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THE BEHAVIOR OF CAPILLARY SURFACES WHEN GRAVITY GOES TO ZERO

by L.-F. TAM (Stanford University)

1. PRELIMINARY.

Let Ω be a bounded (smooth or piecewise smooth) domain in R^n, $n \geq 2$. Consider the following boundary value problem:

$$(1.1)\qquad \begin{aligned} &\operatorname{div}(Tw) = \operatorname{div}\left(Dw/\sqrt{1+|Dw|^2}\right) = H + Bw \quad \text{in } \Omega \\ &Tw \cdot \nu = \cos\gamma \quad \text{on } \partial\Omega \end{aligned}$$

where $B > 0$, $\pi/2 > \gamma \geq 0$ are constants, $H = \dfrac{|\partial\Omega|}{|\Omega|}\cos\gamma$ and ν = outward normal of $\partial\Omega$.

The solution of (1.1) corresponds to capillary surface with gravity. We are interested in the behavior of w when gravity goes to zero, i.e. when B tends to zero. So we compare w with the solution of

$$(1.2)\qquad \begin{aligned} &\operatorname{div}(Tv) = H \quad \text{in } \Omega \\ &Tv \cdot \nu = \cos\gamma \quad \text{on } \partial\Omega \ . \end{aligned}$$

(1.2) may not have a solution. If (1.2) has a bounded solution, $\gamma > 0$ and Ω is smooth, then it is proved by Siegel in [18] that there exists a constant C which is independent of B such that $\sup_{\Omega}|w - v| \leq C \cdot B$ where v is the solution of (1.2) normalized by $\int_\Omega v\,dx = 0$.

In this paper we are going to investigate the case when Ω is piecewise smooth, the case when Ω is smooth but $\gamma = 0$ and the case when (1.2) has no solution. We shall use the idea of generalized solutions introduced by Miranda [17], see also Giusti [10].

It is known that if v is a bounded solution of (1.2) where H is replaced by any bounded measurable function $H(x)$, then v is a variational solution of

$$(1.3)\qquad \mathcal{F}(\Omega;v) = \int_\Omega \sqrt{1+|Dv|^2} + \int_\Omega H(x)v(x)\,dx - \cos\gamma\int_{\partial\Omega} v(x)\,dH_{n-1}$$

for $v \in BV(\Omega)$.

We introduce another functional:

$$(1.4)\qquad F(\Omega;U) = \int_{\Omega\times R} |D\chi_U| + \int_{\Omega\times R} H(x)\chi_U(x,t)\,dxdt - \cos\gamma\int_{\partial\Omega\times R} \chi_U(x,t)\,dH_n$$

where $U\subset\Omega\times R$ is a Caccioppoli set, χ_U is the characteristic function of U.

In (1.3) and (1.4) we do not assume Ω to be bounded.

Definition 1.1. $U\subset\Omega\times R$ is said to be a solution of (1.4) if and only if for any compact set K in R^{n+1} and any Caccioppoli set V of $\Omega\times R$ such that $\mathrm{spt}(\chi_U-\chi_V)\subset K$, then $F_K(\Omega;U) \le F_K(\Omega;V)$ where

$$(1.5)\qquad F_K(\Omega;W) = \int_{\Omega\times R\cap K} |D\chi_W| + \int_{\Omega\times R\cap K} H(x)\chi_W(x,t)\,dxdt - \cos\gamma\int_{\partial\Omega\times R\cap K} \chi_W(x,t)\,dH_n$$

We also introduce two subsidiary functionals:

$$(1.6)\qquad G_1(\Omega;A) = \int_\Omega |D\chi_A| + \int_\Omega H(x)\chi_A(x)\,dx - \cos\gamma\int_{\partial\Omega}\chi_A(x)\,dH_{n-1}$$

and

$$(1.7)\qquad G_2(\Omega;A) = \int_\Omega |D\chi_A| - \int_\Omega H(x)\chi_A(x)\,dx + \cos\gamma\int_{\partial\Omega}\chi_A(x)\,dH_{n-1}$$

for $A\subset\Omega$. Solutions of (1.6) and (1.7) are defined similarly.

Definition 1.2. A function $u:\Omega\to[-\infty,+\infty]$ is a generalized solution of (1.3) if its subgraph $U = \{(x,t)\in\Omega\times R \mid t<u(x)\}$ is a solution of (1.4).

Theorem 1.1. *Let* Ω *be a bounded piecewise smooth domain, and* $u\in BV(\Omega)$*, then* u *is a solution of* (1.3) *if and only if* u *is a generalized solution of* (1.3).

2. CASE WHEN Ω IS PIECEWISE SMOOTH.

In this section we make the following assumptions:

(2.1) Ω is a bounded piecewise smooth domain in R^2 ;

(2.2) let $2 \cdot \bar{\alpha}$ = minimum of interior angles of Ω , then $\pi/2 - \gamma < \bar{\alpha} < \pi/2$;

(2.3) (1.2) has a bounded solution v which is normalized by $\int_\Omega v(x)dx = 0$.

We also assume $0 < B < 1$.

Theorem 2.1. *Under the above assumptions, there exists a constant* C *which is indipendent of* B *such that*

(2.4) $$\sup_\Omega |w - v| \le C \cdot B \quad .$$

Before we prove the theorem, we have several lemmata. In what follows C_i will denote constants independent of B.

Lemma 2.1. *There is a constant* C_1 *such that*

(2.5) $$|w| \le C_1 \quad .$$

Proof. Use comparison principle as in [18].

q.e.d.

