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Astérisque, tome 118 (1984), p. 55-67
<http://www.numdam.org/item?id=AST_1984__118__55_0>

Société Mathématique de France
Astérisque 118 (1984) p.55 à 67.

FREE BOUNDARY PROBLEM IN FLUID DYNAMICS

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PART I: ONE FLUID PROBLEMS.

Let N be a $C^{1+\alpha}$ curve $x = g(y)$ $(b \le y < \infty)$ with $b > 0$, $g(b) = 0$ and $\lim_{y\to\infty} g(y) = -\infty$. Consider the 2-dimensional symmetric jet problem: Find a function u and a curve Γ in $\{0 < y \le b\}$ satisfying

(1.1) $N \cup \Gamma$ is a continuous curve, such that with G denoting the domain bounded by $N \cup \Gamma$ and the x-axis I_0, the following properties hold:

(1.2) $0 < u < 1$ in G ,

(1.3) $\Delta u = 0$ in G ,

(1.4) $u = 0$ on I_0 ,

(1.5) $u = 1$ on $N \cup \Gamma$,

(1.6) $\frac{\partial u}{\partial \nu} = \lambda$ on $N \cup \Gamma$;

finally, Γ is required to be smooth, say in $C^{1+\delta}$, and

(1.7) $N \cup \Gamma$ is a C^1 curve in $B_\delta(A)$, u is in $C^1(\overline{G} \cap B_\delta(A))$

for some $\delta > 0$, where $A = (0,b)$ and $B_r(X)$ denotes the disc of radius r and center X. The parameter λ is to be determined together with u, Γ.

This problem was solved by Leray [21], Serrin [22] and others; see [11] [20] and the references given there. The methods used by most authors are based on

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This work is partially supported by National Science Foundation Grant MCS 791 5171.

reducing the problem by means of the hodograph variables to a nonlinear integral equation, and then solving the equation with the aid of the Leray-Schauder fixed point theorem. There are however some other methods, based on variational principles; see Garabedian and Spencer [18] and Garabedian, Lewy and Schiffer [17].

A new variational approach was introduced by Alt, Caffarelli and Friedman (ACF) [2]; it exploits the paper of Alt and Caffarelli [1] who studied the following variational problem. Let

$$J(v) = \int_\Omega \left(|\nabla v|^2 + Q(x) I_{\{v>0\}} \right) dx \tag{1.7}$$

and let

$$K = \left\{ v \in H^{1,2}(\Omega),\ v = u^0 \text{ on a part } S \text{ of } \partial\Omega \right\} ,$$

where Q is a given positive C^α function in $\overline{\Omega}$, $\Omega \subset R^n$ and $u^0 \geq 0$. Consider the problem of finding $u \in K$ such that

$$J(u) = \min_{v \in K} J(v) \quad .$$

It is proved in [1] that any minimizer u is Lipschitz continuous, and the free boundary $\Gamma = \partial\{u>0\}$ is smooth if $n = 2$ (analytic if Q is analytic). Further

$$\frac{1}{r} \fint_{B_r(x^0)} u \geq c \quad (c>0) \quad \text{if} \quad x_0 \in \Gamma \quad \text{(non-degeneracy)}$$

<u>Theorem 1.1</u> [2]. *There exists a unique solution of problem* (1.1)-(1.7).

Uniqueness is well known [19] and is based on a comparison argument. To prove existence, set

$$I_1 = \{(x,b);\ x>0\}$$

and denote by Ω the domain bounded by $N \cup I_1$ and I_0. For any $\mu > 0$, set

$$\Omega_\mu = \Omega \cap \{x > -\mu\} \quad ,$$

$$K_\mu = \left\{ v \in H^{1,2}(\Omega_\mu);\ v = 0 \text{ on } I_0,\ v = 1 \text{ on } N \cup I_1,\ v = h_\mu(y) \text{ on } x = -\mu \right\},$$

where $h_\mu(y)$ is increasing, $h_\mu(0) = 0$, $h_\mu = Q$ at $N \cap \{x = -\mu\}$.

