rational functions. We expect that a similar problem should arise when dealing with varieties defined over a ring of p-adic integers and having bad reduction to characteristic p ; our treatement may serve as an introduction to how p-adic analysis should be modified to treat that case.

In section 2 we present an alternative approach to Dwork's duality theory via the notion of cup-product of analytic cohomology classes. We believe that this notion can be widely generalized in the set-up of Monsky-Washnitzer cohomology.

Finally, it is clear that a treatment similar to the one given in this paper for the gamma function could be given for a number of interesting functions, in particular for Bessel functions and for (confluent) hypergeometric functions.

We are indebted to Professor Dwork for suggesting the problem studied here and for guidance in the preparation of this paper.

1. ANALYTIC DIFFERENTIALS.

For $s=1,2,\dots,\infty$, let $\gamma_s \in \Omega$ denote the zero of $\sum_{i=0}^{s} x^{p^i}/p^i$ which is closest to π ; we have in fact $\operatorname{ord}(\gamma_s-\pi) = p-1+\frac{1}{p-1}$ for $s=2,3,\dots,\infty$, while $\gamma_1=\pi$. It follows from section 4,a) of [4], that the field $\mathbb{Q}_p(\gamma_s)$ is independent of $\underline{s}$ and that it coincides with the field obtained by adjoining to $\mathbb{Q}_p$ the p-th roots of 1. Let us denote this field by K and limit ourselves to the consideration of objects (e.g. analytic functions, analytic differentials,...) which are defined over K. Notice that $\operatorname{Gal}(K/\mathbb{Q}_p)$ is isomorphic to the group of the roots of unity of order p-1 : if ε denoted such a root the corresponding element of $\operatorname{Gal}(K/\mathbb{Q}_p)$ is ρ_ε, where $\rho_\varepsilon(\gamma_s) = \varepsilon\gamma_s$, for $s=1,2,\dots,\infty$. In particular, ρ_{-1} is denoted by a symbol of conjugation : $\rho_{-1}x = \bar{x}$, for $x \in K$. Let L = {functions analytic in an annulus $1-\delta<|x|<1+\delta$, with unspecified $\delta>0$}, a K-vector space endowed with the norm $\|f\|_L = |f|_0(1) = \sup_{|x|=1} |f(x)|$, for $f \in L$. Let $U = \mathbb{Q}\cap\mathbb{Z}_p\setminus\mathbb{Z}$; $\mathbb{Z}$ operates on U by translations : for $a\in U$, $\bar{a}$ will denote the orbit of a under $\mathbb{Z}$. For real numbers b, c, we define :

$$L(b,c) = \{\sum_{i=0}^{\infty} a_i x^i \mid a_i \in K,\ \operatorname{ord} a_i \geqslant ib+c,\ \text{for}\ i=0,1,\dots\}$$

$L(b) = \bigcup_{c\in\mathbb{R}} L(b,c)$. Then L(b) is the space of functions analytic and bounded on $D(0,p^{b-})$ and is therefore naturally a Banach space in the supnorm over that ball.

For $s=1,2,\dots,\infty$, we define (see [4], § 4, a)) :

$$a_s = (p-1)^{-1} - p^{-s}(s+(p-1)^{-1}) \qquad (a_\infty = (p-1)^{-1}),$$

$$\theta_s(t) = \exp\{\sum_{j=0}^{s} (t\gamma_s)^{p^j}/p^j\} = \sum_{j=0}^{\infty} c_j^{(s)} t^j \in L(a_{s+1},0)$$

$$\gamma_{s,j} = \sum_{i=0}^{j} \gamma^{p^i}_s / p^i , \qquad \text{for} \quad j = 0,1,\ldots,$$

$$\hat{\theta}_s(t) = \exp \{ \sum_{j=0}^{s-1} \gamma_{s,j} \; t^{p^j} \}$$

$$\hat{\theta}_s(t)/\hat{\theta}_1(t) = \sum_{j=0}^{\infty} \beta_j^{(s)} \; t^j \in L(1-\frac{1}{p}, 0) .$$

For $a \in U$ we also define :

