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VARIATIONAL FORMULATION FOR THE FOKKER-PLANCK KINETIC EQUATION IN EQUILIBRIUM PROBLEMS

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1. INTRODUCTION.

Variational approaches have been widely employed in rarified gas dynamics [2, 4] and plasma dynamics [13,16,18,19] to investigate boundary-value problems for the linearized Boltzmann and Fokker-Planck equations. In both cases the interest of such methods lies in the possibility of evaluating macroscopic quantities of direct physical significance (fluxes through isobaric surfaces, wave frequencies and growth rates for linear "turbulent" perturbations, etc.). In fact, even in linear problems (i.e., for which it sufficies to consider linearized approximations of the previous kinetic equations), a good accuracy in estimating the effect of two-particle Coulomb collisions is difficult to achieve due to the complicate nature of the collision operator, even in its linearized version.

As far as what concerns the application to rarified gas dynamics variational methods have been found extremely satisfactory (for a review see Ref. [4]). In particular, Cercignani [2] developed a variational formulation for a non-self-adjoint Boltzmann equation showing how a trial-function technique could be adopted.

Analogous methods have been adopted in plasma kinetic theory to investigate both collisional transport problems in quiescent magnetoplasmas (i.e., in which turbulent perturbations are negligeable) [15,19] and linear stability problems in weakly turbulent systems [14,16]. For the first type of problems, Rosenbluth,

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Hazeltine and Hinton [15] were able to develop an "asymptotic theory", i.e. a variational theory based on an asymptotic estimate for the trial function which was determined only to lowest significant order with respect to a small parameter δ ($\delta = \langle (1 - B/B_{max})^{1/2} \rangle_S \ll 1$, with B_{max} the absolute maximum of B on a given isobaric surface S , with $\underline{B}$ the magnetic field, and the brackets "$\langle \; \rangle_S$" denoting an appropriately weighted average on the same isobaric surface). Their method was specifically intended for applications to a special class of hydromagnetic equilibria, i.e. those exibiting toroidal axisymmetry and subject to "ad hoc" assumptions on the relevant physical parameters which are known as "neoclassical ordering" [17]; it concerned, in addition, only so-called weakly-collisional magnetoplasmas, i.e. subject to the asymptotic condition $\rho_s = \nu_{s,eff}/\omega_b \ll 1$ (with $\nu_{s,eff}$ an effective collition frequency to be appropriately defined and ω_b the bounce or transit frequency characterizing the unperturbed particle motion long a magnetic flux line). A generalization of their technique, which - rather than on an asymptotic estimate for the trial function - is based on a consistent perturbative expansion w.r. to an appropriate adimensional parameter Δ ($\Delta = \delta|1 - \delta|^2$) has been more recently proposed by the present author [18] and its application to the investigation of collisional transport in a multi-species plasma has been worked out [19].

Similar techniques, on the other hand, were developed by Rosenbluth and other authors [16] for the linear stability analysis of a magnetoplasma subject to linear electrostatic perturbations of the drift type and in the presence of two-particle Coulomb collisions. In this case, a simplified model collision operator, i.e. a so-called pitch-angle scattering approximation, was, in particular, adopted for the construction of a variational principle for the linearized Fokker-Planck equation.

All such theories, apart Ref. [13] - where, however, a different ordering scheme is assumed for the linearized Fokker-Planck equation - refer to a magnetoplasma in the so-called weakly collisional regime, i.e. $\rho_s \ll 1$. On the contrary, investigations of collisional $(\rho_s \sim 0(1))$ or strongly collisional $(\rho_s >> 1)$ magnetoplasmas have not been based up to now, to the author's knowledge, on a systematic investigation able to encompass the various plasma "regimes" of possible interest. In particular, for the investigation of regimes of collisional or strongly collisional plasmas, respectively, a variety of approaches have been proposed which are founded either on hydrodynamical models [1], moment equations [8], expansions in orthogonal functions [6,10] and finally approaches based on the construction of model collision operators (see also Ref. [10] and references therein cited).