Consider the truncated functional

$$(1.8) \qquad J_{\lambda,\mu}(v) = \int_{\Omega_\mu} |\nabla v - \lambda I_{\{v<1\}\cap E} e|^2 dx\,dy$$

where $e = (0,1)$, $E = \{(x,y);\ -\infty < x < \infty,\ 0 < y < b\}$. If $\lambda > 1/b$ then we can construct a v_0 in K_μ with

$$v_0 = \min\{\lambda y, 1\} \quad \text{if} \quad x > 1 \quad .$$

For such v_0 , $J_{\lambda,\mu}(v_0) < \infty$. Consequently, there exists a minimizer $u_{\lambda,\mu}$ of

$$v \to J_{\lambda,\mu}(v) \quad , \quad v \in K_\mu \quad .$$

Lemma 1.2. *The minimizer is unique.*

Indeed, suppose u_1, u_2 are two minimizers and introduce $u_1^\varepsilon(x,y) = u_1(x-\varepsilon,y)$ and

$$v_1 = u_1^\varepsilon \wedge u_2 \ , \quad v_2 = u_1^\varepsilon \vee u_2 \quad .$$

Denote by J^ε the functional $J = J_{\lambda,\mu}$ corresponding to the translation $x \to x + \varepsilon$ of Ω_μ , K_μ. One verifies that

$$J^\varepsilon(u_1^\varepsilon) + J(u_2) = J^\varepsilon(v_1) + J(v_2) \quad ,$$

which implies that $J(u_2) = J(v_2)$, i.e., $u_1^\varepsilon \vee u_2$ is a minimizer. Consequently $u_1^\varepsilon \vee u_2$ is smooth, which implies that either $u_1^\varepsilon \geq u_2$ or $u_1^\varepsilon \leq u_2$ everywhere. Since $u_1^\varepsilon < u_2$ near the boundary, we deduce that $u_1^\varepsilon < u_2$ throughout the domain. Taking $\varepsilon \to 0$ we get $u_1 \leq u_2$, and similarly $u_2 \leq u_1$.

Taking $u_1 = u_2$ in the above argument we get:

$$(1.9) \qquad \frac{\partial}{\partial x} u_{\lambda,\mu} \geq 0 \quad .$$

Thus the analytic free boundary $\Gamma = \Gamma_{\lambda,\mu}$ has the form

(1.10) $\Gamma_{\lambda,\mu} : x = f_{\lambda,\mu}(y)$.

Next we have:

Lemma 1.3. $f_{\lambda,\mu}(y)$ *is continuous and finite if and only if* $h < y < b$, *where* $h = 1/\lambda$.

Lemma 1.4. $\lambda \to f_{\lambda,\mu}(b)$ *is continuous.*

Using the Lipschitz continuity estimate on $u_{\lambda,\mu}$ and non-degeneracy, we establish:

Lemma 1.5. *If* λ *is sufficiently large then* $f_{\lambda,\mu}(b) < 0$; *if* $\lambda > 1/b$ *and* $\lambda - 1/b$ *is small enough, then* $f_{\lambda,\mu}(b) > 0$.

From Lemmas 1.4, 1.5 we deduce that there is a value $\lambda = \lambda(\mu)$ such that $f_{\lambda,\mu}(b) = 0$, i.e., there is a "continuous fit" at A. For this value of λ , $(u_{\lambda,\mu}, \Gamma_{\lambda,\mu})$ "almost" solve the jet problem. In order to complete the construction of a solution we let $\mu \to \infty$, $\lambda(\mu) \to \lambda$ and denote the limiting $u_{\lambda,\mu}$, $\Gamma_{\lambda,\mu}$ by u, Γ. Then (1.1)-(1.6) are satisfied. Finally:

Lemma 1.6. *Continuous fit implies "smooth fit", i.e., it implies* (1.7).

The above outline of the proof of Theorem 1.1 is a special case of ACF [2] who actually considered the corresponding 3-dimensional axially symmetric jet problem, whereby one replaces

$$\Delta u \quad \text{by} \quad \mathcal{L}u \equiv \Delta u - \frac{1}{y} u_y \quad ,$$

and

$$\frac{\partial u}{\partial \nu} = \lambda \quad \text{by} \quad \frac{1}{y}\frac{\partial u}{\partial \nu} = \lambda \quad .$$

For this problem the classical methods do not apply.