$\Omega^0_{\overline{a}} = x^a \hat{\theta}_s L$ (independent of $s = 1,2,\ldots,\infty$ and of $a \in \overline{a}$), a normed vector space isomorphic to L via :

$$\varphi^0_{s,a} : L \longrightarrow \Omega^0_{\overline{a}}$$

$$\xi \longrightarrow x^a \hat{\theta}_s \xi , \tag{1.1}$$

$\Omega^1_{\overline{a}} = \Omega^0_{\overline{a}} \, dx$, again isomorphic to L via :

$$\varphi^1_{s,a} : L \longrightarrow \Omega^1_{\overline{a}}$$

$$\xi \longrightarrow x^a \hat{\theta}_s \xi \, \frac{dx}{x} . \tag{1.2}$$

Let $E = xd/dx$; if we define, for a, s as before :

$$D_{s,a} = E + a + \sum_{j=0}^{s-1} \gamma_{s,j} p^j x^{p^j} \tag{1.3}$$

then the diagram :

$$\begin{tikzcd} L \arrow[r, "\varphi^0_{s,a}"] \arrow[d, "D_{s,a}"'] & \Omega^0_{\overline{a}} \arrow[d, "d"] \\ L \arrow[r, "\varphi^1_{s,a}"] & \Omega^1_{\overline{a}} \end{tikzcd} \tag{1.4}$$

where d denotes exterior differentiation, commutes.

We set $W(\bar{a}) = \Omega^1_{\bar{a}} / d\Omega^0_{\bar{a}}$, $[\omega]$ = class of $\omega \in \Omega^1_{\bar{a}}$, modulo $d\Omega^0_{\bar{a}}$,

$$\omega_{s,a} = x^a \hat{\theta}_s \frac{dx}{x} \in \Omega^1_{\bar{a}} . \tag{1.5}$$

From (1.1) - (1.4) we obtain an isomorphism :

$$\varphi_{s,a} : L/D_{s,a}L \xrightarrow{\sim} W(\bar{a}) . \tag{1.6}$$

The theory depends upon the initial choice of $\pi = \gamma_1$. Replacing π by $\varepsilon\pi$, $\varepsilon^{p-1} = 1$, produces a "conjugate" theory. If $\varepsilon = -1$, the objects of this new theory are denoted with a symbol of conjugaison ; for example

$$\bar{\omega}_{s,a} = x^a \hat{\theta}_s^{-1} \frac{dx}{x} \in \bar{\Omega}^1_{\bar{a}}$$

or

$$\bar{D}_{s,a} = E + a - \sum_{j=0}^{s-1} \gamma_{s,j} p^j x^{p^j} .$$

We rely upon the results of Dwork (section 21 of 2) expressed by him in terms of the operator $D_{1,a}$, and in particular upon the fact that $\dim_K W(\bar{a}) = 1$, to deduce relations among cohomolgy classes.

We have :

$$[\omega_{s,a}] = [\hat{\theta}_s/\hat{\theta}_1 \, \omega_{1,a}] = [\sum_{j=0}^{\infty} \beta_j^{(s)} x^j \omega_{1,a}] = \sum_{j=0}^{\infty} \beta_j^{(s)} [\omega_{1,a+j}] =$$

$$= \sum_{j=0}^{\infty} \beta_j^{(s)} \frac{(a)_j}{(-\pi)^j} [\omega_{1,a}] , \quad \text{where for } j = 1,2,\ldots$$

$(a)_j = a(a+1)\ldots(a+j-1)$, while $(a)_0 = 1$.

Therefore, for $a \in U$:

$$[\omega_{s,a}] = R_s(a) \, [\omega_{1,a}] \tag{1.7}$$

where :

$$R_s(y) = \sum_{j=0}^{\infty} \beta_j^{(s)} (y)_j / (-\pi)^j \in L(1 - \tfrac{1}{p} - \tfrac{1}{p-1}, 0) . \tag{1.8}$$

In fact R_s is invertible in $L(1-\frac{1}{p}-\frac{1}{p-1}, 0)$; its inverse will be written down later ((1.15)). Let us now define functions $\Lambda_s(a)$, $s = 1,2,\dots,\infty$, $a \in U$, by means of :

$$(1.9) \qquad [\omega_{s,a+1}] = a/(-\gamma_s)\Lambda_s(a)\ [\omega_{s,a}] ,$$

so that :

$$(1.10) \qquad [\omega_{s,a+i}] = (a_i)/(-\gamma_s)^i\ \Lambda_s(a)\Lambda_s(a+1)\dots\Lambda_s(a+i-1)\ [\omega_{s,a}] ,$$

for $i = 1,2,\dots$.