It is the purpose of this Note to propose a variational formulation of this type, i.e. applying to a magnetoplasma of "arbitrary collisionality", for which, more precisely, the adimensional parameter ρ_s is not requested to be small. In particular, we intend to show how the variational principle may be given in a form useful for the development of an approximate solution technique based on orthogonal function expansions. The method, here developed for a special example (a quiescent magnetoplasma), can be applied also to problems of weak turbulence (magnetoplasma subject to linear electromagnetic perturbations). It appears, therefore, of remarkable generality. In addition, since its accuracy seems to be limited, in principle, only by the choice of the (numbers of terms in the) approximating sequence for the solution, a highly accurate evaluation of the relevant macroscopic quantities should be possible. Such a method should be, for this reason, useful for actual applications to transport problems in plasma dynamics.

2. VARIATIONAL FORMULATION FOR THE STEADY FOKKER-PLANCK EQUATION.

We shall consider here the case of a quiescent magnetoplasma, namely a plasma in which turbulent perturbations are negligeable. For definitess, we consider a model analogous to that of Ref.s [18] and [19]. Thus we assume a magnetoplasma embedded in a toroidal and axisymmetric magnetic configuration and subject to the so-called neoclassical ordering [17]. In this case a straightforward Larmor-radius expansion for all the physically relevant quantities, and in particular for the one-particle distribution function of the s - th particle species $f_s(\underline{r},\underline{v},t)$ (with s = 1,n , being n the number of particle species present in the system), delivers for the first-order perturbation $\overline{f}_{1,s}$ the so-called drift Fokker-Planck equation:

$$(1) \qquad L_k(h) = F_k \quad (k = 1,r)$$

where L_k is the linear operator $L_k(h) = L_{Ak}h_k - C_k(f_o|h)$, with $L_{Ak} = v_{||}\hat{n}\cdot\nabla$, and the source term F_k reads $F_k = -\underline{v}_{D,k}\cdot\nabla f_{o,k} - L_{Ak}v_{||}G_k f_{o,k}E_{||}^{rot}$, with standard notations (see Ref. [19]). Thus, in particular, $\hat{n} = \underline{B}/B, v_{||} = \underline{v}\cdot\hat{n}, \underline{v} = \underline{v}_{\perp} + v_{||}\hat{n}, \underline{v}_{D,k}$ is the diamagnetic drift velocity of the species k and finally $C(f_o|h)$ is the linearized Fokker-Planck collision operator in the Landau form, i.e., with standard notations [18]:

$$(2) \qquad \begin{aligned} C_k(f_o|h) = \sum_{s=1,r} q_{ks}\frac{\partial}{\partial\underline{v}}\cdot\int d^3v'\frac{\partial^2 u}{\partial\underline{v}\partial\underline{v}}\cdot\Bigg\{&\frac{\partial}{\partial\underline{v}}h_s(\underline{v})f_{o,k}(\underline{v}') + \frac{\partial}{\partial\underline{v}}f_{o,s}(\underline{v})h_k(\underline{v}') - \\ &- \frac{m_k}{m_s}\frac{\partial}{\partial\underline{v}'}f_{o,k}(\underline{v}')h_s(\underline{v}) + \frac{m_k}{m_s}\frac{\partial}{\partial\underline{v}'}h_k(\underline{v}')f_{o,s}(\underline{v})\Bigg\} \quad ; \end{aligned}$$

Eq. (1) holds under the assumption of identifying the "equilibrium" distribution $f_{o,k}$ with a local maxwellian distribution constant on a given isobaric surface and subject to the condition of temperature equilibration $T_{o,k} = T_{o,s}$ for k,s = 1,r. Furthermore, h_s is related to the first-order perturbation $\overline{f}_{1,k}$, in terms of the equation $\overline{f}_{1,k} = h_k + v_{||}G_k f_{o,k}E_{||}^{rot}$, with G_k the function of Spitzer-Härm, solution of the equation:

$$(3) \qquad C_k(f_o|v_{||}f_oG) = -\frac{e_k}{T_{o,k}}v_{||}f_{o,k}$$

and $\underline{E}^{rot}$ the inductive part of the electric field $\underline{E}$.

