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MINIMAL SURFACES WITH FREE BOUNDARIES AND RELATED PROBLEMS

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1. INTRODUCTION.

These lectures provide a survey on joint work of the author with Nitsche [14] and with Grüter and Nitsche [8], [9]. We are concerned with free boundary value problems for minimal surfaces and related questions. Already in 1816, Gergonne [5] posed a problem of this kind:
"Couper un cube en deux parties, de telle manière que la section vienne se terminer aux diagonales inverses de deux faces opposées, et que l'aire de cette section, terminée à la surface du cube, soit un minimum. Donner, en outre, l'équation de la courbe suivant laquelle la surface coupante coupe chacune des autres faces de ce cube".
This problem remained unsolved for more than half a century. It was H.A. Schwarz [19] who noted in 1872 that a solution of Gergonne's problem must not only have mean curvature zero but has to meet the two faces of the cube, on which its boundary is not preassigned, under a right angle. More generally, he considered a surface M minimizing area among all surfaces bounded in part by given curves Γ, while the rest of its boundary lies on given surfaces $\mathcal{S}$. By applying Gauss' formula for partial integration, he found that M has to intersect the "supporting" surfaces $\mathcal{S}$ orthogonally in a system of curves which one might call the "free trace" Σ of M on $\mathcal{S}$.

Accordingly, Schwarz formulated Gergonne's problem somewhat more generally as follows:

Determine the surfaces of mean curvature zero (i.e., minimal surfaces) bounded by two opposite faces $\mathcal{S}_1$, $\mathcal{S}_2$ of a cube and by a pair of straight arcs Γ_1, Γ_2 connecting four end points of these faces cf. [19] , Tafel 4 , which intersect $\mathcal{S}_1$ and $\mathcal{S}_2$ orthogonally.

He found denumerably many simply connected minimal surfaces without singularities satisfying these boundary conditions. This might seem surprising since it is still not known as to whether there exist closed Jordan curves bounding more than finitely many simply connected minimal surfaces. On the other hand, one can easily find boundary configurations spanning more than denumerably many minimal surfaces which intersect the free part $\mathcal{S}$ orthogonally. For instance, a sphere, a cylinder, or a torus furnish an example, or, more generally, each surface $\mathcal{S}$ which is symmetrical with respect to a 1-parameter group of motions G and which bounds a minimal surface orthogonal to $\mathcal{S}$ but not invariant under G. The boundary configuration of Gergonne's problem is not invariant under a continuous group of motions. Therefore the infinitely many solutions of Schwarz exhibit a rather interesting phenomenon.

Schwarz also described a further rigid configuration possessing infinitely many helicoids as solutions of the corresponding boundary value problem. In a cartesian system of coordinates x, y, z he considers a configuration $\langle\Gamma_1, \Gamma_2, \mathcal{S}\rangle$ consisting of the cylinder surface

$$\mathcal{S} = \{x^2 + y^2 = R^2 \ , \ |z| \leq \pi/4\}$$

and of the two straight arcs

$$\Gamma_1 = \{x = y, \ z = \pi/4\} \ , \quad \Gamma_2 = \{x = -y, \ z = -\pi/4\} \quad .$$

Then the parts of the helicoids

$$M_n^+ = \left\{ \operatorname{tang}(2n+1)z = (-1)^n \frac{y}{x} \right\}$$

and

$$M_n^- = \left\{ \tan g(2n+1)z = (-1)^n \frac{x}{y} \right\} ,$$

$n = 0, \pm 1, \pm 2, \ldots$, which are contained in the solid cylinder formed by the convex hull of $\mathcal{S}$, are bounded by $<\Gamma_1, \Gamma_2, \mathcal{S}>$ and meet $\mathcal{S}$ under a right angle. Their areas $A_n^\pm$ are given by

$$A_n^\pm = \pi \int_0^R \sqrt{1 + (2n+1)^2 r^2}\, dr$$

and are minimal for $n = 0$.

Since the fundamental investigations of Schwarz, a multitude of boundary value problems for minimal surfaces has been investigated, of which free boundary value problems play a prominent role. This does not only include the problem of finding minimal surfaces the boundary of which (or part of it) is left free on supporting manifolds, but also minimal surfaces with movable boundary curves of prescribed length, obstacle problems, and systems of minimal surfaces meeting along one or more curves which are not preassigned.

We do not attempt to describe the variety of results on free boundary problems for minimal surfaces available in the literature. Instead we refer the reader to Chapter VI of J.C.C. Nitsche's lectures on minimal surfaces [18], to the papers [20], [21] and [11], [12], [13], [14] of Jean Taylor and of Hildebrandt-Nitsche.

Here we shall restrict ourselves mainly to the study of minimal surfaces having free or partially free boundaries on prescribed supporting surfaces. In the last section we shall touch upon a free boundary value problem for surfaces of constant mean curvature.