We have :

$$[\omega_{s,a+1}] = R_s(a+1)\ [\omega_{1,a+1}] = R_s(a+1)\ a/(-\gamma_1)\ [\omega_{1,a}]$$

and

$$[\omega_{s,a+i}] = a/(-\gamma_s)\Lambda_s(a)\ [\omega_{s,a}] = a/(-\gamma_s)\Lambda_s(a)R_s(a)\ [\omega_{1,a}] .$$

We conclude that :

$$(1.11) \qquad \Lambda_s(a) = \gamma_s/\pi\ R_s(a+1)/R_s(a) \quad \text{for} \quad a \in U .$$

The right hand side of (1.11) represents a function of $\underline{a}$ which belongs to $L(1-\frac{1}{p}-\frac{1}{p-1}, 0)$; therefore $a \longrightarrow \Lambda_s(a)$, $a \in U$, admits an interpolating function $\Lambda_s \in L(1-\frac{1}{p}-\frac{1}{p-1}, 0)$. By (1.4) we have :

$$0 = [dx^a\hat{\theta}_s] = [D_{s,a}(1)\omega_{s,a}] = [a\omega_{s,a} + \sum_{j=0}^{s-1} \gamma_{s,j}\ p^j \omega_{s,a+p^j}] =$$

$$= (a + \sum_{j=0}^{s-1} \gamma_{s,j}\ p^j\ \frac{(a)_{p^j}}{(-\gamma_s)^{p^j}}\ \Lambda_s(a)\Lambda_s(a+1)\dots\Lambda_s(a+p^j-1))\ [\omega_{s,a}] .$$

Therefore :

$$(1.12) \qquad 1-\Lambda_s(a) = \sum_{j=1}^{s-1} \gamma_{s,j}p^j\ \frac{(a+1)_{p^j-1}}{(-\gamma_s)^{p^j}}\ \Lambda_s(a)\Lambda_s(a+1)\dots\Lambda_s(a+p^j-1) .$$

For $1-\frac{1}{p}-\frac{1}{p-1} \leqslant r \leqslant 1$ consider the map

$$\eta_s^{(r)} : L(r,0) \longrightarrow L(r,0) \tag{1.13}$$

defined by :

$$\eta_s^{(r)} : \xi(y) \longrightarrow 1 - \sum_{j=1}^{s-1} \gamma_{s,j} p^j \frac{\binom{(y+1)}{p^j-1}}{(-\gamma_s)^{p^j}} \xi(y)\xi(y+1)\ldots\xi(y+p^j-1) \ .$$

Clearly, $\eta_s^{(r)}$ is a contraction of $L(r,0)$: its unique fixed point is $\Lambda_s(y)$ which therefore belongs to $L(1,0)$. Since the map $\eta_s^{(r)}$ is invariant under the replacement $\gamma_s \mapsto \varepsilon\gamma_s$, $\varepsilon^{p-1} = 1$, we conclude :

LEMMA 1.4. <u>The function</u> Λ_s <u>appearing in</u> (1.9) <u>is represented by a power series</u> $\Lambda_s(y) \in \mathbb{Z}_p[[y]]$, <u>such that</u> $\Lambda_s(y) = \sum_{i=0}^{\infty} \lambda_i^{(s)} y^i$, $\lambda_i^{(s)} \in p^i\,\mathbb{Z}_p$.

We can now write down explicitly the inverse of $R_s(y)$ in $L(1-\frac{1}{p}-\frac{1}{p-1}, 0)$; it is :

$$R_s(y)^{-1} = \sum_{j=0}^{\infty} \bar{\beta}_j^{(s)}(y)_j/(-\gamma_s)^j\, \Lambda_s(y)\Lambda_s(y+1)\ldots\Lambda_s(y+j-1) \ . \tag{1.15}$$

For a, $b \in U$, $pb-a \in \mathbb{Z}$, the K-linear maps (cfr. [2] , § 21) :

$$\begin{aligned} &\alpha_{\bar a}^1 : \Omega_{\bar a}^1 \longrightarrow \Omega_{\bar b}^1 \\ &\xi\omega_{s,a} \longmapsto \psi(\xi x^{a-pb}\theta_s)\,\omega_{s,b} \\ &\alpha_{\bar a}^0 : \Omega_{\bar a}^0 \longrightarrow \Omega_{\bar b}^0 \\ &x^a\hat\theta_s\xi \longmapsto x^b\hat\theta_s\psi(x^{a-pb}\theta_s\xi) \end{aligned} \tag{1.16}$$

for $\xi \in L$, are independent of $s = 1,2,\ldots,\infty$ and of $a \in \bar a$, $b \in \bar b$. We remind the reader that $\psi : L \longrightarrow L$ is defined by
$\psi(\sum_{i\in\mathbb{Z}} a_i x^i) = \sum_{i\in\mathbb{Z}} a_{pi} x^i$.

