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THE FREE BOUNDARY OF A SEMILINEAR ELLIPTIC EQUATION

by D. PHILLIPS (Purdue University)

1. INTRODUCTION.

We consider the Dirichlet problem

$$(1.1)\qquad \begin{aligned} \Delta u &= \lambda f(u) \quad \text{in } \Omega \quad (\lambda > 0)\\ u &= 1 \quad \text{in } \partial\Omega \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^{2+\bar{\alpha}}$, $f(s) = 0$ if $s \leq 0$, $f(s) = s^p\hat{f}(s)$ if $s > 0$ $(0 < p < 1)$ with $\hat{f} \in C^2([0,\infty))$ and $0 < m \leq \hat{f} \leq M < \infty$.

A solution, u_λ, satisfies $0 \leq u_\lambda < 1$ in Ω. Our objective is to study the geometric structure and location of the level set $\{u_\lambda = 0\} \equiv N_\lambda$.

The motivation for this work stems from the theory of reaction-diffusion in porous pellets. The region Ω plays the role of the pellet which is partially comprised of a catalyst. The function, u, represents the concentration of a gaseous impurity normalized to 1 on the boundary. And the parameter λ is a modeling coefficient related to the reaction rate.

The pellet is a filter in the sense that the impurity reacts with the catalyst and is removed as the total fluid passes through. As a result N_λ is the portion of the pellet not utilized (see [1]).

One of our main results is the following:

Theorem 1. *If* Ω *is a two-dimensional convex domain and* f *satisfies*

$$(1.2)\qquad f'(s) + \frac{f(s)}{1-s} > 0 \quad \textit{for} \quad 0 < s \leq 1$$

then (1.1) *has a unique solution for each* $\lambda \geq 0$. *Moreover there is a* $\lambda_* > 0$ *so that*

$$N_\lambda = \phi \qquad \textit{if} \quad \lambda < \lambda_*$$

$$N_\lambda \ \textit{is a single point if} \quad \lambda = \lambda_*$$

$$N_\lambda \ \textit{is a convex domain if} \quad \lambda > \lambda_* \ .$$

The process is assumed to be isothermal. This allows us to consider one equation as opposed to a coupled system in u and a temperature field t.

A model found in the literature [1] for isothermal processes has

$$f(s) = s^p \exp\left(\frac{-\nu}{\beta + 1 - \beta s}\right) \quad (\nu, \beta \ \text{constants}, \ \nu > 0, \ \beta > 0 \ ).$$

This equation for f satisfies (1.2) for a certain range of the parameters ν and β.

The research described in this paper has been done jointly with Avner Friedman [3].

2. EXISTENCE AND UNIQUENESS RESULTS.

From the structure of $f(s)$ and the boundary conditions one can show that classical solutions to (1.1) exist ($C^{2+\alpha}$, $\alpha = \min(\bar{\alpha}, p)$). This can be done by minimizing the functional

$$(2.1) \qquad J_\lambda(v) = \int_\Omega \left(\frac{1}{2}|\nabla v|^2 + \lambda F(v)\right)dx$$

$$F(s) = \int_0^s f(t)\,dt \ , \quad \text{subject to the b.c.} \quad v = 1 \quad .$$

With $\Delta u \geq 0$, from the maximum principle we get $u < 1$ in Ω. And since $f(s) = 0$ for $s \leq 0$, $0 \leq u$. As $f(s)$ is only Hölder continuous near $s = 0$ (not Lipschitz), N may be nontrivial for a particular solution.

The function $f(s)$ is not assumed to be increasing and as a result (1.1) has in general more than one solution. Nonetheless (1.2) is a sufficient condition for uniqueness to the n-dimensional problem. This was shown in [2] and in [3] using a different argument.

3. COMPARISON THEOREMS.

Although a number of our results pertain to an arbitrary solution of (1.1), we are mainly interested in solutions that are elements of a family for which we have comparison theorems. Such families are minimums of (2.1), maximal, and minimal solutions of (1.1).

<u>Definition</u>. A solution $\overline{u}_\lambda$ of (1.1) is a *maximal solution* if for any other solution u_λ, $u_\lambda(x) \leq \overline{u}_\lambda(x)$ in Ω.

<u>Theorem 2</u>. *Let* $\lambda_1 < \lambda_2$ *and* $\overline{u}_{\lambda_i}$ $(\underline{u}_{\lambda_i})$ *be respective maximal (minimal) solutions. Then*

$$\overline{u}_{\lambda_2} < \overline{u}_{\lambda_1} \quad \text{on} \quad \{\overline{u}_{\lambda_2} > 0\} \quad ,$$

$$\underline{u}_{\lambda_2} < \underline{u}_{\lambda_1} \quad \text{on} \quad \{\underline{u}_{\lambda_2} > 0\} \quad ,$$

and $\{\overline{u}_{\lambda_1} = 0\}$ $(\{\underline{u}_{\lambda_1} = 0\})$ *is contained in the interior of*

$$\{\overline{u}_{\lambda_2} = 0\} \quad (\{\underline{u}_{\lambda_2} = 0\}) \quad .$$

An analogous comparison result is true for minimizers of (2.1) as well.