Assuming, in addition, an equilibrium magnetic field $\underline{B} = \underline{B}_T + \underline{B}_P$ with toroidal axisymmetry (B_T and B_P being, respectively, the toroidal and poloidal components of $\underline{B}$) one obtains for $\overline{f}_{1,k}$ the decomposition $\overline{f}_{1,k} = g_k + g_k^{(D)}$, where

(4) $$L_k g_k^{(D)} = -\underline{v}_{D,k} \cdot \nabla f_{o,k}$$

with $\underline{v}_{D,k}$ the diamagnetic drift and a right-handed and orthogonal curvilinear coordinate system (θ,χ,ψ) (corresponding to the vectors $\underline{B}_T,\underline{B}_P$ and $\underline{B}_T x \underline{B}_P$) has been adopted.

From Eq. (1) an integral equation can immediately be obtained by performing an appropriate average on an isobaric surface, thus delivering the so-called integral drift equation:

(5) $$\langle \frac{B}{v_{||}} C_s(f_o|h) \rangle_{S(\lambda)} = 0$$

where for circulating particles ($0 \leq \lambda < 1/B_{max}$, with λ the pitch-angle variable $\lambda = 2\mu/v^2$ and $\mu = v_{\perp}^2/2B$ the magnetic moment per unit mass) $S(\lambda)$ coincides with a given isobaric surface S , while for trapped particles ($1/B_{max} \leq \lambda \leq 1/B$) is the subdomain of S where $v_{||} \geq 0$.

As usual, solutions of Eq.s (1) and (5) shall be seeked under standard boundary conditions (implied by periodicity on a given toroidal isobaric surface and, possibly, of boundness of $\overline{f}_{1,s}$ together with $\frac{\partial}{\partial \underline{v}} \overline{f}_{1,s}$) and regularity requests ($\overline{f}_{1,s}$ of class $C^2(\Omega_v)$, being $\Omega_v \equiv R_v^{(3)}$ the whole $\underline{v}$-space, and of class $C^1(\Omega_r)$ with Ω_r a torus in configuration space with boundary $\delta\Omega_r$, coinciding with the material wall of the discharge chamber). The validity of such regularity conditions is needed, in particular, to assure that the Fokker-Planck collision operator fulfills the usual conservation laws (namely, particle, momentum and kinetic energy conservation).

In order to obtain a local variational formulation [5] for the previous problems (i.e., either Eq. (1) or (5) equipped with the previous boundary and regularity conditions), we notice that, at first sight, they seem to require quite different mathematical approaches. In fact, the linear operator resulting in Eq.

(1) is - unlike for Eq. (5) - non-symmetric. Thus a variational formulation in the strict sense [21] for the first problem can only be given in the form of a "constrained" variational principle, namely by limiting appropriately the class of admissible variations, as shall now be clarified.

It is convenient to introduce the vector linear operator $L = \{L_1,...,L_r\}$ acting on the vector $k = \{k_1,...,k_r\}$. By defining the scalar product $\langle h|k\rangle$:

(6) $\langle h|k\rangle = \Sigma_s \left\langle \int d^3v\hat{h}_s k_s \right\rangle_S$ (with $\hat{h}_s = h_s/f_{o,s}$ and $\|h\| = (\langle h|h\rangle)^{1/2}$ the norm)

one obtains from Eq. (1)

(7) $\langle h|Lk\rangle = -\langle h|Ck\rangle + \langle h|L_A k\rangle + \langle h|L_A(g^{(D)} + v_{\|}Gf_oE_{\|}^{rot})\rangle$

where L and C are the linear operators $L = \{L_1,...,L_r\}$ and $C = \{C_1,...,C_r\}$ with:

(8) $C_k(h) = C_k(f_o|h) \quad (k = 1,r)$.

The two operators C and L_A result respectively symmetric and antisymmetric, in fact $\langle h|Ck\rangle = \langle Ch|k\rangle$, while $\langle h|L_A k\rangle = -\langle L_A h|k\rangle$.

On the other hand, a necessary condition for the existence of a variational formulation in the strict sense (Volterra theorem [22]) requires the symmetry of the operator involved in the equation (here L), and in addition - in order to make this condition sufficient - also the convexity of its domain of definition. It follows that a variational formulation for the problem related to Eq. (1), to be intended in the previous strict sense, can only be given provided the class of admissible variations corresponds to a convex domain: in other words the variations must be appropriately constrained.