Since $\alpha_{\bar a}^1 \circ d = pd \circ \alpha_{\bar a}^0$, we get an induced map, denoted by $\alpha_{\bar a}$:

$$\alpha_{\bar a} : W(\bar a) \longrightarrow W(\bar b) \tag{1.17}$$

We now define, for $pb-a \in \mathbb{Z}$,

$$(1.18) \qquad \alpha_{\underline{a}}([\omega_{s,a}]) = \gamma^{(s)}(a,b)\,[\omega_{s,b}]$$

so that :

$$(1.19) \quad \gamma^{(s)}(a,b) = \sum_{i=\frac{a-pb}{p}}^{\infty} c^{(s)}_{pb-a+pi} \frac{(b)_i}{(-\gamma_s)^i} \Lambda_s(b)\Lambda_s(b+1)\ldots\Lambda_s(b+i-1) \ .$$

We have, for $m,\ n = 0,1,\ldots$:

$$(1.20) \quad \gamma^{(s)}(a+m,b+n) = \gamma^{(s)}(a,b)\ \frac{(a)_m}{(b)_n}(-\gamma_s)^{n-m}\ \frac{\Lambda_s(a)\Lambda_s(a+1)\ldots\Lambda_s(a+m-1)}{\Lambda_s(b)\Lambda_s(b+1)\ldots\Lambda_s(b+n-1)} .$$

We define the s-<u>th Dwork gamma function</u> $\Gamma_{D,s}$, by :

$$(1.21) \quad \gamma_s^t\ \Gamma_{D,s}(-t+py) = \sum_{i=0}^{\infty} c^{(s)}_{pi+t}\ \frac{(y)_i}{(-\gamma_s)^i} \Lambda_s(y)\Lambda_s(y+1)\ldots\Lambda_s(y+i-1)$$

where $t \in S$ and $\operatorname{ord} y \geqslant -e_s$, and

$$(1.22) \qquad e_s = pa_{s+1} - (p-1)^{-1} = 1-p^{-s}(s+1+(p-1)^{-1}) \ .$$

Clearly :

$$(1.23) \qquad \Gamma_{D,s}(-t+py) \in L(e_s\ ,\ -tp^{-s-1}(s+1+(p-1)^{-1}))$$

so that $\Gamma_{D,s}$ is analytic and bounded on $\bigcup_{t\in S} D(t,(p^{e_s-1})^-)$. If $\varepsilon^{p-1} = 1, \theta_s^{\rho_\varepsilon}(x) = \theta_s(\varepsilon x) = \sum_{i=0}^{\infty} c_i^{(s)}\varepsilon^i x^i$, so that $(c^{(s)}_{pi+t})^{\rho_\varepsilon} = \varepsilon^t c^{(s)}_{pi+t}$ and $\gamma_s^{-t}\ c^{(s)}_{pi+t} \in \mathbb{Q}_p$. We conclude that $\Gamma_{D,s}(-t+py) \in \mathbb{Q}_p[[y]]$. If $a,\ b \in U$, $pb-a = \mu_a \in S$, we have :

$$(1.24) \qquad \gamma_s^{\mu_a}\ \Gamma_{D,s}(a) = \gamma^{(s)}(a,b)$$

so that we obtain the following <u>functional equation of translation</u> for $\Gamma_{D,s}$:

$$(1.25)\qquad \Gamma_{D,s}(a+1) = \begin{cases} -\, a\, \Lambda_s(a)\Gamma_{D,s}(a)\,, & \text{if } |a| = 1 \\ p/\gamma_s^{p-1}\; \Lambda_s(a)/\Lambda_s(a/p)\; \Gamma_{D,s}(a)\,, & \text{if } |a| < 1 \end{cases}$$

valid a priori for $a \in U$, but extended to $\bigcup_{t \in s} D(t,p^{e_s-1})^-)$, by analytic continuation.

Now, if $pb - a \in \mathbf{Z}$:

$\alpha_{\bar{a}}([\omega_{s,a}]) = \gamma^{(s)}(a,b)[\omega_{s,b}] = \gamma^{(s)}(a,b)\; R_s(b)\; [\omega_{1,b}]$

and, on the other hand :

$\alpha_{\bar{a}}([\omega_{s,a}]) = R_s(a)\alpha_{\bar{a}}([\omega_{1,a}]) = R_s(a)\gamma^{(1)}(a,b)\; [\omega_{1,b}]$

so that :

$$(1.26)\qquad \gamma^{(s)}(a,b) = R_s(a)\,/\,R_s(b)\;\gamma^{(1)}(a,b)\,.$$

From (1.26) and the identification ([2], § 21) of $\Gamma_{D,1}$ with the Morita gamma function Γ_p, it follows that for $t \in S$ and $x \in D(-t\,,\,(p^{-1/p\,-\,1/(p-1)})^-)$:

$$(1.27)\qquad \Gamma_{D,s}(x) = (\pi/\gamma_s)^t\; \frac{R_s(x)}{R_s(\frac{t+x}{p})}\; \Gamma_p(x)\;.$$