Satisfactory results exist regarding the existence[1] of solutions, and to the behavior of a solution surface near the fixed arcs of its boundary[2], while the behavior of a solution surface at its free boundary had only been studied for

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[1] Cf. [2], chapt. VI, pp. 199-223, and [18], Kap. VI, pp. 431-474.

[2] Cf. [18], Kap. V, pp. 281-348.

absolute minima of the area[3] but not for stationary solutions in general. Recent investigations by Grüter-Hildebrandt-Nitsche [8] and by Dziuk [3] have filled the gap, and the present survey will describe some of the results obtained in those papers. Moreover, we shall state an estimate for the length of the trace of a minimal surface on the free part of the boundary, derived by Hildebrandt-Nitsche [14].

2. REGULARITY AT THE BOUNDARY.

The investigation of the boundary behavior of minimal surfaces in $\mathbb{R}^3$ with a free boundary can be reduced to the study of mixed boundary value problems for vector-valued functions

$$X(u,v) = (x^1(u,v),\ x^2(u,v),\ x^3(u,v))$$

satisfying a system of equations

$$\Delta x^\ell + \Gamma^\ell_{ik}(X)\{x^i_u x^k_u + x^i_v x^k_v\} = 0 \tag{2.1}$$

where $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is the ordinary Laplacian, and Γ^ℓ_{ik} are the Christoffel symbols of second kind with respect to a symmetric positive definite matrix (g_{ik}). It turns out that the system (2.1) are the Euler equations of an integral of the form

$$\mathcal{I}_\Omega(X) = \int_\Omega f(u,v,X,\nabla X)\,du\,dv\ ,\quad \Omega \subset \mathbb{R}^2\ , \tag{2.2}$$

where $f(u,v,x,p)$ is a continuous function of its variables (u,v,x,p) such that

$$m_1|p|^2 - m_o \le f(u,v,x,p) \le m_2|p|^2 + m_o,\ 0 < m_1 \le m_2,\ m_o \ge 0 \tag{2.3}$$

and

[3] Cf. [18], Kap. VI, pp. 447-474, and [11], [12], [13].

$$(2.4)\qquad \begin{aligned} & f_{p^i_\alpha p^j_\beta}(u,v,x,p)\ \xi^i_\alpha \xi^j_\beta \geq \lambda|\xi|^2 \ , \quad \lambda > 0 \ , \\ & |f_{p^i_\alpha p^j_\beta}| \leq \mu \end{aligned}$$

holds for all $(u,v) \in \Omega$, $x \in \mathbb{R}^3$, $p,\xi \in \mathbb{R}^6$ $(p = (p^i_\alpha),\ \ \xi = (\xi^i_\alpha),\ 1 \leq i \leq 3,\ 1 \leq \alpha \leq 2)$.

In fact, the integrand f associated with (2.1) is

$$(2.5)\qquad f(u,v,x,p) = \frac{1}{2}\, g_{ik}(x)\left\{p^i_1 p^k_1 + p^i_2 p^k_2\right\} = \frac{1}{2}\, g_{iK}(x) p^i_\alpha p^k_\alpha \ .$$

Before we turn to the boundary regularity of stationary surfaces X of (2.2), we shall briefly review the situation with respect to interior regularity.

Morrey [17] has proved the following celebrated result:

Suppose that f *satisfies* (2.3), *and that* $X \in H^1_2(\Omega,\mathbb{R}^3)$ *fulfils*

$$\mathcal{I}_\Omega(X) \leq \mathcal{I}_\Omega(X+\Phi) \quad \textit{for all} \quad \Phi \in H^1_2(\Omega',\mathbb{R}^3)\ ,\ \Omega' \subset\subset \Omega \ .$$

Then X *is* Hölder *continuous in* Ω.

Starting from this point, one can prove higher regularity for every minimum X of $\mathcal{I}_\Omega$.

However, it is impossible to carry over Morrey's theorem to stationary points of (2.1) in general since Frehse [4] has constructed an integrand f satisfying (2.3) as well as the ellipticity condition (2.4), for which the integral (2.1) possesses a critical point X of class H^1_2. Thus it was unclear whether "weak" minimal surfaces in a Riemannian manifold are regular, i.e., are classical minimal surfaces.

However, the integral

$$(2.6)\qquad E_\Omega(X) = \frac{1}{2}\int_\Omega g_{ik}(X)\left\{x^i_u x^k_u + x^i_v x^k_v\right\} du\, dv$$

is invariant with respect to conformal transformations of the independent variables. Thus it is well known that critical points of E_Ω with respect to boundary conditions of the Plateau type or to free boundary conditions are a. e. on Ω conformally parametrized, i. e., we have

$$(2.7) \qquad g_{ik}(X)\, x^i_u x^k_u = g_{ik}(X)\, x^i_v x^k_v \;,\; g_{ik}(X)\, x^i_u x^k_v = 0 \quad \text{a. e. in } \Omega \;.$$

Thus weak minimal surfaces in a Riemannian manifold are weak solutions of (2.1), contained in H^1_2, and satisfying in addition (2.7).