The advantage of considering a branch of solutions with the monotonicity relations above is that if $\lambda_1 < \lambda < \lambda_2$ then

$$N_{\lambda_2} \subset\subset N_\lambda \subset\subset N_{\lambda_2} \quad . \tag{3.1}$$

Thus we are able to use N_{λ_2} and N_{λ_1} as barriers to obtain properties of ∂N_λ.

4. ASYMPTOTIC ESTIMATES.

For large values of the parameter λ the reaction occurs near $\partial\Omega$. We have the following theorem.

Theorem 3. *There are positive constants* λ_0, γ_0, *and* c *so that if* $\lambda \geq \lambda_0$ *and* u_λ *is any solution to* (1.1) *then*

$$\left\{x \in \Omega \,\middle|\, \mathrm{dist}(x,\partial\Omega) > \frac{\gamma_0}{\sqrt{\lambda}} + \frac{c}{\lambda}\right\} \subset N_\lambda \subset \left\{x \in \Omega \,\middle|\, \mathrm{dist}(x,\partial\Omega) > \frac{\gamma_0}{\sqrt{\lambda}} - \frac{c}{\lambda}\right\} . \tag{4.1}$$

Moreover if $x = h(t)$ is a local parameterization of $\partial\Omega$ and $\nu(t)$ is the normal pointing into Ω then there is a function $K(t)$ (depending on u_λ) so that ∂N_λ can be represented in the form

$$x = h(t) + K(t)\nu(t) \quad , \tag{4.2}$$

with

$$\left|K(t) - \frac{\gamma_0}{\sqrt{\lambda}}\right| < \frac{c}{\lambda} \quad ,$$

$$|K(t)|_{C^{1+\delta}} < c \quad .$$

5. CONVEXITY PROPERTIES.

The function u_λ grows away from the level set N_λ at a prescribed rate. In particular if we set

$$g(u_\lambda(x)) = \int_0^{u_\lambda(x)} \frac{ds}{\sqrt{2F(s)}} \ , \ F(s) = \lambda \int_0^s f(t)\,dt$$

then $g(u_\lambda(x))$ is Lipschitz in Ω and in a weak sense

$$|\nabla(g(u_\lambda))| = 1 \quad \text{on} \quad \partial N_\lambda \quad .$$

Moreover if Ω is convex one can show that (see [4])

$$|\nabla g(u_\lambda)| \leq 1 \quad \text{in} \quad \Omega \quad . \tag{5.1}$$

And with this it can be shown that

$$\Delta g(u_\lambda) \geq 0 \quad \text{in} \quad \Omega \quad .$$

Now let K be a smooth subdomain of Ω with $H^{n-1}(\partial K \cap \partial N_\lambda) = 0$.

We can apply Green's Theorem to $K \cap \{u_\lambda > 0\}$. We get

$$0 \leq \int_{K \cap \{u_\lambda > 0\}} \Delta g(u_\lambda) = -\int_{\partial N_\lambda \cap K} 1 dH^{n-1} + \int_{K \cap \{u_\lambda > 0\}} \nabla g(u_\lambda) \cdot \nu dH^{n-1} \quad .$$

Using (5.1) we find that

$$H^{n-1}(\partial N_\lambda \cap K) \leq H^{n-1}(\partial K \cap \{u_\lambda > 0\}) \quad .$$

That is ∂N_λ minimizes surface area subject to variations on oneside. Such a boundary can be considered a surface of generalized positive mean curvature. When Ω is two-dimensional this becomes a statement of convexity.

We have the following theorem:

<u>Theorem 4</u>. *If* Ω *is convex,* $\Omega \subset \mathbb{R}^2$, *and* u_λ *any solution to* (1.1) *then each component of* N_λ *with nonempty interior is a convex domain.*

We can now describe the basic idea behind theorem 1. From section 1 (assuming (1.2)) the map, $\lambda \to u_\lambda(x)$, is well defined. Using (3.1) we see the sets, N_λ, are nested. One then must show that the map, $\lambda \to N_\lambda$, deforms continuously. Using this and the fact that for $\lambda \geq \lambda_0$, N_λ is a convex domain (theorems 3 and 4) one can show that N_λ is convex as long as $\text{int}(N_\lambda) \neq \phi$. Setting

$$\lambda_* = \inf\{\lambda \,|\, \text{int}(N_{\tilde{\lambda}}) \neq \phi \quad \text{for} \quad \tilde{\lambda} > \lambda\}$$

one then proves that N_{λ_*} consists of one point.

Analogous theorems are shown for the Robins condition

$$\frac{\partial u}{\partial \nu} + \mu(u-1) = 0 \quad \text{on} \quad \partial\Omega \quad (\mu > 0)$$

instead of $u = 1$ on $\partial\Omega$.

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