Either from (1.27) or from (1.21), we deduce that

$$(1.28)\qquad \Gamma_{D,s}(0) = 1\,.$$

Since (1.12) shows that $\Lambda_s(-1) = \Lambda_s(-2) = \ldots = \Lambda_s(1-p) = 1$, it follows from (1.25) that :

$$(1.29)\qquad \Gamma_{D,s}(-i) = 1/i! \quad \text{for} \quad i = 1,2,\ldots,p-1\,.$$

If ord $\Gamma_{D,s}(-t+py)$ were to change in the ball $D(0,(p^{e_s})^-)$, then $\Gamma_{D,s}(-t+py)$ would have a zero there, i.e. $\Gamma_{D,s}(x)$ would have a zero in $D(-t\,,\,(p^{e_s-1})^-)$ for some $t\in S$. From this and the functional equation (1.25) we could then show that $\Gamma_{D,s}(x)$ has infinite zeros in $D(t\,,\,(p^{e_s-1})^-)$ for all $t\in S$, hence $\Gamma_{D,s}$ would be identically zero. This contradiction prove that $\Gamma_{D,s}(x)$ is a unit for all $x \in \bigcup_{t\in S} D(t\,,\,(p^{e_s-1})^-)$.

We conclude :

THEOREM 1.30. <u>The s-th Dwork gamma function</u> $\Gamma_{D,s}(x)$ <u>is analytic on</u> $\bigcup_{t\in S} D(t\,,\,(p^{e_s-1})^-)$ <u>where</u> e_s <u>is given by</u> (1.22), <u>and assumes only unit values</u>. <u>On</u> $D(-t\,,\,(p^{e_s-1})^-)$, $\Gamma_{D,s}(x)=\Gamma_{D,s}(-t+py)$, <u>where</u> $\Gamma_{D,s}(-t+py)\in Z_p[[y]]\cap L(e_s,0)$.

2. DUALITY.

We explain here in different words Dwork's duality theory (see sections 2 and 4 of [2]). Let R_0' (resp. R_∞') denote the ring of Laurent series in x that represent analytic functions in an annulus $1-\varepsilon<|x|<1$ (resp. $1<|x|<1+\varepsilon$) for unspecifiec $\varepsilon>0$. Let R_0 (resp. R_∞) denote the ring of functions analytic in $D(0,1^-)$ (resp. $D(\infty,1^-)$, vanishing at ∞). We view R_0 as contained in R_0' (resp. R_∞ in R_∞') and $R=R_0\oplus R_\infty$ as contained in $R'=R_0'\oplus R_\infty'$.

We observe that L is contained in both R_0' and R_∞' ; we will therefore regard L as embedded diagonally in R'. We have in fact $R'=R\oplus L$ and denote by γ_- (resp. γ_+) the projection of R' onto the first (resp : second) factor. Let $a\in U$, $\omega\in\Omega^1_{\bar a}$, $\eta\in\bar\Omega^1_{-\bar a}$. We define a non-degenerate alternating pairing (which we call "cup-product") :

$$(2.1)\qquad \begin{array}{ccc} W(\bar a)\times\bar W(-\bar a) & \longrightarrow & K \\ ([\omega]\,,\,[\eta]) & \longmapsto & <[\omega]\,,\,[\eta]> \end{array}$$

as follows. Let $\omega=f\omega_{1,a}$ and $=g\bar\omega_{1,1-a}$ with $f, g\in L$; we want to show that there exists a unique $\xi = (\xi_0, \xi_\infty) \in R'$ such that :

$$(2.2) \qquad \overline{D}_{1,1-a}(\xi_i/f) = g\,, \qquad i = 0,\infty\,.$$

This is a consequence of the following lemma.

LEMMA 2.9. Let $a \in U$. The operator $D_{1,a} = E + a + \pi x$ operates bijectively on both R'_0 and R'_∞.