Grüter has proved in his remarkable thesis [6, 7] that one can derive from this fact the *interior regularity of each minimal surface in a Riemannian manifold*[4]. The papers [8] and [3] employ Grüter's technique to tackle the regularity of minimal surfaces at a free boundary. The approach of [8] is somewhat more flexible and permits also the discussion of obstacle problems as considered in [11], [12], [13], while the method of [3] is based on the reflection principle introduced by Jäger so that is cannot be applied to the obstacle problem. On the other hand, Dziuk's technique yields Hölder continuity of ∇X assuming only that the supporting surface $\mathcal{S}$ is of class $C^{1,1}$.

In the following we shall restrict ourselves to the consideration of minimal surfaces with a partially free boundary. Since our approach will contain all the essential ideas, it can as well be applied to other free boundary value problems. Thus we consider the following type of boundary configurations in $\mathbb{R}^3$:

Let Γ be a regular arc in $\mathbb{R}^3$ having its two end points P_1 and P_2, $P_1 \neq P_2$, on a two-dimensional surface $\mathcal{S}$ of $\mathbb{R}^3$, but which has no other points in common with $\mathcal{S}$.

We identify the two-dimensional Euclidean space $\mathbb{R}^2$ with $\mathbb{C}$, and write accordingly $w = (u,v) = u + iv$ for the points of $\mathbb{R}^2$. We shall choose the open semi-disc

$$B = \{w : |w| < 1,\ v > 0\}$$

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[4] A more general result has recently been obtained by R. Schoen.

as parameter domain of the surfaces $X = X(w)$ which will be considered.

Denote by C the closed circular arc $\{w : |w| = 1,\ v \geq 0\}$ and by I the open interval $\{w : |w| < 1,\ v = 0\}$, so that $\partial B = C \cup I$.

Then we introduce the class $\mathcal{C} = \mathcal{C}(\Gamma, \mathcal{S})$ of *admissible surfaces* $X = (x^1(w), x^2(w), x^3(w))$ as set of mappings $X \in H_2^1(B, \mathbb{R}^3)$ which are bounded by the configuration $\langle \Gamma, \mathcal{S} \rangle$ in the following sense: For $X \in \mathcal{C}$, let X_C and X_I be the L_2-traces of X on C and I, correspondingly. Then X_C maps C continuously and in a weakly monotonic manner onto Γ such that $X_C(-1) = P_1$ and $X_C(1) = P_2$, while $X_I(w) \in \mathcal{S}$ L^1-almost everywhere on I.

For $X \in H_2^1(B, \mathbb{R}^3)$ we introduce the Dirichlet integral by

$$D_B(X) := \frac{1}{2} \int_B |\nabla X|^2 du\, dv$$

where $\nabla X = (X_u, X_v)$ is the weak gradient of X, and

$$|\nabla X| = \left(|X_u|^2 + |X_v|^2\right)^{1/2} = \left(x_u^i x_u^i + x_v^i x_v^i\right)^{1/2}$$

denotes its Euclidean length.

As usual, a mapping $X : B \to \mathbb{R}^3$ is said to be a *minimal surface* (parametrized on the domain B) if it is real analytic, and if it satisfies Laplace's equation

$$\Delta X = 0$$

as well as the conformity relations

$$|X_u|^2 = |X_v|^2 , \quad X_u \cdot X_v = 0$$

on B, and $X(w) \not\equiv \text{const}$ on B.

Furthermore we define an *admissible variation* of a surface $X \in \mathcal{C}$ as family $\{X_\varepsilon\}_{|\varepsilon| < \varepsilon_o}$, $\varepsilon_o > 0$, of surfaces $X_\varepsilon \in \mathcal{C}$ of the form $X_\varepsilon(w) = X(w) + \varepsilon\Psi(w, \varepsilon)$ such that $D_B(\Psi_\varepsilon)$ is bounded independently of ε, and that $\lim_{\varepsilon \to 0} \Psi_\varepsilon(w)$ exist for almost all $w \in B$, where $\Psi_\varepsilon = \Psi(\cdot, \varepsilon)$.

A surface $X \in \mathcal{C}$ is said to be *stationary in* $\mathcal{C}$ if

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[D_B(X_\varepsilon) - D_B(X)\right] = 0$$

holds for all admissible variations of X.

It is known that a nonconstant surface $X \in \mathcal{C}$, which is stationary in $\mathcal{C}$, has to be a minimal surface (parametrized on B). We call it a *minimal surface which is stationary in* $\mathcal{C}$. It turns out that a stationary minimal surface intersects $\mathcal{S}$ orthogonally if it is of class C^1 at its free boundary.