In order to obtain a nontrivial solution to this problem it is necessary that the constraint equation(s) to be determined have mathematical (and obviously also physical) significance. By that we mean that the class of solutions thus determined from a constrained variational principle must coincide with that of the given problem. Although, in the author's opinion, there is no general rule for constructing such a type of variational formulation, it seems clear that the constraint equation(s) must be obtained constructing appropriate bilinear forms in terms of the given equation (i.e., Eq. (1)). Since h_s can be decomposed in terms of its odd and even parts w.r. to $v_{||}$, namely respectively Dh_s and Ph_s ($h_s = Dh_s + Ph_s$), the simplest bilinear forms which can be constructed in terms of Eq. (1) read:

$$(9) \qquad Q_{1,s}(h_s|h_s) = <\int d^3v\, D\hat{h}_s\{L_s Ph_s - C_s(f_o|Dh)\}> = 0$$

$$(10) \qquad Q_{2,s}(h_s|h_s) = <\int d^3v\, P\hat{h}_s\{L_s Dh_s - C_s(f_o|Ph) + L_s\{v_{||}G_s f_{o,s}E_{||}^{rot} + g_s^{(D)}\}> = 0$$

for each particle species, or simply:

$$(9') \qquad Q_1(h|h) = \Sigma_s Q_{1,s}(h_s|h_s) = 0$$

$$(10') \qquad Q_2(h|h) = \Sigma_s Q_{2,s}(h_s|h_s) = 0 \quad .$$

A possible choice for the variational functional turns out to be:

$$(11) \qquad W(h|h) = -<h|Ch> \quad .$$

Thus a variational principle for the previous problem, related to Eq. (1), is delivered - as previously pointed out by the author [20] - by the Euler equation:

$$(12) \qquad \delta_s W(h|h) = 0 \quad (s = 1,r)$$

where the variations are constrained by Eq.s (9') and (10').

We recall, in addition, that a variational formulation for the integral drift Fokker-Planck equation (Eq. (5)), is delivered by the previous Euler equation by constraining the class of admissible variations *only* with the equations:

$$(13) \qquad \hat{n} \cdot \nabla \delta h_s = 0 \quad (s = 1,r) \quad .$$

We mention that both variational principles are "minimal", in the sense that the solution of Eq. (1) (and analogously of the integral equation (5)) minimizes the functional $W(h|h)$ both w.r. to Dh_s and Ph_s.

3. EXPANSION IN ORTHOGONAL FUNCTIONS.

From the previous discussion it emerges clearly that in order to determine the (macroscopic) fluxes, which are needed for the macroscopic description of a plasma, the search of approximate solutions of the drift F - P equation can be limited to weakly convergent series expansions, in particular in the sense of energy convergence:

(14) $$\lim_{n\to\infty} \|u_n - h\|_C = 0$$

where $\{u_n, n \in N\}$ is some sequence of functions with finite energy, namely:

(15) $$\|u_n\|_C^2 = -\langle u_n | Cu_n \rangle < +\infty$$

and $\|u_n\|_C$ may be defined as the energy or the energy norm of u_n w.r. to the operator C (notice that C results positive definite in an appropriate functional class, from which are excluded local maxwellian distributions, since for them results $C(h) = 0$ identically).

Let us therefore introduce a set of "coordinate" functions $\{\phi_{i,s}(\underline{r},\underline{v}), i=1,n, s=1,r\}$, for each particle species, orthonormal in energy, in the sense:

(16) $$\|\phi_i\|_C^2 = 1 \quad (i = 1,n) \quad \text{and} \quad \langle \phi_i | C\phi_j \rangle = 0 \quad (\text{for } i,j=1,n \text{ with } i \neq j)$$

and complete in the sense of energy convergence w.r. to the operator C [11].