Proof. We consider the action of $D_{1,a}$ on R'_0. Clearly $D_{1,0}$ has no kernel on R'_0. Let $\sum_{i \in \mathbb{Z}} a_i x^i \in R'_0$; we look for $\sum_{i \in \mathbb{Z}} b_i x^i \in R'_0$ such that :

$$(2.3.1) \qquad (E + a + \pi x) \sum_{i \in \mathbb{Z}} b_i x^i = \sum_{i \in \mathbb{Z}} a_i x^i\,.$$

We then have :

$$(2.3.2) \qquad (E + a + \pi x) \sum_{i < 0} b_i x^i = b_{-1}\pi + \sum_{i < 0} a_i x^i\,.$$

But $\sum_{i<0} a_i x^i \in L$ and $E + a + \pi x$ is known ([2], § 21) to have index -1 and 0-Kernel as an operator on L. We conclude that (2.3.2) can be satisfied for precisely one value of b_{-1} and that $\sum_{i<0} b_i x^i \in L$. We then solve formally :

$$(2.3.3) \qquad (E + a + \pi x) \sum_{i \geq 0} b_i x^i = -b_{-1}\pi + \sum_{i \geq 0} a_i x^i\,.$$

Since :

$$(2.3.4) \qquad b_i = \frac{a_i - \pi b_{i-1}}{a + i}\,, \qquad i > 0$$

and

$$(2.3.5) \qquad \lim_{i \to +\infty} |a_i| r^i = 0 \qquad r \in (0,1)\,,$$

we conclude that

$$(2.3.6) \qquad \lim_{i \to +\infty} |b_i| r^i = 0 \qquad r \in (0,1)\,,$$

and therefore that $\sum_{i\geq 0} b_i x^i \in R_0$. Therefore $\sum_{i\in\mathbb{Z}} b_i x^i \in R'_0$. A similar proof for R'_∞. Q.E.D. .

We then define :

$$\langle\omega,\eta\rangle = \sum_{i=0,\infty} \operatorname{Res}_i \xi_i \, dx \ . \tag{2.4}$$

Notice that, formally, $\langle\omega,\eta\rangle = \sum_{i=0,\infty} \operatorname{Res}_i \omega\int\eta$.

We have :

LEMMA 2.5. <u>If</u> $\eta \in d\,\overline{\Omega}^0_{-\underline{a}}$,

$$\langle\omega,\eta\rangle = 0 \ .$$

<u>Proof</u>. Suppose that $\eta = x^{1-a}\hat{\theta}_1^{-1} g \frac{dx}{x} = d(x^{1-a}\hat{\theta}_1^{-1} G)$, $G\in L$. We then have, if $\omega = x^a \hat{\theta}_1 f \frac{dx}{x}$: $\langle\omega,\eta\rangle = \sum_{i=0,\infty} \operatorname{Res}_i \xi_i \, dx$, where $\xi_i = \xi = xfG \in L$. But if $h\in L$, $\sum_{i=0,\infty} \operatorname{Res}_i h dx = 0$. Q.E.D. .

LEMMA 2.6. <u>If</u> $\omega \in \Omega^1_{\underline{a}}$, $\eta \in \overline{\Omega}^1_{-\underline{a}}$:

$$\langle\omega,\eta\rangle = -\langle\eta,\omega\rangle \ .$$

<u>Proof</u>. It is sufficient to prove the assertion for

$$\omega = x^a \hat{\theta}_1 \frac{dx}{x} , \quad \eta = x^{k-a}\hat{\theta}_1^{-1} \frac{dx}{x} , \quad k \in \mathbb{Z} \ .$$

We solve :

$$\begin{array}{ll} \overline{D}_{1,k-a}(\xi_i/x^k) = 1 & \\ & i = 0,\infty \\ D_{1,a}(\xi_i/x^k) = 1 & \end{array} \tag{2.6.1}$$

and prove that $\operatorname{Res}_0(\xi_0 + \tilde{\xi}_0)\frac{dx}{x} = \operatorname{Res}\,(\xi_\infty + \tilde{\xi}_\infty)\frac{dx}{x} = 0$.

Explicitly :

(2.6.2)
$$\xi_0 = \sum_{s=0}^{\infty} \frac{(\pi x)^s x^k}{(k-a)_{s+1}}, \quad \xi_\infty = \sum_{s=1}^{\infty} \frac{(1-k+a)_{s-1}}{(-\pi x)^s} x^k$$
$$\tilde{\xi}_0 = \sum_{s=0}^{\infty} \frac{(-\pi x)^s x^k}{(a)_{s+1}}, \quad \tilde{\xi}_\infty = \sum_{s=1}^{\infty} \frac{(1-a)_{s-1}}{(\pi x)^s} x^k .$$

Then the assertion is obvious. Q.E.D. .