In the following, we shall describe the behavior of minimal surfaces $X \in \mathcal{C}$, which are stationary in $\mathcal{C}$, at their *free boundary* I. For this purpose, we have at least to assume that $\mathcal{S}$ is a regular two-dimensional surface in $\mathbb{R}^3$, without self-intersections and without boundary, which is of class C^3. Moreover, we have to impose an assumption (V) which is a uniformity condition at infinity. This assumption will automatically be satisfied if $\mathcal{S}$ is also compact.

Assumption (V). $\mathcal{S}$ is a two-dimensional manifold of class C^2, imbedded into $\mathbb{R}^3$ and without boundary, for which there exist numbers $\rho_o > 0$, $K \geq 0$, and K_1, K_2 with $0 < K_1 \leq K_2$ such that the following holds:
For each $f \in \mathcal{S}$, there exist a neighborhood U of f in $\mathbb{R}^3$ and a C^3-diffeomorphism h of $\mathbb{R}^3$ onto itself such that the inverse h^{-1} maps f onto 0, and U onto the open ball $\{y : |y| < \rho_o\}$ such that $\mathcal{S} \cap U$ is mapped onto the set $\{y : |y| < \rho_o,\ y^3 = 0\}$ of the hyperplane $\{y^3 = 0\}$.

Moreover, if $g_{ik}(y) := h^l_{y^i}(y) h^l_{y^k}(y)$, (summation with respect to l from 1 to 3!), then we have

$$K_1 |\xi|^2 \leq g_{ik}(y) \xi^i \xi^k \leq K_2 |\xi|^2$$

for all $\xi \in \mathbb{R}^3$ and all $y \in \mathbb{R}^3$, and also

$$\left| \frac{\partial g_{ik}(y)}{\partial y^l} \right| \leq K$$

for all $y \in \mathbb{R}^3$ and all $i,k,l \in \{1,2,3\}$.

Finally, for every point $x^* \in \mathbb{R}^3$, there exists a point $f \in \mathcal{S}$, such that $|x^* - f| = \operatorname{dist}(\mathcal{S}, x^*)$ provided that $\operatorname{dist}(\mathcal{S}, x^*) < \frac{1}{4} \rho_o \sqrt{K_2}$.

Now we can formulate the main result of this section:

Theorem 1. *Suppose that* $X : B \to \mathbb{R}^3$ *is a minimal surface of class* $\mathcal{C}(\Gamma, \mathcal{S})$ *which is stationary in this class. Furthermore, let* $\mathcal{S}$ *be a supporting surface which satisfies assumption* (V). *Then* X *is of class* $C^{2,\beta}(B \cup I, \mathbb{R}^3)$ *for every* $\beta \in (0,1)$. *Moreover, if* $w_o \in I$ *is a branch point of* X *on the free boundary, i.e.* $X_w(w_o) = 0$, *then there exist a vector* $b = (b^1, b^2, b^3) \in \mathbb{C}^3$ *with* $b \neq 0$ *and* $b \cdot b = 0$, *and an integer* $\nu \geq 1$, *such that*

$$X_w(w) = b \cdot (w - w_o)^\nu + o(|w - w_o|^\nu) \quad \text{as} \quad w \to w_o \quad ,$$

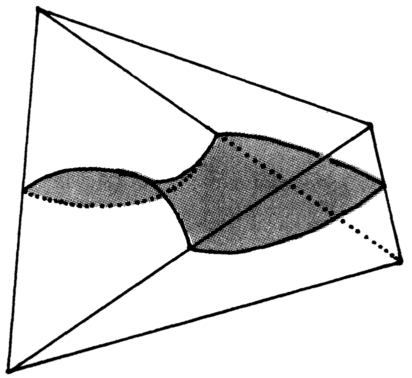
where $X_w = \frac{1}{2}(X_u - iX_v)$. *Consequently, the surface normal*

$$N(w) = \frac{X_u(w) \wedge X_v(w)}{|X_u(w) \wedge X_v(w)|}$$

tends to a limit vector as $w \to w_o$. *That is, the tangent plane of* X *tends to a limiting position as* w *tends to a branch point on the free boundary. Moreover, the nonoriented tangent of the trace* $\{X(w) : w \in I\}$ *of the minimal surface on* $\mathcal{S}$ *moves continuously through a boundary branch point. The oriented tangent is continuous at branch points* w_o *of even order* ν, *but, for branch points of odd order, the tangent direction jumps by* 180^o *degrees.*

Finally, X *is of class* $C^{s,\alpha}(B \cup I, \mathbb{R}^3)$ *if also* $\mathcal{S} \in C^{s,\alpha}$, $s \geq 2$, $0 < \alpha < 1$, *and* X *is real analytic on* $B \cup I$, *if* $\mathcal{S}$ *is real analytic.*

An interesting, non-planar, and not area minimizing but stationary minimal surface with boundary on $\mathcal{S}$ has been exhibited by H.A. Schwarz, Gesammelte Math. Abhandlungen I, pp. 149-150. We present the picture of this surface, due to Schwarz, in figure 1:



We sketch the proof of Theorem 1, starting with the following fundamental observation:

Lemma 1. *Let* $X : B \to \mathbb{R}^3$ *be a minimal surface. Then, for each open subset* Ω *of* B *and for every point* $w^* \in \Omega$ *, we get*

$$\limsup_{\sigma \to 0} \frac{1}{\sigma^2} \int_{\{w \in \Omega : |X(w) - X(w^*)| < \sigma\}} |\nabla X|^2 \, du \, dv \geq 2\pi \quad .$$

The *proof* of this formula can easily be derived from the well known asymptotic expansion of minimal surfaces; cf. [18], §361.