The variational approach of ACF [2] has been extended to more difficult jet problems. We shall mention a few. In ACF [3] the plane asymmetric jet problem is considered, whereby the nozzle N is made up of two curves N_1 and N_2, given by $y = g_i(x)$ $(-\infty < x \leq 0)$ and $g_1(x) > g_2(x)$. Setting $A_i = (0, g_i(0))$, one seeks two curves Γ_1, Γ_2 of the form $\Gamma_i : y = f_i(x)$ $(0 \leq x < \infty)$ with $f_1(x) > f_2(x)$, such that,

$$f_i(0) = g_i(0) \quad ,$$

and, with G denoting the domain bounded by $N_1 \cup \Gamma_1$, $N_2 \cup \Gamma_2$,

$$-1 < u < 1 \ , \quad \Delta u = 0 \quad \text{in } G \quad ,$$

$$u = 1 \quad \text{on } \Gamma_1 \ , \quad u = -1 \quad \text{on } \Gamma_2 \quad ,$$

$$\left|\frac{\partial u}{\partial \nu}\right| = \lambda \quad \text{on } \Gamma_1 \cup \Gamma_2 \quad ,$$

$$\lim_{x \to \infty} f_1'(x) = \lim_{x \to \infty} f_2'(x) \quad ,$$

and both limits exist, and, finally, there is smooth fit at A_1 and A_2.

In this problem both λ and the direction of the jet at $x = \infty$ are not given, and are to be found together with u, Γ_1, Γ_2.

One works with the truncated functional

$$\int_{\Omega_\mu} |\nabla v - \lambda I_{\{|v| < 1\} \cap E} e^{\perp}|^2 dx\, dy$$

where $E = \{x > 0\}$, e is unspecified unit vector (e_1, e_2) with $e_1 > 0$, and $e^{\perp}$ is obtained by rotating e counterclockwise by $\pi/2$.

Notice that the present situation is a 2-parameter problem, with parameters λ, e; these must be chosen so as to achieve continuous (and smooth) fit at A_1 and A_2. For details see ACF [3].

In ACF [4] a jet in a gravity field is considered. Here, roughly speaking,

$$\left|\frac{\partial u}{\partial \nu}\right| = \lambda\sqrt{|y|}$$

on the free boundary, which means that we replace λ by $\lambda\sqrt{|y|}$ in the variational functional. An interesting "threshold" phenomenon occurs: If $\alpha \le 120^{\circ}$ then there exists a $Q_0 > 0$ such that a solution exists if and only if the flux Q is $> Q_0$ (for $\alpha > 120^{\circ}$ there exists a solution for any $Q > 0$); see Figure 1.

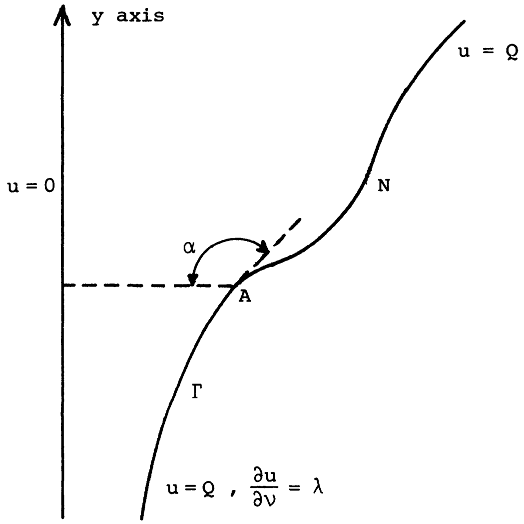

Figure 1

We turn to the axially symmetric cavity problem past a nose $N : x = g(y)$ $(0 \le y \le 1)$, with $g(y) < 0$, $g(0) = -a$. Set $I_- = \{(x,0),\ x \le -a\}$. We seek a curve Γ initiating at $(0,b)$ and lying in $\{x > 0,\ y \ge b\}$ such that, if G is the domain bounded by $N \cup \Gamma$ and I_-, then:

$$\begin{aligned} &Lu = 0,\quad u > 0 \quad \text{in } G, \\ (1.11)\qquad &u = 0 \quad \text{on } \partial G, \\ &\frac{1}{y}\cdot\frac{\partial u}{\partial \nu} = 1 \quad \text{on } \Gamma, \end{aligned}$$

$$\frac{\nabla u}{y} \to (0,1) \quad \text{if} \quad x^2+y^2 \to \infty, \ (x,y) \in G$$

and there is smooth fit at $(0,b)$.