We are then in a position to define, for $\omega \in \Omega^1_{\underline{a}}$, $\eta \in \overline{\Omega}^1_{\underline{-a}}$,

(2.7)
$$< [\omega] , [\eta] > = <\omega,\eta> .$$

We have :

LEMMA 2.8. Let $\omega \in \Omega^1_{\underline{a}}$, $\eta \in \overline{\Omega}^1_{\underline{-a}}$. Then

$$< \alpha_{\underline{a}}([\omega]) , \overline{\alpha}_{\underline{-a}}([\eta]) > = p< [\omega] , [\eta] > .$$

Proof. For $\omega = \omega_{1,a}$, $\eta = \overline{\omega}_{1,1-a}$, we compute :

$$< [\omega_{1,a}] , [\overline{\omega}_{1,1-a}] > = \sum_{i=0,\infty} \mathrm{Res}_i \, \xi_i \frac{dx}{x}$$

where
$$(E + 1-a - \pi x)(\xi_i/x) = 1 , \qquad i = 0,\infty .$$

Therefore

$$\xi_0 = \frac{1}{\pi} \sum_{s=1}^{\infty} \frac{(\pi x)^s}{(1-a)_s}, \quad \xi_\infty = -\frac{1}{\pi} \sum_{s=0}^{\infty} \frac{(a)_s}{(-\pi x)^s} .$$

So :

(2.8.1)
$$< [\omega_{1,a}] , [\overline{\omega}_{1,1-a}] = \frac{1}{\pi} , \text{ independent of } \underline{a} .$$

On the other hand if $pb - a = t \in S$:

$$<\alpha_{\bar{a}}([\omega_{1,a}])\ ,\ \bar{\alpha}_{-\bar{a}}([\bar{\omega}_{1,1-a}])> = \gamma^{(1)}(a,b)\ \overline{\gamma^{(1)}(1-a\ ,\ 1-b)}$$

$$<[\omega_{1,b}]\ ,\ [\bar{\omega}_{1,1-b}]> = \frac{1}{\pi}\ \gamma^{(1)}(a,b)\ \overline{\gamma^{(1)}(1-a\ ,\ 1-b)} =$$

$$= \frac{1}{\pi}\ \pi^{t}\ \Gamma_{D,1}(a)(-\pi)^{p-1-t}\ \Gamma_{D,1}(1-a) =$$

$$= (-1)^{1-t}\ \frac{p}{\pi}\ \Gamma_{D,1}(a)\ \Gamma_{D,1}(1-a) = -(-1)^{t}\ \frac{p}{\pi}\ \Gamma_{p}(a)\Gamma_{p}(1-a) =$$

$$= p<[\omega_{1,a}]\ ,\ [\bar{\omega}_{1,1-a}]>\ ,$$

since $\Gamma_p(a)\Gamma_p(1-a) = -(-1)^t$.

Since $[\omega_{1,a}]$ and $[\bar{\omega}_{1,1-a}]$ span $W(\bar{a})$, $W(-\bar{a})$, respectively, the theorem is proved. Q.E.D. .

We now compute some important cup-products. For $s = 1,2,\dots,\infty$ let us define

$$M_s(y) = 1 + \sum_{j=1}^{s-1} \gamma_{s,j} p^j \sum_{i=0}^{p^j-1} \frac{(y)_{p^j-i-1}(1-y)_i}{\gamma_s^{p^j}} \cdot \tag{2.9}$$

$$\cdot\ \Lambda_s(y)\Lambda_s(y+1)\dots\Lambda_s(y+p^j-i-2)\Lambda_s(1-y)\Lambda_s(2-y)\dots\Lambda_s(i-y)\ ;$$

then M_s is an invertible element of $L(1,0)$.

We have :

LEMMA 2.10. _Let_ $a \in U$ _and_ $s \in \{1,2,\dots,\infty\}$. _Then_

$$<\omega_{s,a}\ ,\ \bar{\omega}_{s,1-a}> = \gamma_s^{-1}\ M_s(a)^{-1}\ .$$

Proof. If

$$\bar{D}_{s,1-a}\ \xi_i = 1\ ,\qquad i = 0,\infty\ ,\qquad \xi_i \in R_i'\ , \tag{2.10.1}$$

then

$$<\omega_{s,a}\ ,\ \bar{\omega}_{s,1-a}> = \sum_{i=0,\infty} \mathrm{Res}_i\ \xi_i\ dx\ . \tag{2.10.2}$$

Therefore, subject to (2.10.1), for $\xi = (\xi_0, \xi_\infty)$:

$$(2.10.3) \qquad \langle \omega_{s,a}, \overline{\omega}_{s,1-a} \rangle = \sum_{i=0,\infty} \mathrm{Res}_i \gamma_-(\xi)_i dx = \mathrm{Res}_\infty \gamma_-(\xi)_\infty \, dx .$$