The next estimate is essentially due to Courant and Lebesgue (cf. [2], p. 102). We introduce the following notations:

Let $w = (u,v) = u + iv$ be a point of $\mathbb{R}^2 \cong \mathbb{C}$, and set $S_r(w_o) = \{w : |w - w_o| <$ $< r,\ v > 0\}$, $C_r(w_o) := \{w : |w - w_o| = r,\ v > 0\}$, $I := \{w : |w| < 1,\ v = 0\}$.

Lemma 2. *For each* $X \in C^1(B, \mathbb{R}^3)$ *, for every* $w_o \in I$ *, and for each* $R_o \in (0, 1 - |w_o|)$ *, there is a number* $r \in [R_o/2, R_o]$ *such that*

$$\operatorname{osc}_{C_r(w_o)} X \leq \sqrt{\pi/\log 2} \left\{ \int_{S_{R_o}(w_o)} |\nabla X|^2 \, du \, dv \right\}^{1/2} \quad .$$

The following estimate follows from a simple application of the triangle inequality:

Lemma 3. *Let* $w_o \in I$, $r \in (0,1-|w_o|)$ *, and suppose that* X *is a surface of class* $C^1(B,\mathbb{R}^3)$ *. Assume also that, for some positive numbers* α_1 *and* α_2 *,*

$$\operatorname{osc}_{C_r(w_o)} X \leq \alpha_1$$

and

$$\sup_{w^* \in S_r(w_o)} \inf_{w \in C_r(w_o)} |X(w) - X(w^*)| \leq \alpha_2 \quad .$$

Then

$$\operatorname{osc}_{S_r(w_o)} X \leq 2\alpha_1 + 2\alpha_2 \quad .$$

The crucial estimate of our regularity result is contained in the following

Lemma 4. *Let* $w_o \in I$ *, and suppose that* $X : B \to \mathbb{R}^3$ *is a minimal surface of class* $\mathfrak{C} = \mathfrak{C}(\Gamma,\mathcal{S})$ *which is stationary in* $\mathfrak{C}$*. Assume also that the supporting surface* $\mathcal{S}$ *satisfies the assumption* (V) *with constants* ρ_o, K, K_1, K_2*. Then, for* $\rho_1 := \rho_o\sqrt{K_2}$ *and for some number* K_3 *depending only on* ρ_o, K, K_1 *and on* K_2 *, the following holds:*

If $w^* \in S_r(w_o)$, $0 < r < 1 - |w_o|$, $0 < R < \rho_1$ *, and if*

$$\inf_{w \in C_r(w_o)} |X(w) - X(w^*)| > R \quad ,$$

then

$$R \leq K_3\sqrt{e(w_o,r)}$$

where we have set

$$e = e(w_o,r) := \int_{S_r(w_o)} |\nabla X|^2 \, du \, dv \quad .$$

The proof of this lemma is rather complicated. We shall briefly indicate the main ideas of the proof at the end of this section and proceed presently with the verification of Theorem 1:

Choose an arbitrary point $w_o \in I$ and an arbitrary number R with $0 < R < \rho_1$.

Since $D_B(X) < \infty$, we can find a number $R_o \in (0,1-|w_o|)$ such that

$$R > K_3\sqrt{e(w_o,R_o)} \quad .$$

Then we infer from Lemma 4 that

$$\sup_{w^* \in S_r(w_o)} \inf_{w \in C_r(w_o)} |X(w) - X(w^*)| \leq R$$

for every $r \in (0,R_o]$.

Moreover, in virtue of Lemma 2, there exists a number $r \in \left[\frac{1}{2} R_o , R_o\right]$ such that

$$\operatorname{osc}_{C_r(w_o)} X \leq K_4\sqrt{e(w_o,R_o)} < (K_4/K_3)R$$

where $K_4 := \sqrt{\pi/\log 2}$.

On account of Lemma 3, we obtain that

$$\operatorname{osc}_{S_r(w_o)} X \leq 2(1+K_4/K_3)R \quad .$$

That is,

$$\lim_{r\to 0} \operatorname{osc}_{S_r(w_o)} X = 0 \quad .$$

Thus we have proved that X is continuous on $B \cup I$.

Next one proves by a "hole-filling" device that $X \in C^{o,\mu}(B \cup I, \mathbb{R}^3)$. This is by now more or less standard. For details, we refer the reader to [8], pp. 19-21.