This problem was solved by Caffarelli and Friedman [12] (see also [14] and [9]). One first considers a truncated problem, such as the one obtained by looking at the same problem in a channel of width $2H$, setting $u = Qy^2/2$ on the boundary of the channel. Then, with a suitable $Q = Q(H)$ one can achieve a solution with smooth fit. Finally one takes $H \to \infty$ and shows that $Q(H) \to 1$. This fact is used in order to prove the last condition in (1.11) for the limiting (u,Γ). In this manner we obtain a solution of (1.11) with smooth fit at $(0,b)$; further, for large x, Γ has the form

$$\Gamma : y = f(x) \quad \text{with} \quad f'(x) \to 0 \quad \text{if} \quad x \to \infty \ .$$

The above results have recently been extended in ACF [9] to compressible flows; the corresponding quasi-linear functional was studied in ACF [8].

We finally mention that axially symmetric cavities for rotational flow in a channel were studied by Friedman [15], and irrotational flow in a channel with oscillatory wall was studied by Friedman and Vogel [16].

PART II: TWO FLUID PROBLEMS.

Let $N_1 : y = H$, I_0 the x-axis, and $N_2 : y = g(x)$ $(-\infty < x \leq 0)$ with $0 < g(x) < H$, and set $A = (0,g(0))$. Consider the following problem: Find a curve Γ,

(2.1) $\quad \Gamma : y = f(x) \quad (0 \leq x < \infty)$, $f(0) = g(0)$, f continuous, $0 < f(x) < H$,

and a function u, $u = u_1$ in the domain Ω_1 bounded by N_1 and $N_2 \cup \Gamma$, $u = u_2$ in the domain Ω_2 bounded by I_0 and $N_2 \cup \Gamma$, such that the following is true:

$$\begin{cases} u_1 = Q \text{ on } N_1 \\ u_1 = u_2 = 0 \text{ on } N_2 \cup \Gamma \ , \\ u_2 = -1 \text{ on } I_0 \ , \\ 0 < u_1 < Q \text{ in } \Omega_1 \ , \\ -1 < u_2 < 0 \text{ in } \Omega_2 \ , \\ \Delta u_i = 0 \text{ in } \Omega_i \ ; \end{cases}$$

further,

(2.3) ∇u is bounded in $B_\delta(A)$, $N_2 \cup \Gamma$ has tangent at A

for some $\delta > 0$, and

(2.4) $|\nabla u_1|^2 - |\nabla u_2|^2 = \lambda$ on Γ .

Here λ is a constant which must be found together with u and Γ.

The jump relation (2.4) is the new feature of this problem. In ACF [5] the functional

$$J(v) = \int\left[|\nabla v|^2 + Q^2(x)\lambda^2(v)\right]dx$$

was introduced with

$$\lambda^2(v) = \begin{cases} \lambda_1^2 & \text{if } v < 0 \\ \lambda_2^2 & \text{if } v > 0 \end{cases}$$

and $\lambda_1^2 - \lambda_2^2 \neq 0$, with the idea of applying it to study 2-fluid problems, such as (2.1)-(2.4). It was proved that any minimizer is Lipschitz continuous, a non-degeneracy property holds, and the free boundary is in C^1 in case $n = 2$. These results are used in order to establish:

Theorem 2.1 (ACF [6]). *There exists a unique solution of the problem* (2.1)-(2.4), *and* Γ *is in* C^1.

To prove the theorem we introduce a truncated functional

$$J_{\lambda,\mu}(v) = \int_{\Omega_\mu} \left|\nabla v - \left(\lambda_2 I_{\{v<0\}} + \lambda_1 I_{\{v>0\}} + \lambda_0 I_{\{v=0\}}\right) I_{\{x>0\}} e\right|^2 dx\, dy$$

where $e = (0,1)$, $\Omega_\mu = \{-\mu < x < \infty , 0 < y < H\}$,

$$\lambda_2 = \frac{1}{h} , \lambda_1 = \frac{Q}{H-h} , \lambda = \lambda_2^2 - \lambda_1^2 , \lambda_0 = \min(\lambda_1, \lambda_2)$$

and introduce the admissible set

$$K_\mu = \left\{ v \in H^{1,2}(\Omega_\mu);\ v = -1 \text{ on } I_0,\ v = 0 \text{ on } N_1,\ -1 \le v \le 0 \text{ below } N_2, \right.$$
$$\left. 0 \le v \le Q \text{ above } N_2,\ v = h(y) \text{ on } x = -\mu \right\}$$

with $h(y)$ suitable monotone increasing function. The problem

$$v \to \min J_{\lambda,\mu}(v) , \quad v \in K_\mu$$

has a unique solution u with $u_y \ge 0$ (cf. Lemma 1.2). Thus the free boundary is a curve

$$\Gamma_{\lambda,\mu} : y = f_{\lambda,\mu}(x) .$$

Notice that h is the asymptotic height of the free boundary at $x = \infty$.