So, if we set $\langle \omega_{s,a}, \overline{\omega}_{s,1-a} \rangle = \beta \in k$, and

$$\gamma_-(\xi) = (\gamma_-(\xi)_0 , \gamma_-(\xi)_\infty) = (\sum_{i=0}^{\infty} a_i x^i , \sum_{i=1}^{\infty} b_i/x^i) ,$$

we have, for $j \geq 1$:

$$\langle \omega_{s,a+j}, \overline{\omega}_{s,1-a} \rangle = (a)_j/(-\gamma_s)^j \Lambda_s(a)\Lambda_s(a+1)\dots\Lambda_s(a+j-1)\beta =$$

$$= \mathrm{Res}_\infty x^j \gamma_-(\xi)_\infty \, dx = -b_{j+1} , \quad \text{while} \quad \beta = -b_1 .$$

Therefore :

$$(2.10.4) \qquad b_i = -\beta \frac{(a)_{i-1}}{(-\gamma_s)^{i-1}} \Lambda_s(a)\Lambda_s(a+1)\dots\Lambda_s(a+i-2)$$

for $i \geqslant 2$, while $b_1 = -\beta$. We now compute :

$$\overline{D}_{s,1-a}(\gamma_-(\xi)) = \overline{D}_{s,1-a}(\xi) - \overline{D}_{s,1-a}(\gamma_+(\xi)) = 1 - \overline{D}_{s,1-a}(\gamma_+(\xi)) \in L .$$

Therefore :

$$\overline{D}_{s,1-a}(\gamma_-(\xi)) = \overline{D}_{s,1-a}(\gamma_-(\xi)_\infty) = \overline{D}_{s,1-a} (\sum_{i=1}^{\infty} b_i/x^i) =$$

$$= \sum_{i=1}^{\infty} \left[(1-a-i)b_i - \sum_{j=0}^{s-1} \gamma_{s,j} p^j b_{i+p^j}\right] / x^i -$$

$$- \sum_{j=0}^{s-1} \gamma_{s,j} p^j \sum_{i=0}^{p^j-1} b_{-i+p^j} x^i = - \sum_{j=0}^{s-1} \gamma_{s,j} p^j \sum_{i=0}^{p^j-1} b_{p^j-i} x^i ,$$

by (2.10.4) and (1.12). We conclude that

$$\overline{D}_{s,1-a}(\gamma_+(\xi)) = 1 + \sum_{j=0}^{s-1} \gamma_{s,j} p^j \sum_{i=0}^{p^j-1} b_{p^j-i} x^i$$

and, since $\gamma_+(\xi) \in L$, that the differential

$$\eta = \frac{1}{\beta}\left(\overline{\omega}_{s,1-a} + \sum_{j=0}^{s-1} \gamma_{s,j} p^j \sum_{i=0}^{p^j-1} b_{p^j-1}\, \overline{\omega}_{s,1-a+i}\right)$$

is exact, i.e. belongs to $d\,\overline{\Omega}^0_{-\overline{a}}$. Therefore :

$$0 = \langle \omega_{s,a}, \eta \rangle = 1 - \sum_{j=0}^{s-1} \gamma_{s,j} p^j \sum_{j=0}^{p^j-1} \frac{(a)_{p^j-i-1}}{(-\gamma_s)^{p^j-i-1}} \Lambda_s(a)\Lambda_s(a+1)\ldots$$

$$\ldots \Lambda_s(a+p^j-i-2) \frac{(1-a)_i}{\gamma_s^i} \Lambda_s(1-a)\Lambda_s(2-a)\ldots\Lambda_s(i-a) \ .$$

From this we deduce the formula in the statement. Q.E.D. .

THEOREM 2.11. <u>Let</u> $a, b \in U$, $pb - a \in \mathbb{Z}$, <u>and</u> $s \in \{1,2,\ldots,\infty\}$. <u>Then</u> :

$$\gamma^{(s)}(a,b)\overline{\gamma^{(s)}(1-a,1-b)} = p\,\frac{M_s(b)}{M_s(a)} \ .$$

COROLLARY 2.12. (<u>Functional equation of reflection for</u> $\Gamma_{D,s}$) .
<u>Let</u> $s \in \{1,2,\ldots,\infty\}$ <u>and</u> $x \in D(-t,(p^{e_s-1})^-)$ <u>for</u> $t \in S$; <u>we have</u> :

$$\Gamma_{D,s}(x)\Gamma_{D,s}(1-x) = (-1)^t p / \gamma_s^{p-1} \ M_s\left(\frac{t+x}{p}\right) / M_s(x) \ .$$

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