From here on, well known techniques furnish the statement of Theorem 1; cf. [18], pp. 447-474 and p. 707 for references.

The proof of Lemma 4 proceeds as follows:

Let $w^* \in S_r(w_o)$, $0<R<\rho_1$, and set $x^* = X(w^*)$, $\delta(w^*) = \operatorname{dist}(S,x^*)$.

If $\delta(x^*) > 0$ we choose

$$\eta(w) = \begin{cases} 0 & \text{if } w \notin \overline{S_r(w_o)} \\ \lambda(\rho - |X(w) - x^*|)\{X(w) - x^*\} & \text{if } w \in \overline{S_r(w_o)} \end{cases}$$

where $\lambda = \lambda_\varepsilon \in C^1(\mathbb{R},\mathbb{R})$, $\lambda' \geq 0$, $\lambda(t) = 0$ for $t \leq 0$, and $\lambda(t) = 1$ if $t \geq \tilde{\varepsilon}$.

It turns out that $X_\varepsilon = X + \varepsilon\eta$ is an admissible variation of X so that

$$0 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left\{ D_B(X_\varepsilon) - D_B(X) \right\} = \int_{S_r(w_o)} \nabla X \cdot \nabla \eta \, du \, dv \quad .$$

Employing the conformality relations for X, letting $\tilde{\varepsilon}$ tend to zero, and taking Lemma 1 into account, we may infer that

$$2\pi \leq R^{*-2} \int_{S_r(w_o) \cap K_{R^*}(x^*)} |\nabla X|^2 du \, dv$$

where we have set

$$R^* = \min\{\delta(x^*), d^2R\} \ , \quad K_\tau(x^*) = \{w \in B : |X(w) - x^*| < \tau\} \quad .$$

Hence

$$\text{(2.8)} \qquad R \leq \left\{ \frac{1}{2\pi d^4} \int_{S_r(w_o)} |\nabla X|^2 \, du \, dv \right\}^{1/2} \quad \text{if} \quad R^* = d^2R \quad (\text{i.e.} \quad d^2R = \delta(x^*)) \quad .$$

If $\delta(x^*) < d^2R$ (which will also include the case $\delta(x^*) = 0$) we have already found that

$$\text{(2.9)} \qquad 2\pi \leq \delta(x^*)^{-2} \int_{S_r(w_o) \cap K_{\delta(x^*)}(x^*)} |\nabla X|^2 \, du \, dv \quad .$$

Then there exists a point $f \in \mathcal{S}$ such that

$$|f - x^*| = \delta(x^*) < d^2R \leq \frac{1}{4} R \quad .$$

We choose f as center of a new system of coordinates as indicated in assumption (V), with the defining diffeomorphism h, and we introduce $Y := h^{-1} \circ X$. Let $g_{ij}(Y) = h^\ell_{y^i}(Y) h^\ell_{y^j}(Y)$ be the components of the associated fundamental tensor, and set

$$||Y(w)||^2 := g_{ij}(Y(w)) y^i(y) y^j(w) \quad .$$

If $d^{-1}\delta(x^*) < p < dR$, we define

$$\eta(w) = \begin{cases} 0 & \text{if } w \notin B - \overline{S_r(w_o)} \\ \lambda(\rho - \|Y(w)\|)Y(w) & \phantom{\text{if }} w \in \overline{S_r(w)} \end{cases}$$

For sufficiently small $|\varepsilon|$, the family of surfaces

$$X_\varepsilon(w) = h(Y(w) + \varepsilon\eta(w))$$

forms an admissible variation of $X(w)$. Hence we infer from

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left\{ D_B(X_\varepsilon) - D_B(X) \right\} = 0$$

that

$$\int_{S_r(w_o)} \left\{ g_{ik}(Y) D_\alpha y^i D_\alpha \eta^k + \frac{1}{2} g_{ik,y^\ell}(Y) D_\alpha y^i D_\alpha y^k \eta^\ell \right\} du\, dv = 0$$

where $u^1 = u$, $u^2 = v$, $D_\alpha = \dfrac{\partial}{\partial u^\alpha}$, and $Y(w) = (y^1(w), y^2(w), y^3(w))$.

By a similar reasoning as before, we obtain that

$$\frac{1}{\delta^2(x^*)} \int_{S_r(w_o) \cap K_{2\delta(x^*)}(f)} |\nabla X|^2\, du\, dv$$

(2.10)

$$\leq \frac{C(R)}{d^4} \frac{1}{R^2} \int_{S_r(w_o)} |\nabla X|^2\, du\, dv \quad .$$

By virtue of $K_{\delta(x^*)}(x^*) \subset K_{2\delta(x^*)}(f)$, we derive from (2.9) and (2.10) the inequality

$$R^2 \leq \frac{C(R)}{2\pi d^2} \int_{S_r(w_o)} |\nabla X|^2\, du\, dv \quad \text{if} \quad 0 < \delta(x^*) < d^2 R \quad . \tag{2.11}$$

In the case $\delta(x^*) = 0$ (i.e., $x^* = f$), we arrive at

$$R^2 \leq \frac{C(R)}{8\pi d^4} \int_{S_r(w_o)} |\nabla X|^2\, du\, dv \quad \text{if} \quad \delta(x^*) = 0 \quad . \tag{2.12}$$

From (2.8), (2.11), and (2.12), we infer that $R \leq K_3\sqrt{e}$ for some number K_3 as described in the assertion of Lemma 4, q.e.d.