Approximate solutions in the form of expansions in terms of an orthonormal basis can be found adopting standard methods. In particular we look for a sequence $\{u_{n,s}(\underline{r},\underline{v}), n \in N\}$ of functions, defined in the same domain of $h_s(\underline{r},\underline{v})$ and fulfilling the same boundary conditions as well as the same regularity properties, which is "extremal" for $W(h|h)$ in the sense

(17) $$\lim_{n\to\infty} \|Du_n\|_C^2 = m \qquad \lim_{n\to\infty} \|Pu_n\|_C^2 = M$$

with

$$m = \inf \|Dh\|_C^2 \tag{18}$$
$$M = \inf \|Ph\|_C^2$$

and Du_n, Pu_n, as well as Dh and Ph, fulfill the constraint equations (11). By an appropriate restriction of the choice of equilibria (i.e., $\underline{B}$ and $f_{o,s}$) results in particular $m, M > 0$, thus the operator $-C$ is positive definite and bounded from below.

It is then immediate to prove that if Eq. (10) with constraints (11) has a solution, say h_o, with finite energy ($\|h_o\|_C^2 < +\infty$) any sequence $\{u_n(\underline{r},\underline{v}), n \in N\}$ which is extremal for $W(u_n|u_n)$ converges in energy to h_o. In fact results:

$$\inf \|Dh\|_C = \|Dh_o\|_C \tag{19}$$
$$\inf \|Ph\|_C = \|Ph_o\|_C \ .$$

Thus an energy convergent sequence can be constructed adopting a technique which resembles the wellknown Rayleigh-Ritz direct solution method. It can be obtained by introducing the sequences (for each particle species s):

$$Pu_{n,s} = \sum_{i=1,n} a_{Pi}\ \phi_{Pi,s}(\underline{r},\underline{v}) \tag{20}$$
$$Du_{n,s} = \sum_{i=1,n} a_{Di}\ \phi_{Di,s}(\underline{r},\underline{v}) \qquad (n \in N)$$

where $\{\phi_{Pi,s}(\underline{r},\underline{v}), i \in N, s=1,r\}$ and $\{\phi_{Di,s}(\underline{r},\underline{v}), i \in N, s=1,r\}$ are two orthonormal bases in energy, while the coefficients a_{Pi}, a_{Di} read:

$$a_{Pi} = \langle \phi_{Pi} | Pu_n \rangle \tag{21}$$
$$a_{Di} = \langle \phi_{Di} | Du_n \rangle$$

are chosen in such a way that $W(u_n|u_n)$ has a conditional extremum constrained by the equations:

$$Q_1(u_n|u_n) = \langle Du_n|LPu_n\rangle - \langle Du_n|CDu_n\rangle = 0 \tag{22}$$
$$Q_2(u_n|u_n) = \langle Pu_n|LDu_n\rangle - \langle Pu_n|CPu_n\rangle - \langle Pu_n|L\ v_{\|} GE_{\|}^{rot}\ f_o\rangle - \langle Pu_n|Lg^{(D)}\rangle = 0$$

and

(23) $$\hat{n}\cdot\nabla a_{Pi} = \hat{n}\cdot\nabla a_{Di} = 0 \quad (i = 1,n \text{ and } n \in N) \quad .$$

The unknown coefficients can thus be determined, adopting the method of Lagrange multipliers, yielding the system of linear algebraic equations:

(24) $$\begin{aligned} &\frac{\partial}{\partial a_{Pi}}\left\{W(u_n|u_n) + \lambda_1 Q_1(u_n|u_n) + \lambda_2 Q_2(u_n|u_n)\right\} = 0 \\ &\frac{\partial}{\partial a_{Di}}\left\{W(u_n|u_n) + \lambda_1 Q_1(u_n|u_n) + \lambda_2 Q_2(u_n|u_n)\right\} = 0 \end{aligned}$$

which has to be solved together with the system of Eq.s (22).

4. CONCLUSION.

An approximate solution method based on an expansion in othogonal functions has been worked out, and its application to the investigation of collisional transport has been mentioned.

Such a method seems sufficiently simple to require a relatively small amount of calculation and to assure, at least in the case of collisional or strongly collisional magnetoplasmas, a good accuracy for the computation of the relevant macroscopic quantities (i.e. fluxes, growth rates and real mode-frequencies). Thus it seems potentially useful for applications in plasma dynamics. These applications will be the object of subsequent papers.

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