We now let h vary and show that for some value of h , say $h = h(\mu)$, $f_{\lambda,\mu}(0) = b$, that is, there is a continuous fit at A. We finally let $\mu \to \infty$ in order to complete the existence proof; for more details see ACF [6].

In principle, all the steps in the proof of Theorem 2.1 have their counterparts in the proof of Theorem 1.1, but they require a more refined analysis.

A significantly harder problem occurs when the upper fluid is not everywhere confined by $\{y = H\}$, that is, N_1 consists of $\{y = H, x \le 0\}$ and one seeks, in addition to Γ , another free boundary Γ_1 initiating at $(0,H)$ such that

$$\frac{\partial u_1}{\partial \nu} = \bar{\lambda} \quad \text{on} \quad \Gamma_1 \quad .$$

This is a 2-parameter problem (with parameters λ, $\bar{\lambda}$) and they must be chosen in such a way as to achieve a continuous fit at both A and (0,H).

This problem was solved in ACF [7]. The proof involves quantitative estimates on the location of the free boundaries Γ and Γ_1 , depending on the relations between λ and $\bar{\lambda}$. Some of these estimates are derived by comparison arguments; the flows used in the comparison are constructed by solving one-fluid jet problems.

Some of the techniques developed for 2-fluid jet problems can be applied to other 2-fluid problems. We briefly mention the 2-fluid dam problem (see Figure 2).

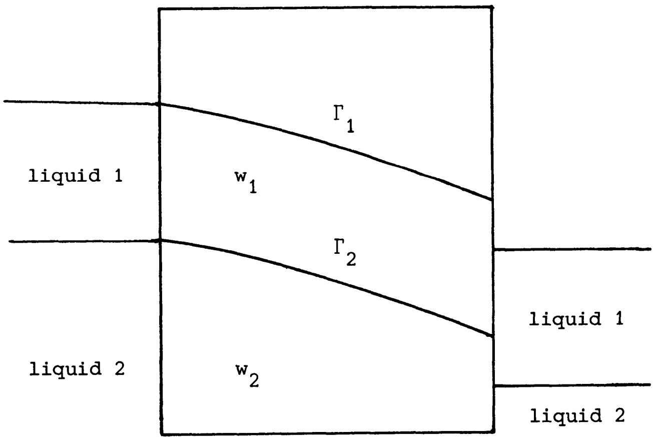

Figure 2

Denote by $\phi_i = p + \delta_i y$ the piezometric head of the i-th fluid and assume that $\delta_2 > \delta_1$. We introduce the stream function ψ (that is, $\phi - i\psi$ is holomorphic). Then in terms of ψ the problem reduces to

$$\Delta\psi = \frac{\partial}{\partial x} H(\psi) \quad \text{in the dam,}$$

where

$$H(t) = \begin{cases} 0 & \text{if } t < 0 , \\ -\delta_2 + \delta_1 & \text{if } 0 < t < Q_1 , \\ -\delta_2 & \text{if } t \geq Q_1 ; \end{cases}$$

here $\psi_2 = -Q_2$ at the bottom of the dam, and $\psi = Q_1$ at the top. On the top free boundary $\Gamma_{0,1}$:

$$\psi = Q_1 \quad \text{and} \quad \frac{\partial \psi_1}{\partial \nu} = \delta_1 \frac{\partial x}{\partial \nu} ,$$

and on the intermediate free boundary $\Gamma_{1,2}$:

$$\psi_i = 0 , \quad \frac{\partial \psi_1}{\partial \nu} - \frac{\partial \psi_2}{\partial \nu} = (\delta_1 - \delta_2) \frac{\partial x}{\partial \nu} .$$

ψ also satisfies suitable Dirichlet and Neumann conditions.

Theorem 2.2 (ACF [10]). *There exists a solution* $(\psi, \Gamma_{0,1}, \Gamma_{1,2})$ *of the 2-fluid dam problem with* $\Gamma_{0,1}$, $\Gamma_{1,2}$ *analytic and* C^1 *respectively, and*

$$\Gamma_{0,1} : y = f_1(x) , \quad \Gamma_{1,2} : y = f_2(x)$$

where $f_i(x)$ *are Lipschitz continuous and strictly monotone decreasing functions.*

Partial results are obtained in ACF [10] for general 2-dimensional dams.

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