Remarks.

1) If the supporting surface $\mathcal{S}$ is a manifold with boundary, then any stationary minimal surface X in $\mathcal{C}$ is of class $C^{1,1/2}(B \cup I, \mathbb{R}^3)$, and this result is, in general, optimal; cf. [12] and [13].

2) One should note that our regularity proof does not yield a priori estimates for minimal surfaces X stationary in $\mathcal{C}$, since one needs to know how fast

$$\int_{S_R(w_o)} |\nabla X|^2 \, du \, dv$$

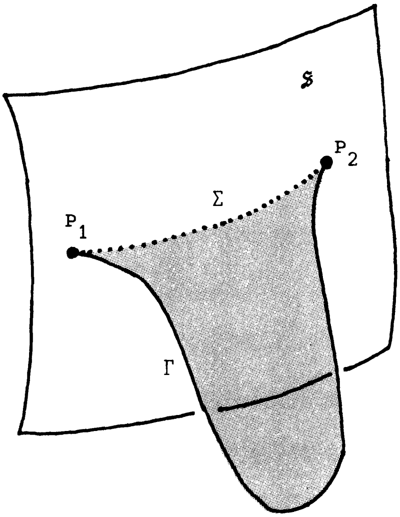
tends to zero as $R \to 0$.

In order to obtain such estimates we have to specify the conformal representation X of every minimal surface by fixing a 3-point-condition. For instance, we may look for estimates in the class $\mathcal{C}^* := \{X \in \mathcal{C} : X(i) = P_3\}$ where P_3 is a fixed point of Γ between P_1 and P_2. Yet we shall be disappointed since Schwarz' helicoid example from section 1 exhibits a boundary configuration $\langle \Gamma_1, \Gamma_2, \mathcal{S} \rangle$ bounding infinitely many "really different" stationary minimal surfaces, the modulus of continuity of which can obviously not be bounded, nor exists a bound on their area. Thus the best one can hope for are estimates for $X \in \mathcal{C}^*$ which only depend on Γ, $\mathcal{S}$, P_3, and on the area $D(X)$.

Such a result has recently been proved by Ye for various classes of supporting surfaces $\mathcal{S}$, assuming that X has no branch points of odd order on I (cf. section 3). In particular, one obtains a priori estimates if $\mathcal{S}$ is smooth and oriented, and if there exist a constant number $\sigma > 0$ and a constant unit vector n_o such that $n \cdot n_o \geq \sigma$ holds for some field of unit normals n on $\mathcal{S}$.

3. GEOMETRIC PROPERTIES OF THE TRACE.

We consider in the following a boundary configuration $\langle \Gamma, \mathcal{S} \rangle$ with the same properties as in section 2. Let $L(\Gamma) < \infty$ be the length of the arc Γ. Moreover, let $\mathcal{S}$ be a regular orientable surface of class C^3 such that



(i) $\mathbb{R}^3 - \mathcal{S}$ consists of two disjoint opens sets;

(ii) For each point P of $\mathcal{S}$, there are two spheres of radius R, one on each side of $\mathcal{S}$, such that $\mathcal{S}$ has no points in common with the interior of these spheres.

Finally we assume that X is a minimal surface as in section 2, parametrized over the semidisc B, bounded by $\langle\Gamma,\mathcal{S}\rangle$ and stationary for the Dirichlet integral in the class $\mathcal{C}(\Gamma,\mathcal{S})$. Then the following holds:

Theorem 2. *For* $c = 2$ *, the length* $L(\Sigma) = \int_I |dX|$ *of the trace* $\Sigma = \{X(w) : w \in I\}$ *of the free boundary of* X *can be estimated by*

$$(3.1) \qquad L(\Sigma) \leq L(\Gamma) + \frac{c}{R} D_B(X)$$

provided that X *possesses no branch points of odd order on* I*. Here* $L(\Gamma)$ *denotes the length of the arc* Γ*. Moreover, the constant* $c = 2$ *is optimal.*

Remarks.

1) The estimate (3.1) has been proved by Hildebrandt-Nitsche [14] for $c = 7$. By sharpening the estimates of [14], Küster [15] has established the optimal result $c = 2$.