The next crucial step is to obtain a uniform bound for the gradients of w and v. If Ω is smooth, then it immediately follows from [7]. If Ω is only piecewise smooth, then by [7], [13] and [20] we can always get uniform bound for the gradients away from the corners. So it remains to find a bound near the corners. Without loss of generality we may assume a corner is at $(0,0)$ and near it Ω consists of two segments on $\Theta = -\alpha$ and $\Theta = \alpha$. Let $\bar{w} = w$ + constant such that $(0,0,0) \in R^3$ belongs to the closure of the graph of $\bar{w}$. Here w is a solution of (1.1) or (1.2).

Lemma 2.2. *Let* $\bar{U}$ *be the subgraph of* $\bar{w}$. *There exists constants* $C_2 > 0$ *and* $R_o > 0$ *which are independent of* B , *such that for any* $(x_o,t_o) \in \bar{\Omega} \times R$ *and let* $C_r(x_o,t_o) = \{(x,t) \in R^3 \mid |x - x_o| < r$ *and* $|t - t_o| < r\}$ *the following are true:*

(1) *if* $|\bar{U}_r| = |C_r(x_o,t_o) \cap \bar{U}| > 0$ *for all* $r > 0$ *then* $|U_r| \geq C_2 r^3$ *for all* $r \leq R_o$;

(2) *if* $|\bar{U}'_r| = |C_r(x_o,t_o) - \bar{U}| > 0$ *for all* $r > 0$ *then* $|\bar{U}'_r| \geq C_2 r^3$ *for all* $r \leq R_o$.

Lemma 2.3. *There exists a constant* C_3 *such that*

(2.6) $|Dw| \leq C_3$, *where* w *is the solution of* (1.1) *or* (1.2).

Proof. Take any sequence $B_k \geq 0$ (not necessarily distinct) and take any sequence of positive numbers $\varepsilon_k > 0$. Let

$$\bar{w}_{k,\varepsilon_k} = \frac{1}{\varepsilon_k} \bar{w}_k(\varepsilon_k x)$$

where w_k is the solution corresponding to B_k. We can then find a subsequence of $\bar{w}_{k,\varepsilon_k}$ which tends to a generalized solution u of (1.3) with $H(x) \equiv 0$ in the domain

$$\Omega_\infty = \lim_{k\to\infty} \frac{1}{\varepsilon_k} \Omega \quad .$$

Let $P = \{x \mid u(x) = \infty\}$ and let $N = \{x \mid u(x) = -\infty\}$. Then P is a solution of $G_1(\Omega_\infty, A)$ with $H(x) \equiv 0$. Use assumption (2.2) we can prove that $P = \phi$ or Ω_∞. By Lemma 2.2 we conclude that $P = \phi$. Similarly $N = \phi$. From these and lemma 2.2 we can prove that $\bar{w}_{k,\varepsilon_k}$ are uniformly bounded in $\{x \in \Omega_\infty \mid 1 \leq |x| \leq 2\}$ if k is large enough. From [7], [13] and [20], the lemma follows.

q.e.d.

Now we can proceed as in [18] to get a proof of Theorem 2.1.

3. CASE WHEN $\gamma = 0$.

Let Ω be a smooth domain in R^n , $n \geq 2$. If $\gamma = 0$, solution of (1.2) may not exist, or may exist but fail to be bounded. See [9]. Suppose (1.2) has a solution v , then we have the following

Theorem 3.1. *Either* (1) $v \in L^1(\Omega)$ *and* $\lim_{B\to 0} w = v + C$ *in* Ω *for some constant* C ; *or*

(2) $v \notin L^1(\Omega)$ *and* $\lim_{B \to 0} w = -\infty$ *in* Ω .

The proof of Theorem 3.1 is obtained by using the idea of generalized solution and comparison principle.

Theorem 3.2. *We can find a function* $C(B)$ *such that* $\lim_{B \to 0} (w + C(B)) = v$ *in* Ω.

The proof of Theorem 3.2 is also obtained by using the idea of generalized solution and the following lemma.

Lemma 3.1. *For any* $B_k \to 0$, *we can find a subsequence* B_{k_j} *such that* $\lim_{j \to \infty} B_{k_j} w_{k_j} = 0$, *where* w_k *is the solution of* (1.1) *corresponding to* B_k.

Corollary 3.1. $\lim_{B \to 0} Dw = Dv$ *in* Ω.

Note that all convergences are uniform in compact subset of Ω.

4. CASE WHEN (1.2) DOES NOT HAVE A SOLUTION.

We make the following assumptions:

(3.1) Ω is a piecewise smooth domain in R^2 such that every interior angle 2α satisfies $\pi/2 > \alpha \geq \pi/2 - \gamma$;

(3.2) Ω satisfies internal sphere condition for some radius $\delta > 0$ and angle γ in the sense of [6];

(3.3) $G_1(\Omega;A) \geq 0$ for all $A \subset \Omega$ where $H(x) \equiv H$, and there is a unique set P such that $P \neq \phi$ or Ω and $G_1(\Omega;P) = 0$.

Lemma 3.1 is still true in this case and we have:

Theorem 4.1. *There are functions* $C_1(B)$ *and* $C_2(B)$ *such that:*

(1) $w + C_1(B)$ *tends to a classical solution of* $\operatorname{div}(Tu) = H$ *in the interior of* N *and tends to positive infinity in the interior of* P ;

(2) $w + C_2(B)$ *tends to a classical solution of* $\operatorname{div}(Tu) = H$ *in the interior of* P *and tends to negative infinity in the interior of* N.

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