2) Küster has also pointed out that neither an estimate of the form

$$L(\Sigma) \leq L(\Gamma) + c \cdot \sqrt{K_S} \cdot D(X)$$

nor of the form

$$L(\Sigma) \leq L(\Gamma) + c \cdot H_S \cdot D(X)$$

can hold, where K_S and H_S are upper bounds for the absolute value of the Gauss curvature K and of the mean curvature H of $\mathcal{S}$, respectively. The first estimate can be disproved by taking $\mathcal{S}$ as an appropriate part of a cylinder surface, the second one by some part of a catenoid.

3) In [14], also the following has been proved:

(i) X has no branch points on I if it lies on one side of $\mathcal{S}$. This is, for instance, the case if $\mathcal{S}$ is the boundary of an open star-shaped convex or H-convex set (i.e., $H \geq 0$), and Γ is contained in $\Omega \cup \mathcal{S}$.

(ii) X has no branch points of odd order on I if it minimizes area in $\mathcal{C}$.

(iii) X has no branch points at all if $\mathcal{S}$ is real analytic and X minimizes area in $\mathcal{C}$.

4) An immediate consequence of Theorem 2 is the following result:

$X(u,v)$ *is continuous on* $\overline{B} = B \cup C \cup I$ *if it does not possess any branch points of odd order on* I.

The proof of Theorem 2 follows from Gauss' formula

$$\int_B \nabla X \cdot \nabla \eta \, du\, dv = -\int_B \eta \cdot \Delta X \, du\, dv + \int_{\partial B} \frac{\partial X}{\partial \nu} \cdot \eta \, d\mathcal{H}^1$$

by inserting the test function $\eta = \zeta(X)$, where ζ denotes a smooth vector field $\zeta(x)$ on $\mathbb{R}^3$ which is on $\mathcal{S}$ of length one and orthogonal to $\mathcal{S}$. Then we obtain on I that

$$\frac{\partial X}{\partial \nu} \cdot \eta = |X_v| = |X_u|$$

whence

$$\int_I \frac{\partial X}{\partial \nu} \cdot \eta \, d\mathcal{H}^1 = \int_I |X_u| \, du = L(\Sigma) \quad .$$

If we assume also $|\zeta| \leq 1$ on $\mathbb{R}^3$, we get

$$\int_C \frac{\partial X}{\partial \nu} \cdot \eta \, d\mathcal{H}^1 \leq L(\Sigma)$$

and therefore

$$L(\Sigma) \leq L(\Gamma) + \text{const } D_B(X) \quad .$$

A suitable choice of the vector field ζ leads to the more precise estimate stated in Theorem 2.

4. A PARTITIONING PROBLEM.

Let K be a convex body in $\mathbb{R}^3$ with boundary $\mathcal{S}$. Then we consider the following partitioning problem:

Determine a rectifiable surface $\overline{F}$ of minimal or at least of stationary area, with boundary Σ on $\mathcal{S}$ which divides K into two parts K_1 and K_2 such that

$$\text{(4.1)} \qquad \text{meas}K_1 = \sigma \cdot \text{meas}K \;, \quad \text{meas}K_2 = (1-\sigma)\text{meas}K$$

where σ denotes a preassigned number with $0 < \sigma < 1$.

We shall not discuss the existence of solutions to this problem but shall restrict ourselves to the question of regularity at the boundary.

In order to have a clear cut situation, we shall assume that the "inner part" F of $\overline{F}$ is a regular C^1-surface of the type of the disc which is given by a conformal parameter representation $X : B \to \mathbb{R}^3$ of class $C^1(B,\mathbb{R}^3)$ on the unit disc and has finite area, i.e., $X \in H^1_2(B,\mathbb{R}^3)$. We suppose that

$$\Sigma := X(\partial B) \subset S$$

(i.e. the trace $X/\partial B$ maps a.a. points of ∂B into $\mathcal{S}$), but $X(B)$ omits a neighborhood of some point of $\mathcal{S}$.

Finally we assume that X embeds B into the interior of K such that

$$\text{int}K - F = \Omega_1 \cup \Omega_2 \;, \quad \Omega_1 \cap \Omega_2 = \phi \quad ,$$

where Ω_1 and Ω_2 are simply connected and of measure $\sigma \cdot \text{meas}K$ and $(1-\sigma) \cdot \text{meas}K$, respectively.

Then the following result is a special case of a theorem proved in [9].

Theorem 3. *Let* S *be a regular surface of class* $C^{s,\alpha}$, $s \geq 3$, $0 < \alpha < 1$ *, or real analytic, respectively. Moreover, let* $X : B \to F \subset \mathbb{R}^3$ *be stationary for the area functional in the class of disc type surfaces with boundary on* S *which decompose* K *into two parts of measure* $\sigma \cdot \mathrm{meas}K$ *and* $(1-\sigma) \cdot \mathrm{meas}K$.
Then X *is a real analytic surface of constant mean curvature on* B *which is of class* $C^{s,\alpha}(\bar{B},\mathbb{R}^3)$ *, or real analytic on* $\bar{B}$ *, respectively, and which intersects* S *orthogonally in* Σ.

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